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Continuous Monotonic Orderings in Euclidean n-Space[1]

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(Note: This paper is to be submitted to a mathematical journal. The economic significance of the results will be discussed in a separate discussion paper. Your comments are appreciated.)

1. Assumptions and Definitions

The problem discussed in this paper arises in connection with the economic problem of determining individuals' tastes by means of questionnaires.

We consider a (weak) ordering relation R on the non-negative orthant of a Euclidean n-space. The symbols $\underline{x}$, $\underline{y}$, etc. will denote points of this orthant; x_i will denote a coordinate of $\underline{x}$. Then $x_i \geq 0$.

Postulate I. If $\underline{x}\ R\ \underline{y}$ and $\underline{y}\ R\ \underline{z}$, $\underline{x}\ R\ \underline{z}$. (Transitivity)

Postulate II. For all $\underline{x}$ and $\underline{y}$, either $\underline{x}\ R\ \underline{y}$ or $\underline{y}\ R\ \underline{x}$. (Connexity)

It is not exluded that $\underline{x}\ R\ \underline{y}$ and $\underline{y}\ R\ \underline{x}$ both hold for some $\underline{x} \neq \underline{y}$.

Definition 1. $\underline{x}\ I\ \underline{y}$ (read, "$\underline{x}$ is indifferent to $\underline{y}$") means $\underline{x}\ R\ \underline{y}$ and $\underline{y}\ R\ \underline{x}$.

Definition 2. $\underline{x}\ P\ \underline{y}$ (read, "$\underline{x}$ is preferred to $\underline{y}$") means $\underline{x}\ R\ \underline{y}$ and not $\underline{y}\ I\ \underline{x}$.

The following lemma sums up some self-evident consequences of the above postulates and definitions.

Lemma 1. (a) For all $\underline{x}$ and $\underline{y}$, one and only one of the following holds: $\underline{x}\ P\ \underline{y}$, $\underline{x}\ I\ \underline{y}$, $\underline{y}\ P\ \underline{x}$.

(b) For all $\underline{x}$, $\underline{x}\ I\ \underline{x}$; if $\underline{x}\ I\ \underline{y}$, then $\underline{y}\ I\ \underline{x}$; if $\underline{x}\ I\ \underline{y}$ and $\underline{y}\ I\ \underline{z}$, then $\underline{x}\ I\ \underline{z}$.

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Certain other properties will be demanded of the relation R.

Postulate III. If $x_i \geq y_i$ for all i and $x_j > y_j$ for some j, then $\underline{x}$ P $\underline{y}$. (Monotonicity)

Under the assumption of Postulate III, the ordering R in a one-dimensional space simply coincides with the usual ordering of real numbers; hereafter, we will assume $n \geq 2$.

Postulate IV. If $\underline{x}$ R $\underline{y}$ and $\underline{y}$ R $\underline{z}$, then there is a real number t such that $0 \leq t \leq 1$ and $\underline{y}$ I $t\underline{x} + (1-t)\,\underline{z}$. (Continuity)

By a hyperplane, we shall mean a set on which one or more coordinates are fixed and all others are allowed to vary freely. In particular, a plane is a hyperplane on which n - 2 coordinates are fixed. The following notation will be employed: P_i (a) is the hyperplane defined by the condition $x_i = a$; $P_{ij}(a, b)$ is the (n-2)-dimensional hyperplane on which $x_i = a$ and $x_j = b$.

It will be shown that the above postulates so restrict the possible orderings that a knowledge of the ordering of the points on each plane separately suffices to determine the ordering in the space as a whole.

Lemma 2. Postulates I-IV hold on each hyperplane; hence, all theorems derived for the entire space will hold equally well on each hyperplane.

This is obvious for Postulates I-III. For Postulate IV, it need only be noted that if $\underline{x}$ and $\underline{z}$ lie in a given hyperplane, so does $t\underline{x} + (1-t)\,\underline{z}$.

By an indifference set I is meant the set of all points indifferent to a given point.

Lemma 3. Any two points in I are indifferent to each other; if $\underline{x}$ is not in I, either x is preferred to every point of I or every point of I is preferred to $\underline{x}$.

Lemma 3 follows immediately from Lemma 1.

If A and B are sets, $A \cap B$ will mean their intersection. We will define A_i to be the set of real numbers a for which $P_i(a) \cap I$ is non-null; i.e., A_i is the projection of I on the x_i-axis.

As indicated by Lemma 2, we will sometimes be interested in considering the restricted space of a hyperplane; in particular, attention will be paid to the hyperplanes $P_n(a)$. $P_n(a) \cap I$ is an indifference set in $P_n(a)$ and has all the properties attributed to indifference sets by Lemma 3. Define $A_i(b)$ to be the A_i on the restricted space $P_n(b)$; i.e., $A_i(b)$ is the set of real numbers a for which $P_{in}(a, b)$ intersects I.

