20 Lectures on Income, Employment, and Price Level
(Given at the University of Chicago)
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Cowles Commission for Research in Economics
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PROBLEMS

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Lecture 1.

This is a course in macro-economics. It deals with aggregates such as the total expenditure on consumption goods, total income, total demand for labor, etc., rather than with the demand or supply of single firms or families for single commodities. The relations between aggregates have to be consistent, to be sure, with our knowledge of the behavior of single firms or households with regard to single goods.

Macro-economic analysis helps to judge the effect of policies upon some particularly important aggregates. During the course considerable attention will be devoted, for example, to the controversy about the effect of fiscal, monetary and wage policies upon real national income. Roughly speaking, the "Keynesian" approach emphasizes the possibility of affecting real income by fiscal and monetary policies, under certain conditions, and tends to minimize the effect of money wage rates upon real income. The pre-Keynesian economics, on the other hand, largely neglected the effect of fiscal and monetary policy upon real income, and expected economic recovery from cuts in money wage rates. Crudely,

<table>
<thead>
<tr>
<th>Table 1.I</th>
<th>Effect upon real income</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Keynes</td>
</tr>
<tr>
<td>of government demand</td>
<td>+</td>
</tr>
<tr>
<td>of money quantity</td>
<td>-</td>
</tr>
<tr>
<td>of money wage-rate</td>
<td>0</td>
</tr>
</tbody>
</table>

Generally, policy consists in choosing a set of actions \( (A) \), that will give the best set of results \( (R) \), given a set of uncontrolled conditions \( (C) \). In symbols,

\[
(1.1) \quad R = f(A, C),
\]

where \( R = \text{set} \{ r'(\text{income, say}), r''(\text{price-level}), r'''(\text{inequality of incomes}), \ldots \} \);

\[
(1.2) \quad A = \text{set} \{ a'(\text{fiscal policy}), a''(\text{money policy}), \ldots \};
\]

\[
C = \text{set} \{ c'(\text{tastes}), c''(\text{technology}), c'''(\text{resources}), \ldots \}.
\]
A policy matrix:

<table>
<thead>
<tr>
<th></th>
<th>C₁</th>
<th>C₂</th>
<th>C₃</th>
<th>⋮</th>
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<tbody>
<tr>
<td>A₁</td>
<td>R₁₁</td>
<td>R₁₂</td>
<td>R₁₃</td>
<td>⋮</td>
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<tr>
<td>A₂</td>
<td>R₂₁</td>
<td>R₂₂</td>
<td>R₂₃</td>
<td>⋮</td>
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<tr>
<td>A₃</td>
<td></td>
<td></td>
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<td>⋮</td>
</tr>
</tbody>
</table>

Suppose conditions are C₂. We choose the best result among R₁₁, R₁₂, R₁₃, ⋮; if the best is R₁₂, we choose A₁. This is the essence of any practical action. In a competitive market, for example, the prices and technology are given, (C), and the entrepreneur chooses the output (A) such as would give him the highest profit (R); more generally, R indicates the best combination of profit and power; or of profits of successive years, rather than a single profit-figure.

Table 1.1 (or, say, its "Keynesian" column) gave a detailed aspect of the policy matrix: viz., the effect of single policies (certain elements of the whole set of policies) upon a single element of the results, viz., the real income. Denote real income by y, and the government demand by G. Then the expression ∂y/∂G measures the effect of a unit change of government/ upon real income. Mathematicians call this measure a partial derivative. Economists use the term "marginal effect" or, in certain cases, "multiplier". We can study the effect of changing policies as well as the effect of changing customs. Remembering the notations (1.2) we have (as a general form of Table 1.1):

<table>
<thead>
<tr>
<th>a'</th>
<th>r'</th>
<th>r''</th>
<th>r''''</th>
</tr>
</thead>
<tbody>
<tr>
<td>∂r'</td>
<td>∂r''</td>
<td>∂r''''</td>
<td>⋮</td>
</tr>
<tr>
<td>a''</td>
<td>∂r'</td>
<td>∂r''</td>
<td>⋮</td>
</tr>
<tr>
<td>c'</td>
<td>⋮</td>
<td>⋮</td>
<td>⋮</td>
</tr>
<tr>
<td>c''</td>
<td>⋮</td>
<td>⋮</td>
<td>⋮</td>
</tr>
</tbody>
</table>

The practical need is to know the elements in each cell, or at least to know whether they are zero, positive, or negative. If economic history had supplied us with a large number and variety of policies and conditions, or if we could experiment, we might use statistical estimation without further theorizing. Thus the effect of (controlled) fertilizing and (uncontrolled) weather upon the growth of plant is estimated from observations directly (method of multiple regression...
Unfortunately, economic history gives only a small amount and variety of observations. We need additional information. This is drawn from scattered observations on the behavior of individuals, and is added to the more systematic data on economic aggregates. The name of economic theory is usually given to knowledge derived from information on individual behavior. In this sense, economic theory is needed to supplement aggregative data in order to estimate the effects of policies.

<table>
<thead>
<tr>
<th>Policies and conditions:</th>
<th>Results:</th>
<th>Yield per acre (y)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Amount of nitrogen (N)</td>
<td>θy/θN</td>
<td></td>
</tr>
<tr>
<td>Humidity (H)</td>
<td>θy/θH</td>
<td></td>
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A series of examples will illustrate how theories on the behavior of individuals can supplement aggregative data in deriving the effect of given policies and conditions upon economic aggregates.

It has been asserted that for every individual, (1), the ratio of his income \(Y_1\) to his cash \(M_1\) is a constant;

\[
\frac{Y_1}{M_1} = v_1 \text{ is a constant;}
\]

\[
X_1 = M_1 v_1
\]

Therefore, (aggregate)

\[
Y_2 = X_1 Y_1 = \sum M_1 v_1
\]

(but aggregate) \(N = \sum M_1\) Now define the "average velocity of circulation", \(v\):

\[
v = \frac{\sum M_1 v_1}{\sum M_1} = \frac{\sum M_1 v_1}{M}
\]

Then \(Y/N = v\)

(2.1) \(Y = N v\)

We shall use Roman capital letters for quantities of dollars (as in \(M\)), or quantities of dollars per unit of time (as in \(Y\)), or per unit of commodity or service (as in the case of prices or wage rates).

\(v\) is the "average velocity of circulation," weighted by the individual's cash holdings. It is constant as long as each individual's share in total cash remains unchanged; or, more generally, as long as the cash share of individuals having the same cash holding habits (expressed by \(v_1\) and depending on the frequency of income receipts and rent-payments and on other institutional factors such as holidays) remains unchanged.

that \(v\) is constant.

From the verifiable statement/the marginal effect of \(N\) on \(Y\) is easily derived. A dollar increase in \(N\) increases \(Y\) by \(v\) dollars;

\[
dY/dMv
\]

The influence of \(N\) upon real income, \(y\), is a more important thing to know. The measure or direction of this influence cannot be derived from (2.1) which does not involve real income. There is a definitional relation between real income, \(y\), "price level", \(P\); and money income, \(Y\).
(2.2) \( P = \frac{y}{y'} \).

This adds to our theory one equation but two unknowns. We shall need one more equation if we want to explain how \( y \) (and \( P \)) is determined by \( M \).

Until further notice, we shall make a simplifying assumption that the prices of all products that constitute national output always change in the same proportion. That is, we are not interested in changes of relative prices between the products, and neglect such changes, thereby incurring, of course, a possibly sizable error. The assumption permits us to treat the whole output (real income) as a single commodity, measured in physical units, viz., in "dollar units of a basis-year."

A theory consisting of the two relations, (2.1) and (2.2), permits us to find, in terms of the known \( M \) and \( v \), a single value of \( Y \) but not a single pair \( (P, y) \). We have, instead, a relation between (or "restriction upon") \( P \) and \( y \), viz.,

\[(2.1) \quad Py = Mv,\]

known as the equation of exchange: see Graph 2:I

![Graph 2:I](image)

Relation between \( P \) and \( y \), at constant \( v \) and varying \( M \).

Each curve (it may be called "demand curve for all goods") is a "constant outlay curve" (= "demand curve of unit elasticity" = a rectangular hyperbola). As long as \( M \) and \( v \) are constant, \( P \) and \( y \) are connected by the relation which the curve represents: the area \( \text{FAYD} \) is the same for all values \( P \), \( Y \) of \( P, y \). If either \( M \) or \( v \) rises, the curve "shifts upward."

The particular value which \( P \) and \( y \) take depends on \( M \), \( v \), and further conditions. A few examples of such an additional condition can be given:

(a). **Price control.** The government fixes \( P \), that is, we have the relations:
(2.3) $P_y = MV$; (2.4a) $P = \bar{P}$.

Real income, $y$, must then be

$y = MV/\bar{P}$  \hspace{1cm} \text{(On Graph 2:I, see intersection of curve with horizontal line.)}

(b). Output Control. $y$ is fixed.

(2.3) $P_y = MV$; (2.4b) $y = \bar{y}$.

Price must then settle at

$P = MV/\bar{y}$.  \hspace{1cm} \text{(On Graph 2:I, see intersection of curve with vertical line.)}

(c). Labor theory of value. \hspace{0.5cm} \text{(example suggested by Mr. Weil.)}

(2.3) $P_y = MV$; (2.4c) $P = c\bar{W}$; (2.4c') $W = \bar{W}$,

where $\bar{W}$ is the level at which government or unions fix the money wage rate $\bar{W}$; and $c$ is a constant. On Graph 2:I this case appears as a variant of case (a), with the P-line shifting upward as $\bar{W}$ rises.

(d). A "supply curve for all goods," based on the idea that marginal product declines as output rises, and that real wage rate ($W/P$) offered by employers equals marginal product. (This superficial statement will be explored later more critically.)

(2.3) $P_y = MV$; (2.4d) $y = \sigma (P/\bar{W})$; (2.4d') $W = \bar{W}$,

where $\sigma$ is an increasing function of $P/\bar{W}$, and $\bar{W}$ is again assumed fixed "politically" at $\bar{W}$. On Graph 2:II, the two falling curves are the same as on 2:I;

Graph 2:II

Introducing "supply curve for all goods" (depending on money-wage rate, $\bar{W}$).

The two rising curves represent the supply relation between $P$ and $y$ at two different levels of money wage rate, $\bar{W}$.
assumed to be fixed arbitrarily from outside; the higher \( W \), the less is produced at a given price.

From the graphs, the sign of the effect of policies could be estimated, if we could assume that it is possible to fix \( M, \overline{W} \) (and, as the case may be, \( P, \overline{y} \)) while maintaining \( v \) and \( c \) unchanged. As long as we cannot assert that it is possible, the above theories are very weak, indeed. They have served to illustrate the logic of the problem. The following summary completes this illustration:

<table>
<thead>
<tr>
<th>Table 2.1</th>
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<tbody>
<tr>
<td>Effect upon occupational income</td>
</tr>
<tr>
<td>of raising ( M )</td>
</tr>
<tr>
<td>( \overline{W} )</td>
</tr>
</tbody>
</table>

Under each of the above theories the two variables were determined as depending on certain givens: viz., the "politically fixed" \( M, \overline{W} \), and possibly \( P, \overline{y} \); and on psychologically, sociologically, or technologically given quantities or functions, \( v, c \), and \( \sigma \). For example, under theory \( d \), we can write

\[
(5.6) \quad y = f(M, \overline{W}, v, P).
\]

We shall always use \( f \) to mean "depends on the values of symbols contained between the parentheses, and on nothing else."

Each of the four theories was logically complete and consistent; each contained as many unknowns as independent relations; and a solution for each of the unknowns—such as (5.6) for \( y \)—can be found. A complete and consistent theory can still be false, i.e., it may contradict facts. E.g., suppose it is known that money-wage rates are not fixed arbitrarily but are always related to, say, output. Then, in theories \( a \) and \( d \), the equation \( W = \overline{W} \) becomes false. If this equation is simply dropped, we obtain an incomplete theory. We have, rather, to search for some replacement for the dropped relation \( W = \overline{W} \).
Lecture 3.

In the theories of the 1930's, the equation of exchange was derived from observations among individuals on the habit or necessity of holding money relative to the income of the years; \( y \), being a constant characteristic of the individual. It followed that the aggregate \( M \) depends on the aggregate \( Y \). The conclusion that \( Y \) can be manipulated by arbitrarily changing \( M \) required a logical jump: \( Y \) and \( y \) were interpreted as the demand for goods (in dollars or physical units respectively) that demand had to equate supply determined by certain additional factors (e.g., fixed output, technology, fixed wages, etc.). But, it may be questioned whether income and demand (pot, being measured either in dollars or in physical units) are really the same thing.

The Keynesian analysis distinguishes demand for consumers' goods (\( C \) dollars, or a physical unit per year) and demand for investment goods (\( I \) dollars, or 1 physical units, per year). Neglect provisionally the government as a buyer and tax collector. Unlike the equation of exchange, the following functions are supposed to be derived from observations on decisions: demand for goods not from observations on decisions to hold cash:

\[
(3.1) \quad C = \alpha(Y) \quad (3.2) \quad I = I,
\]

where \( \alpha(\cdot) \) is a function depending on the distribution of income and of consumption habits, and where \( I \) is a constant.

(3.1) might be completed to include the effect of available cash upon the decision to consume

\[
(3.1') \quad C = \alpha(Y, \text{M}),
\]

but we postpone this refinement of the consumer demand equation. We also postpone the discussion whether it is permissible to have a system without an equation of exchange or some other equation expressing the decision to hold cash. \( (3.1') \) is not such an equation, but \( (3.1) \) is.

The symbol \( I \) in \( (3.2) \) is the investment level decided upon by firms and allegedly independent of other economic variables. Verbally, \( (3.2) \) is equivalent to the statement:

"I is exogenous (predetermined, autonomous)."

Again, various refinements of \( (3.2) \) will be discussed later.

A theory alternative to \( (3.1) \), \( (3.2) \) is

\[
(3.1') \quad c = \alpha^*(y) \quad (3.2') \quad i = i^*,
\]

describing all decisions and the decision factor in physical rather than money terms. The important difference was overlooked in the discussion: physical units are not a variable. In the
In the present lecture, only the theories (3.1), (3.2) will be used. These two equations have three unknowns: \( C, Y, I \). The resulting equation is

\[
(3.3) \quad C + I = Y;
\]

on the left hand we have total demand (in dollars); on the right hand we have total income, i.e., the sum of incomes: wages, interest, rents, profits paid out to factors (including profit-receivers) producing the output; this sum is identical with net output (not double-counting of raw materials, and of wear and tear), or supply, in dollars per year. (3.3) says that total supply equals total demand, both in dollars per year. (3.3) can be thought of as expressing the fact that if demand exceeds supply (both in dollars per year) then either physical output or price level or both are raised by the businessmen very quickly until the excess demand vanishes. Thus, (3.3) is an "equilibrium condition": a departure from it is possible but must be short-lived. Thus demand and income are not the same thing; they rarely tend to be equal. (See first paragraph of this lecture.) The supply and demand are the same thing.

