Mathematical Appendix to
"Involuntary Unemployment and the Keynesian Supply Function"[1]

by Don Patinkin
February 24, 1949

The nature of the Keynesian model which figures so prominently in §§ 2-7 can best be appreciated if it is actually aggregated from the demand and supply function of traditional economic theory. This we shall now proceed to do.

Consider an economy consisting of n goods: the first n - 1 being finished goods; and the n-th, paper money. Let D_i, S_i, and p_i ($i = 1, \ldots, n$) equal the amount demanded, the amount supplied, and the price of the i-th good, respectively. By definition $p_n = 1$. Let $\mathcal{Y}$ money national income. Then we can write the following (modified) Casselian system:

$$(1) \quad D_i = f_i\,(p_1, \ldots, p_{n-1}, \mathcal{Y})$$

$$(2) \quad S_i = g_i\,(p_1, \ldots, p_{n-1}, \mathcal{Y}) \qquad (i = 1, \ldots, n)$$

$$(3) \quad D_i = S_i$$

$$(4) \quad \mathcal{Y} = \sum_{i=1}^{n-1} p_i S_i$$

Equations (1) and (2) are the demand and supply equations, respectively. These are assumed to depend on national income as well as the prices of all goods. They tell us how much of each good the individuals of the economy desire to buy at different sets of prices and income.

[1] This is a mathematical appendix to the article "Involuntary Unemployment and Keynesian Supply Function by D. Patinkin from the Economic Journal Vol. 59, No. 235, September 1949. This article is reprinted as Cowles Commission Paper, New Series, No. 38.

Equations (3) are the equilibrium conditions of the system. The last equation defines the money national income as the value of the total output of finished goods.

We now define aggregate money expenditure $\mathcal{E}$ as the sum of expenditures on all goods, so that the aggregate money expenditure function becomes

$$(5)\ \mathcal{E} = \sum_{i=1}^{n-1} p_i f_i\ (p_1, \ldots, p_{n-1}, \mathcal{Y}).$$

The function (5) gives the desired aggregate money expenditure on goods and services corresponding to given levels of prices and money income. It must be emphasized that $\mathcal{E}$ includes investment as well as consumer expenditures. Assume that the functions f_i are of such a form that aggregate expenditure, $\mathcal{E}$, is a function of only money income, $\mathcal{Y}$, and a suitable general price index, p, derived from the aggregation process (5). That is, assume the f_i are of such ~~a~~ form that the following relationship holds:

$$(6)\ \mathcal{E} = \sum_{i=1}^{n-1} p_i f_i\ (p_1, \ldots, p_{n-1}, \mathcal{Y}) = H(p, \mathcal{Y}).$$

Assume further that the function $H(p, \mathcal{Y})$ is of such a form that a given percentage change in the general price level and in money incomes results in the same percentage change (in the same direction) in aggregate money expenditures. (That is, H is assumed to be homogeneous of degree 1 in p and $\mathcal{Y}$.) Under this assumption we can then rewrite the aggregate expenditure function as

$$(7)\quad \frac{\mathcal{E}}{p} = H(1, \frac{\mathcal{Y}}{p}) = F(\frac{\mathcal{Y}}{p}).$$

Let real expenditures $\frac{\mathcal{E}}{p}$ be represented by E; similarly let real income $\frac{\mathcal{Y}}{p}$ be represented by Y. Then we can rewrite (7) as

$$(8) \quad E = F(Y).$$

This is the aggregate real expenditure function of Figure 2.

As an example of the preceding development, consider the case where the f_i are of the form

$$(9) \quad D_i = \sum_{j=1}^{n-1} \gamma_{ij} \frac{p_j}{p_i} + \delta_i \frac{\mathcal{Y}}{p_i} \qquad (i = 1, \ldots, n-1)$$

where the γ_{ij} and δ_i are known constants. Then, by (6), the aggregate expenditure function has the form

$$(10) \quad \mathcal{E} = \sum_{i=1}^{n-1} p_i \left[\sum_{j=1}^{n-1} \gamma_{ij} \frac{p_j}{p_i} + \delta_i \frac{\mathcal{Y}}{p_i} \right]$$

$$(11) \quad = \sum_{j=1}^{n-1} p_j \sum_{i=1}^{n-1} \gamma_{ij} + \sum_{i=1}^{n-1} \delta_i \mathcal{Y}$$

Let p be a general price index of the form

$$(12) \quad p = \sum_{j=1}^{n-1} \omega_j p_j$$

where the weights, ω_j, are determined by the equations

$$(13) \quad \omega_j = \sum_{i=1}^{n-1} \gamma_{ij} / \upsilon \qquad (j=1, \ldots, n-1)$$

where υ is a known constant. Also define a new constant

$$(14) \quad \delta = \sum_{i=1}^{n-1} \delta_i .$$

Substituting from (12), and (13), and (14) into (11) we obtain

$$(15)\quad \mathcal{E} = \nu p + \delta \mathcal{Y}.$$

Dividing through both sides by p we obtain

$$(16)\quad E = \nu + \delta Y$$

which is clearly of the form (8).