Finally define a_i' to be inf A_i, a_i to be sup A_i, $a_i'(b)$ to be inf $A_i(b)$, $a_i(b)$ to be sup $A_i(b)$. These quantities will be considered to be undefined if A_i or $A_i(b)$ is empty; a_i and $a_i(b)$ will be permitted to take on the values $+\infty$. Clearly,

$$a_i \geq 0, \ a_i(b) \geq 0. \tag{1}$$

2. Some Lemmas. A number of properties of the indifference and preference relations will be established in order to prove the main results of this paper. Many of these lemmas can be generalized.

Lemma 4. If $a_1 < a_2 < a_3$ and I intersects both $P_i(a_1)$ and $P_i(a_3)$, then I intersects $P_i(a_2)$.

Proof: Let $\underline{x}$ belong to $I \cap P_i(a_1)$, $\underline{x}'$ to $I \cap P_i(a_3)$. Define $\underline{y}$ and $\underline{y}'$ as follows: $y_j = x_j$ $(j \neq i)$, $y_i = a_2$, $y_j' = x_j'$ $(j \neq i)$, $y_i' = a_2$. As $a_1 < a_2 < a_3$, $\underline{y}\ P\ \underline{x}$, and $\underline{x}'\ P\ \underline{y}'$. But $\underline{x}$ and $\underline{x}'$ both belong to I; by Lemma 3, $\underline{x}\ I\ \underline{x}'$. By Lemma 1(c), $\underline{x}\ P\ \underline{y}'$; hence, by Definition 2, $\underline{y}\ R\ \underline{x}$ and $\underline{x}\ R\ \underline{y}'$. There is a t such that $\underline{z} = t\underline{y} + (1-t)\underline{y}'$ is indifferent to $\underline{x}$ (Postulate IV). As $\underline{z}$ is indifferent to an element of I, $\underline{z}$ must belong to I (Lemma 3). But $z_i = ty_i + (1-t)y_i' = a_2$. Hence, z belongs to both I and $P_i(a_2)$. Q.E.D.

Corollary 1. A_i is an interval.

Proof: If A_i contains no elements or one element, it is an interval by definition. If A_i contains two numbers a_1 and a_2, it contains all intermediate numbers by Lemma 4.

Corollary 2. If $b_1 < b_2 < b_3$, and $a_i(b_1)$ and $a_i(b_3)$ are defined, then $a_i(b_2)$ is defined.

Proof: $a_i(b)$ is defined if and only if $A_i(b)$ is non-null and therefore if and only if $P_n(b)$ intersects I. The corollary follows by letting $i = n$ in Lemma 4.

Lemma 5. If $n = 2$, the following properties hold for I:

(a) There is a function $f(a)$ defined over A_1 such that the set of points $[a, f(a)]$ is precisely I;

(b) $f(a)$ is strictly decreasing;

(c) I is a closed set;

(d) $f(a)$ is continuous;

(e) if $a_1 < +\infty$, then the point $(a_1, 0)$ belongs to I.

Proof (a) If a belongs to A_1, there is a point $\underline{x}$ belonging to I with $x_1 = a$. Suppose there is another point $\underline{x}'$ with the same properties. Then $\underline{x}$ I $\underline{x}'$, as both belong to I(Lemma 3). But $x_1 = x_1' = a$; if $x_2 \neq x_2'$, then $\underline{x}$ P $\underline{x}'$ [or $\underline{x}'$ P $\underline{x}$] (Postulate III), which is impossible by Lemma 1(a). Hence, there is one point (a, x_2) in I. Define $x_2 = f(a)$.

(b) If $a < a'$ and $f(a) \leq f(a')$, $(a', f(a'))$ P $(a, f(a))$ by Postulate III, contrary to the definition of $f(a)$.

(c) Let $\underline{x}$ be a limit point of I. There is a sequence $\underline{x}^{(k)}$ of points of I such that $\lim \underline{x}^{(k)} = \underline{x}$. Consider the three subsequences formed by requiring $x_1^{(k)} < x_1$, $x_1^{(k)} = x_1$, $x_1^{(k)} > x_1$, respectively. Since these three subsequences exhaust the original sequence, at least one of them must be infinite. Hence, we may say that there is a sequence $\underline{x}^{(k)}$ approaching $\underline{x}$ such that one of the following conditions hold: $x_1^{(k)} < x_1$, for all k; $x_1^{(k)} = x_1$ for all k; or $x_1^{(k)} > x_1$ for all k.

(1) $x_1^{(k)} < x_1$ for all k: For each k, either $\underline{x}$ R $\underline{x}^{(k)}$ or $\underline{x}^{(k)}$ R $\underline{x}$. These define two subsequences, at least one of which must be infinite. Hence, in this case, there is a sequence $\underline{x}^{(k)}$ approaching $\underline{x}$ such that $x_1^{(k)} < x_1$ for all k and such that one of the following conditions holds for all k: $\underline{x}$ R $\underline{x}^{(k)}$ or $\underline{x}^{(k)}$ R $\underline{x}$.