The difference \( Y = (C + I) = 0 \) consists of produced but not demanded goods: the "undesired inventories" (the desired inventories are part of \( I \)). In equilibrium they vanish:

\[
(3.3a) \quad Y = C + I = 0.
\]

\( Y = C \) is called "savings"; hence, in equilibrium "savings equal investment." That is, a difference between savings and investment can be only short-lived and is therefore neglected, rendering a more exact restatement.

The system (3.1), (3.2), (3.3) would explain the determination of \( I \) as a function of the two given: viz., of the fixed investment level \( I \), and of the function \( \alpha \):

\[
(3.4) \quad Y = \Phi (I; \alpha), \text{ say.}
\]

For an arithmetic example, assume \( \alpha \) linear. Measuring \( I, C, I \) in Billions a year, suppose:

\[
(3.1.1) \quad C = 0.5X + 5,
(3.2.1) \quad I = 35,
(3.3.1) \quad C + I = Y
\]

by (3.3) and (3.2.1), \( C = Y = 35; Y = 35 = 0.5X + 5 \); hence

\[
(3.4.1) \quad Y = \frac{40}{1 - 0.8} = 200, \text{ a special case of (3.4).}
\]

For more useful (because more general) results, we need algebra:

\[
(3.1.2) \quad C = \epsilon I, Y = \alpha I, \text{ say;}
(3.2.2) \quad I = I
(3.3.2) \quad C + I = Y
\]
Solving as before:

\[ Y = \frac{T}{1 - \alpha} + \frac{c_0}{1 - \alpha}, \]

a special case of (3.4) \((c_0, \alpha, \beta, \gamma)\) are the two parameters of a linear function \(\alpha\) and describe it completely. (3.4.B) implies that a rise in intended investment by 1 unit results in a rise of income by

\[ \frac{1}{1 - \alpha}, \text{ units.} \]

The expression \(1(1 - \alpha)^{-1}\) is thus the derivative of \(Y\) with respect to \(I\), \(\frac{\partial Y}{\partial I}\), it being understood that the other givens \((\alpha, c_0)\) are not affected by the change in \(I\). This expression is called "the investment multiplier".
Geometrical proof that \( M = 1/(1 - \alpha_1) \), where

\[ M = \frac{\partial Y}{\partial I} = \text{"investment multiplier," and where} \]

\[ \alpha_1 = \frac{d\alpha(Y)}{dy} = \text{"marginal propensity to consume."} \]

(The function \( \alpha(Y) \) is the "consumption function.")

Consumption and Total Demand Curves are linear.

The slopes of angles ROI and STQ are, respectively, 1 and \( \alpha_1 \).

When investment increases by \( \overline{RS} = 1 \), income increases by

\[ QT = RT = m = 1/(1 - \alpha_1). \]

Proof:

\[ m = QT = RS + ST = 1 + m\alpha_1. \]

\[ D = \alpha(Y) + I + 1. \]

Graph 1:1

*Remarks on notation. In Lecture 1, the effects of changes in various policies or in non-controlled conditions upon an economic variable were written out as partial derivatives, e.g., \( \partial Y/\partial Q \). This emphasized that several policies or non-controlled conditions may change simultaneously, and that we have to take them up one by one. In the present lecture, investment level, \( I \), is considered as the only condition that is susceptible to change. This justifies the use of the total derivative symbol, \( dY/dI \), for the investment multiplier. We concentrate attention upon the effects of changes in investment level and deliberately forget changes in other conditions. If changes in the function \( \alpha \) (e.g., in the linear case, changes in the slope \( \alpha_1 \), or the intercept \( \alpha_0 \)) were to be discussed, the partial derivative symbol would be more helpful since quantities such as \( \partial Y/\partial I \), \( \partial Y/\partial \alpha \) would have to enter.

Similarly, the marginal propensity to consume can be denoted as a total derivative \( d\alpha(Y)/dy \)--or, briefly, \( d\alpha/dY \)--since \( Y \) is here supposed to be the only argument of the consumption function. If consumers' demand depended, in addition, upon, say, money stock \( M \), the more appropriate notation would be \( d\alpha(Y, M)/dY \), or briefly, \( d\alpha/\partial Y \).
Suppose first that the consumption function, \( C(\dot{Y}) \), is linear, with slope \( < - 1 \). It is represented on Graph 4:I by the line through \( P_i \); this renders the equation 3.1.B. If, as in equation 3.2, investment, \( I_i \), is fixed at level \( \bar{I} \), independently of income, total demand is represented by the line \( D = \alpha (Y) + \bar{I} \). If \( I \) is raised by \( RS = \bar{I} \) to \( \bar{I} + 1 \), total demand line becomes \( D = \alpha (Y) + \bar{I} + 1 \). The equilibrium condition, 3.3, is expressed by the line \( \alpha \), passing through the origin at a 45 degree angle to the Y-axis. When \( I = \bar{I} \), equilibrium income \( \bar{Y} \) is \( \bar{Q} \); when \( I = \bar{I} + 1 \), equilibrium income \( \bar{Y} + \bar{Q} \). The multiplier \( m = \frac{\bar{Q}}{\bar{S}} = \frac{\bar{R} + \bar{S}}{\bar{R}} \). Hence \( m = \frac{1}{1 - \alpha} \).

Note: when \( \bar{I} = 0 \), the equilibrium income \( \bar{Y} \) is called "rock-bottom income." This is not the lowest possible income, since \( \bar{I} \) may be negative when a nation lives on its capital, i.e., depletes inventories and leaves equipment in disrepair (remark by Mr. . . . ).

The proof is valid even if the curve, \( C(Y) \), is not a straight line, provided the change in \( I \) is small, so that the relevant segment \( QS \) of the total demand curve can be approximated by a straight-line segment. That is, the marginal propensity to consume, i.e., the slope of \( \alpha \), \( = d C(Y) / dY \), may change with the income, but its change within the considered range of income must be negligible.

Graph 4:II

Non-linear consumption and total demand curves.

On Graph 4:II, the consumption curve \( \alpha (Y) \) and hence the total demand curve, \( \alpha (Y) + \bar{I} \), has its slope decreasing as \( Y \) increases; for example, the tangential straight line drawn to the \( D \)-curve at \( Q \) would be steeper than the tangential straight line drawn to the same curve at \( S \). However, if \( RS \) (and consequently \( \bar{Q} \)) is small, the two tangential straight lines almost coincide with each other. (The arc \( QS \) almost coincides with its core.)

Note: On Graph 4:II, the curved consumption line was restricted by the following conditions, deemed realistic:
4.1 $0 < \alpha < 1$

4.2 $\alpha$ falls as $Y$ rises (it is almost 1 for a very poor nation, and almost 0 for a very rich one);

4.3 $\alpha(Y) > 0$.

These conditions were assumed valid for any positive $Y$. But this is too restrictive. For the theorem, $m = 1/(1 - \alpha)$, to be valid over some relevant range, it is sufficient to assume these conditions for that range only, and let the curve outside of that range behave in any fancy way.

The system 3.1, 3.2, 3.3, if true, permits us to predict income, $Y$, for any given investment, $I$, provided the consumption function, $\alpha(Y)$, is known. If not all properties of $\alpha(Y)$ are known but only its derivative, $\alpha'$, for the relevant range of $Y$, it is still possible to predict the increment of $Y$ resulting from a given small increase in investment, $\Delta I$: i.e., $\Delta Y = \Delta I/(1 - \alpha)$. Thus, if the initial income $Y_0$ is known, it is possible to predict $Y = Y_0 + \Delta Y$, which will prevail as the result of a given change in investment, even though the knowledge of the consumption function is limited to that of the marginal propensity to consume, $\alpha$.

Such predictions are of practical importance and were attempted during 1939-1945, especially when forecasts of the post-war situation were attempted. Suppose, for example, that

4.4 $\alpha(Y) = .5Y + 5$.

Then (see Lecture 3), any pairs of values for $Y$, $I$ are eligible provided they satisfy the relation

4.5 $Y = 6I + 25$. Examples of such value-pairs are: 10 and 75; 20 and 125; 50 and 175; 40 and 225, etc. The complete set of eligible pairs is given by all points on the straight line, 4.5. One can also obtain this complete set by shifting the total-demand-line on Graphs 4:II or 4:II by varying amounts and marking the equilibrium points. (In the case of Graph 4:II, a curved consumption line, the relation between $Y$ and $I$ will also be curvilinear.)

Instead of these quick and exhaustive methods, the method of piecemeal trial and error has been often applied in the literature of 1939-1945. A worker would try several pairs of values for $Y$, $I$ to see whether they satisfy the condition, $Y = I + \alpha(Y)$. Each pair that satisfied this test, together with figures for real income, employment, and unemployment that were derived from $Y$, constituted a "model". (To derive real income, an additional assumption was made regarding the price-level; employment was then derived from real income on the basis of some labor-productivity assumption. Unemployment was defined as the difference between employment and "labor force" or "maximum employment," based essentially on demographic data. A complication introduced by taxes will occupy us later.)

*) In what follows, we shall occasionally use the term, "model", always identically with "system" or "theory." E.g., 1.1, 1.2, 1.3 constitute a model. We shall not use any word to denote a particular set of figures that satisfies a model. Once the theory is formulated as a system of equations, the obtaining of eligible numerical values is trivial. Much of the literature on the subject lacks a clear formulation of the theory and the objective because the authors failed to write out the equations they had in mind.
Problem 3. (To be inserted after Lecture 4)

Assume that the following variables (all measured in dollars per year) are predetermined: net private investment, government expenditure, tax receipts. Assume the following behavior relations:

(1) Consumption demand (dollars per year) = linear function of disposable national money income;
(2) Supply value (dollars per year) = demand value (dollars per year);

Write the following identities:

(3) Supply value = national money income;
(4) Demand value = consumption demand plus net private investment and government demand.

Express national money income as a function of predetermined variables only (write $c'$, and $d'$, for the slope and intercept of the linear "propensity to consume function").

Problem 4.

In Problem 3, assume $c'_c = 2/3$, $c'_o = 20$ bill a year, private net investment = $20$ bill a year. Calculate the tax receipts and the government deficit, if the desired national money income is $200$ bill, and the government has committed itself to an expenditure of

a) $20$ bill or
b) $40$ bill or
c) $60$ bill

Problem 5.

Construct a diagram showing the results of Problem 4 and, in addition, the general relationship between government expenditures and tax receipts, if money income is to be $200$ bill and if the propensity to consume function and the private net investment are as in Problem 4. (Hint: express the tax receipts as a function of government expenditure, when all other magnitudes are given; insert from Problem 4 the numerical values of those givens; then plot the relationship of government expenditure to tax receipts.)

Problem 6.

Same as Problem 5, but the desired national money income is $160$ bill. Plot the result on the same diagram as Problem 5.

Problem 7.

Same as Problem 5, but assume $c'_o$(marginal propensity to consume) = 4/5.

Problem 8.

Same as Problem 5, but assume private net investment = $10$ bill.
Lecture 5

Before proceeding further with the systematic development of macro-economic models, an interlude of three lecture periods served to discuss the relation of such models to the behavior of single individuals: the problem of "aggregation" or "transition from micro- to macro-economics. This was illustrated by two examples: (1) aggregate demand for cash; and (2) aggregate demand for consumers' goods, both as functions of aggregate income and, possibly, of other variables. Example (1) was the subject of an assignment given to students after Lecture 2, and requested them to check the statement that, in the U.S.A., the ratio \( \frac{Y}{M} \) was constant over the period 1920-1948. (2) was assigned after Lecture 7, requesting them to estimate the relationship between \( C \) and \( Y \) under various alternative assumptions as to the changes in the frequency distribution of family incomes (changing cash family income by the same amount; or by the same percentage; or by transferring income from upper to lower brackets), using a sample of urban families with two or more persons, U.S. 1944 (Statistical Abstract, 1947, Table 307).

Inability to make experiments puts the economist at a disadvantage when compared with the natural scientist. This is partially redeemed by the economist's power of introspection into the plausible, or "understandable" (Max Weber) behavior of the smallest unit, the individual. It is plausible that the individual should determine his cash amount and his consumption expenditure as functions of his money income. Furthermore, it is possible to define general principles of consistent "rational behavior" (such as "maximizing the utility") which would imply the existence of those functions; this is the subject of "micro-economics."

Consider the hypothesis

\[
M_v = k_v \nu \tau, \quad \nu, \tau, \nu \tau \geq 0, \quad t;
\]

where \( \nu \) identifies the individual and \( \tau \) the point of time, so that the ratio \( \frac{M_v \tau}{\nu \tau} = k_v \) depends on the individual but not on time. \( k_v \) is identical with \( \frac{1}{\nu} \) of Lecture 2, where the aggregation of the hypothesis 5.1 over all individuals, using an appropriate definition of "average velocity of circulation" was discussed. However 5.1 may be too narrow a case of the general relation

\[
M_v \tau = \lambda_v (\nu \tau),
\]

where \( \lambda_v \) is some function characteristic of the \( v \)-th individual. To approximate 5.2 by a straight line for some relevant range of values -- as on Graph 5.1 -- we may have to introduce an intercept, \( \hat{E}_v \) (positive or negative), in addition to the slope \( k_v \):

\[
M_v \tau = k_v \nu \tau + \hat{E}_v,
\]

even though the exact function itself is known to pass through the origin; i.e., even if we should believe that people with zero-income demand zero-cash.
Approximating a non-linear relation by a straight line.
Lecture 6

Another direction in which the special hypothesis, 5.1, may have to be generalized to explain the lack of constancy in the individual's "velocity of circulation" (or its reciprocal, \( k_v \)), is to take into account further relevant variables, in addition to his income. For example, the interest rate—say, \( r \)—may also affect his cash demand:

\[ m_{vt} = k_v (y_{vt}, 4) \]

the so-called "liquidity-preference function" for the individual \( y \) \( r \), using again a linear approximation:

\[ m_{vt} = k_v y_{vt} + m_r r + \ell_v \]

where \( k_v, m_r, \ell_v \) are characteristics of the \( v \)-th individual. Graphically, 6.2 can be represented by a family of parallel straight lines: in the \((y_M)\)-plane, with \( k_v \) as the slope and with \( r \) responsible for "shifts"; or in the \((r_M)\)-plane, with \( m_r \) as the slope and with \( y \) responsible for "shifts"; or in the \((y_r)\)-plane. Graph 6.1 uses the first way of presentation, the changing intercept \( = m_r r + \ell_v \).

\[ \text{Graph 6.1} \]

Liquidity-preference equation 6.2

Summing up 6.2 over all individuals, \( v=1, \ldots, n \), we obtain

\[ m = \sum k_v y_{vt} + \sum m_r r + \sum \ell_v \]

provided the aggregates \( M, Y \), and the "macro-economic" constants \( k, m, \ell \) are defined as follows (the summation sign \( \sum \) meaning here summation over individuals, not over time):

\[ m = \sum m_{vt} \]

\[ y = \sum y_{vt} \]

\[ r = \sum r \]

\[ \ell = \sum \ell_v \]

\[ m = \sum m_v \]

\[ k = \frac{\sum k_v y_{vt}}{\sum y_{vt} r} \]

\[ (\text{average of all } k \text{'s, weighted by corresponding incomes}) \]
Lecture 7.

