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Measurable Utility and the Theory of Assets

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I. INTRODUCTION

1. The Problem

It is proposed to show that if the economists' theory of assets (Ch. III) is completed by a certain plausible axiom on rational human behavior (Ch. IV), then utility can be defined as a quantity unique up to a linear transformation, and not merely up to any monotone transformation. That is (Ch. V), utility becomes measurable in the same sense in which the (non-absolute) temperature is measurable, and not merely in the same sense in which students' intelligence is measurable. This result gives some justification to the advice of statistical theory, to choose rules of behavior in such a way as to maximize the mathematical expectation of utility ("minimize the average loss", in A. Wald's terminology): in Ch. VII below, a suggestion for further research in this direction is made. The measurability of utility and the rule of maximizing its expected value greatly simplifies economic theory: see VIII below.

2. Acknowledgements

These results are, of course, inspired by (and, in some points, give a mere paraphrase of) von Neumann's and Morgenstern's Theory of Games and Economic Behavior. It will be attempted (in Ch. VI) to sketch some relations between their approach and the present one.

Parts of the paper were discussed with Herman Chernoff and Herman Rubin: see specific acknowledgments below.

3. Behavior: advisable and actual

We must distinguish between (a) advisable behavior and (b) actual behavior. Both are types of responses to environment. Without trying to give here close definitions, we can assume general agreement that both kinds of behavior deserve study. For example, a government (or a firm), has to (a) choose the advisable course of its own responses to environment; (b) know, as a part of its environment, something about the actual behavior of tax-payers, consumers, competitors, etc. We shall start with advisable behavior. It is also called the behavior of a rational man. For brevity, we shall speak of "the man".

4. Complete information

We have further to distinguish between two cases: the case when the man thinks he knows certain relevant probability distributions

and the case when the man does not think so. We call the former case, complete information, and the latter, incomplete information. The theory of rational behavior under incomplete information is not presented here. It would be related to the studies on the rationale of sampling, started by Neyman and Pearson and developed in particular by Wald. The case of complete information is a limiting one: it is approached as the number of observations which have been made by the man increases.

A special case of complete information is that of certainty: in this case all probabilities have values 0 or 1.

II. DEFINITIONS

5. Sequences X

Let $x_{(1)}(\tau)$, $x_{(2)}(\tau)$,... denote the quartity of commodities 1, 2,.. at time τ. $x(\tau) = \{x_{(1)}(\tau), x_{(2)}(\tau), \ldots\}$ is a point in the commodity-space (a "budget") at time τ. $X(t) = \{x(1), x(2), \ldots, x(t)\}$ is a time-sequence of commodity-points (a "history"). We shall enumerate such sequences as $X_1(t)$, $X_2(t)$,...., $X_N(t)$ or simply X_1, X_2...X_N. Inasmuch as the man may treat time and the amounts of some goods as continuous, or as not bounded, the number N of sequences may be infinite. This does not exclude that some of the amounts of other goods may be discrete and/or bounded, and may even assume only the values 0 or 1. Thus the identical time-sequence of combinations of foods, clothes etc., in a totalitarian, a democratic, and some other kind of society, would be represented by three distinct points in the space X.

6. Prospects P

We call a prospect (of first order) P^i the set (distribution) of probabilities $p_1^i, \ldots, p_N^i$.

P^i is thus a function on X subject to the restriction

$$(6.1) \qquad \sum_n p_n^i = 1, \; p_n^i \geqq 0, \quad n = 1, \ldots, N.$$

We can also write P^i as a matrix

$$(6.2) \qquad P^i \quad \left\| \begin{matrix} \bar{p}^i \\ X \end{matrix} \right\|, \text{ where } \bar{p}^i = \left\{ p_1^i, \ldots, p_N^i \right\}.$$

The set of all prospects, $P = \left\{ P^1, \ldots, P^i, \ldots \right\}$ is infinite (regardless of whether N is or is not infinite) because each probability p_n^i can take all real values from 0 to 1.

7. Balance sheet

Under our assumption of complete information, each balance sheet (list of assets and liabilities) corresponds to a prospect. A balance sheet can also be regarded as a lottery ticket; or as a collection of various lottery tickets, winnings from one lottery possibly depending upon the possession of tickets to other lotteries (complementarity of assets). The winnings consist either of goods or of other "tickets".

