COWLES COMMISSION DISCUSSION PAPER: ECONOMICS: NO. 253

# Is the Production Function Redundant?

By Jacob Marschak
February 28, 1949

1. In two papers by Carl Christ (Cowles Commission Discussion Papers: Economics 241 and 241A) the question was discussed whether in a system describing the behavior of a firm maximizing its profits under perfect competition, the production function

$$(1.1) \quad x = f(y)$$

(where x is the output and $y \equiv \{y_i\}$, $i = 1, \ldots, n$ are the inputs), is independent of the "demand-for-inputs equations"

$$(1.2) \quad y_i = D_i(q_1/p, \ldots, q_n/p)$$

(where p is the output price, and $q \equiv \{q_i\}$, $i = 1, \ldots, n$ are the input prices). The independence was questioned because (1.2) is derived from (1.1) and the condition that the profit

$$(1.3) \quad \pi = px - \sum_1^n q_i y_i$$

be maximized. This question will be discussed here.

2. It is assumed that f(y) has first and second derivatives, to be denoted, respectively by $f_i(y)$ and $f_{ij}(y)$, $i, j = 1, \ldots, n$. (The unpleasant implications of linear homogeneity that will be shown do not necessarily apply to non-differentiable production functions treated in "linear programming.")

3. There are, I think, two definitions of perfect competition: 1) p, q are constants; 2) p, q are affected by entry and exit of other firms in such a way that $\pi = 0$. We deal here with the first case only ("short run").

4. Lemma (probably well known). The determinant

$$(4.1) \quad \left| f_{ij}(y) \right| = 0, \quad i,j = 1, \ldots, n$$

for all values of $y$ if and only if $f$ is homogenous linear in $y$.

Proof: multiply the first, ..., n-th column by $y_1, \ldots, y_n$ respectively, and add. The i-th element of the resulting column, $\sum_{j=1}^{n} f_{ij}\, y_j$ vanishes, by Euler's theorem, if and only if $f_i$ is homogenous of zero degree; but this is the case if and only if $f$ is homogenous of first degree.

5. Note that if $f(y)$ is homogenous linear in $y$, then and only then, is

$$(5.1) \quad \pi = pf(y) - \sum_{1}^{n} q_i y_i \equiv \pi(y)$$

also homogenous linear in $y$; and then and only then the determinant

$$(5.2) \quad \left| \pi_{ij}(y) \right| = 0,$$

where $\pi_{ij}(y) = \partial^2 \pi / \partial y_i \partial y_j$.

6. To see when the system (1.1), (1.2) is consistent and independent, consider first the system

$$(6.1) \quad x = f(y)$$

$$(6.2) \quad f_i(y) = q_i/p, \; i = 1, \ldots, n.$$

(6.1), (6.2) are necessary conditions for maximizing $\pi(y)$ as defined in (5.1). Equations (6.2) are "marginal-productivity equations" from which the "demand-for-factor" equations (1.2) are derived. The Jacobian of (6.1), (6.2)

$$(6.3) \quad \begin{vmatrix} 1 & f_1(y) \;\; \ldots \;\; f_n(y) \\ 0 & \\ - & - \;\; \| f_{ij} \| \\ 0 & \end{vmatrix} = \left| f_{ij} \right|$$

7. Suppose $f$ is homogenous linear. Then and only then, by Section 4,

$\left| f_{ij} \right| = 0$ for all values of y, and consequently the Jacobian in (6.3) vanishes; that is, the equations (6.1), (6.2) are independent provided f is not homogenous linear.

8. The "<u>demand-for-factor</u>" equations (1.2) are obtained by solving (6.2) for y. But the set (6.2), taken by itself, has Jacobian $\left| f_{ij} \right|$ which vanishes if and only if f is homogenous linear. Hence, for a set of equations (1.2) to exist, it is sufficient and necessary that f be not homogenous linear.

9. It might seem that the same result would be obtained from second-order conditions because of (5.2). Here, however, the answer to Christ is more Pilatian than in previous sections. Let $\hat{y}$ be a value of y that satisfies (6.1), (6.2). Then:

> (9.1) If f is homogenous linear, then and only then $\left\| \pi_{ij}(y) \right\|$ is singular for all values of y.
>
> (9.2) If $\left\| \pi_{ij}(y) \right\|$ is negative definite for $y = \hat{y}$, then $\pi(\hat{y})$ is a maximum.