We have seen that the lack of constancy of the ratio M/Y can be explained by the narrowness of the hypothesis (6.1) for each individual. His demand for cash may depend on income in ways other than simple proportionality, and it may depend on further variables, such as interest rate, price-level, number of children.

The number of variables (such as income, interest rate, etc.), each of which produces a sizeable so-called "systematic" effect upon M is limited. But in addition there is the erratic or "random" effect caused by simultaneous action of a host of further variables, each of which is responsible for a very small effect only. This is why the same individual may, at two points of time (t = 1, and t = 2) demand different cash even though his income, the interest rate, etc., are the same. This statistical ("stochastic", "random") nature of his behavior may be approximated by the following hypothesis:

\[ M_t = k_t Y_t + m_t r_t + l_t + u_{\nu \tau} : \nu = 1, \ldots, n; \tau = 1, \ldots, T, \]

where \( u_{\nu \tau} \) is a "random deviation". It is further assumed that while the individual's behavior thus fluctuates from day to day, these fluctuations have certain constant features: \( u_{\nu \tau} \) takes certain values with certain probabilities. The long-run average of \( u_{\nu \tau} \) (the statistician's expectation, or "expected value") is zero, and \( u_{\nu \tau} \) may, for example, have the following probability distribution:

\[ \begin{align*}
  u_{\nu \tau} & = -1 ; 0 ; 1 \\
  \text{with probabilities:} & 0.2 ; 0.6 ; 0.2
\end{align*} \]

Sometimes it may be possible to characterize the probability distribution of \( u_{\nu \tau} \) by a single number, e.g., the standard deviation, a measure of "fickleness" of the individual in his cash-holding behavior. Such a constant (or, if necessary, two or more such "statistical parameters"), together with the coefficients \( k_t, m_t, l_t \), would give a full picture of the individual's cash-holding behavior, i.e., of the way his demand for cash responds to changes in his income, in the interest rate, etc. It enables us to make predictions, i.e., to tell the probability with which his cash-demand will fall into a given interval.

The above statement that the long-run average of \( u_{\nu \tau} \) is zero is limited in the following assumption: that the value taken by \( u_{\nu \tau} \) at any time \( \tau \) is independent of any of its previous or subsequent values. For example, in the case of the probability distribution (7.2), negative and positive deviations, occurring haphazardly, would tend to offset each other in the long-run. Suppose now a society consists of a very large number of individuals. Suppose that each man \( \nu (\nu = 1, \ldots, n) \) is characterized by the same probability distribution (7.2) and fluctuates in his behavior independently of other members of the society. Then,
by the same reasoning as before, the average of all deviations occurring at
the same time, say \( u_1 + u_2 + \cdots + u_n, u_1 \) will tend to be zero:
the larger \( n \), the closer will \( u_1 \) approach zero. The same applies even if the
individuals have different probability distributions (e.g., different standard
deviations) of their behavior, provided that each fluctuates in his behavior
independently of his neighbor. We would then have, for very large aggregates
of men, exact functions such as (6.5), with no random term attached; while for
small aggregates--e.g., the aggregate of the half-dozen or so automobile produc-
ers, even if considered mutually independent in their decisions--we would have
sizeable random fluctuations from one time point to the next, and would have
to write:

\[
\gamma = \kappa_0 \gamma + \kappa_1 \gamma + \cdots + \kappa_t, t = 1, \ldots, t,
\]

where the values of \( u_\gamma \) for various points of time would be independent and
would, in the long run, average out into zero. But at every single point of
time, the average deviation \( u_\gamma \) would, in general, be positive or negative,
not zero. Only if the aggregate is large (i.e., \( n \) is large) would \( u_\gamma \) at
each time \( t \) be negligible, under our assumption of men making their decisions
independently.

Suppose, however, that men do not make their decisions independently but
are "keeping up with the Joneses." Then the average \( u_\gamma \) will be sizeable even
though the society may be large. For example, if a fit of hoarding behavior
affecting Mr. Jones is, more likely than not, accompanied by similar behavior
of Mr. Smith, the positive deviation of Jones will not be offset by a negative
deviation of Smith. Thus the deviation \( u_\gamma \) of the aggregate demand for cash
from some exact function of aggregate income, interest rate, etc., will be
larger, the larger the degree of dependence between the fluctuations in the be-
havior of individuals. (As an extreme example, consider the case when all
individual deviations are "perfectly correlated", e.g., retain constant propor-
tions to each other!)

Macro-economic relations are usually "stochastic". They look like (7.3)
rather than (6.3); even after all relevant variables have been accounted for,
there remains a sizeable "unexplained residual". This is the case even when
the aggregates considered are very large, and is then due to the lack of in-
dependence in the fluctuations of men's behavior, as between individuals
("imitation", "infection", "fashion").

Though exemplified in particular behavior relations, vis; those explain-
ing the demand for cash, the discussion of Lectures 5, 6, 7 was intended to apply
also to other relations which will play a role in the course.

*) In more specific terms: to derive the aggregate equation (7.3) from the
micro-equation (7.1) it is not sufficient to know the \( n \) probability distribu-
tions (7.2), referring to each of the \( n \) individuals. It is necessary to know the \"joint
probability distribution"; e.g., to know the probability that a certain deviation
(say, \( -1 \)) of the first individual is accompanied by specified deviations of the
second, third, \ldots, \( n \)-th individual. Or, in terms of \"statistical parameters":
even if each single distribution such as (7.2) could be completely described
by its standard deviation, we need also know the correlation coefficients for
each pair of individuals.
Lecture 8.

The models of Lectures 2-4 are static. That is, they cannot explain the chain of events in time except through changes in external variables. For example, if the external variables (parameters) I, \( \alpha_c \), \( \alpha_r \) in 3.1.B, 3.2.B, 3.3.B are fixed at certain levels, then certain values of the three economic variables, Y, C, I satisfy those three conditions. These values are called the "solution(s)" and depend on the external variables only:

\[
\begin{align*}
(8.1) Y &= (I + \alpha_c)/(1 - \alpha_r) \quad \text{(cf. 3.4.B)} \\
(8.2) C &= (\alpha_r - I + \alpha_c)/(1 - \alpha_r) \\
(8.3) I &= I
\end{align*}
\]

If I is replaced by a new value, \( I + \Delta I \), while \( \alpha_c, \alpha_r \) remain unchanged, Y will also change, viz., by the amount

\[\Delta I/(1 - \alpha_r)\]

(Similarly one can derive the effect of a change in \( \alpha_r \)).

But the transition of Y from its old to its new level can be explained only in a dynamic model: a model in which at least one of the internal variables enters at two distinct points of time, thus linking the past and the present, the present and the future. Replace, for example, the system (3.15, 3.2, 3.3) by

\[
\begin{align*}
(8.4) Y_{t+\theta} &= \alpha(Y_t) + I_t \quad \text{where } t \text{ indicates time (e.g., } Y_t \text{ = annual rate of income at time point } t), \text{ and } \theta \text{ is the time-lag in the businessmen's reaction. This lag is due to psychology as well as technology. If this lag is negligible, (3.3) becomes a tolerable approximation of the more realistic (8.3). In this case, the variables Y, C are (almost) constant through time as long as } I_t \text{ and the function } \alpha(\cdot) \text{ are constant. But if } \theta \text{ is sizeable, } Y \text{ and } C \text{ change even if the external variables are fixed. We have, by substituting (8.1) and (8.2) into (8.3)}
\end{align*}
\]

Thus income at any time will depend on an earlier income, but will not, in general, be equal to it. To give a "solution" for Y means now, not to give a certain constant value for Y but to construct the path of Y through time, beginning with a given initial value \( I_0 \). Such construction is possible, by (8.4):

\[Y_t = \alpha(Y_0) + I_t \]

\[Y_0 = \alpha(Y_t) + I_0 \]

\[\text{etc...} \quad \text{The path will depend on the function } \alpha \text{ and on the externally prescibed path of } I.\]

In the particular case of a stable dynamic model each variable converges to some constant value as time goes on. If our dynamic model (8.1), (8.2), (8.3) is stable, the differences \( (Y_{t+\theta} - Y_t) \), 

\[(C_{t+\theta} - C_t) \]

converge to zero as time goes on (as \( t \to \infty \)). If so
denote the "equilibrium values", i.e., the constants to which \( Y_t, G_t \) converge, by \( Y_\infty, G_\infty \), and the externally fixed value of \( I \) by \( I_\infty \), then (8.5) implies

\[
Y_\infty = C_\infty + I_\infty
\]

since as time goes on, the error of replacing \( Y_t \) by \( Y_\infty \) tends to vanish. We recognize in (8.5) the "equilibrium condition" (3.3) now properly specified as being valid "in the long run" only.

As an example of a stable dynamic model, assume the linear consumption function as in (3.1.3), restricted by the condition

\[
0 < \alpha, < 1 \quad \text{(cf., 4.1)}
\]

This example will illustrate the "dynamic theory of the multiplier". (8.4) becomes

\[
Y_{t+\theta} = \alpha_t Y_t + \alpha'_t + I_t
\]

Suppose that until time \( t = 0 \) investment was fixed at \( I_0 \) and the system was in equilibrium, so that \( Y_0 = Y_{-\theta} \). That is, by (8.7) and denoting \( I_0 + \alpha_0 Y_0 \) by \( Y_0 \),

\[
Y_0 = \alpha_0 Y_0 + \alpha_0 + I_0 = \alpha_0 Y_0 + \alpha_0 + I_0
\]

(8.8.0)

Suppose that at time \( t = 0 \), investment is raised by 1 unit; then, by (8.7)

\[
\begin{align*}
Y_0 &= \alpha_0 Y_0 + \alpha_0 + I_0 + 1 = \alpha_0 Y_0 + \alpha_0 + I_0 + 1
\end{align*}
\]

(8.8.1)

Thus differences between successive income levels are 1, \( \alpha, \alpha^2, \ldots \) converging to zero as time goes on, since, by (8.6), \( \alpha \) was assumed to be a proper fraction. We can represent the process graphically as a race between the income (or supply) \( Y_t \) and demand \( D_t = \alpha Y_t + \gamma \). The excess demand \( D_t - Y_t \) is zero initially, is raised to 1 at time 0, and diminishes progressively as both \( D_t \) and \( Y_t \) converge to a new equilibrium value which is \( 1/(1-\alpha) \) above the initial one. On Graph 8:1, \( \alpha = 2/3 \)

\[
\text{Demand} \quad \text{Supply} \quad \text{Graph 8:1. Race between demand and supply (in dollars per year), when } \alpha = 2/3 \text{ and investment is fixed at 1 unit above its initial level.}
\]

*And remains at this new level indefinitely.*
Another geometrical representation (reminiscent of that of the "cobweb theorem") uses the static framework of Graph 9.1 where \( Q \) denotes the initial, and \( R \) the final equilibrium combinations of Demand and Supply, and where \( Q', Q'', \ldots \) are intermediate steps.

Graph 9.1 (Compare Graph 4.1)

\( \text{slope } SQT = \alpha_1; \text{ slope } ROY = 1; \text{ QQ' } = \text{ change in investment; } QT = \text{ ultimate change in income.} \)

In Lecture 8, we assumed a time lag in the behavior of producers only: 9 time units elapse between \( Q' \) and \( Q'' \), again between \( Q''' \) and \( Q'\text{IV} \). But the jumps from \( Q'' \) to \( Q''\text{IV} \) (change in demand in response to higher income payments to workers and others) is instantaneous. We can now introduce a time-lag for consumers also; say, 9 time units between income raise and the rise in demand for consumers' goods (difference between positions \( Q'' \) and \( Q''\text{IV} \)). The model becomes

9.1 \[ C_t = \alpha(Y_{t-\theta}) \]
9.2 \[ I_t = \bar{I}_t \]
9.3 \[ Y_{t+\theta} = C_t + I_t \]

Income at any time is therefore derivable from earlier income by the relation

9.4 \[ Y_{t+\theta} = \alpha(Y_{t-\theta}) + \bar{I}_t \]

In the linear case, we can also write

9.5 \[ Y_{t+\theta} = \alpha_1 Y_t + \alpha_0 + \bar{I}_t + \theta \]
This is similar to 3.7: income at any time depends on lagged income and on lagged investment; but income is now lagged by \( \theta + \theta' \), i.e., the delays in production and consumption are added. The solution, i.e., the path of \( Y \) through time is similar to that of Lecture 3; the equilibrium value is exactly the same but successive increases of \( Y \) by \( 1, \alpha, \alpha^2, \ldots \), are separated by a longer time.

The comparison of the two models, \( \theta' = 0 \) (Lecture 3) and \( \theta' > 0 \) (present lecture) is a typical problem in comparative dynamics. In Lecture 1, problems in comparative statics were formulated; to measure the effect of the change in a (controlled or non-controlled) parameter of a static model upon the values of the economic variables, i.e., upon the "solution" of the static model. Analogously, comparative dynamics studies the effect of a change in a parameter of the model (such as \( \theta' \)) upon the "solution", i.e., upon the path (or "time-shape") of each economic variable. The path itself is characterized by certain parameters, e.g., the equilibrium value (if any: i.e., if the model is stable, as ours is); the time-distance between steps (if such steps can be defined, as is the case with our model); etc. If the path were "cyclical", i.e., showed periodicity, the length of each wave would be another important path-parameter. Comparative dynamics, then, studies the effect of a change in a given parameter of the system upon each of the important parameters of the path; and the matrix \( I, \alpha \) can be re-labelled accordingly. Problems in business cycles policy are of this nature; we want to know how a change in policy (i.e., in a controlled parameter) affects, say, the time-length of the cycle.