8. Higher order prospects. Sure prospects

We can define a prospect of second order - say Q^j - analogously to (6.2):

$$(8.1) \quad Q^j = \left\| \begin{matrix} p_1^j, \ldots, p_i^j, \ldots \\ P_1, \ldots, P_i, \ldots \end{matrix} \right\| = \left\| \begin{matrix} \bar{p}^j \\ P \end{matrix} \right\|;$$

but since under prospect Q^j the probability of X_n is $\sum_i p_i^j \, p_n^i$, Q^j is identical with the prospect of first order

$$(8.2) \qquad \left\| \begin{matrix} \bar{p}^j \, \bar{p}_n^* \\ X \end{matrix} \right\|,$$

where $\bar{p}_n^*$ is the transpose of $\bar{p}_n$. Similarly any prospect of still higher order is identical with some prospect of first order, i.e., some point in the space of P.

Any sequence X_n itself can also be regarded as a prospect of first order, viz., as a "sure prospect":

$$(8.3)\quad X_n = \left\| \begin{matrix} 0,\ldots, & 1,\ldots 0 \\ X_1,\ldots, & X_n,\ldots X_N \end{matrix} \right\| = \left\| \delta^n_X \right\| .$$

The man has thus to choose among the points in the P-space only, i.e., among matrices such as (6.2). We shall regard $X = (X_1,\ldots,X_N)$ as exhausting all possible sequences of commodity combinations and therefore as fixed. Thus every prospect is fully described by the upper row in (6.2), i.e., the probability distribution of sequences X.

9. Environment

Denote by E_m the set of allprospects that the man can (in his opinion) achieve by some action of his. E_m is thus a region in the P-space. It can be called environment. It is entirely determined by factors like the technology of the man's farm or factory, his ability to borrow, etc. The set of all environments--i.e., of certain, generally overlapping, regions of the P-space--will be denoted by $E = (E_1,\ldots, E_m,\ldots)$.

The man's decision consists in picking out a certain prospect P_i within a given E_m.

10. Geometrical presentation

Because of the restriction (6.1) it is convenient to represent a prospect P^i geometrically not by the N-dimensional vector $\left\{ p^i_1,\ldots, p^i_N \right\}$ but the (N-1) dimensional vector $\left\{ p^i_2,\ldots p_N \right\}$, remembering that

$$(10.1)\quad \sum_{n>1} p^i_n \leqq 1,\ p^i_n \geqq 0,\ n = 2,\ldots, N.$$

On Fig. 1, N = 3, and all imaginable prospects lie on or within the triangle 1 0 1, because of (10.1). All achievable prospects lie within the shaded area representing the environment E_m; its boundary (apart from the axes) is, in the economists' language, an "opportunity line" or "transformation line".

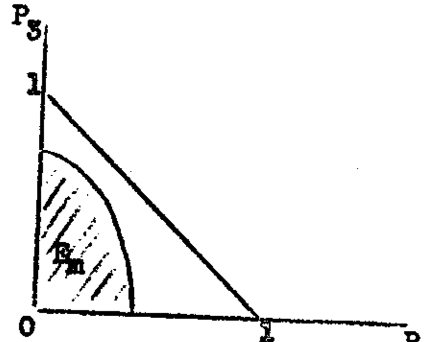

FIG. 1

III. AXIOM A AND THE OLD THEORY OF ASSETS

11. "The man chooses among all achievable prospects". This phrase implies both of two things: that some prospects are preferable to others, and that some are achievable and others not. These two implications will be dealt with separately in 12. and 13.

12. The set of all prospects is completely ordered by his preferences. We shall rephrase this as

AXIOM A. There exists a set of sets $\Omega = (\omega^{(1)}, \ldots)$ and a relation called "preferred to", with the following properties:

(12.1) Each element of a subset $\omega^{(r)}$ of Ω is some prospect P.

(12.2) Each prospect P belongs to one and only one subset such as $\omega^{(r)}$.

(12.3) For any two distinct elements of Ω, --say $\omega^{(r)}, \omega^{(s)}$, -- either $\omega^{(r)} > \omega^{(s)}$, (read $\omega^{(r)}$ preferred to $\omega^{(s)}$) or $\omega^{(s)} > \omega^{(r)}$.

(12.4) If $\omega^{(r)} > \omega^{(s)}$ and $\omega^{(s)} > \omega^{(t)}$, then $\omega^{(r)} > \omega^{(t)}$

(12.5) If $\omega^{(r)} > \omega^{(s)} > \omega^{(t)}$ then there exist two positive numbers a, b (a + b = 1), and two prospects, P^i in $\omega^{(r)}$ and P^k in $\omega^{(t)}$, such that prospect $(aP^i + bP^k)$ belongs to $\omega^{(s)}$.