Until now we have discussed only the demand side of the macrosystem. There is also a supply side, built up from equations (2). By a curious asymmetry this side of the macrosystem was rarely considered by Keynesian analysis. But it is clear that (completely analogous to aggregate money demand) we can define aggregate money supply, $\mathcal{S}$, and the aggregate money supply function

$$(17)\quad \mathcal{S} = \sum_{i=1}^{n-1} p_i\, g_i\,(p_1, \ldots, p_{n-1}, \mathcal{Y}).$$

This function gives us the amount people desire to supply (in dollars) at any given set of prices and national income. Making similar assumptions to those made in (6) above, rewrite (17) as

$$(18)\quad \mathcal{S} = J(p', \mathcal{Y})$$

where p' is again a general price index arising out of the aggregating process. If we again assume homogeneity of degree 1 for the function J, (18) can be rewritten as

$$(19)\quad \frac{\mathcal{S}}{p'} = J(1, \frac{\mathcal{Y}}{p'}) = Q(\frac{\mathcal{Y}}{p'}).$$

If we assume that p and p' are the same,[2] then (19) can be rewritten as

---

2/ p is essentially a price index of finished goods and services; while p' is an index of the prices of productive services as well. The assumption that they are the same implies that these two sets of prices always move proportionately. In other words, the real rate of return to productive services is constant. These reflections would seem to strengthen the speculations below that the aggregate supply function (under these assumptions) has the form (21).

$$(20) \quad S = Q(Y).$$

This is the aggregate real supply function of Figure 4.

The assumption that p and p' are the same is heroic, to say the least. But as emphasized in the text, our main concern is to demonstrate the existence of an aggregate supply function; its particular form need not concern us here.

In Figure 4, however, one assumption is made about the shape of the supply function. Specifically, it is assumed that its slope with respect to Y is smaller than that of the expenditure function. The reason for this distinction lies back in equations (1) and (2). The dependence of the demand function on income is a familiar result of the theory lying behind equation (1)—a theory which considers individuals maximizing their utilities subject to the restraint of a fixed income. Due to this restraint, income enters the demand functions. But no such theory exists on the supply side; nor is it clear that these equations depend on national income. In the extreme case where $\mathcal{Y}$ does not appear in any of the supply equations, (2), the aggregate real supply function in (20) is independent of Y; in other words, aggregate real supply is equal to a constant, say,

$$(21) \quad S = \eta.$$

In this extreme case, the supply curve in Figure 4 would be a horizontal line at the height η.

So far we have made use of only equations (1) and (2) of the Casselian system. We have yet to form the macroeconomic counterparts of equations (3) and (4). This is readily done. Multiply each side of (3) by p_i and sum up over all n-1 finished goods in the economy; then divide through by the general price level. This provides our macroeconomic equilibrium condition

(22) $E = S$.

In a similar way divide both sides of (4) by p to obtain

(23) $Y = S$.

Here, too, we have used the assumption that p and p' are the same.

Thus our Keynesian macrosystem consists of the four equations, (8), (20), (22), and (23), in the three variables E, S, and Y: the system is overdetermined. The meaning of this overdeterminacy has been presented in the analysis of Figure 4 where it was shown that there existed no level of income which would simultaneously equilibrate both the demand and supply sides of the market. In other words, the system is inconsistent.

The entire preceding development has been concerned with building up a Keynesian macrosystem dealing only with the real variables of the system. If in addition we would want to have a system explaining the absolute price level, it would be necessary to include those equations of (1) - (3) which were ignored in the construction of (8), (20), and (22). Specifically, we would have to include the excess demand equation for money derived from (1) and (2) for $i = n$.

Under the Pigou assumption the expenditure function (8) depends also on the absolute price level p.3/ That is,

(24) $E = G(Y, p)$.

Our Keynesian macrosystem (24), (20), (22), and (23) would then have an equal number of variables and equations. However, this does not insure its consistency.4/ Even if it is consistent, the concept of involuntary unemployment enters in a dynamic sense as in §§11 and 12 above.

3/ On this whole paragraph, cf. "Price Flexibility," §§1-8, especially §8.

4/ Cf. above, § 20.