In every dynamic model of this kind the preceding lecture, restoration of equilibrium takes "infinite time." The duration of full adjustment could not therefore be used as a path-parameter. Instead one can ask: How long does it take to achieve one-half -- or 75\%, 90\%, or any preassigned fraction \( k \) -- of the adjustment? The final gain in income in response to a unit increase in investment is the difference

\[
\Delta Y = Y_0 = 1/(1-\alpha_1);
\]

an intermediate income gain --say, at time of the \( n \)-th step, i.e., \( n (\theta + \theta') \) time units after the disturbance, is

\[
1 + \alpha_1 + \alpha_2 + \ldots + \alpha_1^{n-1} = (1-\alpha_1^n) / (1-\alpha_1).
\]

The ratio between the intermediate and the final gain is, therefore, \( 1-\alpha_1^n \)

We ask what is the "half-life" (or "75\%" "90\%" etc.) of this adjustment. Put

\[
1 - \alpha_1^n = 1/2 \quad (\text{or} \ 75\%, \ 90\% \ldots), \text{ hence }
\]

\[
\alpha_1^n = 1/2 \quad (\text{or} \ 25\%, \ 10\% \ldots).
\]

We find, for example, that if \( \alpha_1 = 3/4 \), three steps achieve half of the full adjustment, since \( (3/4)^2 = 9/16, (3/4)^3 < 1/2 \). The needed number of steps is the larger, the smaller \( \alpha_1 \) (and hence the smaller the multiplier). Therefore, the increase in \( \alpha_1 \) (e.g., through transfer of incomes from the rich to the poor) is not conducive to "stability" of the system if the degree of stability is defined as the period necessary to achieve a given fraction of adjustment. (Or: "necessary to overcome a given fraction of the disturbance", the disturbance being measured as the difference between demand and supply, \( D_t - Y_t \); see Graph 8.1).
Obviously, an increase in one or both of the lags \( \theta \) or \( \theta' \) has also a "destabilizing" effect in the sense just defined.

Question: "Comment on the following statement: Both the velocity of circulation, \( v = Y/M \), and the multiplier, \( \partial Y/\partial I \), turn out to lie around 3. This suggests that they are identical. In fact, the effect of a change in investment works out in the following steps: additional money is paid out to workers and others; it is spent by them \( 1/v \) time units later, thus creating new demand for labor and other factors of production, etc. The greater \( v \), the larger the effect of given investment."

Comment. (1) The two quantities cannot be identical; \( v \) is measured in "times per year", while the multiplier is a pure number. (2) However, the time-length \( 1/v \) would be partly reflected in lag \( \theta' \) if consumers would revise their purchases not at the time of the revision of their income contracts (e.g., being hired, or being granted higher wages) but at the time of receiving cash. The time-path of \( Y \) is then affected by a change in \( v \)--not in the sense that \( Y \) and the multiplier, \( \partial Y/\partial I \), are changed--but in the sense that the duration of cash adjustment step, \( \theta + \theta' \), and therefore the "half-life" of adjustment is changed. (3) Accordingly, the velocity of circulation would enter the model as a parameter of two independent equations: the demand for cash equation (with interest \( r \) as a further variable)

\[
9.6 \quad M = \lambda(Y_t; v)
\]
and the consumption equation

\[
9.7 \quad C = \alpha (Y_t - \theta'(v))
\]

the function \( \theta' (\cdot) \) indicating the way in which a change in \( v \) affects the lag between income and consumption.
Lecture 10.

10.1

As an example of aggregation, Problem 2 was discussed: "Using Table 308, Statistical Abstract 1947, estimate the relationship between per-family income and per-family consumption if the changes in the income of each family obey one of the following conditions:

10.A - All incomes change by the same amount of dollars;
10.B - All incomes change by the same percentage."

The demand of a single family for consumption goods (in dollars) was assumed to be a function of this family's money income only ("personal consumption function");

\[ C_r = \alpha (Y_r), \quad r = 1, \ldots, n \]

The relation to be derived is that between per-family (or average) consumption demand, \( C \), and per-family (or average) money income \( Y \); the "collective consumption function":

\[ C = \bar{\alpha}(\bar{Y}), \quad \bar{C} = \frac{1}{n} \sum_{i} C_r, \quad \bar{Y} = \frac{1}{n} \sum_{i} Y_r. \]

It was made clear that hypotheses alternative to 10.1 were excluded, for example, the following hypotheses,

10.1.1 \[ C_r = \alpha (Y_r, \gamma_{r}^c), \quad \text{where } \gamma \text{ is a lag}; \]

10.1.2 \[ C_r = \alpha (Y_r, Y_{r-1}), \quad \text{where } Y_r \text{ is the highest income reached at any time before } r, \gamma_r^c. \]

Thus 10.1.1 expresses the slowness of adjustment of consumption both upward and downward, while 10.1.2 expresses the slowness of people's adjustment downward, to new poverty. Appropriate combinations of these hypotheses could be devised. Also, 10.1.2 could be amplified to include as a further variable the time elapsed since the last peak of income.

These refinements were excluded from consideration. Excluded also was the fact that, given the income, consumption depends on the size of the family. It was understood that having two (or three, four, ...) instead of one independent variable in 10.1 would require data in a form different from the simple "single entry" table of the Statistical Abstract, which gives consumption and relative frequencies for each bracket of current income. The hypothesis 10.1.1, for example, would require a double-entry table, showing the consumption and the relative frequency for each "cell" corresponding to a given current and given past income-bracket.

To fix the idea, assume 10.1 to be a quadratic:

\[ C_r = \alpha + \alpha_r Y_r + \alpha_r^2 Y_r^2 = \alpha (Y_r) \]

then, by the definitions of the averages, \( C \) and \( Y \), we have

\[ \bar{C} = \alpha + \bar{\alpha}_1 \bar{Y} + \bar{\alpha}_2 \bar{Y}^2 = \alpha (\bar{Y}) \]

now define the variance (\( \sigma \), square of standard deviation) of family incomes:

\[ \sigma = \frac{1}{n} \sum_{i} (Y_r - \bar{Y})^2 \]

\[ 10.5' - \sigma = \frac{1}{n} \sum_{i} (Y_r - \bar{Y})^2 \]
+ obeys an identity taught in elementary statistics:
\[ \sum_{i} X_i^2 - \bar{X}^2 = \sigma^2 \]

Since 10.1 becomes
\[ \bar{X} = \alpha + \epsilon \]

10.6 \[ \alpha(Y) = \gamma(Y) + \epsilon \]

If the "personal consumption function" \( \alpha(Y) \) is a straight line, then \( \gamma \) in 10.3 vanishes; and 10.6 shows that in this case, the personal and the collective consumption functions coincide. This also follows, of course, directly by forming averages on both sides of the equation.

\[ C = \alpha + \epsilon Y \]

In this case no transfer of money between poor and rich has any effect upon average (and therefore total) consumption as long as the total (and therefore average) income, \( Y \), remains the same. This is obvious, since, in this case, if the transfer of a dollar reduces the consumption of the rich man by, say, \( \frac{1}{2} \) dollar, and increases the consumption of the poor by exactly the same amount. (Note that the linearity of consumption function means that the marginal, but not necessarily the average propensity to consume is independent of income!)

We have, in general, \( \alpha > 0 \). The rich may consume a smaller proportion of their income than the poor do of theirs. The relevant question is whether they consume a small proportion of any added, or subtracted, income.

In general, \( \epsilon \) is not zero, but presumably negative. The function \( \alpha(Y) \) is convex with respect to the \( C \) axis, i.e., the slope, or marginal propensity, falls as income rises. 10.6 implies, therefore, that \( \alpha(Y) \) lies below \( \gamma(Y) \); that is, for any \( Y \),

10.8 \[ \alpha(Y) < \gamma(Y) \]

It will be shown in Lecture 11 that this result is general, for any convex curve \( \gamma(Y) \), and not only a quadratic one. We also see immediately that, quite generally, if personal marginal propensity to consume is the larger the income, then income-equalization raises consumption. The transfer of $1 from a man with 0.5 marginal propensity to consume causes 0.5 increase of consumption. See Graph 10.1 to a man with 0.8 propensity to consume.
(Effect of income-equalization upon consumption. Average consumption of two families \((\gamma = 1, 2)\) is, before equalization,

\[
\bar{Y}_3 \bar{D} = \frac{(Y_1 C_1 + Y_2 C_2)}{2}, \quad \text{where} \quad Y_3 = \frac{(Y_1 + Y_2)}{2}.
\]

After equalization, the average consumption is \(\bar{Y}_3 \bar{D}'\).

We have \(\bar{Y}_3 \bar{D}' > \bar{Y}_3 \bar{D}\) if the curve is convex; \(\bar{Y}_3 \bar{D}' = \bar{Y}_3 \bar{D}\) if it is linear.
Other properties of the collective consumption function, \( \alpha(Y) \), given the personal consumption function \( \alpha(Y) \), depend on what assumptions are made about the way in which income-distribution changes as \( Y \) changes. For example, our alternative assumptions 10.1, 10.3 are two such ways:

Note: In the economists' language, case 10.1 is one of "unchanged distribution"; for the statistician, distribution is defined by the relative frequencies assigned to the variable--in this case to income--and therefore the changes in case 10.3 as well as 10.1. Case 10.3 might be called that of "constant proportionate shares." In both cases, one distribution parameter, viz. the average \( Y \), changes. But in case 10.1, the variance \( \sigma^2 \) (defined in 10.3) is unchanged since, for any family \( Y \), the deviation of its income from the average, \( Y - \bar{Y} \), is unchanged. In case 10.3, an unchanging character for each family is its relative deviation, \( (Y - \bar{Y})/\bar{Y} \); consequently, the "coefficient of variation" (squared here for convenience)

\[
11.1 \quad (1/n) \sum (Y_i - \bar{Y})^2/\bar{Y}^2 = \sigma^2/\bar{Y}^2 = v^2
\]

is unchanged as \( Y \) changes.

By 10.6, the (quadratic) personal consumption function lies above its collective counterpart, the distance being \( \alpha \sigma^2 \). This distance is constant in case 10.1. In case 10.3, this distance grows with \( Y \), since \( \sigma^2 = v \bar{Y}^2 \) and since, in this case, \( v^2 \) is constant.

[In each of these two graphs, the upper line represents a (quadratic) personal consumption function, and the lower line, the collective consumption function. A change in average income \( Y \) is accompanied, in case A, by equal dollar increments for each family; in case B, by equal percentage increments for each family.]

Graphs 11.1: \( C_Y + C \)
Moreover, the inequality \( \sigma \) does not depend on these special assumptions, provided the personal marginal propensity to consume does increase with income, \( \bar{C}_0 = \bar{a} + \bar{b} \). Given any income distribution, the average income \( \bar{Y} \) can be regarded as marked by the center of gravity of points on the \( \bar{X} \)-axis, the \( \bar{X} \) axis being conceived as a wire having different density at different parts (corresponding to the different relative frequencies at different income brackets). Similarly, the average consumption \( \bar{C} \) can be regarded as the center of gravity of points on the \( \bar{Y} \)-axis. Consequently, the point \( (\bar{C}, \bar{Y}) \) is the center of gravity of the current wire representing the equation \( \bar{Y} = \alpha(\bar{C}) \), and, with that curve present, the center of gravity will lie "inside" and therefore below \( \alpha(\bar{Y}) \). If the distribution (of densities, \( \bar{C}_0 \), of income) is changed in some prescribed way, the center of gravity will shift, tracing out the \( \alpha(\bar{Y}) \)-curve (possibly with loops, \( \partial \bar{C} / \partial \bar{Y} \)), which will lie below the \( \alpha(\bar{Y}) \) curve.

Graph 11:XXI

Graphs 11:XX, A and B, illustrate the economic effects of inequality of incomes. Suppose we measure inequality by \( \sigma \). The following two distinct questions can be asked and are often confused: What is the effect of a unit change in inequality upon

(a) the average (and therefore total) income \( \bar{Y} \),
investment level \( \bar{I} \) being constant;

(b) upon the multiplier, \( \partial \bar{Y} / \partial \bar{I} \)
the initial level of income \( \bar{Y} \) being given. We thus measure

11.a \( \partial \bar{Y} / \partial \sigma \);  \hspace{1cm} 11.b \( \partial (\partial \bar{Y} / \partial \bar{I}) / \partial \sigma \).

Since \( \bar{Y} = \bar{C} + \bar{I} \), the expression 11.a is identical with \( \partial \bar{Y} / \partial \sigma \); and we see that, in case A (left part of the Graph 11:XXI) average consumption \( \bar{C} \) falls as inequality increases; the curve \( \alpha(\bar{Y}) \) shifts further down, parallel to itself. But the slope of the collective consumption curve, at a given \( \bar{Y} \), is not affected, in case A, by any shift caused by a change in \( \sigma \). Therefore, the expression 11.b is
The income multiplier does not depend on income elasticity as measured by \( e \); if the societies differ only with respect to "inequality" \( \sigma \), and have equal average income and equal taxes, a billion rise in investment will have the same effect on average income in both societies.

On the other hand, if we consider case B and measure inequality by the "coefficient of variation" \( \nu \), we obtain a different result (right-hand part of Graph 11:1) since the slope is now related to the degree of inequality, as is seen by drawing a new \( \epsilon(\ ) \)-curve, with a larger \( \nu \). An expression such as 11:6 may be said to measure the effect of inequality upon the degree of economic stability. The higher the multiplier, the "less stable" is the society, in the sense that a shock of a given size (a change in investment level 1) affects income strongly, whether upward or downward. Confusion is created by sometimes describing expression 11:6 also as a measure of stability; high income (e.g., "full employment income") is confused with stable income.
12.1

The main data used in the consumer expenditure theory concerns the relation between disposable personal income and the population's income (net of personal and corporate income after personal taxes). Consequently, the collective consumption function derived from it, should be written, in the notation of the earlier lectures,

12.1. \[ C = \alpha(Y - T), \]

where \( \alpha \) is a function, \( Y \) is personal income after taxes, and \( T \) is the government's tax receipts per family. (We shall disregard all other than personal taxes.) Since we consider population as given, all that follows can be easily translated into terms of total consumption, total income, total tax-receipts instead of the per-family quantities, provided the function \( \alpha(\cdot) \) is given. Throughout the rest of the course, we shall deal with totals, unless stated otherwise.