The subsets $\omega^{(1)}, \ldots, \omega^{(r)}, \ldots$ are represented, in the economists' language, by indifference surfaces,* as will be drawn on Fig.2 p.7. There exists

*That (12.5) is necessary for the conclusions of 19.1 below was pointed out by Rubin.

a "utility function" $u(P)$ such that

(12.6) $u(P^i) >, <, = u(P^j)$ according to whether P^i is preferred to P^k, P^k is preferred to P^i, or the two prospects are interchangeable;

that is (in the language of Axiom I), according to whether $\omega^{(r)}$ (which contains P^i) is preferred to $\omega^{(s)}$ (which contains P^k); or $\omega^{(s)}$ is preferred to $\omega^{(r)}$, or the two are identical. The real numbers, ("utilities") $u(P^i)$, $u(P^j)$ then correspond to the sets (or surfaces) $\omega^{(r)}$, $\omega^{(s)}$. The utility function $u(P)$ is determined up to a monotone transformation; that is, if two functions $u(P)$ and $v(P)$ have the property (12.6) then there exists a monotonically increasing function F such that (12.7) $v(P) = F(u(P))$

<u>Maximizing utility</u>

13. The man chooses P^i in E_m, such that for any P^k in E_m, $i \neq k$, $\omega^{(r)}$ (which contains P^i) is either preferable to; or identical with $\omega^{(s)}$ (which contains P^k). Or, in the language of the utility function:

(13.1) $u(P^i) \geq u(P^k)$,

(13.2) $u(P^i) = \text{Max } u(P \text{ in } E_m)$.

<u>Decision functions</u>

14. The content of 13. might also be rendered by the use of Wald's concept of decision functions, which will be treated in Ch.VII. below.

<u>Indifference surfaces</u>

15. The geometrical representation of 11., 12. is familiar. On Fig.2, the equation

(15.1) $\mu = u(P)$

is represented by a one-parametric family of indifference surfaces*, with as parameter. u has a maximum at a point P^i where the indifference line is tangent to the boundary of E_m.

*The last part - (12.5)- of Axiom I guarantees that (in the case N=3, of Fig.12) they are lines and not points; or (for N=4) that they are surfaces and not lines or points, etc. See footnote to 12.

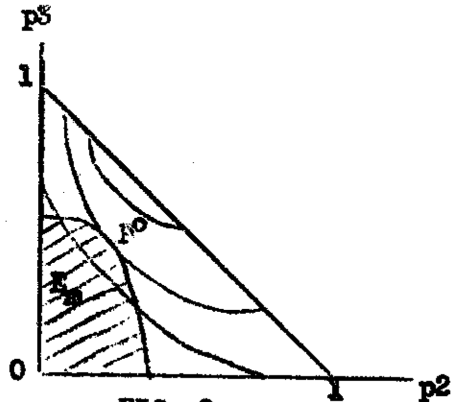

FIG. 2

Theory of Assets

16. This is, essentially, as far as the old theory of assets went (apart from the pricing of assets). Most writers described, however, a prospect P^1 not as a point in the space of probabilities of the mutually exclusive time sequences X but as a point in the space of parameters (e.g. means, standard deviations, correlations), of the joint probability distribution of the amounts of various commodities at various time-points, i.e., of the variables $x_h(\tau)$ $(h = 1,\ldots; \tau = 1,\ldots)$ of 5. above. These parameters themselves were regarded as being the "goods" ("bads") in the eyes of the individual. Tintner's approach was closer to the present one; he pointed out that the operation (13.2) was the maximization of a functional (since P is a function on X).

The mean and other parameters.

17. It has been common to all these writers (including the present one) to insist that the means of the variables $x_h(\tau)$ are not the only parameters of their joint distribution that are relevant to the man's decisions, and possibly not the most important ones; and attempts were made to specify which additional, or alternative, parameters--e.g. the higher moments--should be considered.* The statement that the average amounts of goods are not, or not alone relevant to the man's decision, is not invalidated when it is shown below that the average utility is maximized by (rational) man. The latter proposition gives,

*As a rival to the mean, the mode was suggested by Lange ( ). As a measure of riskiness the standard deviation was suggested by I. Fisher as early as 1906; (continued on page 8).

furthermore, a guidance in selecting the relevant distribution parameters: see Ch.VIII. below.

