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Computational Suggestions for Maximizing a Linear Function Subject to Linear Inequalities

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Let the function to be maximized be

(1) $$y = \pi'_* \cdot x$$

(y scalar, π_* and x vectors), subject to linear inequalities

(2) $$\phi_k + \pi'_k \cdot x \geq 0 \qquad k = 1, \ldots, K, \quad \phi_k \text{ scalar, } \pi_k \text{ vector.}$$

We distinguish two main cases (A and B).

A. <u>When one possesses an initial point</u> x_o <u>satisfying</u> (2).

A.1 Traversal Method (Koopmans' Suggestion).

Find the largest value θ_o of the scalar θ such that

(3) $$x = x_o + \theta\pi_*$$

satisfies (2) and write

(4) $$x_1 = x_o + \theta_o\pi_*.$$

Then for at least one value, k_1 say, of k

(5) $$\phi_{k_1} + \pi_{k_1}' \cdot x_1 = 0.$$

If it is true for only one value of k, determine scalars λ_1, μ_1, such that

(6) $$\pi' \cdot (\lambda_1 \pi_{k_1} + \mu_1 \pi_*) = 0.$$

This is impossible only if $\pi_* = c\pi_{k_1}$, c scalar, in which case x_1 already maximizes y. Determine θ_1 as the largest value of θ for which

(7) $$x = x_1 + \vartheta(\lambda_1 n_{k_1} + \mu_1 n_*)$$

satisfies (2) and write

(8) $$x_2 = x_1 + \tfrac{1}{2}\vartheta_1(\lambda_1 n_{k_1} + \mu_1 n_*).$$

Proceed with x_2 as previously with x_0. If at the n-th step more than one k_n satisfies an equation like (5), select one arbitrarily, or use an average of all n_{k_n} that satisfy (5). (The latter procedure depends on the normalization used for the vector n_k.)

A.2 Plane Intersection Method (G. Brown's Suggestion).

Having obtained (4), intersect the plane

(9) $$x = x_1 + \lambda n_{k_1} + \mu n_*$$

(λ and μ freely variable scalar parameters) successively with each of the hyperplanes

(10) $$\phi_k + n_k' \cdot x = 0.$$

The intersections consist of K straight lines inside (9). The segments of these lines on which (2) is satisfied form a convex polygon. On that polygon select a point on which y reaches its maximum. Generally there is just one such point, for which write x_2. Now there are two variants.

A.2a Plane determined by normal to the convex set (2). If x_2 is unique there are at least two values of k for which (10) is satisfied. Take any one of these, or take their average, as n_{k_2} and proceed as in (9) with 1 replaced by 2.

A.2b Plane determined by normal within boundary of convex set (2).

Having arrived close to the maximum, it may be desirable to attempt not to lose any of the equalities (10) once they are satisfied. Let x_n be such that

$$\phi_k + \pi_k' \cdot x_n = 0 \quad \text{for } k = k_1, k_2, \ldots, k_{r_n}. \tag{11}$$

In the space of the vectors ψ such that

$$\psi' \cdot \pi_{k_r} = 0, \qquad r = 1, \ldots, r_n \tag{12}$$

choose the vector of steepest ascent, i.e., the vector satisfying

$$\psi' \cdot \psi = 0 \qquad \psi' \cdot \pi_* = \text{maximum}, \tag{13}$$

and use that vector as π_{k_n} in (9). This will have been wasted effort if the resulting x_{n+1} fails to satisfy (11). In conversation, Von Neumann made objections to this variant if the number of dimensions is large. It would seem that each step becomes computationally more expensive as r_n grows.

B. <u>How to obtain an initial point satisfying (2).</u>

B.1 Selective Penetration Method (G. Brown's Suggestion).

Take an arbitrary initial point x_0. This point partitions the set S of inequalities (2) into two subsets S_0 and S_0', those of S_0 being satisfied by x_0, those of S_0' not being satisfied by x_0. If S_0' is empty, the goal has been achieved. If it is not, select arbitrarily an inequality of S_0', numbered k_0, say. Use π_{k_0} as the vector π_* in (1) indicating the "desired direction," use the inequalities of S_0 instead of the full set of conditions (2) and apply method A until a point is reached in which the inequality numbered k_o is satisfied. Call that point x_1 and proceed with a new partition S_1. Obviously

$$S_o \in S_1 \in S_2 \ldots.$$

The method can fail only if no internal point exists.

B.2 Guided Penetration Method (Koopmans' Suggestion)

Instead of selecting an arbitrary π_k of S_o' to be the π_* in (1), take

$$\pi_* = \sum_{k \in S_o'} (\phi_k + \pi_k' \cdot x) \frac{\pi_k}{\pi_k' \cdot \pi_k}.$$

This is the vector sum of the normals dropped from x_o onto the planes

$$(\phi_k + \pi_k' \cdot x) = 0 \qquad k \in S'.$$

B.2a Keep the π_* so selected constant while making a number of iterative improvements to x_o by method A, always requiring that the inequalities S_o be preserved.

B.2b Be willing to sacrifice some inequalities of S_o if thereby a larger number of inequalities of S_o' can be satisfied. In this case one determines θ_o in (4) in such a way as to minimize the number of inequalities in S_1'.

For neither of these variants certain attainment of the objective (if attainable) has been proved. They might, however, work faster than the selection method. The second alternative is suspect if the convex (2) is not bounded.

C. General Observations

The general idea underlying the foregoing method is to make big jumps rather than "crawling along the edges" of the convex (2). All methods indicated are based on some idea of steepest ascent, and thus depend on the units of measurement of the variables x.

The determination of the "normal" n_{k_1} in (5)--by any of the methods suggested--is likely to present practical difficulties in the neighborhood of the maximum. It may then be useful to take the vector sum of normals of unit length to all planes $n_k \cdot x' = 0$ to which x_o has a distance less than a small amount ε.