From these two propositions it does not follow that linear homogeneity of f is either a sufficient or a necessary condition for the non-existence of profit-maximum.

10. The nature of $\pi(\hat{y})$ in the case of f homogenous linear can be ascertained by determining the ratios $\left\{ \hat{y}_i / f(\hat{y}) \right\} = \left\{ \hat{r}_i \right\} = \hat{r}$, satisfying (6.1), (6.2). Then $\hat{r}$ does not depend on x, and $\pi(\hat{y}) = x \cdot \pi(\hat{r})$ is proportionate to x. But $\pi(\hat{r})$ may or may not be a maximum value of $\pi(r)$.

11. Comparison with utility function. In the analysis of consumers' demand, we have to

$$\text{maximize } I[U(a)], \quad a = \left\{ a_1, \ldots, a_n \right\}$$

$$\text{subject to } \sum_1^n p_i a_i = \text{constant},$$

where I is an arbitrary monotonic function.

To obtain an analogous statement of production theory with m outputs and n-m inputs, write the transformation functions

$$V_h(a) = 0; \quad h = 1, \ldots, m < n$$

of this, (1.1) is a special case, with $a=(x, y_1, \ldots, y_n)$, $m=1$. Enter the inputs with negative signs so that profit = $\sum p_i a_i$. Suppose $V(a)$ is subjected to a monotonic transformation J. (We consider the case $m=1$ and write $V \equiv V_1$.)

$$\text{Maximize } \sum_1^n p_i a_i$$

$$\text{subject to } J[V(a)] = 0$$

In the utility case, the first order conditions are

$$(11.1) \quad \sum p_i a_i = \text{constant}$$

$$(11.2) \quad U_i/U_1 = p_i/p_1 \quad i = 2, \ldots, n,$$

with $I'$ cancelled out in the second line. But in the production case the first order conditions are

$$(11.3) \quad J[V(a)] = 0$$

$$(11.4) \quad V_i/V_1 = p_i/p_1 \quad i = 2, \ldots, n,$$

here $J'$ cancels from the second line but J remains in the first line. The only admissible class of functions for J is defined by:

$$J(x) = 0 \text{ if and only if } x = 0$$

(as also pointed out by Arrow).

Thus, utility function <u>but not transformation function</u> can be submitted to arbitrary monotonic transformations without affecting the corresponding demand functions (which are the solutions for <u>a</u> of the equations (11.2) and (11.4), respectively).

COWLES COMMISSION DISCUSSION PAPER: ECONOMICS: NO. 253-A

# Corrections and Further Comments to Economics 253

By Jacob Marschak
March 30, 1949

It was stated in Economics 253 that

(1) a system consisting of a production equation of a firm [ $x = f(y_1, \ldots, y_n)$ ] and its marginal productivity equations in a competitive market, [ $f_i$ = price $p_i$, $i = 1, \ldots, n$ ] is dependent if and only if the Hessian of the production function, det $\| f_{ij} \|$ vanishes; and that

(2) det $\| f_{ij} \|$ vanishes if and only if f is homogenous linear.

C. Christ and K. May have pointed out to me that in (2) the words "and only if" must be cancelled. Consider, for example $f = ay_1 + b + g(y_2, \ldots, y_n)$. Generalizing their examples somewhat, consider the case

$$(3)\quad f(y_1, \ldots, y_n) = g^{(1)}(y_1, \ldots, y_{n_1}) + g^{(2)}(y_{n_1+1}, \ldots, y_{n_2}) + \ldots + g^{(k)}(y_{n_{k-1}}, \ldots, y_{n_k}),\quad n_k = n,$$

where at least one of the functions $g^{(\ )}$ is and one <u>is not</u> homogenous linear. Yet det $\| f_{ij} \| = 0$.

Thus enlarged, the class of cases that lead to indeterminate cases may be defined as follows:

> The firm's activities can be grouped into $k \geq 1$ "plants" that are independent in the sense of (3) and of which at least one operates under constant returns to scale, apart from an additive constant.

However, this may not be the largest class of cases. Hence the

Question: What are the general properties of functions whose Hessian vanishes, and what economic interpretation can be given to those properties?