IV. AXIOM B AND THE LINEAR UTILITY FUNCTION OF PROBABILITIES

18. We shall now supplement Axiom A of the traditional theory by

AXIOM B. If P^i, P^j both belong to $\omega^{(r)}$ then, for any P^k, there exist two positive numbers a, b (a + b = 1) and a set $\omega^{(s)}$ such that the prospects $(aP^i + bP^k)$, $(aP^j + bP^k)$ both belong to $\omega^{(s)}$

We consider separately the cases (1) k = j; and (2) $k \neq j$.

(18.1) If k = j, Axiom B says that the second-order prospect $(aP^i + bP^j)$ belongs to the same set as P^j (and P^i). In words: if P^i and P^j are interchangeable, then a prospect of getting with certainty either P^i or P^j is interchangeable with any of these two prospects.

This can hardly be refuted. If P^i and P^j are tickets to two lotteries--one offering some chance of a car and the other some chance of a house, then either of these tickets is as desirable as a ticket to the lottery which offers with certainty the winning of a prize consisting in a ticket to either the one or the other of the first two lotteries.

(18.2) If $k \neq j$, Axiom B says that $(aP^i + bP^k)$ and $(aP^j + bP^k)$ belong to the same set. In words: if P^i and P^j are interchangeable with each other but more (or less) desirable than P^k, then the alternative of getting either P^k or P^i with given chances is as desirable as the alternative of getting P^k or P^j with the same chances.

This, too, is not refutable. To use further the example of (18.1), imagine three more lotteries: one with a cow as the prize; one offering the odds a: (1-a) for a ticket in the car lottery and against a ticket in the cow

*Footnote continued from p.7.
while Cramér recommended the probability of losses (   ) as a matter of actuarial practice--a suggestion which (according to Somers) may be the meaning of the one made by Fellner (   ). Of course, at least some of these suggestions intend to describe actual rather than advisable behavior (see 3 above).

lottery; and one with the odds a: (1-a) for a house-lottery ticket and against a cow-lottery ticket. Clearly there is no reason to prefer either one of these two "mixed" lotteries.

An Implication: Geometrical proof

19. We shall now conclude geometrically

1) from (18.1) (case $k = j$) that each indifference surface is a hyperplane;

2) from (18.2) (case $k \neq j$) that all indifference surfaces are parallel hyperplanes.

19.1 On Fig.3., probabilities p_4, p_5.... are fixed at p_4', p_5',..., say: and $a = 1/3$. The three points P^i, P^j, $ap^i + bp^j$ are collinear. Hence (18.1) implies that the contour of the indifference surface in the plane ($p_4 = p_4'$, $p_5 = p_5'$,...) is a straight line. Since this applies when p_4, p_5... are given some other values, the indifference surfaces are linear.*

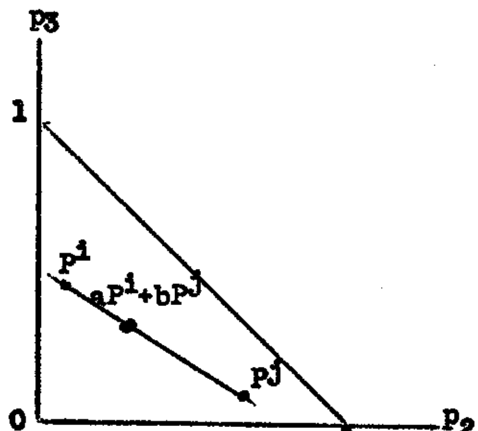

FIG. 3

19.2. On Fig.4 (where again $p_n = p_n'$, $n = 4, 5,...$), all points interchangeable with P^i, P^j are on a straight line, by 19.1. The two points $Q^i = aP^i+bP^k$ and $Q^j = aP^j + bP^k$ are by 18.2, interchangeable and lie therefore also on a straight line. But these two straight lines are parallel, since

*See also footnote to 15.

$$a = \frac{\text{segm. } P^iQ^i}{\text{segm. } P^iP^k} = \frac{\text{segm. } P^jQ^j}{\text{segm. } P^jP^k}$$

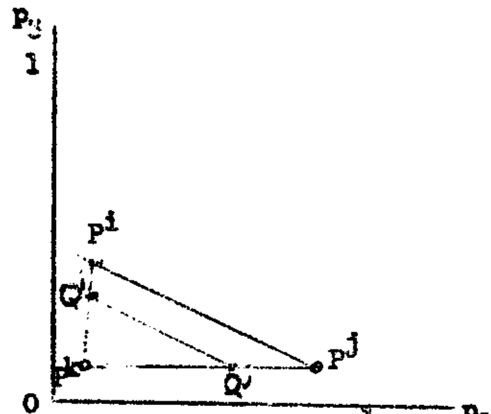

FIG. 4.