(i) $\underline{x}$ R $\underline{x}^{(k)}$ for all k: By part (a) of this lemma, $x_2^{(k)} = f(x_1^{(k)})$. Since $\underline{x}^{(k)}$ approaches $\underline{x}$,

$$\lim_{k \to \infty} f(x_1^{(k)}) = x_2. \tag{2}$$

Suppose for some m, $x_2^{(m)} < x_2$. As $x_1^{(m)} < x_1$, $x_2^{(k)} = f(x_1^{(k)}) \leq x_2^{(m)}$ for all k such that $x_1^{(m)} < x_1^{(k)} < x_1$, since f is monotonic by part (b) of this lemma; hence, $\lim_{k \to \infty} x_2^{(k)} \leq x_2^{(m)} < x_2$, contrary to (2). Therefore,

$$x_2^{(k)} \geq x_2 \text{ for all k.} \tag{3}$$

From (3), it follows that $\underline{x}^{(k)}$ R $(x_1^{(k)}, x_2)$. As $\underline{x}$ R $\underline{x}^{(k)}$ by assumption, there is a number $y_1^{(k)}$ such that

$$x_1^{(k)} \leq y_1^{(k)} \leq x_1 \tag{4}$$

and $(y_1^{(k)}, x_2)$ is indifferent to $\underline{x}^{(k)}$ (Postulate IV) and therefore belongs to I. By part (a) of this lemma, if we interchange the two coordinates, there is at most one number y_1 such that (y_1, x_2) belongs to I. Therefore, $y_1^{(k)} = y_1$ for all k. But, if k approaches infinity, $y_1^{(k)}$ approaches x_1 by (4). Hence, $y_1 = x_1$, and $\underline{x}$ belongs to I.

(ii) $\underline{x}^{(k)}$ R $\underline{x}$ for all k: the proof follows the same lines, using the point $(x_1, x_2^{(k)})$ instead of the point $x_1^{(k)}, x_2)$.

(2) $x_1^{(k)} = x_1$ for all k: Then $x_2^{(k)} = f(x_1^{(k)}) = f(x_1)$, a constant, for all k. By (2), $x_2 = f(x_1)$, so that $\underline{x}$ belongs to I.

(3) $x_1^{(k)} > x_1$ for all k: the proof follows the same lines as in (1). Hence, part (c) of this lemma has been proved.

(d) By part (b) of this lemma, f(a) is a decreasing function so that the right-hand limit exists at each point of interval A_1 (Corollary 1 of Lemma 1) except the left-hand endpoint. Let r be the right-hand limit at the point a. Then (a, r) is a limit-point of I, so that (a, r) belongs to I by part (c) of this lemma. Hence, r = f(a), by part (a) of this lemma. f(a) is then continuous on the right wherever the condition makes sense; continuity on the left holds in a similar fashion, so that f(a) is everywhere continuous.

(e) As f(a) is a decreasing function, while $f(a) \geq 0$, $\lim_{a \to a_1} f(a)$ exists and is non-negative. Let $b = \lim_{a \to a_1} f(a)$. If $a_1 < +\infty$, the point (a_1, b) is a limit point of I and belongs to I by part (c) of this lemma. Suppose b > 0. Then

$$(a_1, b)\ P\ (a_1, b/2), \tag{5}$$

$(a_1 + 1, b)\ P\ (a_1, b)$. By Postulate IV, there is a real number t such that $t \geq 0$, and $(a_1 + t, (1/2 + t/2)b)$ is indifferent to (a_1, b) and hence belongs to I. If t = 0, $(a_1, b)\ I (a_1, b/2)$ contrary to (5); if t > 0, I contains the point $(a_1 + t, (1/2 + t/2)b)$, so that $a_1 + t$ belongs to A_1, contrary to the definition of a_1. Hence b = 0, and $(a_1, 0)$ belongs to I.

<u>Lemma 6</u>. If $a < +\infty$, a necessary and sufficient condition that $a_i = a$ is that I contain the point $\underline{x}$, where $x_j = 0$ $(j \neq i)$, $x_i = a$.

<u>Lemma 7</u>. $a_i(b)$ is monotonic decreasing in its domain of definition and

strictly decreasing in its domain of finiteness.

Proof of Lemma 6:

(1) Sufficiency: Let $\underline{y}$ be any point in I distinct from $\underline{x}$. Suppose $y_i > a_i$; as $y_j \geq x_j$, for $j \neq i$, $\underline{y}$ P $\underline{x}$, which is a contradiction. Therefore, $y_i \leq a$ for all $\underline{y}$ in I, so that $a_i = a$.

(2) Necessity: This proof will run by an induction parallel with the proof of Lemma 7. The necessity for n = 2 has already been established in Lemma 5, part (e); the case i = 2 can be obtained by relabeling coordinates. Assume that the necessity has been established for n - 1, and that Lemma 7 has been established for n.