Denote by \( G \) the government expenditure, while \( I \) will be, from now on, used to denote the private sector investment only. If we assume \( I, G, \) and \( T \) exogenous, and assume/consumption function linear over the relevant interval, the system of Lecture 3 becomes

12.1. \[ C = \alpha(Y - T), \]
12.2. \[ Y = G + I + C, \]
12.3-5. \[ I = I, \ G = G, \ T = T, \]

but instead of writing out the last three equations we can note verbally that \( I, G, T \) are exogenous, i.e., given and independent of the other variables of the system, and thus spare the symbols \( I, G, T, \)

we can solve 12.1 and 12.2, i.e., express \( Y \) and \( C \) in terms of the given:

12.6. \[ Y = \frac{I + G + \alpha}{1 - \alpha}, \ T. \]

We note that the "government expenditure multiplier"

12.7. \[ \frac{\partial Y}{\partial G} = \frac{I}{1 - \alpha}, \quad (e.g., \alpha = 3 \text{ if } \alpha = 2/3), \]
while the "tax receipts multiplier"

12.8. \[ \frac{\partial Y}{\partial T} = -\frac{\alpha}{1 - \alpha}, \quad (e.g., \alpha = -2 \text{ if } \alpha = 2/3). \]

Thus if government expenditure is increased by \( \Delta G \), (say, 4 billion), while tax receipts are increased by \( \Delta T \), (say, 1 billion), the income increment is

\[ \Delta Y = \frac{\partial Y}{\partial G} \cdot \Delta G + \frac{\partial Y}{\partial T} \cdot \Delta T. \]

If the increase in government expenditure is fully balanced by an increase in taxes, i.e., if both are increased by \( \Delta G \), then

12.9. \[ \Delta Y = \frac{1}{1 - \alpha} \cdot \Delta G - \frac{\alpha}{1 - \alpha} \Delta G = \Delta G, \]

that is, income increases by \( \alpha \) for each dollar increase in tax-financed government expenditure. This result has nothing to do with the redistributive effect of progressive taxes, or of certain types of public works. It would hold also if the taxes were proportionate; or if all incomes were equal; or if the personal consumption function were linear (making-as shown in Lecture 10- total consumption inde-
pendent of income distribution). The result 12.9 appears less para-
doxical if one remembers that the government increases its demand by
the full amount of tax, while the consumers diminish their demand by
only a fraction \( \alpha' \) of the tax. Consider successive steps ("dynamic
approach"; Lectures 8-9):

Adding \$1 to tax receipts changes demand by \(- \alpha, - \alpha', - \alpha'', - \ldots\)

Adding \$1 to government expenditure changes demand by \(+ \beta', + \beta'', + \ldots\)

Cancelling \(- \alpha', \) and \(+ \alpha', \), \(- \alpha'', \) and \(+ \alpha'', \), etc., we obtain a net total of \$1

Dissimilar results are observed if I or G or T are not exogenous.
For example, it is claimed that entrepreneurs may be "scared" by
government expenditure into an "investment strike"; say,

12.3.e  \( I = \beta - \beta' G \)  \( (\beta' > 0) \)

Inserting this into 12.6 we have

12.6.a  \[ Y = \frac{\left(\frac{1}{1 - \beta'} \right) + \alpha' + \beta}{1 - \beta'} - \frac{\alpha'}{1 - \alpha'} \cdot T \]

in this case the multiplier of government spending is

12.7.a  \[ \frac{\partial Y}{\partial G} = \frac{1 - \beta'}{1 - \alpha'} \]

and is smaller than when I is exogenous, as was the case in 12.7.
Other hypotheses about non-exogenous investment will be studied later.

It is certain that T is not exogenous: the government fixes the
tax-schedule (a functional relationship between individual income and
tax), not the tax-receipts. As a simple example, assume proportional
income tax, hence

12.1.b  \[ T = Y \cdot \tau \]

where \( \tau \) means "tax-rate". Instead of 12.1 we then have

12.1.b  \[ C = \alpha_c + \alpha(1 - \tau)Y \]

where \( \alpha_c = \alpha_c (1 - \tau) \) can be called the "net marginal propensity to
consume". 12.6 becomes

12.6.b  \[ Y = \frac{I + G + \alpha_c}{1 - \alpha''} \]

An increase in tax-rate is thus equivalent to a certain decrease in
the multiplier.

As in Lecture 11, we can ask two questions:

1) What is the effect of tax-rate increase upon equilibrium
income, given the level of investment and government spending? Obviously

12.8  \[ \frac{\partial Y}{\partial \tau} < 0, \text{ since } \frac{\partial Y}{\partial \tau^2} > 0. \]

In this sense, tax-rate increase is "deflationary" (and not "inflationary").
2) The effect of an income increase upon the multiplier, i.e., upon \( \frac{\partial Y}{\partial G} \), i.e., upon the size of income change produced by a given change in government spending (or investment). As already remarked, the multiplier is the larger, the larger \( \frac{\partial Y}{\partial G} \), i.e., the smaller \( \frac{\partial Y}{\partial G} \). 

\[ 12.3 \quad \frac{\partial Y}{\partial G} < 0. \]

In this sense, an increase in tax-rate is "stabilizing" (and not de-stabilizing).
As a step to make the model of Lecture 3 more realistic, we drop the assumption that investment is exogenous. In the version of the "Washington-Keynesian" model, investment depends on national income.

A major reason for this may be that entrepreneurs decide to enlarge plant if they expect high future sales or high future profits; that high future profits or sales are expected when current profits or sales are high; and that current profits and sales are strongly correlated with national income. However, precisely what causes the firm executive to expect high profits (say), how this expectation moves him to choose a particular investment level and how such individual responses are to be aggregated into a macroeconomic relation, we shall not attempt to analyze here. (In the case of consumption, the aggregation step was explained in some detail in lectures 10-11, but the preceding steps—e.g., deriving the personal consumption function from the principle of utility maximization—were also not attempted.) We have, then

13.1 \( C = \alpha(Y) \) or, in \( 13.1.L \), \( C = \alpha + \alpha Y \)

13.2 \( I = \beta(Y) \) or, in \( 13.2.L \), \( I = \beta + \beta Y \)

13.3 \( Y = C + I + G \) or, in \( 13.3.L \), \( Y = C + I + G \)

The exogenous variable \( G \) will here indicate deficit-financed government expenditure; we neglect taxes to simplify the presentation—but they could be easily introduced on the lines of Lecture 12.

If we denote \( C + I \) by \( D \) (for private demand), and the sum of functions \( \alpha(Y) I + \beta(Y) \) by a function \( \delta(Y) \) ("propensity to spend function"), or—in the linear case—with \( \alpha \) and \( \beta \) ("marginal propensity to consume and invest"), we have again a two-equations model

13.4 \( D = \delta(Y) \)

13.5 \( Y = D + G \)

or

13.4.L \( D = \delta + \delta Y \)

13.5.L \( Y = D + G \)

It follows that \( Y = (\delta + G)/\delta + G \); the "multiplier" is \( 1/(1 - \delta) \).

Graphs 4.1 and 9.1 are applicable to our new system with appropriate changes; instead of drawing the line \( \alpha(Y) \) and shifting it by a given amount of investment (which included government demand), we now draw the line \( \delta(Y) \) and shift it by the amount of government demand.

Note that if \( G = 0 \), we have

13.6 \( D = \delta(Y) \)

13.7 \( D = X \)

or

13.6.L \( D = \delta + \delta Y \)

13.7.L \( D = Y \)

a system of two equations in two variables which has been repeatedly misunderstood in economic literature. One hears, for example, that because of 13.7, the sum of the marginal propensities to consume and to invest is equal to unity. Individuals can either consume or not consume the product, and the not consumed (saved) part is added to the nation's plant and inventories, \( I \), is identical with investment. The investment so defined is not the variable quantity; that obe ys
the behavior relation is only a particular
the variable \( g \) has value \( b \) satisfies only 13.2
but also the other equations of the system \( \text{v.i.} \quad 13-7 \).
This particular value is called equilibrium value because 13.7 is an
equilibrium condition; 13-7 is actually a more approximation of some
dynamic statement such as

\[
13-9 \quad (Y_t - D_t) \to 0 \quad \text{as} \quad t \to \infty.
\]
Thus, equation 13-6; \( L \) and 13-7; \( L \) are quite distinct. From the valid-
dity of 13-7; \( L \) it does not follow, not even in the "long-run" that \( \bar{g} \),
\( g \to 0 \) (Analogy: from the requirement that demand be equal supply,
it does not follow that the demand curve and the supply curve are iden-
tical curves).

The equilibrium values of economic variables have also been called
their "ex-post" values; the term "ex-ante" would then apply to values
that are taken by economic variables before the equilibrium has been
achieved. It is, of course, the "ex-ante" values that are meant when
response equations such as 13.4 are written. Another term for "ex-
ante" is "intended". E.g., the quantities plotted on the demand curve
or on the supply curve are "intended" quantities; only at one parti-
cular price these quantities take values that satisfy the equilibrium
conditions \( \text{v.i.} \) the condition that they be equal.

The difference \( Y_t - D_t \) in 13.8 has been occasionally called
"hoarding" (of money)—an unhappy terminology since it may strongly
suggest an increase in the national stock of money. There \( \bar{D} \),
be sure, an (unintended) increase in the stock of goods. However,
this increase may not fully coincide with \( Y_t - D_t \) since not all goods
are storeable: when the demand for electricity falls short of capac-
ity of power plants, the difference \( Y_t - D_t \) shows itself in the
continued fixed charges and (usually) not in accumulation of produced
electricity. Thus the statement in Lecture 3 that identified \( Y_t - \bar{D}_t \)
with "unintended increase in inventories" has to be corrected.
14.1

Equation 1.0.2 may be criticized for not including interest rate among the variables affecting the decision to invest. The cheaper a firm can borrow money, the more likely it is to expand its plant and equipment, given the rate of profits (the marginal efficiency of capital) that are expected to be yielded by additional plant or equipment. Using linear approximations,

\[ \Delta X = \beta_0 + \beta_1 \Delta Y. \]

As the interest rate also affects consumption (and saving) decisions, we write

\[ \Delta C = \alpha_0 + \alpha_1 \Delta Y. \]

We recognize in \( \alpha_1 \) the marginal propensity to consume, previously denoted by \( \alpha \); we have \( \Delta C < \Delta Y \).

As to the sign of \( \alpha \), the effect of interest rate upon consumption "impatience" (preferring consumption in the present to consumption in the future) would suggest that \( \alpha > 0 \). On the other hand, as pointed out long ago by Marshall, Canon, and others, \( \alpha \approx 0 \) if society consists of people who plan to save, in the course of their active life, a fixed total sum. The higher the interest rate, the smaller the annual saving (e.g., life insurance premiums) needed to compound a fixed desired total. Finally, a large class of consumers is probably indifferent to interest rate (making \( \alpha \approx 0 \)) at least as long as the interest rate has the low levels characteristic of Western civilization in the last hundred years or more. Empirical evidence collected by Paul N. Douglas and by F. A. Radice suggests that, for the aggregate of consumers in the U.S.A. or Great Britain, \( \alpha \) is near zero. It is almost certainly smaller in absolute value than \( \beta \).

Businessmen when questioned by economists (of Oxford and of Harvard around 1938) have often stated that the interest rate plays too small a part in their cost to affect their investment plans, although this would not be true of the residential building industry which contributes a large share of total investment of the nation.

On the balance, the sum \( \alpha + \beta \) is probably positive, though much further research, including surveys of businessmen's attitudes, is needed. For further discussion of our model, it is this sum,

\[ \Delta C = \alpha + \beta \Delta Y, \]

that will matter. Denote total private (i.e., non-governmental) demand by \( D \) and write

\[ D = I + C; \quad \Delta D = \Delta I + \Delta C; \quad \Delta D = \Delta I + \alpha \Delta Y + \beta \Delta Y. \]

so that 14.1, 14.2 become

\[ 14.7.3 \quad \Delta D = \Delta I + \alpha \Delta Y + \beta \Delta Y, \]

where \( \Delta I, \Delta C, \Delta D \) are all positive.

Note that in 14.1 and 14.2--and therefore also in 14.3--we have neglected taxes. On the lines of Lecture 12, we ought to
reduce income \( X \) by the disposable income of individuals and of firms through introducing the exogenous variable \( T \) (tax revenue) and known tax rate. But different tax revenues and rates would be relevant to the demand of consumers and to that of firms. We don't want to go into detail here, and since the general method of studying the tax effect has been discussed in Lecture 13, we now permit ourselves to neglect taxes altogether and to uphold 14.3.

As before, we have the equilibrium condition for the market of goods: total (private and governmental) demand \( D + G \) equals total supply \( Y \):

\[
14.4: \quad D + G = Y.
\]

Since we have neglected taxes, \( T = 0 \), the symbol \( G \) stands at the same time for government demand and government "deficit-spending." It would not be difficult to re-introduce taxation into the system.
15.1

With a new variable (the interest rate, \( r \)) now introduced into the system, an additional equation is needed, unless \( r \) is exogenous, i.e., determined outside the system. The latter would be the case if the interest rate were directly controlled by the government. We would then write

\[ 15.1 \quad M = r, \]

where \( r \) is a constant; or we would indicate verbally that \( r \) is exogenous. It is somewhat more realistic to describe our monetary institutions by saying that

(1) public authorities (the Federal Reserve System in vague cooperation with the Treasury Department, directed by the President and loosely supervised by Congress) determine the supply of money, \( M^S \),

\[ 15.1 \quad M^S = M \] (say)

(2) firms and consumers desire to keep aggregate money stocks which are

(a) the larger, the larger their aggregate annual flow of receipts and expenditures, roughly proportional to aggregate income; this is the "transaction motive" for cash holding, the same one that was considered by the classical "circulation velocity" theory (see Lectures 2 and 5) and

(b) the larger, the lower the interest rate. Money's advantage over other assets (durable goods, securities) consists in its costless convertibility into any desirable asset at short notice. If times are uncertain, this advantage is important and is worth the sacrifice of a part of interest receipts. A high interest rate can, however, lure the individual into more daring decisions, i.e., into holding less cash and more securities or durable goods. [On "Money and the Theory of Assets", see Marschak in Economica 1938 (with Helen Makower) and in Economica 1938.] Hence we have the "liquidity preference equation"

\[ 15.2.1 \quad M = \lambda_1 Y \]

\[ 15.2.2 \quad M^S = \lambda_2 - \lambda_3 + \lambda_4 Y \] (using linear approximations)

where \( \lambda_1, \lambda_2, \lambda_4 \) are all positive. The "transaction motive" would account for at least a part of the term \( \lambda_4 Y \); so that \( \lambda_4 \) is roughly equal to or somewhat larger than the "Cambridge \( k \)"\((=1/\nu\), the reciprocal of the velocity of circulation);?

(3) The demand for money cannot, for any appreciable time, exceed or fall short of money supply:

\[ 15.3 \quad M = M^S \]

This is an "equilibrium condition" analogous to the one we have met in Lecture 8. Like 9.5, the equation 15.3 is merely an approximation for some dynamic equation that would describe the process which takes place when supply jumps over or dives under the level of demand. Also, institutions are imaginable under which supply for money exceeds demand for a sizable time. This was perhaps the case during the war when durable goods were rationed so that people were forced to hold larger money stocks that they would hold if they had free choice between cash and cars or houses.
We can combine Eqs. (15.1) and (15.2) into a single equation
accompanied by the statement that \( M \), the "equilibrium amount of money" (demand which is equal to supply), is exogenous.
The system arrived at in the previous lectures is

16.1 \[ D = \delta_y - \delta_y \gamma \]  
16.2 \[ D = Y - G \]  
16.3 \[ M = \lambda_y - \lambda_y r + \lambda_y y \]  
(This is 14.1.)  
(This is 14.4.)  
(This is 15.1.)