20. <u>Utility as a linear function of probabilities.</u> The fact that all indifference surfaces are parallel hyperplanes is expressed by specifying the general equation (15.1) as

$$\sum c_n p_n = u\ ,$$

where the c's are constants, the same for all utility surfaces, and the parameter u, changing from one indifference surface to another represents the utility. Utility is, then, a linear function of the probabilities (of the alternative time sequences). This implication of the Axioms A and B will now be proved algebraically.

21. <u>Algebraic proof.</u> The Theorem to prove is: there exists a real vector $C = \{c_1, \ldots, c_n\}$ and, for every set $\omega = \omega^{(s)}$, a constant $u^{(s)}$ such that

(α): if P is in $\omega^{(s)}$ then

$$\sum c_n p_n = u^{(s)}; \text{ and}$$

(β): if $\omega^{(r)} > \omega^{(s)} > \omega^{(t)}$ then

either $u^{(r)} > u^{(s)} > u^{(t)}$ or

$u^{(r)} < u^{(s)} < u^{(t)}$;

(i.e. either the constants defined in (α), or their negatives, can be used as utility indicators.)

The steps 21.1, 21.2 will correspond to 19.1, 19.2 of the geometrical proof above, and prove (α). (β) is proved in 21.3.

21.1 Consider P^i, P^j, both in $\omega^{(r)}$, $P^i \neq P^j$. Then by Axiom B (case k = j), $(aP^i + bP^j)$ is in $\omega^{(r)}$. Hence there exists a hyperplane

(21.1.1) $\sum_n c_n^{(r)} p_n = 1$, say,

satisfied by all points in $\omega^{(r)}$. In particular

$$\sum_n c_n^{(r)} p_n^i = 1.$$

21.2 Consider now $P^{(k)}$ in $\omega^{(t)} \neq \omega^{(r)}$. By Axiom B (case $k \neq j$), the two points $(aP^i + bP^k)$ and $(aP^j + bP^k)$ belong to one and the same set; call it $\omega^{(s)}$. Therefore, applying (21.1.1) to this set,

$$\sum_n c_n^{(s)}(ap_n^i + bp_n^k) = \sum_n c_n^{(s)}(ap_n^j + bp_n^k) = 1. \text{ Hence}$$

(21.2.1) $\sum_n c_n^{(s)} p_n^i = \sum_n c_n^{(s)} p^j$. But since both P^i and P^j are in $\omega^{(r)}$, we also have, by (21.1.1)

(21.2.2) $\sum_n c_n^{(r)} p_n^i = \sum_n c_n^{(r)} p_n^j = 1.$

This is consistent with (21.2.1) only if

$$c_n^{(s)}/c_n^{(r)} = \text{a constant}; \; n = 1, \ldots, N.$$

Call the proportionality constant, $u^{(s)}$;

(21.2.3) $c_n^{(s)} = c_n^{(r)} \cdot u^{(s)}$

Applying this and (21.1.1) to the set $\omega^{(s)}$ we obtain

(21.2.4) $\sum c_n p_n = u^{(s)}$,

where $c_n = c_n^{(r)}$ (i.e. $\omega^{(r)}$ is chosen as a "fundamental" set; we also note that by (21.2.3), $u^{(r)} = 1$). This completes the proof of part (α) of our theorem.

21.3. To prove part (β) of the theorem*, consider P^i in $\omega^{(r)}$, and P^k in $\omega^{(t)}$. Then, by (21.2.4) that we have just proved, $u^{(r)} = \sum_n c_n p_n^i$, $u^{(t)} = \sum_n c_n p_n^k$. We have to prove that $u^{(s)}$ lies between $u^{(r)}$ and $u^{(t)}$.

*Both the formulation and the proof of (β) are by H. Rubin.

By (12.5) of Axiom A, there exists in $\omega^{(s)}$ a prospect $P^j = aP^i + bP^k$ determined by a proper choice of two positive numbers, a, b = 1 - a. Hence $u^{(s)} = \sum c_n p_n^j = a \sum c_n p_n^i + b \sum c_n p_n^k = au^{(r)} + bu^{(t)}$; that is, $u^{(s)}$ lies between $u^{(r)}$ and $u^{(t)}$.

