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Logical Relations between Production Function and Demand Equations for Factor

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I. This paper is a supplement to Cowles Commission Discussion Paper: Economics: 241A, and is written to replace an incorrect formulation which I made there in section IIIA, from the last 3 lines on page 11 to the end of page 13[1]. Assuming a production function of the form

(1) $$x = \phi(y_1, \ldots, y_m),$$

I stated without proof that it is not possible to derive this production function uniquely from the demand-for-factor equations, i.e. that there is a family of production functions, all of which lead (under the assumption of profit maximization) to the same demand-for-factor equations. Therefore, I argued, a production function (or something from which it can be derived) should be explicitly introduced into Klein's model.

II. Let x be output, $y_1, \ldots, y_m$ be amounts of inputs, and ϕ be the production function as in (1). Also let p be output price, and $q_1, \ldots, q_m$ be factor prices. Recall that under competition the demand-for-factor equations

(2) $$y_j = h_j(p, q_1, \ldots, q_m), j = 1, \ldots, m$$

are obtained by solving simultaneously the set of profit-maximizing equations

(3) $$\frac{\partial g}{\partial y_j} = 0, j = 1, \ldots, m$$

where g is the function to be maximized (g is obtained by substituting the production function into the profit equation):

I. The presence of an error was first suggested by Koopmans during a staff meeting, and then the matter was clarified in subsequent discussion with Georgescu-Roegen, Arrow, Marschak, Rubin, and Mrs. Bronfenbrenner.

(4) $$g = p\phi(y_1, \ldots, y_m) - \sum_1^m q_j y_j$$

The argument was based first on the fact that there is an infinity of sets of equations, besides (3), of which the demand-for-factor equations (2) are the simultaneous solution, and second on the incorrect conjecture that it is impossible to recognize the set (3) from among the infinity of such sets. But due to the fact that the profit function is of the form

(5) $$\pi = px - \sum q_j y_j$$

where each q_j appears only once and always as the coefficient of its y_j, we can be sure that the equations (3) must look like this:

(6) $$\frac{\partial g}{\partial y_j} = p\frac{\partial \phi}{\partial y_j} - q_j = 0, \; j = 1, \ldots, m$$

where $\partial\phi/\partial y_j$ contains no prices p or q_j, because it is a derivative of the production function. This means that the equations (3) [or (6)] can be found by solving the demand-for-factors equations simultaneously for the factor-prices q_j and then transposing q_j to the other side of the equality sign. Better yet, the demand-for-factor equations can be solved for the relative factor prices q_j / p to get

(7) $$\frac{q_j}{p} = \frac{\partial \phi}{\partial y_j} \qquad j = 1, \ldots, m$$

Now equations (7) can be directly integrated[2] to obtain the production function (1(1) uniquely except for an additive constant. This additive constant can then be determined from the boundary condition that zero inputs yield zero output,[2a] i.e. that

(8) $$\phi(0_1, \ldots, 0_m) = 0$$

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2. Subject of course to integrability conditions requiring cross-partials to be independent of the order of differentiation. These conditions will be satisfied if we have the correct demand-for-factor equations to begin with and if entreprenaurs really behave as if they were maximizing profits.

2a. If it is not desired to assume that the production function passes through the origin (due to indivisibilities small non-zero inputs might yield zero outputs), then one single observation will determine the constant.

This completes the proof that if the production function is of the form (1), i.e. with only one output, and if competitive markets and profit-maximizing are assumed, then the production function

is uniquely implied in the demand-for-factor equations (2) and the profit function (5).

III. If the assumptions about competitive markets and profit-maximizing are retained, but now several outputs $x_1, \ldots, x_n$ assumed instead of one, we can use a production function like this:

$$\psi(x_1, \ldots, x_n, y_1, \ldots, y_m) = 0 \qquad (9)$$

where $\partial\psi/\partial x_i$ are either all positive or all negative, $i = 1, \ldots, n$, and $\partial\psi/\partial y_j$ all have opposite sign from $\partial\psi/\partial x_i$, $j = 1, \ldots, m$. Observe that if $n = 1$, equation (9) reduces to equation (1) as a special case.

By an argument similar to that used above in section II we can see that if we are given the firm's demand equations for all the factors and also its supply equations for all but one of the outputs, then it is again possible to derive the production function[3]. This is because, as before, it is possible to solve the $m + n - 1$ supply and demand equations simultaneously for $m + n - 1$ price ratios to obtain directly integrable equations [2] analogous to (7).

IV. In the case of a monopolistic-monopsonistic firm which faces competitive factor-sellers and output-buyers, there exist no output-supply curves and input-demand curves as functions of price; rather there exist optimum values of outputs and inputs which depend on the demand curves for output and on the supply curves for input. If these dependences and the factor-supply curves and the product-demand curves are known, then the analogs of equations (7) are again obtainable and directly integrable, [2] though they will now contain elasticities.

V. These results as they stand apply to a single firm; they have not yet

3. Recall that competitive markets and profit-maximizing are still assumed here.

been aggregated. But even if it should turn out that the aggregate production function can be obtained by integrating the demand-for-factor equations, it will be desirable to include it explicitly in order to get more efficient estimates of the parameters of the model.

If the production function is introduced explicitly, then the number of endogenous variables may be increased by one provided that as a result the Jacobian (of the transformation from observable endogenous variables to disturbances, i.e. of the model) is not made to vanish. The Jacobian does not in fact vanish, unless the production function is of a very special and unlikely form. Thus the inclusion in the model of any of the ordinarily used types of production function permits the addition of another endogenous variable. Intuitively it seems that this variable should be one of those appearing in the production function.