Let P be the plane defined by $x_j = 0$ $(j \neq i, n)$. As a_i is defined, A_i is non-null, and $A_i(b)$ must then be non-null for at least one value of b. Then $a_i(b)$ must be defined for that value of b. As $A_i(b)$ is a subset of A_i, $a_i(b) \leq a_i$, and therefore $a_i(b) < +\infty$. $P_n(b)$ is a space of n - 1 dimensions; by the necessity part of Lemma 6 for n - 1, there is a point in $I \cap P_n(b)$ with $x_j = 0$ $(j \neq i, n)$, $x_i = a_i(b)$, $x_n = b$. This point lies in P, so that $I \cap P$ is non-null. Let A_i' be the set of real numbers a for which $x_i = a$ for some point in $I \cap P$; let $a_i'' = \sup A_i'$. A_i' is non-null, so that a_i'' is defined; A_i' is a subset of A_i, so that $a_i'' \leq a_i < +\infty$. As P is a two-dimensional space, it follows from part (e) of Lemma 5 that there is a point in $I \cap P$ with $x_i = a_i''$, $x_n = 0$. This point belongs to $P_n(0)$; as, for this point, $x_j = 0$ $(j \neq i)$, $x_i = a_i''$, it follows from the sufficiency part of Lemma 6 applied to the space $P_n(0)$ that $a_i(0) = a_i''$. But, by Lemma 7, $a_i(b) \leq a_i(0)$, so that $a_i = \sup a_i(b) = a_i(0) = a_i''$. Hence, I contains the point with $x_j = 0 (j \neq i)$, $x_i = a_i$.

Proof of Lemma 7:

(1) n = 2: By part (a) of Lemma 5, $a_2(b)$ coincides with f(b), and hence is strictly decreasing by part (2) of Lemma 2. The case i = 1 can be handled by relabeling the coordinates.

(2) $n > 2$: Lemma 7 will be proved for general n, assuming that Lemma 6 has been proved for $n - 1$. This proof, in conjunction with the proof of the necessity part of Lemma 6, forms a complete inductive proof.

Suppose that b_1 and b_2 are both in the domain of definition of $a_i(b)$, and $b_1 < b_2$. It is required to prove that $a_i(b_1) \geq a_i(b_2)$. If $a_i(b_1) = +\infty$, this inequality obviously holds, so that we may assume $a_i(b_1) < +\infty$ and seek to prove $a_i(b_2) < a_i(b_1)$. If the necessity part of Lemma 6 holds for (n-1)-dimensional spaces, it may be applied to the space $P_n(b_1)$. There is a point $\underline{x}$ in $I \cap P_n(b_1)$ with $x_j = 0$ $(j \neq i, n)$, $x_i = a_i(b_1)$, $x_n = b_1$. If P is defined as in the proof of Lemma 6, $\underline{x}$ lies in P, so that $I \cap P$ is non-null. Define A_n' as the set of real numbers a for which there exists a point in $I \cap P$ such that $x_n = a$, $a_n'' = \sup A_n'$. A_n' is non-null, and therefore a_n'' is defined.

Suppose $a_n'' < b_2$; then there is a point $\underline{y}$ in $I \cap P$ with $y_i = 0$, $y_n = a_n''$ (Lemma 6 applied to the two-dimensional space P), and therefore a point $\underline{y}$ in I with $y_j = 0 (j \neq n)$, $y_n = a_n''$. As $a_i(b_2)$ is defined, $I \cap P_n(b_2)$ is non-null. Let $\underline{z}$ belong to $I \cap P_n(b_2)$; then $z_j \geq y_j (j \neq n)$, $z_n = b_2 > a_n'' = y_n$ so that $\underline{z}$ P $\underline{y}$, which is impossible since both belong to I. Therefore, $a_n'' \geq b_2$.

By Corollary 1 to Lemma 4, A_n' is an interval. As $\underline{x}$ belongs to $I \cap P$, b_1, which is less than b_2, belongs to A_n'. If $a_n'' = +\infty$, then b_2 belongs to A_n'; if $a_n'' < +\infty$, then a_n'' belongs to A_n' (by the necessity part of Lemma 6 in the case $n = 2$), and, as $a_n'' \geq b_2$, b_2 belongs to A_n', in this case also. By the definition of A_n', this means that there is a point $\underline{w}$ of $I \cap P$ with $w_n = b_2$. $\underline{w}$ and $\underline{x}$ both belong to $I \cap P$, and P is a two-dimensional surface; as $w_n = b_2 > b_1 = x_n$, we must have $w_i < x_i = a_i(b_1)$ by part (b) of Lemma 5. But $\underline{w}$ belongs to $I \cap P$, so that $w_j = 0 (j \neq i, n)$; and $\underline{w}$ belongs to $I \cap P_n(b_2)$. By the sufficiency part of Lemma 6, $a_i(b_2) = w_i < a_i(b_1)$. Q.E.D.

