A Remark on Arrow's "Homogenous System in Mathematical Economics: A Comment" (C.C.D.P. Ec. 242)

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February 17, 1949

Arrow's approach has two advantages over Tintner's: 1) simplicity and 2) generality. Generality consists in the extension to non-differentiable functions and to discrete variables. Simplicity is achieved by not using the Euler property of homogenous functions but using, instead, that property which customarily is regarded as their definition, viz., "f is (positive) homogenous of degree K in z if, for any $t > 0$, $f(tz) = t^K f(z)$."

It may be interesting to note that the use of the Euler property has made Tintner's proof unnecessarily complicated even for his special case, viz., that of differentiable functions. Using the customary definition only, the assumptions of Tintner's Theorem 6 imply that there exist numbers K, H, and a non-zero vector $\lambda \equiv \{\lambda^{(k)}\}$ such that, for any positive s, t,

(A) $$g(x, x^*, x^{**}) = t^{-K} g(x, x^*, tx^{**}),$$

(B) $$0 = h^{(k)}(x, x^*, x^{**}) = s^{-H} h^{(k)}(x, x^*, sx^{**}), \; k = 1, \ldots, N_i$$

(C) $$\frac{\partial}{\partial x_i} g(x, x^*, x^{**}) + \sum_k \lambda^{(k)} \frac{\partial}{\partial x_i} h^k(x, x^*, x^{**}) = 0, \; i = 1, \ldots, p$$

Differentiate (A), (B) with respect to x_i, (note that x^*, x^{**} are independent of x) and insert the resulting right-hand member into (C):

(D) $$t^{-K} \frac{\partial}{\partial x_i} g(x, x^*, tx^{**}) + \sum s^{-H} \frac{\partial}{\partial x_i} h^{(k)}(x, x^*, sx^{**}) \lambda^{(k)} = 0,$$

$i = 1, \ldots, p$, for any $t, s > 0$. Hence the solution of (D), (B),

$$x_i = x_i(x^*, tx^{**}, sx^{**}, t, s), \; i = 1, \ldots, p,$$

holds good for $t = 1$, $s = 1$, as well as for any other positive t, s. That is, x is homogenous or zero-degree in x^{**}.

As indicated by Lange, Hurwicz, and others, an important application to economics is, of course, this: Maximize $u = f(x)$ subject to $h = y - \sum p_i x_i = 0$, where $x \equiv \{x_i\}$ is the consumption vector, $p \equiv \{p_i\}$ the price vector, and y the income. Since both u and h are homogenous (of zero and first degree respectively) in $x^{**} \equiv (y, p)$, the maximizing value of x is homogenous of zero degree in x^{**}: the case of "no money illusion."