All these are behavior equations: (16.1) describes decisions on buying; the "equilibrium condition" for the market of goods, (16.2) is an approximation of some dynamic equation stating the response of sellers to excess demand; and (16.3) describes decisions on holding of cash. More precisely, (16.3) is not itself a behavior equation but is already the result of combining the demand-for-cash equation (15.2) with an equilibrium condition for the money market, (15.5). It is then understood that the following quantities in the system are exogenous: \( M \) (money supply) and \( G \) (government demand for goods) assumed to be financed by deficits only; and the coefficients (the \( \delta \)'s and \( \lambda \)'s) are also determined outside of the system. The three endogenous variables are \( Y, D, \) and \( r \). To evaluate the effect of policies, \( (G \) and \( M) \) upon, say, \( Y \), we have to solve the system for \( Y \): that is, to express \( Y \) as a function of \( G \) and \( M \), and not as a function of other endogenous variables. That is, we have to "eliminate" the other two endogenous variables \( (D \) and \( r) \). We can do this in two steps: 1) first eliminate \( D \) by using (16.1) and (16.2), and obtaining

16.4 \[ (1 - \delta_y) Y + \delta_y r = G + \delta_y \]

2) then solve (16.4) jointly with (16.3), rewritten as

16.5 \[ \lambda_y \cdot Y - \lambda_y \cdot r = M - \lambda_y \cdot \gamma \]

These two steps are represented on graphs 16.1A and B.
Part A of the two cases equates a solid line, for two different levels of interest rates, and equation (16.2) as a dotted line, for two different levels of government deficit. The resulting equilibrium points A1, A2, A3, A4 are re-plotted on chart B as points B1, B2, B3, B4. The negatively sloped line B-B0 and E-E0 express equation (16.4) for the two levels of G. The negatively sloped line expresses equation (16.5) for two different levels of the equilibrium point A. Each intersection point gives that equilibrium pair of Y and r that is determined by each of the four considered pairs of M and G (0 and 20; 0 and 50; 10 and 30; 10 and 50).

Algebraically, the joint solution of (16.4) and (16.5) gives

\[ 16.6 \quad Y = G \cdot \left( \frac{\partial Y/\partial G}{1 - \delta y \lambda Y} \right) = \left( \frac{\partial Y/\partial M}{1 - \delta y \lambda Y} \right) \times \text{a constant term, where} \]

\[ 16.7 \quad \frac{\partial Y/\partial G}{1 - \delta y \lambda Y} = \frac{(1 - \delta y \lambda Y) + \lambda Y \sigma Y}{\sigma Y}, \quad \text{and} \]

\[ 16.8 \quad \frac{\partial Y/\partial M}{1 - \delta y \lambda Y} = \frac{(1 - \delta y \lambda Y) + \lambda Y \sigma Y}{\sigma Y}, \quad \text{the constant term} \quad \frac{(1 - \delta y \lambda Y) + \lambda Y \sigma Y}{\sigma Y}. \]

Unless one is certain the original equations of the system are linear the resulting equation (16.6) must not be considered linear either, except as an approximation. The constant term therefore of little interest. But the partial derivatives \( \frac{\partial Y/\partial G}{\partial r/\partial M} \) are important even for non-linear systems, provided the changes considered are small. If the increments \( \Delta G \) and \( \Delta M \) are small,

\[ 16.10 \quad \Delta Y = \Delta G \cdot \left( \frac{\partial Y/\partial G}{1 - \delta y \lambda Y} \right) + \Delta M \cdot \left( \frac{\partial Y/\partial M}{1 - \delta y \lambda Y} \right). \]

Similarly, we can solve (16.4), (16.5) with respect to \( r \), and obtain, for the strictly linear case, a solution analogous to (16.6). The important part of the result, valid for small changes, also in non-linear cases, is analogous to (16.10).

\[ 16.11 \quad \Delta r = \Delta G \cdot \left( \frac{\partial r/\partial G}{1 - \delta y \lambda Y} \right) + \Delta M \cdot \left( \frac{\partial r/\partial M}{1 - \delta y \lambda Y} \right) \],

where \( \frac{\partial r/\partial G}{\partial r/\partial M} \) a positive denominator.

\[ 16.13 \quad \frac{\partial r/\partial M}{1 - \delta y \lambda Y} = \frac{(1 - \delta y \lambda Y) + \lambda Y \sigma Y}{\sigma Y} \],

the denominator being the same as in (16.7), (16.8). Thus, a rise in \( G \) raises \( Y \) as well as \( r \); a rise in \( M \) raises \( Y \) and depresses \( r \).

(16.7), (16.8), (16.11), (16.13) answer the question: how does a (small) change in \( Y \) and \( r \) affect national income and interest rate? One might also form expressions like \( \partial Y/\partial \lambda Y \), or \( \partial Y/\partial \lambda Y \) to express the effect upon income, or the "marginal propensity to demand", or in the "coefficient of liquidity preference", etc.

But it is meaningless if our model is valid to ask the question: "What is the effect of a change in the interest rate upon the income", (or "What is the elasticity of income with respect to interest rate"), since both are endogenous. To change any of them, at least one of the exogenous variables, \( M \) or \( G \) (or one of the Greek letters expressing psychosociological conditions) has to change. If the change in \( Y \) and \( r \) is due to a change in \( M \) only, while \( G \) is constant, \( Y \) and \( r \) move in opposite direction.
This is shown by a line such as B1, B2, on the equation (16.4); the latter equation shows that if C is to be kept constant, an increase in Y must be compensated by a decrease in r. On the other hand, if Y and r change not because of a change in C, but because of a change in G, then Y and r move in the same direction: this is clear from (16.5) or from looking at one of the upward-sloping lines on Part B of the Graph—say the line labelled "r = 30".

Finally, if both G and M change, then Y and r may move either in the same or in opposite directions (or may both stay unchanged), depending on the size and sign of changes in G and M. The policy instrument may be such as to tie G and M together, for example if deficit is managed by increasing the stock of money. Then, roughly, the increase of the annual deficit by $2 billion a year will increase money stock at the end of the first month by $0.12 billion, it will raise the year's average by $0.1 billion, and on the year's average by $0.1 billion. Therefore, by (16.10), (16.11), income and interest rate will be changed, on the year's average by, respectively

$$\Delta Y = 2^* \frac{\partial Y}{\partial G} \Delta G + \frac{\partial Y}{\partial r} \Delta r$$

$$\Delta r = 2^* \frac{\partial r}{\partial G} \Delta G + \frac{\partial r}{\partial r} \Delta r$$

Since the denominators in (16.7), (16.8), (16.12), (16.13), are the same, the ratio between the increments of income and interest rate is

$$\Delta Y/\Delta r = \left(2^* \frac{\partial Y}{\partial G}/(2^* \partial Y) \right) \Delta G$$

$$\Delta r$$

Whether this is positive or negative (i.e., whether Y and r move in the same or opposite directions) depends on whether $$2^* \frac{\partial Y}{\partial G}$$ is larger or smaller than (1 - $$\delta Y$$): i.e., it depends on the way in which demand for cash and demand for goods respond to income changes.

On our Graph (Part B), the tie between changes in G and M is expressed in the prescription that any shift in the negatively sloped line must be accompanied by a definite (i.e., not an independent) shift in the positively sloped one. Whether the resulting intersection points will align themselves along a positively or a negatively sloped line will then depend on the slopes of the original lines in Parts A and B. This geometrical result remains less definite than the algebraic result (16.14) unless one goes to more trouble.

Another case to consider is that of borrowing from the public. Then M and G could be fixed independently of each other. But this case (analogous to taxes, except when repayment is due), would involve reformulating (16.1) in terms of "disposable" income (see end of Lecture 14). The algebraic analysis of these cases is left to the interested students.

Equations such as (16.6) (and a corresponding one for r) have been called "reduced form." They express each endogenous variable as a function of exogenous variables only, and are therefore useful for discussion of policies. They do not, however, constitute by themselves an "economic theory": the latter is given by the original behavior equations, describing the behavior of people by relations which may involve interdependence between endogenous variables: e.g., consumers' demand depend on income. Given the system of such behavior equations, the "reduced form" can be derived.
Lecture 17.

Two special cases deserve discussion.

First, what is the relation between our result 16.6, and the "circulation velocity" theory of money which says

\[ Y = Mv \]

17.1 implies that \( \frac{\partial Y}{\partial G} = 0 \), that is, (by 16.7)

\[ \lambda = 0 \]

The "circulation velocity" theory of money thus neglects the effect of interest rate upon cash holding. If 17.2 is accepted then \( \frac{\partial Y}{\partial M} \) becomes simply \( \frac{1}{\lambda} \). And since 17.1 also means that the constant term 16.9 equals zero, we have \( \lambda / \lambda y = 0 \) and therefore \( \lambda = 0 \). That is, the demand-for-cash equation 16.3 coincides with the reduced form 16.6, both being expressed by 17.1. The level of money income is fully determined by money stock, via the (emasculated) demand-for-cash equation; it is entirely independent of interest rate \( r \), and of the propensity of consumers and entrepreneurs to demand goods! Keynes' contribution was to draw attention to the propensities to consume and invest, and to relate demand for cash to interest rate.

From the extreme "anti-Keynesianism" of 17.1 we now proceed to a second case under consideration, to an hypothesis constituting a bit of "extreme Keynesianism". This consists in stating that demand for cash tends to become infinitely elastic with respect to interest rate, as the interest rate approaches lower levels. Such a demand-for-cash equation 16.3 is drawn in Graph 17:1. (Note that we use here the \( M, r \)-plane, and give two different levels of \( Y \); while on Graph 16:1:B the demand-for-cash equation was represented by the two, positively sloped, lines in the \( Y, r \)-plane, for two different levels of \( M \).

![Graph 17:1](Equation 16.2:a)

\[ Y = 200 \]
\[ Y = 100 \]

Graph 17:1

The asserted property is

\[ \lambda \rightarrow \infty \quad \text{as} \quad M \rightarrow \infty \]

\( \lambda \rightarrow 0 \quad \text{as} \quad M \rightarrow \infty \)
The monetary policy has a less strong effect upon money income as money stock reaches higher levels, and interest reaches lower levels. Hence, money injection is neither useful (to overcome deflation) nor harmful (as a threat to price level in times of full employment). If interest rate is low, the assumption that money stock is determined only by the money market, that interest rate can never fall below a certain positive level, since uncertainty about the future (including the possibility of rising interest rates) will always induce people to compete for cash (while the lenders will always charge a certain rate to cover handling costs).

The policy implication 17.4 was opposed around the end of war II by economists who rightly foreshadowed inflationary effects of the high money stock accumulated during the war. In this argument one should distinguish between two independent hypotheses:

1. That 17.3 is false;

2. That 16.1 should be reformulated to include money stock among the factors determining the demand for goods. Thus:

\[ D = \dot{S}_m + S + M, \quad S_m \geq 0 \]  

(Compare with (3.1a).)

If 17.5 is accepted, then the "Keynesian" policy implication 17.4 becomes false even though the "Keynesian" assumption 17.3 (and the Graph 17.1) should be true. (The algebraic proof is left to students).

Crudely, even if money stock should be unable to influence demand indirectly via interest rate (by cheapening the interest rate and thus inducing the businessmen to borrow for plant expansion) it is perhaps still able to influence demand directly. However, little is known about the size of the relevant coefficient \( \gamma \) in 17.5.

With demand for goods related to cash stock as in 17.5, our system has two relations involving cash stock and income:

1. 17.6, describing the decision to demand goods, and especially (in the case of consumers) the decision to consume rather than not to consume;

2. 16.3, describing the decision to hold cash rather than securities.

The rationale of this behavior calls for closer microeconomic analysis.

Assumptions 17.3 and 17.5 have played a major role, not only in the recent discussions of inflation, but also in the discussion of wage policy. This will be taken up later.
Problem 9
(To be inserted after Lecture 17)

(Given as quarterly test at the end of the course.)

1. "The higher the national debt the higher the price level." Comment.

2. What are the conditions under which the relation between the quantity of money and the national money income, as derived from the equations expressing the behavior of consumers, businessmen and general cash-holders, would degenerate into the "equation of exchange"? (The "equation of exchange" says that national money income is proportional to the quantity of money.)
Lecture 18.

18.1

Generalize our system, as given in the beginning of Lecture 16 in the following three directions simultaneously:

1) Buyers are affected not only by income and interest rate but also by cash stock, \( Y' \), as in 17.5. 
2) Demand for goods is affected by disposable income, \( Y(1-\tau) \) (where \( \tau \) is tax rate) rather than by total income \( Y \); or still more generally, the demand for goods, and possibly also the demand for cash, is affected by both \( Y \) and \( \tau \). 
3) The functions involved are not necessarily linear; they will be denoted by Greek letters; \( \delta, \lambda, \ldots \); their partial derivatives will be denoted by \( \delta_x, \lambda_y, \ldots \).

We have then:

\[
\begin{align*}
\nu &= \delta(r, Y, \mu, \tau), \\
D &= Y - G, \\
M &= Y(r, Y, \mu, \tau).
\end{align*}
\]

It is understood that the following variables are exogenous: tax rate \( \tau \), government expenditure, \( G \), and cash stock, \( M \). The three equations then determine the three endogenous variables: total demand, \( D \) (of consumers as well as businessmen) for goods, total income, \( Y \), interest rate, \( r \). Thus income, \( Y \), (and also each of the other two endogenous variables) depends entirely on the following controlled conditions: \( \tau \), \( G \), \( M \); and on the following uncontrolled conditions: the behavior functions \( \delta \) and \( \lambda \). We shall express this by writing the "solution of the system":

\[
\begin{align*}
Y &= \mathcal{f}(\tau, G, M; \delta, \lambda), \\
D &= Y(\tau, G, M; \delta, \lambda) \\
r &= X(\tau, G, M; \delta, \lambda),
\end{align*}
\]

where the symbols \( \mathcal{f}(\ldots) \) \( Y \) are equivalent to the words "depends on the quantities or functions listed in parentheses". A solution of a system of economic behavior relations -- such as (18)--, with respect to each of the endogenous variables, is also called the "reduced form" of the system. It shows the effect of each of the conditions upon each of the endogenous variables. It is thus a guide to policies (compare also Lecture 1), provided something is known about the form and parameters of the functions (such as \( \delta \), \( \lambda \)) involved.

The model (18) and consequently the reduced form (18.a) assume an unduly simplified form of government decisions. This can be corrected to a certain extent, still retaining great simplicity in the system.