<u>Lemma 8</u>. If $b_1 < b_2$, $a_i(b_1) < +\infty$, and $a_i(b_2) < +\infty$, then $a_i(b)$ is

continuous in the closed interval $\langle b_1, b_2 \rangle$.

Proof: By Corollary 2 to Lemma 4, $a_i(b)$ is defined throughout the closed $\langle b_1, b_2 \rangle$. By Lemma 7, $a_i(b) \leqslant a_i(b_1) < +\infty$ in this interval. Then, for each b, there is a point in I with $x_j = 0$ $(j \neq i, n)$, $x_i = a_i(b)$, $x_n = b$, by Lemma 6. These points lie on the plane P described in the proof of Lemma 6, and form a segment of $I \wedge P$. In the closed interval $\langle b_1, b_2 \rangle$, $a_i(b)$ coincides with the function f(b) described in part (a) of Lemma 5, if x_1 and x_2 there, are relabeled x_n and x_i, respectively. Hence, by part (d) of Lemma 5, $a_i(b)$ is a continuous function.

<u>Lemma 9.</u> If $b_1 < b_2$, $a_i(b_1) = +\infty$, and $a_i(b_2) < +\infty$, then there is a real number b_3 such that $b_1 \leqslant b_3 < b_2$, $a_i(b) = +\infty$ for $b_1 \leqslant b \leqslant b_3$, $a_i(b) < +\infty$ for $b_2 \geqslant b > b_3$, and $\lim_{b \to b_3 + 0} a_i(b) = +\infty$.

Proof: Divide the real numbers in the closed interval $\langle b_1, b_2 \rangle$ into two classes according as $a_i(b) < +\infty$ or $a_i(b) = +\infty$. By assumption, neither class is null; by Lemma 7, these classes constitute a Dedekind cut. Hence, there is a b_3 such that $a_i(b) = +\infty$ for $b_1 \leqslant b < b_3$, and $a_i(b) < +\infty$ for $b_2 \geqslant b > b_3$.

For each b such that $b_2 \geqslant b > b_3$, there is a point in $I \wedge P$ such that $x_i = a_i(b)$, $x_n = b$, where P is defined as in the proof of Lemma 6.
From Lemma 7, $\lim_{b \to b_3 + 0} a_i(b) = c$ exists. Suppose $c < +\infty$.
Then the point on P for which $x_i = c$, $x_n = b_3$ is a limit point of $I \cap P$, and hence belongs to $I \cap P$ by part (c) of Lemma 5. As this point also belongs to $P_n(b_3)$, $a_i(b_3) = c$, by Lemma 6. If $b_3 = b_2$, then certainly $a_i(b_3) = c < +\infty$.

Suppose that for some $b < b_3$, there is a $\underline{y}$ in $I \cap P$ such that $y_n = b$. Then $\underline{y}$ also belongs to $P_n(b)$, and by Lemma 6 $a_i(b) = y_i < +\infty$; but, either $b < b_1$, in which case $a_i(b) \geqslant a_i(b_1) = +\infty$, or $b_1 \leqslant b < b_3$, in which case $a_i(b) = +\infty$ by the first part of this Lemma. Hence, there are no points of $I \cap P$ for which

$x_n < b_3$; for all points of $I \cap P$ for which $x_n \geqslant b_3$ and $a_i(x_n)$ is defined, $x_i = a_i(x_n) \leqslant a_i(b_3) = c$. Therefore, $a_i'' = c$; but then there is a point of $I \cap P$ for which $x_i = c$, $x_n = 0$. As there is a point of $I \cap P$ such that $x_i = c$, $x_n = b_3$, it follows from part (a) of Lemma 5 that $b_3 = 0$, and hence that $b_1 = 0 = b_3$, since $0 \leqslant b_1 \leqslant b_3$. But $a_i(b_1) = +\infty$ by assumption, while $a_i(b_3) < +\infty$, which is a contradiction. Hence $c = +\infty$, or $\lim_{b \to b_3 + 0} a_i(b) = +\infty$.

In the course of the proof, it has been shown that $a_i(b_3) = +\infty$, so that we must have $b_3 < b_2$ (not merely $b_3 \leqslant b_2$), since $a_i(b_2) < +\infty$; also, $a_i(b) = +\infty$ for $b_1 \leqslant b \leqslant b_3$, not merely $b_1 \leqslant b < b_3$.

B. The Basic Theorems. To show that a knowledge of the preference and indifference relations on every plane implies a knowledge of these relations everywhere, we proceed by induction. It suffices to show that such knowledge on every (n-1)-dimensional hyperplane of an n-dimensional space implies a knowledge of the relation everywhere in the larger space. The general line of reasoning is as follows: Consider any two points in the n-dimensional space. Each lies on a hyperplane containing all points whose n^{th} coordinate has the same value as the given point. If the point on the "upper" hyperplane is in fact indifferent to a point on the "lower" hyperplane, the two original points can be compared by considering the preference and indifference relations in the "lower" hyperplane only. If the point in the "upper" hyperplane is not in fact indifferent to any point in the "lower" hyperplane, then it may be easily shown that the "upper" point is preferred to every point in the "lower" hyperplane. The problem reduces to showing that it can be determined, from a knowledge of the relations on every (n-1)-dinemsional hyperplane, whether or not there is a point on the "lower" hyperplane indifferent to the given "upper" point, and, if there is, which points on the "lower" hyperplane have this property.