22. Utilities of sure prospects. The coefficients $c_1, \ldots, c_N$ of the Theorem in 21. have the following economic meaning: the coefficient c_n $(n=1, \ldots, N)$ is the utility of the sure prospect (compare (8.3) above) $P^{[n]}$, defined by

$$p_m^{[n]} = \left.\begin{array}{l} 1 \text{ for } m = n \\ 0 \text{ for } m \neq n \end{array}\right\} \qquad m = 1, \ldots, N.$$

By replacing in (21.2.4) (s) by $[n]$ we obtain, in fact,

(22.1) $$c_n = u^{[n]}.$$

One can (and does) also call $u^{[n]}$ the utility of the sequence X_n. Under the regime of certainty (end of 4. above) all utilities would belong to the set

(22.2) $$u^{[1]}, \ldots, u^{[N]};$$

and all indifference surfaces would belong to the set

(22.3) $$\omega^{[1]}, \ldots, \omega^{[N]}.$$

In our general case, however, there are utilities other than those of sure prospects. (22.3) is a subset of the set Ω defined in Axiom A.

23. "Average utility", or mathematical expectation of utility, of a prospect P^i is defined as

$$\sum_n u^{[n]} p_n^i .$$

We now see by (22.1) and by the theorem of 21., that if P^i is in $\omega^{(r)}$, then

(23.1) $$\sum_n u^{[n]} p_n^i = u^{(r)}$$

That is, the "average utility" of a prospect is identical with its utility.

V. MEASURABILITY OF UTILITY

24. Definition. By measurability we mean here "uniqueness up to a linear transformation." Suppose that to the sets of prospects,

$\omega^{(1)}, \ldots,$

correspond two sets of numbers,

(24.1) $u^{(1)}, \ldots,$ and

(24.2) $v^{(1)}, \ldots,$, such that

(24.5) if $\omega^{(r)} > \omega^{(s)} > \omega^{(t)}$

then $u^{(r)} > u^{(s)} > u^{(t)}$

and $v^{(r)} > v^{(s)} > v^{(t)}$

Then utilities are unique up to a monotone transformation (cf. (12.7) above); there exists a monotonically increasing function

$$u^{(r)} = F(v^{(r)}),\ r = 1, \ldots$$

We are going to prove that F must be linear: there exist two numbers g_0, g_1 ($g_1 > 0$) such that

$$u^{(r)} = g_1 v^{(r)} + g_0$$

25. Proof. Consider the subset (22.2) of (24.1)

$$u^{[1]}, \ldots, u^{[N]},$$

and the corresponding subset of (24.2):

$$v^{[1]}, \ldots, v^{[N]}.$$

Both correspond to the subset (22.3) of indifference surfaces

$$\omega^{[1]}, \ldots, \omega^{[N]}.$$

Therefore, by (23.1), if P^i is in $\omega^{(r)}$

$$\sum_n u^{[N]} p_n^i - u^{(r)} \cdot 1 = 0$$

$$\sum_n v^{[N]} p_n^i - v^{(r)} \cdot 1 = 0\ ;$$

and always $\sum_n 1 \cdot p_n^i - 1 \cdot 1 = 0$

The homogeneous linear system formed by the last three equations can hold for all values of the p's only if there exist two numbers g_0, g_1 such that

$$u^{[n]} = g_1 v^{[n]} + g_0,\quad n = 1, \ldots, N$$

(25.1) $$u^{(r)} = g_1 v^{(r)} + g_0.$$

The equation (25.1) is more general than the one preceding it since the set $u^{(1)}, \ldots$ contains the subset $u^{[1]}, \ldots, u^{[N]}$. Finally g_1 must be positive, by (24.5).

26. <u>Utility differences strictly measurable</u>. It follows that the difference between two utilities is measurable, in an even stricter sense than the utilities themselves: it is determined up to a constant factor only (like distance). From (25.1),

$$u^{(r)} - u^{(s)} = g_1 \left(v^{(r)} - v^{(s)}\right).$$

If we choose, for example, $u^{(r)}$ as the unit (cf. the end of 21.2), the difference between any two utilities becomes fully determined (like mileage or acreage).