Instead of assuming \( G \) exogenous, we can take account of the fact that a part of the government expenditure -- viz., the unemployment relief payments -- depend on national income, for example

\[
(18.1') \quad G = G_0 + \delta(Y_{\text{max}})
\]

where \( Y_{\text{max}} \) is the national income at which no relief payments would...
be necessary, and \( h \) and \( j \) are certain policy constants depending on the government's previous commitments (such as w.r pensions) and on its past or current views as to the importance of unemployment relief and of other government goals. More generally, we can simply write

\[
G = Y(Y),
\]

where \( Y \) is a function (schedule) controlled by the government within known limits imposed by its previous commitments, the foreign situation, the constitution, etc. We have now one more endogenous variable, \( G \); and one more equation, (18.1).

Similarly, the assumption that cash-stock \( f \) is arbitrarily fixed may be too unrealistic, even if (as in Lecture 15) we replace the statements

\[
\begin{align*}
1 & = \lambda(r, Y), \\
M & = \text{exogenous}
\end{align*}
\]

by the statements

\[
\begin{align*}
\text{demand} & \quad : M = \lambda(r, Y, \pi) \\
\text{supply} & \quad : M = \text{exogenous}
\end{align*}
\]

It may be deemed more realistic to replace the last two statements by a statement such as

\[
W = \mu(r, Y),
\]

where the function \( \mu \) describes the policy (a schedule) which may be chosen by the banking authorities; they have decided that, at a given income level, they will have a certain supply function, i.e., vary the volume of loans concomitantly with the interest rate in a certain way.

How do these amplified descriptions of the fiscal and monetary policy effect our reduced form (15.a)? Simply by replacing the quantities \( G, M \) (which now become endogenous) by the letters \( r, Y \), which describe not quantities, but functions, viz., certain response patterns chosen by government and banking authorities. We have, then,

\[
Y \phi(Y, \pi, \mu, \delta, \lambda),
\]

where the symbols after the semi-colon refer to the response patterns of the public, while those before the semi-colon give the response patterns of the government and are therefore at least partly controlled by the government.

We have thus summarized a model in which the demand for cash and for goods, the supply of cash and the supply of goods (national income) all enter the decisions of individuals and of public authorities as sums of money. We have denoted sums of money by capital Roman letters (see Lecture 2), reserving lower-case roman letters for variables (such as the interest rate \( r \)), which do not depend on the chosen unit of money. We have an identity (definition-related), (2.2) or

\[
Y \phi(Y, \pi, \mu, \delta, \lambda),
\]

so that (18, b) helps to relate real income to the various controlled and uncontrolled conditions as follows:

\[
Y \phi(Y, \pi, \mu, \delta, \lambda),
\]

We denote price level by \( P \), measuring, as it were, the number of dollars per unit of an aggregate of physical goods. Alternatively, one might regard the price level as the average of ratios between prices of various commodities in two years; a pure number, to be denoted by a lower-case letter.
the real rate of interest is represented by graph, such as (2.3), with the rectangular hyperbola (demand curve for all goods) subjected to shifts which are not necessarily by changes in money stock, R, but by changes in any or all of the controlled or uncontrolled conditions. We have here, a curve for all goods is a more general and sophisticated version of the old one, the equation of exchange (see also Lecture 1). First paragraph.

But now assume a different behavior on the part of both the origi- nal people and the government (including monetary authorities): suppose they think entirely in terms of physical goods, similar to the equation (13.19), we have the only to replace in our model the monetary terms (D, Y, W, a) by physical quantities (d, y, ω, m); instead of (13.14) we have, say,

\[ d = \sigma(x, y, m, ω), \] etc.

Obviously, the reduced form (13.2) or (13.3) will be replaced by

\[ y = \sigma(τ, ω, m; δ*, λ*); \] or more generally,

\[ y = \sigma(τ, ω, m; δ*, λ*), \]

where δ*, λ* denote policy schedules with regard to government spending and money supply. In (13.4) real income (quantity of physical goods supplied or level of production) is completely determined by fiscal and monetary policy given people's response patterns. These latter response patterns (δ*, λ*) as well as the decisions of public authorities were assumed "free of money illusion", i.e., independent of any price changes that leave physical quantities unchanged (that is, are offset by proportional changes in money sums.) No wonder that in (13.4) real income is shown to be independent of the price level. In the (p,y)-plane of graph 2.1, the "demand curve for all goods" would be, in this case, represented by a vertical line, subjected to shifts which are due to changes in monetary or fiscal policies.

This extreme case of "froaction from money illusion" is on the part of demanders for and suppliers of goods and cash, is of course, not real. At least the monetary authorities fix their policy in terms of dollars, and not of their physical ("deflated") equivalents. The governmental appropriations and the budgeting of firms' plans are at least to a large part, fixed dollar-wise, although upward adjustments often follow a fast rise in prices. As to the consumers, who can say whether thehousewife acts as in

\[ c = \alpha_y + \alpha_c \] ("no money illusion"),

or as in

\[ c = \alpha_y + \alpha_d \] ("money illusion", viz: "reckoning in dollars only"),

and hence

\[ c = \alpha_y + \alpha_d/p; \]

or, thirdly, as in

\[ c = \alpha_y - \alpha_d p + \alpha_0 \] (another case of money illusion).

Once "money illusion" is assumed on the part of at least one group of individuals, price level enters the system of behavior equations. Consequently (13.3) becomes
where the real income, an endogenous variable, is expressed as a function of exogenous variables (e.g., m) and/or other controlled and uncontrolled conditions (Greek letters), and in addition, of the price-level, p, another endogenous variable. (16.4) is obviously a special case of (16.5), for we have in (16.5) a "demand curve for all goods", but not necessarily in the form of the rectangular hyperbole. (16.5)—or, for that matter, its special case, (16.4)—involves two endogenous variables, y and P. It is not a reduced form. It does not explain how either Y and P are determined by outside conditions. There must exist a further relationship between these same variables. We shall discuss this relationship—"the supply curve for all goods"—in the next lecture, to derive the effect of alternative policies upon y and P, as promised in the first lecture of this course.

"We omit asterisks, as the functions involved may or may not involve "money illusion"."
Each of the relations (16.2) between real income (output) and price level was derived from behavior equations explaining the responses of people who demand goods or cash. With more brevity than profundity we shall call either of the equations (16.5) the "demand curve for all goods"; remembering, however, that it is not a behavior relation like 16.1 or 16.3. Rather, it is obtained by combining several behavior relations in an arbitrary way: viz; by eliminating all endogenous variables except two, (price and real income), whose relations can then be conveniently plotted on a plane under given controlled and uncontrolled conditions. We think of price and output as quantities fully determined by those conditions, there must exist a second relation between price and output, which we shall call, for brevity "supply curve for all goods", and which we shall be able to plot on a plane. Again, this relation will be derived by combining certain behavior relations. We may regard those behavior equations as constituting the "supply sub-set" of the full system of behavior equations; and the equations treated in Chapters 5-16 as elements of the "demand sub-set".

Denote by $n^d$ the aggregate demand of employers for labor, when the money wage rate $W$ and the price level $P$ are given. Denote by $n^s$ the aggregate amount of labor which the individual workers are willing to offer at given $W$ and $P$. Denote by $n$ the actual level of employment. The following "supply sub-set" is worth considering:

(19.1) $n^d = \Delta(W, P)$: labor demand function
(19.2) $n^s = \delta(W, P)$: labor supply function
(19.5) $\Delta = \hat{N}(n)$: short run production function, neglecting the output of the self-employed.

With the help of these 5 equations we can, in general, eliminate 5 out of the 7 variables involved ($n^d, n^s, W, P, y$). By eliminating all variables but $P$ and $y$ we obtain a "supply curve for all goods", under conditions of free labor market.

In the following example we shall assume more specially that employers' demand for labor is a decreasing function of the real wage rate only, while workers' supply of labor is an increasing function of the money wage rate only, finally, the output $y$ increases with employment, but at a decreasing rate ("decreasing marginal returns to labor"). That is, remembering (19.3)-(19.4), our "supply sub-set" of equations becomes:

(19.a.1) $n = \Delta(W/P)$: a decreasing function
(19.a.2) $n = \delta(W)$: an increasing function
(19.a.3) $y = \hat{N}(n)$: a function that increases at decreasing rate.

On Graph 19.1, each of the equations (19.a.1), (19.a.2), is represented in the $(n, P)$-plane by a family of curves, each curve being drawn for one value of $W$. Thus when $W = 2$, employment = 60 mill. and
The locus of each intersection point through the intersection of labor supply and labor demand, \( z_0 \), represents the amount of change in the price level (whatever it may be) that make the price level change. On the right-hand scale for measuring the output \( y = N(z) \) is indicated: output grows slower than employment.

On Graph 19.11, \( y \) is re-plotted on an ordinary scale: we obtain a "supply curve for all goods" which can now be shown together with a "demand curve for all goods" such as the curves of Graph 2:4 which correspond to equation (18.1), or more generally, a curve representing (18.8). The position shifts of the demand curve for all goods was shown in Lecture 18 to depend on certain controlled and uncontrolled conditions, viz: the fiscal and monetary policy and the behavior of those who demand cash or goods:

\[(19.6)\quad y = \Delta (P; \zeta, \gamma, \lambda, \delta, \lambda), \text{ say.}\]

Similarly, the position of the supply curve for all goods depends on the conditions which figured in the "supply sub-set" (19.4); viz: on the behavior of those people who demand or supply labor, and on the production function:

\[(19.7)\quad y = \Delta (P; \Phi, \zeta, \lambda).\]

A change in any of the Greek letters, either in the demand curve, (19.6) or in the supply curve (19.7), leads to a change in the real income and/or price.

In (19.6) the demand for but not the supply of labor was assumed free of money illusion. As in extreme case, it was even assumed in (19.2.a) that workers are interested in money wage rates only and do not pay attention to prices. If, as another extreme, we assume workers interested in real wage rates only, then employment becomes independent of price-level, since the two equations

\[(19.8)\quad n = \Delta (W/P), \quad n = \Delta (W/P),\]

determine \( n \) and \( W/P \). In this case, a change in price results in a proportional change of money wage-rate (and conversely), but does not affect employment. The SS-line of Graph 19.11 becomes horizontal. Therefore no shift of the DD-line (i.e., no change in fiscal-monetary policy or in the propensity to consume, invest or hold cash) can affect real income and employment. This is sometimes described as the "classical" or "anti-Keynesian" theory of the labor market.

On the other hand, to obtain an upward-sloping and not a horizontal DD-line, it is not necessary to make assumptions as narrow as (19.4). Let us first maintain the assumption (19.1.a): labor demand is free from money illusion. But let us generalize (19.2.a) into

\[(19.9)\quad n = \Delta (W/P), \Delta n \text{ an increasing function.}\]

\[(19.10)\quad n = \Delta (W/P), \Delta n \text{ an increasing function of } W, \Delta n \text{ a decreasing function of } P.\]

We still obtain the result that a price rise raises employment, provided the (positive) elasticity of \( \Delta n \) with respect to \( W \) is numerically larger than its (negative) elasticity with respect to \( P \). (Note that if the two elasticities were equal we would have \( n = \Delta (W/P) \), the "classical" case (19.8).) For example, suppose \( P \) is doubled. What will happen to \( W \), and hence to \( W/P \) and to \( n \)? If \( W \) would also double, there would be an excess of labor supply (which we have
Graph 19: I  Free labor market: An example

Vertical lines: labor demand function \( n^d = 40 + 20 P/W \)
Horizontal lines: labor supply function \( n^s = 70 - 10/W \)

Circle points satisfy \( n^d = n^s \).

Scale for \( y \) expresses a production function \( y = \Pi (n^d) \) such that marginal product = real wage offered by employers:
\[
\frac{d \Pi}{dn^d} = \frac{W}{P} = 20/P^d - 4Q
\]

Graph 19: II  Determination of price and output when labor market is free:
D: supply curve for all goods, from Graph 19: I
D: demand curve for all goods, when money illusion is present in at least one equation of demand subset.
The maintaining of (19.c.1)—response of employers to real wages
only—is, however, itself not necessary. It is usually derived from
sertain assumptions which can be partly relaxed. These conditions are:
(1) The only production factor considered is hired labor.
(2) Each firm maximizes its profit.
(3) Money wage-rates and prices are the same for each firm.

Condition (1) may be relaxed in two directions:
(a) An additional variable, capital, can be introduced into (19.a.1)
and (19.a.3), to be defined in an additional (and dynamic) equation,
"capital equals the sum of past investments". We shall not
pursue this here since variations of aggregate capital over
periods of a few years are, in fact, negligible in their effect
upon output.
(b) The labor of the self-employed persons can be separated out,
possibly as an exogenous variable; we shall refer to this fact
in the next lecture.

Condition (2) helps to define the aggregate production function
\( y = \Pi(n) \). This function, and also the aggregate marginal product
d\( \Pi/dn \), is not defined, unless the distribution of labor between firms
is known. If, in particular, condition (3) is replaced by the stronger
statement that there is perfect competition between firms in buying
labor and selling their product, then (2) implies

\[ 19.2 \quad d\Pi/dn = W/P, \]

as exemplified in the last line of the explanation to Graph 19:1.
Under imperfect competition, however, (19.e.8) is, in general, invalid.
But (2) and (3) still make each firm's labor demand, and hence the
sum of these demands depend on \( W/P \), as in (19.a.1); and still make
the distribution of labor of firms, and hence the production function
in (19.c.3), well determined.

For the purpose in hand it is, however, not necessary to main-
tain condition (2), profit-maximization. One may assume instead that
each firm \( i \) (\( i = 1, 2, \ldots \)) has an individual demand function
\( n_i = \Delta_i(W, P) \), and an individual production function \( y_i = \Pi_i(n) \).
Then, (with \( n^d = n = n \))

\[ 19.d.1 \quad n = \Delta_1(W, P) + \Delta_2(W, P) + \ldots + \Delta(W, P) \]
\[ 19.d.2 \quad n = \Pi_1(\Delta_1(W, P)) + \Pi_2(\Delta_2(W, P)) + \ldots \]
\[ 19.d.3 \quad y = \Pi_1(\Delta_1(W, P)) + \Pi_2(\Delta_2(W, P)) + \ldots \]

Thus, in absence of profit maximization and perfect competition,
the aggregate labor demand function is not related to the derivative
(marginal product) of production function, and is not a function of
real wage rate. The "general supply sub-set" (12:1) still results in a "supply curve for all goods." - Graph 19:II; whether this curve is upward-sloping depends, of course, on the properties (e.g., elasticities with respect to \( W \) and \( P \)) of the generalized labor demand and supply functions. Assuming \( T(n) \) always upward-sloping, a necessary and sufficient condition of \( DD \) to be upward-sloping is this: the elasticity of labor supply with respect to \( W \) must exceed its elasticity with respect to \( P \); i.e., a larger proportion than the proportion by which the elasticity of labor demand with respect to \( W \) exceeds its elasticity with respect to \( P \). This condition is likely to be fulfilled; vaguely speaking, while both employers and workers are wage-conscious, the former are relatively more price-conscious than the latter. This is more general than the case studied earlier when the labor demand had equal elasticity with respect to wage and to price, while the wage-elasticity of labor supply was higher than its price-elasticity.