<u>Theorem 1</u>. If $n \geq 3$, $b_1 < b_2$, and $I \cap P_n(b_2)$ is non-null, a necessary and sufficient condition that $I \cap P_n(b_1)$ be non-null is that there exist integers i and j and real numbers b_3, c_1, and c_2, such that $i \leq n-1$, $j \leq n-1$, $b_1 < b_3 < b_2$, and sets I_1, I_2, and I_3, such that I_1 is an indifference set on $P_j(c_1)$, I_2 is an indifference set on $P_i(c_2)$, and I_3 is an indifference set on $P_n(b_3)$, and such that each of the following intersections is non-null: $I_2 \cap (I \cap P_n(b_2))$, $I_2 \cap I_3$, $I_3 \cap I_1$, and $I_1 \cap P_n(b_1)$.

Proof: (1) Sufficiency: Since I_2 intersects I, it follows from the definition of indifference sets and Lemma 3 that I_2 is a subset of I. By repeating the reasoning, it follows that I_1 is a subset of I and hence that I intersects $P_n(b_1)$.

(2) Necessity: The proof here will be broken up into several parts.

(i) If $b_1 \leq b < b_2$, $a_1(b) > 0$.

Proof: By Corollary 2 to Lemma 4, $a_1(b)$ is defined. If $a_1(b) = +\infty$, there is nothing to prove. If $a_1(b) < +\infty$, $a_i(b)$ is strictly decreasing in the closed interval $\langle b, b_2 \rangle$ by Lemma 7, so that $a_1(b) > a_1(b_2) \geq 0$.

(ii) If $b_1 \leqq b < b_2$, $a_1'(b) < a_1(b)$.

Proof: $a_1'(b) < +\infty$ by definition; hence, if $a_1(b) = +\infty$, the inequality is secured. Suppose $a_1(b) = c < +\infty$. By definition, $a_1'(b) \leq a_1(b)$. Suppose $a_1'(b) = a_1(b) = c$; $c > 0$ by part (i) of this proof. Then there is a point in $I \cap P_n(b)$ with $x_1 = c$, $x_j = 0$ $(j \neq 1, n)$. Consider the plane P defined by $x_j = 0$ $(j = 3, \ldots, n-1)$, $x_n = b$. $I \cap P$ contains then a point with $x_1 = c$, $x_2 = 0$; if A_1' is defined as in the proof of Lemma 6, A_1' is non-null. But A_1' is a subset of $A_1(b)$ by definition, and $A_1(b)$ contains but one point by assumption; hence A_1' contains just the one value c. By part (a) of Lemma 5, this implies that $I \cap P$ contains just one point. Then A_2' contains the single value 0, so that sup $A_2' = 0 < +\infty$ and, by Lemma 6, there is a point in $I \cap P$ with $x_1 = 0$, $x_2 = 0$. But

$I \cap P$ contains only one point, so that we must have $c = 0$, contrary to the result of part (i) of this proof.

(iii) b_3 can be so chosen that $b_1 < b_3 < b_2$, and either both $A_1(b_1) \cap A_1(b_3)$ and $A_2(b_2) \cap A_2(b_3)$ are non-null or both $A_1(b_2) \cap A_1(b_3)$ and $A_2(b_1) \cap A_2(b_3)$ are non-null.

Proof: We classify into cases according to the values of $a_1(b_1)$ and $a_1(b_2)$. By Lemma 7, we cannot have $a_1(b_2) = +\infty$, $a_1(b_1) < +\infty$. Hence, there are three possibilities: $a_1(b_1) < +\infty$, $a_1(b_2) < +\infty$; $a_1(b_1) = +\infty$, $a_1(b_2) < +\infty$; $a_1(b_1) = +\infty$, $a_1(b_2) = +\infty$.

(α) $a_1(b_1) < +\infty$, $a_1(b_2) < +\infty$: By Lemma 8, $a_1(b)$ is continuous in the closed interval $< b_1, b_2 >$. Since $a_1(b_1) > a_1'(b_1)$, by part (ii) of this proof, there is a b_3 such that $b_1 < b_3 < b_2$ and $a_1(b_3) > a_1'(b_1)$; also, by part (ii) of this proof, $a_1(b_3) > a_1'(b_3)$. Then we can choose c so that $\max\left[a_1'(b_1), a_1'(b_3)\right] < c < a_1(b_3)$. As $a_1(b_1) \geqslant a_1(b_3)$, $a_1'(b_1) < c < a_1(b_1)$, or c belongs to $A_1(b_1)$; by construction, c belongs to $A_1(b_3)$, so that $A_1(b_1) \cap A_1(b_3)$ is non-null. Since $a_1(b_2) < +\infty$, there is a point in $P_n(b_2)$ with $x_2 = 0$, so that $A_2(b_2)$ contains 0; as $a_1(b_3) < +\infty$, $A_2(b_3)$ also contains the value 0, so that $A_2(b_2) \cap A_2(b_3)$ is non-null.