27. <u>Utility differences and probability distributions</u>. Compare three prospects: P^i in $\omega^{(r)}$, P^j in $\omega^{(s)}$, and P^k in $\omega^{(t)}$, such that

(27.1) $$P^k = aP^i + (1-a)P^j \;, \quad 0 < a < 1.$$

Then, by (23.1)

$$u^{(r)} = \sum u^{[n]} p_n^i$$

$$u^{(t)} = \sum u^{[n]} p_n^k$$

$$u^{(s)} = au^{(r)} + (1-a)u^{(t)}$$

(27.2) $$\frac{u^{(s)} - u^{(t)}}{u^{(r)} - u^{(s)}} = \frac{a}{1-a}$$

For example, if $a = \frac{1}{2}$ (P^j is a fifty-fifty mixture of P^i and P^n) then $u^{(s)}$ lies half-way between $u^{(r)}$ and $u^{(t)}$. For $(u^{(s)} - u^{(t)})$ to exceed $(u^{(r)} - u^{(s)})$, <u>a</u> must exceed $\frac{1}{2}$.

VI. <u>COMPARISON WITH v.NEUMANN AND MORGENSTERN</u>.

28. "<u>Prospect</u>", the word describing in this paper properties of a balance-sheet is also used by von Neumann and Morgenstern (e.g. on the top of p.18) in essentially the same sense; more often, they speak of a "combination of events".

We consider a prospect

as the probability distribution of the time-sequences of all combinations of commodities, while von Neumann and Morgenstern take examples involving two or three commodities at one time-point only.
But this difference is hardly essential.

29. Utility differences. In the verbal presentation (Sec.3.3) of the "Theory of Games", the authors use as a behavior postulate the equivalent of our 27. In the present paper, on the other hand, 27 was derived from the Axioms A and B. von Neumann and Morgenstern (in their verbal presentation) do not have B but have what amounts to A.

Our decision to introduce Axiom B, rather than enouncing 27. as a behavior postulate is dictated by the desire to avoid behavior postulates which while not immediately plausible, yet do not show themselves in easily observable action. Axiom A states essentially that the man chooses between prospects: and we see men actually making choices--determining the balance sheet by buying or selling, or by abstaining from buying and selling assets. Axiom B is hardly a behavior postulate at all, or at any rate a very weak (i.e., very plausible) one; it merely rules out behavior of a kind which most people will call absurd. The statement 27., on the other hand, is neither immediately plausible nor is it susceptible to easy observation. True, a person when interviewed may make comparisons between two utility differences; but the comparison does not show itself in any choice except in the choice to answer a certain question in a certain way. H. Chernoff points out, however, that such a check of utility comparison by an actual choice could be made in the following situation. According to L.J. Savage, the rational strategy in a one-person game under complete ignorance of circumstances, is a strategy that results in minimaxing "regret", i.e., the difference between the utility actually realized and the utility that would have been realized if all circumstances were known. That is, the strategy must be chosen so that even under circumstances which yield for this strategy a regret not smaller than under any other circumstances, this regret is not larger than for any other strategy.* Thus a rational man playing the one-person game under ignorance of future circumstances, has to base his choices on comparing differences between utilities.

30. The chain of implications. In their mathematical presentation, the authors of the Theory of Games do not actually make use of a behavior postulate on utility differences. The behavior postulates used are set up in their Sec. 3.6. An extensive Appendix published in the second edition of the book shows that these postulates result in propositions (3:1:a), (3:1:b) of the text (Sec.3.5). The second of these implies, and is implied by, the linearity of what we called the utility function and appears thus equivalent to our Theorem in 21. From this, measurability of utility can be derived--in (3.5.b), (3.6) of the Theory of Games, and in 25. of this paper.

31. The axioms. The difference between the presentation in this paper, and in the Theory of Games, thus boils down to the difference between our axioms A, B, and the group of axioms (3A), (3B), (3C) of the Theory of Games. Tentatively, I suggest that (3A), (3B) are equivalent to my axiom A; specifically, the first four propositions in A--i.e., (12.1)-(12.4)--would correspond to (3A), and the fifth--i.e., (12.5)--to (3B) of von Neumann and Morgenstern. (I am not too sure of this comparison, however.) The "entities" u, v, w,... of the Theory of Games seem to correspond to our sets of (interchangeable) prospects, $\omega^{(1)}$,... (which correspond to indifference surfaces), but only with the following understanding: when a prospect $P^{(i)}$ in $\omega^{(r)}$ is combined with the prospect $P^{(k)}$ in $\omega^{(t)}$ into a prospect $aP^{(i)} + (1-a)P^{k}$ in $\omega^{(s)}$, the Theory of Games would say that the combination of the entities $\omega^{(r)}$, $\omega^{(t)}$

