COWLES COMMISSION DISCUSSION PAPER: ECONOMICS: NO.246

Index Number Problems in the Empirical Determination of Import Demand Functions

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February 16, 1949

Let
x = quantity of imports

y = quantity of exports

p_x = price of imports

p_y = price of exports

η_x = elasticity of demand for imports

η_y = elasticity of (foreign) demand for exports

ε_x = elasticity of (foreign) supply of imports

ε_y = elasticity of supply of exports

D = percent of depreciation

The effect of a depreciation of D on the demand for foreign currency arising from the purchase of goods in any import "category" is given (in terms of foreign currency) by

(1) $$\frac{(p_x + \Delta p_x)(x + \Delta x) - xp_x}{xp_x} = \frac{x\Delta p_x + p_x\Delta x + \Delta x\Delta p_x}{xp_x} \cong \frac{\Delta p_x}{p_x} + \frac{\Delta x}{x}$$

(2) $$\eta_x = \frac{\Delta x/x}{D + \Delta p_x/p_x}$$ (3) $$\varepsilon_x = \frac{\Delta x/x}{\Delta p_x/p_x}.$$

From these definitions we obtain

$$D\eta_x + \eta_x\frac{\Delta p_x}{p_x} = \varepsilon_x\frac{\Delta p_x}{p_x}$$

(4) $$\frac{\Delta p_x}{p_x} = \frac{D\,\eta_x}{\varepsilon_x - \eta_x}$$

while from (3) and (4), we get

(5) $$\frac{\Delta x}{x} = \frac{D\,\eta_x\ \varepsilon_x}{\varepsilon_x - \eta_x}.$$

The relative change in demand for foreign currency resulting from a depreciation of D can therefore be written

(6) $$\frac{D\,\eta_x\,(\varepsilon_x + 1)}{\varepsilon_x - \eta_x}$$

or if η_x is treated as the absolute value of the price elasticity of demand for the "category" in question,

(6a) $$\frac{-D\ |\eta_x|\ (\varepsilon_x + 1)}{\varepsilon_x + |\eta_x|}.$$

Similarly, the relative change in supply of foreign currency resulting from the effect of depreciation on any export category can be shown to be

(7) $$\frac{D\,\varepsilon_y(1 + \eta_y)}{\eta_y - \varepsilon_y},$$

or

(7a) $$\frac{D\,\varepsilon_y(\ |\eta_y| - 1)}{|\eta_y| + \varepsilon_y}.$$

The improvement in the balance of trade resulting from depreciation, assuming only one export and one import commodity, is given by:

(8) $$\sum D\left[yp_y\ \frac{\varepsilon_y(\ |\eta_y| - 1)}{|\eta_y| + \varepsilon_y} + xp_x\ \frac{|\eta_x|\ (\varepsilon_x + 1)}{\varepsilon_x + |\eta_x|}\right]$$

which if trade was originally balanced and all supply elasticities are infinite, reduces to:

(8a) $$Dxp_x(|\eta_y| - 1 + |\eta_x|)$$

Bias involved involved in use of total import index to obtain an "elasticity of demand for imports" for use in predicting the effects of depreciation:

Assume: (9) no substitution between imported goods,

(10) all foreign elasticities of supply = ∞.

Let e (in general) $= \frac{d(px)}{dp} \cdot \frac{p}{px} = \frac{(x + p\frac{dx}{dp})}{x} = 1 + \eta$

(11) $$\phi = \frac{\Delta V}{V^o} \frac{P^o}{\Delta P}$$ elasticity obtained using index, where V^o refers to initial value of imports, and P^o refers to initial value of price index.

(12) $$e_j = \frac{\Delta v_j}{v_j^o} \frac{p_j^o}{\Delta p_j}$$ "true" elasticity for j^{th} good.

(13) $$e \text{ (in particular)} = \sum \frac{v_j^o e_j}{V^o} = \sum \frac{(\Delta v_j) p_j^o}{V^o \Delta p_j}$$ "true" elasticity of demand for imports.

(14) $$r_j = \frac{P^o}{\Delta P} - \frac{p_j^o}{\Delta p_j}$$

Substituting (14) into (13), we have

(15) $$e = \phi + \frac{\sum (\Delta v_j) r_j}{V^o}.$$

We must now consider whether $\frac{\sum (\Delta v_j) r_j}{V^o}$ is negligible in magnitude.

From (14),

$$\Delta v_j r_j = \Delta v_j \frac{p_j^o}{\Delta p_j} - \Delta v_j \frac{P^o}{\Delta P}.$$

From (12),

$$\Delta v_j r_j = e_j v_j^o - e_j v_j^o \frac{P^o}{\Delta P} \cdot \frac{\Delta p_j}{p_j^o} = e_j v_j^o \left[1 - \frac{P^o \Delta p_j}{\Delta P p_j^o}\right] .$$

Now, let

(16) $$e_j = e + i_j .$$

Then

(17) $$\sum \frac{\Delta v_j r_j}{V^o} = \sum \frac{e v_j^o}{V^o} - \sum \frac{e v_j^o}{V^o} \frac{P^o \Delta p_j}{\Delta P p_j^o} + \sum \frac{i_j v_j^o}{V^o} \left[1 - \frac{P^o \Delta p_j}{\Delta P p_j^o}\right] .$$

Now $\sum \frac{e v_j^o}{V^o} = e$ and

$$\sum \frac{e v_j^o}{V^o} \frac{P^o \Delta p_j}{\Delta P p_j^o} = e \sum \frac{p_j^o x_j^o (\Sigma\, p_j^o x_j^o) \Delta p_j}{(\Sigma\, p_j^o x_j^o)(\Sigma\, x_j^o \Delta p_j) p_j^o} = e$$

(if a Laspeyre price index is used).

Therefore,

(17a) $$\sum \frac{\Delta v_j r_j}{V^o} = \sum \frac{i_j v_j^o}{V^o} \left[1 - \frac{P^o \Delta p_j}{(\Delta P) p_j^o}\right] .$$

Let

(18) $$\frac{i_j v_j^o}{V^o} = \phi_j = \text{ the weighted "elasticity deviation" for each } j$$

and

(19) $$1 - \frac{P^o \Delta p_j}{\Delta P p_j^o} = \pi_j = 1 - \frac{\%\text{ change in } p_j}{\%\text{ change in index}}.$$

Then,

$$(20) \qquad \sum \frac{\Delta v_j r_j}{V^o} = \sum \phi_j \pi_j .$$

Now $\sum \phi_j = 0$,

so that by the theorem $E(xy) = E(x)E(y) + \sigma_x \sigma_y r_{xy}$

we have $\dfrac{\sum \Delta v_j r_j}{V^o} = N \sigma_\phi \sigma_\pi r_{\phi\pi}$.

Equations of the form $\log x = \alpha_0 + \alpha_1 \log \dfrac{P_x}{P_z} + \alpha_2 \log Y$ were derived for the category of spices for the period 1923-1939 (omitting 1930), using different combinations of Laspeyre and Paasche price and quantity indexes, with the following results:

	Price Elasticity	Income Elasticity
Laspeyre P, Paasche x	-.38	.38
Paasche P, Laspeyre x	-.44	.57
Laspeyre P, Laspeyre x	-.46	.69
Paasche P, Paasche x	-.36	.28