(β) $a_1(b_2) < +\infty$, $a_1(b_1) = +\infty$: By Lemma 9, there is a b' such that $a_1(b) < +\infty$ for $b_2 \geqslant b > b'$, $\lim_{b \to b'+0} a_1(b) = +\infty$, $b_1 \leqslant b' < b_2$. Then we can choose b_3 so that $b_2 > b_3 > b'$ and $a_1(b_3) > a_1'(b_1)$. $a_1(b_3) < +\infty$. By repetition of the reasoning in subsection (α) of this part (iii) of the proof, it follows easily that $A_1(b_1) \cap A_1(b_3)$ and $A_2(b_2) \cap A_2(b_3)$ are non-null.

(γ) $a_1(b_1) = a_1(b_2) = +\infty$: Choose b' arbitrarily in the open interval (b_1, b_2). Either $A_2(b') \cap A_2(b_2)$ is non-null or it is null. In the first case, set $b_3 = b'$; as $a_1(b_3) \geqslant a_1(b_2) = +\infty$, $A_1(b_3)$ and $A_1(b_1)$ both have infinite upper limits and both are non-degenerate intervals, so that $A_1(b_3) \cap A_1(b_1)$ is non-null, as is $A_2(b_2) \cap A_2(b_3)$, by assumption.

Suppose now that $A_2(b') \cap A_2(b_2)$ is null. As $a_2(b') \gg a_2(b_2)$, and $A_2(b')$, $A_2(b_2)$ are intervals, we must have $a_2(b_2) \leq a_2'(b') < +\infty$. Then we may interchange the first and second coordinates and apply the results of part (iii), subsections (α) and (β), of this proof to show that there is a b_3 such that $b_1 < b_3 < b_2$ and $A_1(b_2) \cap A_1(b_3)$ and $A_2(b_1) \cap A_2(b_3)$ are both non-null.

(iv) Choose b_3 as in part (iii) of this proof. If $A_1(b_1) \cap A_1(b_3)$ and $A_2(b_2) \cap A_2(b_3)$ are both non-null, choose c_1 in the first of these sets and c_2 in the second, and let $i = 2$, $j = 1$. Let $I_1 = I \cap P_1(c_1)$, $I_2 = I \cap P_2(c_2)$, and $I_3 = I \cap P_n(b_3)$. Since c_2 belongs to both $A_2(b_2)$ and $A_2(b_3)$, $I \cap P_{2n}(c_2, b_2)$ and $I \cap P_{2n}(c_2, b_3)$ are both non-null; but $I \cap P_{2n}(c_2, b_2) = I_2 \cap (I \cap P_n(b_2))$ and $I \cap P_{2n}(c_2, b_3) = I_2 \cap I_3$, so that these latter are non-null. Similarly, $I_3 \cap I_1$ and $I_1 \cap P_n(b_1)$ are non-null.

If, on the other hand, $A_1(b_2) \cap A_1(b_3)$ and $A_2(b_1) \cap A_2(b_3)$ are both non-null, choose c_2 in the first of these sets and c_1 in the second, and let $i = 1$, $j = 2$. Let $I_1 = I \cap P_2(c_1)$, $I_2 = I \cap P_1(c_2)$, and $I_3 = I \cap P_n(b_3)$. Then the conclusion of the theorem follows as in the previous paragraph.

<u>Corollary</u>. If $n \geq 3$, $b_1 < b_2$, and $\underline{x}$ lies on $P_n(b_2)$, the set of all points $\underline{y}$ on $P_n(b_1)$ such that $\underline{x}\ I\ \underline{y}$ is completely determined by a knowledge of the indifference sets on all (n-1)-dimensional hyperplanes.

Proof: Let I be the indifference set in n-dimensional space containing $\underline{x}$. The question is to determine $I \cap P_n(b_1)$ by a knowledge of the indifference sets on all (n-1)-dimensional hyperplanes.