Thus, after dropping the assumptions of maximum profit and of perfect competition, we can still maintain Graph 19:II, and the underlying "demand sub-set" and "supply sub-set" of equations, as an explanation of how the price level and the real income are determined in a free labor market, provided that hired labor is the only varying factor of production.
If there is no "money illusion" in any equation of the demand sub-set, the curve DD of Graph 19:II is horizontal: the "extreme Keynesian" case, implying the inability of wage-cuts to affect real income. If there is no "money illusion" in any equation of the supply sub-set, the curve SS of the same Graph is horizontal: the "classical" case, implying the inability of fiscal-monetary policy to affect real income. Let us inspect the "classical" case a little more closely, starting with a more general one.

On Graph 20:I:a, the labor market equations of Graph 19:I are replotted, using this time the horizontal axis to plot, not the price $P$, but the real wage-rate $w = W/P$. Since on Graph 19:I (as in equations 19.a) labor demand but not labor supply was free of money illusion, the former is now represented, in the $(w,n)$-plane, by a single (downward sloping) curve, but the latter by a family of (upward sloping) curves, each corresponding to a different price level. We recognize the (circled) equilibrium points of Graph 19:I.

Graphs 20:I:a and :b ;Free labor market. No money illusion in labor demand. Downward sloping line: labor demand function $n = n'$. Circled points satisfy $n = n'$.

[Explanation of the Graph continued on page 2.]
Money illusion in labor supply. 

Upright sloping line: labor supply function \( n^d = \frac{70 - 10\log w}{w} \).

This Graph 20:IIa repeats Graph 19:II, but is drawn in \((w,n)\)-plane instead of \((w,P)\)-plane.

Some points where \( n^d = n^s = n \):
- \( n = 50; \ w = 0.25; \ w^d = 2; \ w^s = 0.5; \ w = 1 \)
- \( n = 60; \ w^d = 1; \ w^s = 1; \ w = 2 \)
- \( n = 65; \ w^d = 2.5; \ w = 0.8; \ w = 2 \).

If the still more general assumption of labor demand nor labor supply were free of money illusion, each would be represented by a family of curves, so that to each price-level would correspond a different pair of curves; we should again have a set of intersection points, one for each price level.

On the other hand, Graph 20:IIa represents the "classical" case, equations 19b. In this case, at a given real wage rate neither the demand for nor the supply of labor is affected by price. Consequently each is represented by a unique curve and not by a family of curves. The intersection of these two curves gives the real wage rate and the employment level at which labor demand and labor supply balance.

Graph 20:IIa, b and c.

Determination of price and output when labor market is free.

Accordingly, in the "classical" case employment is independent of price. The supply curve \( SS \) for all goods is, in this case, not upward sloping as in the general case represented on Graph 20:IIa (which is a reproduction of Graph 19:II); instead, the SS-curve is horizontal, as on Graph 20:IIb. In this case, the price level only, but not the real income (and consequently not the employment level) can be influenced by shifts in the demand curve \( DD \) for all goods.
Real income (and employment) can, in this case, be influenced only by shifts in the supply curve (SS). These shifts can originate in this case as in any other case, only in changes of the production function \( x(n) \), the employers' demand schedule \( D \), or the laborers' supply schedule \( S \). In particular, if the workers become willing to offer the same work at a lower wage rate (or more work at the same wage-rate), i.e., if they revise their evaluation of leisure vs. goods, the labor supply curve on Graph 20:IIa will shift to the left; employment will increase while real wage-rate will fall. Thus, it is in the workers' power to increase employment; while monetary and fiscal policies -- the shifting of LD-curve on Graph 20:IIb -- cannot affect employment but can result only in changing the price level. All unemployment is voluntary.

The extreme opposite case occurs if, as on Graph 20:IIc, not the "supply curve (SS) for all goods", but the "demand curve (DD) for all goods" is horizontal. This would be the case if all decisions listed in the "demand sub-set" (Lecture 13) -- private decisions as well as those of fiscal and monetary public authorities -- were made in the absence of money illusion. In this case, real income and hence (via the production function) employment is completely determined by these decisions. The revision of workers' willingness to work, causing a shift in the SS-curve, would change price-level, but not employment. All unemployment is involuntary.

This "extreme Keynesian" position was assumption 19c (labor demand free of money illusion because employers maximize current profits). In this case, if real income and employment are determined by fiscal and monetary policy, the real wage rate is also determined, since to each level of employment corresponds only one real wage rate (equal to the marginal product of labor). But this ceases to be necessarily true under the more general assumption 19d, where \( n^d \) is some function of \( W \) and \( P \), not necessarily of their quotient \( W/P \). Furthermore, the existence of self-employed labor, not subjected to the money wage cuts hitting hired employees, also prevents prices from following the money wage rates exactly, even if the demand for labor were a function of real wage rate. (To prove this, use \( n \) to denote the hired, and \( n' \) the self-employed labor; denote their money rewards per hour by \( W \) and \( W' \) respectively, and study the effect of changing \( W \) when \( W' \) remains constant).

The assumption that the "demand sub-set" but not the "supply sub-set" is free of money illusion leads thus, under conditions of a free labor market, to the proposition that a revision of workers' supply schedule (\( S \)) can lead to no change in employment but only to a change in prices (not necessarily proportional to the money wage cut).

This concept of involuntary unemployment has been used in particular by A. P. Lerner in his various writings on Keynesian theory (partly summarized in his "Economics of Control"). It is in line with Book V of Keynes' "General Theory." However, there also exists quite a different concept of involuntary unemployment: the excess of the labor supply \( n^s \) (number of people willing to work at a given \( W \) and \( P \)) over employment \( n^e \). The equations -- used by us so far -- of a quickly adjusted free labor market (19.4), (19.3) are not compatible with the phenomenon of involuntary
unemployment thus defined. It presupposes a different view of the labor market. As in other markets where the equalization of demand and supply is slowed up or obstructed by technological or institutional causes (e.g., the housing market), we may have to replace the two equations (19.4); (19.3) by a single equation

\[(20.1) \ n = \text{Min}(n^d, n^s)\]

employment is equal to either the demand for or the supply of labor, whichever is smaller. It is understood that \(n^d\) and \(n^s\) are derived as before by aggregating the labor demand schedules of individual employers and the labor supply schedules of individual workers, respectively. In the resulting equations

\[n^d = A(W, P); n^s = \sigma(W, P)\]

the symbols \(A, \sigma\) have the same meaning as in Lecture 19. Equation (20.1) says that if \(n^d < n^s\), then employers cannot press into the workshops more workers than those willing to work at existing \(W\) and \(P\); and if \(n^d > n^s\), the workers willing to work at existing \(W\) and \(P\) but not offered jobs cannot force themselves upon the employers. This is a fair description of our institutions. If \(n^s < n^d\), the difference \(n^d - n^s\) is called labor shortage. If \(n^d > n^s\), the difference \(n^d - n^s\) is called involuntary unemployment in its second sense. This is more or less on the lines of Keynes' Book V, where the willingness of individuals to work at existing wages and prices is emphasized; though it was probably not too clear to Keynes that this concept is really different from that of Book V; nor is it made clear (p. 15) that when \(n\) is exceeded by \(n^s\), \(n\) is not also exceeded by \(n^d\) but is equal to \(n^d\) (equation 20.1). "Involuntary unemployment" in the first sense (inability of wage-cuts to raise employment) is clearly not involuntary in the sense that not all people willing to work at existing \(W\) and \(P\) get jobs. They all do.

If \(n^d = n^s\), then the number of actually filled jobs, \(n\), is \(n = \sigma(W, P)\) the number of jobs wanted at the existing money wage rate \(W\) and existing price level \(P\).

With the single equation (20.1) replacing the two equations \(n^d = n^s\), one equation is lacking to make the system determinate. The failure of labor demand and supply quickly to become equal to each other is explained by the fact that the wage-bargaining is done by unions rather than by individuals. The action of the unions (or of the joint bargaining bodies, possibly including a public arbiter or the government) must be expressed by an equation of the system. As the simplest postulate we may write

\[(20.2) \ W = \text{W}\]

where \(W\) is an exogenous quantity fixed independently of current economic variables; or, possibly more realistically,

\[(20.2a) \ W = \text{W}(n, P)\]

indicating that the outcome of bargaining depends on employment (or real income) and price level. One might be tempted to call (20.2a) --if rewritten with \(n\) on the left hand side-- a "labor supply function of the unions." But this expression would be confusing and would wrongly depict the union decision as that of labor contractors delivering varying quantities of men (or man-hours) depending on varying wage offers.

Graph 20:III uses for simplicity the special labor demand and labor supply functions of Graph 19:1: individual firms react to
real wage rates, while individual workers react to money wage rates only and not to \( \bar{W} \) and \( P \), which would be a more general case of money illusion. But since the labor market is now considered unionized, the condition \( n^d = n^s - \sigma \) is dropped, and union action introduced, thus:

\[
\begin{align*}
n^d & = \frac{\Delta (W/P)}{W/P} \\
n^s & = \sigma (W) \\
(20.5) & \quad n = \min (n^d, n^s)
\end{align*}
\]

**Graph 20:III**

*Unionized labor market (using labor demand and supply functions of Graph 19: I)*

**Graph 20:IV**

*Graph 20:IV*  
Determination of price and output when labor market is unionized.

\[D_D: \text{demand curve for all goods (not free of money illusion)}\]

\[S_S(\text{viz. } S_1S_1, S_2S_2, \ldots \ldots \text{ when } \bar{W} = 1, 2, \ldots )\text{; supply curves for all goods depending on union-fixed money wage rate } \bar{W} \text{ and derived from Graph 20:III.}\]
Graph 20;Il is obtained accordingly from Graph 19;I by cre-ating, for each level of $W$, that part of the Labor Demand curve where demand exceeds supply; and that part of the Labor Supply curve where supply exceeds demand. The remaining segments constitute, for each level of $W$, a relation between employment $n$ and price $P$; a $y$-shaped line, one for each value of $W$ fixed by the unions. To the left of the "kink" there is involuntary unemployment; to the right, labor shortages.

Using now the production function $y = Z(a)$--the same as on Graph 19;I--right-hand scale--we can draw "supply curves for all goods" corresponding to the employment-price relations that were derived on Graph 20;II. We thus obtain Graph 20;IV. Each "supply curve for all goods" corresponds to a certain union-fixed money wage rate $W$. The "demand curve (DD) for all goods" is thus intersected in different points, depending on $W$. Union action is responsible for the shifts of the "supply curve for all goods"; monetary and fiscal policy is (as before) responsible for shifts (not shown on the graph) of the "demand curve for all goods", $DD$.

Should $DD$ be horizontal, (due to absence of money illusion throughout the demand sub-set of the system), the situation already discussed for the case of the free labor market would repeat itself in the present case of the unionized labor market. No action of the unions can affect real income and employment if real income is completely determined by other factors outside of the union's control. In such a case, we would have, in general, involuntary unemployment in both senses: in the sense of ineffectiveness of wage-cuts as well as in the sense that some unemployed people are willing to work at the current wage rate and price level.

It would be desirable to replace assumptions (20;3) by more general ones:

1) the labor demand and supply functions should be generalized into $\Delta(W, P)$ and $\sigma(W, P)$, as in (19;D):

2) the union action should be expressed, not by (20;2), but by the more complicated assumption (20;2:a), which does not regard unions as completely independent of economic conditions. The fact that union bargaining is to a certain extent influenced by government policy can be taken care of by varying the function $\mu$ (the unions' "schedule" of action).

We must conclude here by reminding ourselves of the program set cut in Lecture 1: to evaluate the effect of alternative conditions upon employment $n$ and price level $P$. The controlled conditions (policies) considered were $\zeta$, $\omega$ (or $W$), $\gamma$ (or $\sigma$)--as discussed in Lecture 17; we have now added to them $\Delta$, $\alpha$, $\lambda$ of Lecture 17; and the functions $\Delta$, $\sigma$, $\lambda$ of Lecture 18. Graphs such as 20;II (the case of the free labor market) or 20;IV (for the case of the unionized labor market) summarize the system by conveniently splitting it into a "demand sub-set" and a "supply sub-set", resulting respectively in the demand and the supply curve "for all goods."
The "extreme Keynesian", and the "classical case", in which one of the two curves is horizontal, arise when either the demand sub-set or the supply sub-set of equations is free of "money illusion"; i.e., if the corresponding decisions run in real terms only. These extreme cases deserve attention because of their policy implications, already listed in Table 11:1. According to whether actual facts are nearer the one or the other extreme hypothesis, more stress has to be laid on fiscal-monetary, or on money wage-rate policy."

*) Note: Examine also the attached problem 10, used as a test at the end of the quarter. This problem involves a policy of fixing real wage rates, as attempted recently in the contract between General Motors and the United Automobile Workers.
Problem 10

(Given as quarterly test at the end of the course)

Assume:

Number of physically employable persons = 65 million;
Money income constant at $190 billion;
Demand for labor (in millions) = $d = -2w + 62$, where $w =$ real wage rate in "1950 dollars" per hour.
Supply of labor (in millions)

$n^s = (1/2)(wp) + 60$; where $p =$ price level (1950 = 1);
Employment = $n = n^d$ (neglect the possible case of labor shortage).

Real wage rate fixed by unions from time to time;
Net real output in "1950 dollars" = 3000 n;

Question: What is the effect of raising the real wage rate from 1 to 2 "1950 dollars" per hour, upon

1) "objectiv" unemployment, i.e., difference between the number of physically employable people and the employment;

2) price level;

3) "involuntary" unemployment defined as the difference between the number of people willing to work and the employment.

* * *

Answer

When $w = 1$ and $p = 1$, then $n = n^d = -2 + 62 = 60$.

$n^d = 1/2 + 60 = 60 1/2$

Objective unemployment = 65 - 60 = 5

Involuntary = 60 1/2 - 60 = 1/2

When $w = 2$, then

$n^d = -1 + 62 = 58$;

Objective unemployment = 65 - 58 = 7

Real income

$y = 3n = 174$

$p = 180/174 = 1.034$

$n^s = (1/2)(2)(180/174) + 60 = 61.034$

Involuntary unemployment = 61.034 - 58 = 3.034

Summary

\[
\begin{align*}
  w &= 1 \\
  p &= 1 \\
  n^d - n_{max} &= 5 \\
  n^d - n_{max} &= 7 \\
  n^s - n^d &= 0.5 \\
  n^s - n^d &= 3.034
\end{align*}
\]