$I \cap P_n(b_2)$ is an indifference set on the (n-1)-dimensional hyperplane $P_n(b_2)$ and hence is known by hypothesis. Choose any j and c_2 such that $P_j(c_2)$ intersects $I \cap P_n(b_2)$. There is then a unique known indifference set I_2 on $P_j(c_2)$ which intersects $I \cap P_n(b_2)$. Then choose a b_3 in the open interval (b_1, b_2) such that $P_n(b_3)$ intersects I_2. Again, there is a known unique indifference set I_3 on $P_n(b_3)$ which intersects I_2. Finally, choose any i and c_1 such that $P_i(c_1)$ intersects I_3, and let I_1 be the unique known indifference set on $P_i(c_1)$ which intersects I_3. If I_1 intersects $P_n(b_1)$, then by Theorem 1, I intersects $P_n(b_1)$, and the set $I \cap P_n(b_1)$ can be found since it is the set of points indifferent to any point of $I_1 \cap P_n(b_1)$. If I_1 does not intersect $P_n(b_1)$, the process can be repeated with other values of i, j, c_1, c_2, and b_3; if I_1 intersects $P_n(b_1)$ for any such set of values, the set $I \cap P_n(b_1)$ is determined, while if I_1 does not intersect $P_n(b_1)$ for any such set of values, then it follows from Theorem 1 that the set $I \cap P_n(b_1)$ is null. Note that each step in this process requires knowledge only of the indifference sets on the (n-1)-dimensional hyperplanes.

<u>Theorem 2.</u> If $n \geqslant 3$, $b_1 < b_2$, $\underline{x}$ lies on $P_n(b_2)$, and the set of points $\underline{y}$ on $P_n(b_1)$ for which $\underline{x}$ I $\underline{y}$ is null, then $\underline{x}$ P $\underline{y}$ for all $\underline{y}$ on $P_n(b_1)$.

Proof: By assumption, for every $\underline{y}$ on $P_n(b_1)$, either $\underline{x}$ P $\underline{y}$ or $\underline{y}$ P $\underline{x}$. Suppose for some $\underline{y}$ that we have $\underline{y}$ P $\underline{x}$. Let $\underline{x}'$ be defined as having the coordinates $x_j' = x_j$ $(j = 1,\dots,n-1)$, $x_n = b_1$. Clearly, $\underline{x}$ P $\underline{x}'$. Therefore, there is a point $\underline{z}$ which is a linear combination of $\underline{y}$ and $\underline{x}'$ such that $\underline{x}$ I $\underline{z}$; but $\underline{z}$ lies on $P_n(b_1)$, so that the assumption that $\underline{y}$ P $\underline{x}$ leads to a contradiction.

<u>Theorem 3.</u> If $n \geqslant 2$, the truth of the assertion $\underline{x}$ P $\underline{y}$ for any $\underline{x}$ and $\underline{y}$ is completely determined by a knowledge of the indifference sets on all planes.

Proof: The theorem is obvious for $n = 2$. Assume $n \geqslant 3$, and suppose the theorem proved for n-1. Take $\underline{x}$, $\underline{y}$, so that $x_n > y_n$; it will be shown that the truth or falsity of each of the two assertions, $\underline{x}$ P $\underline{y}$ and $\underline{y}$ P $\underline{x}$, can be determined by a knowledge of the indifference sets on all planes (for convenience, the preceding phrase, beginning with the word "knowledge" will be abbreviated to K.)

By assumption, on each (n-1)-dimensional hyperplane the truth of the assertions $\underline{x}'$ P $\underline{y}'$ and $\underline{y}'$ P $\underline{x}'$ is determined by K; in particular, for any $\underline{x}'$, set of all $\underline{y}'$ for which neither $\underline{x}'$ P $\underline{y}'$ nor $\underline{y}'$ P $\underline{x}'$ is determined by K. That is, all the indifference sets on each (n-1) dimensional hyperplane are determined by K, and therefore, by the Corollary to Theorem 1, substituting y_n for b_1 and x_n for b_2, the set of all points $\underline{z}$ on $P_n(y_n)$ such that $\underline{x}$ I $\underline{z}$ is determined by K. If this set is null, then, by Theorem 2, $\underline{x}$ P $\underline{y}$ is true and $\underline{y}$ P $\underline{x}$ is false. If this set is not null, choose any element $\underline{z}$ in it. Then the assertion $\underline{x}$ P $\underline{y}$ is logically equivalent to $\underline{z}$ P $\underline{y}$, and the assertion $\underline{y}$ P $\underline{x}$ is logically equivalent to $\underline{y}$ P $\underline{z}$; but the assertions $\underline{z}$ P $\underline{y}$ and $\underline{y}$ P $\underline{z}$ involve two points which both lie on the same (n-1)-dimensional hyperplane $P_n(y_n)$, and hence the truth of these assertions and therefore of the assertions $\underline{x}$ P $\underline{y}$ and $\underline{y}$ P $\underline{x}$ is determined by K.

It has been shown thus far that if $x_n > y_n$, the truth of each of the assertions $\underline{x}$ P $\underline{y}$ and $\underline{y}$ P $\underline{x}$ is determined by K. As $\underline{x}$ and $\underline{y}$ can be interchanged in this result, it follows that if $x_n \neq y_n$, the truth of the assertion $\underline{x}$ P $\underline{y}$ is determined by K. If $x_n = y_n$, $\underline{x}$ and $\underline{y}$ both lie on the same (n-1)-dimensional hyperplane, so that again the truth of the assertion $\underline{x}$ P $\underline{y}$ is determined by K. Q. E. D.

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