*Cf. von Neumann and Morgenstern, Sec. 13-15, for two-person zero-sum games; there the rational player obtains the minimax utility, not the minimax difference between two utilities.

with probabilities a, 1-a is equivalent to the entity $\omega^{(s)}$. Since the prospects are real vectors their multiplication with scalars a, (1-a), and subsequent addition, are straight forward operations. Unlike the Theory of Games, we avoid defining or using any operations (whether analogous in any sense, or not, to the operation of multiplication and addition just described) on the sets $\omega^{(\,)}$,... or on any other non-numerical concepts, except the use of the relations " > " ("preferred to"), and " = " ("identical with"), and of the usual one, of "containing" and "belonging to".

If the understanding just stated is correct, the axioms (3:C:a), (3:C:b) of the Theory of Games taken together would be equivalent to our axiom B. In the language of the present paper, (3:C:b) would say: "Consider P^h in $\omega^{(q)}$ and P^k in $\omega^{(t)}$; let a, b, a', b' be positive numbers, a + b = a' + b' = 1. Form a prospect $P^j = aP^h + bP^k$, and call $\omega^{(s)}$ the set containing P^j; form $P^i = a'P^j + b'P^k$, and call $\omega^{(r)}$ the set containing P^i. Consider now P^h_* in $\omega^{(q)}$ and P^k_* in $\omega^{(t)}$. Then the prospect $aa'P^h_* + (1-aa')P^k_*$ belongs to $\omega^{(r)}$. This "double mixing" is indeed implied in our axiom B, as a succession of two single mixings discussed there.* I hesitate to say which (if any) is the stronger axiom. I find it easier to translate axiom B immediately into the language of observable decisions--see the examples in 18.1, 18.2-- than to do so with von Neumann and Morgenstern's (3:C:b), stated as it is in terms of entities not susceptible to ordinary arithmetical operations. To use the phrasing of von Neumann and Morgenstern the present approach seems to me more "transparent" and its axioms have a more "immediate intuitive meaning by which (their) appropriateness may be judged directly".

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*Geometrically, each mixing was the finding of a center of gravity between two masses, a, b located respectively at two prospect points. See Fig. 3, 4. --In my review of the Theory of Games (The Journal of Political Economy, Vol.54, No.2, April 1946, pp. 97-115) I failed to acquaint the reader with the importance of (3:C:b).

32. <u>The proofs</u>. As already mentioned in 30., a part of the route, the part after establishing the proposition on the linearity of the utility function, is common to both presentations. But the previous part of the route, between the axioms and that proposition is vastly different. The present author does not claim to improve upon the minutiousness and rigor of the Appendix to the second edition of the "Games". The Appendix deals in detail with difficulties which are probably inherent even in our axiom A, regardless axiom B is introduced. The usual construction of a utility function, determined up to a monotonous (not necessarily linear) transformation probably glosses over many difficulties (though I cannot identify them), to the removal of which a large part of the Appendix is devoted; only another part of it is then needed to establish linearity. Although the authors thus feel compelled, in effect, to re-work the whole concept of functions determined up to monotonous transformations, they do not object in the body of the book to the indifference curves which Pareto applied to what we call here (8. and 22. above) sure prospects.* The theory of assets consists essentially in applying the technique of indifference curves--as stated in our axiom A--to prospects which are not sure. When axiom B is added, this technique yields in a simple way the Neumann'ian results. The proofs given in this paper are semi-intuitive; the degree of rigor lies somewhere between that of the Appendix to the Theory of Games and its verbal parts.

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*They only take exception to Pareto's having neglected prospects that are not sure: he has thus made utility non-measurable, and had to recoup measurability by assuming that the man can compare the two differences, between the utility of X_1 and X_2, and between the utilities of X_2 and X_3; an assumption which is much stronger than the assumption (see 27. above and Theory of Games, 3.3.2 and 3.4.6) that if the prospect $\frac{1}{2}P^i + \frac{1}{2}P^k$ is preferred to P^j then the difference between the utilities of P^i and of P^j exceeds the difference between the utilities of P^j and of P^k.

VII. WHY MAXIMIZE MATHEMATICAL EXPECTATION OF UTILITY?

33. An implication of the existence of choice between prospects. Consider the prospect P^i in $\omega^{(r)}$,

$$P^i = \begin{Bmatrix} p_1^i, \ldots, p_N^i \\ X_1, \ldots, X_N \end{Bmatrix};$$

its utility is $u^{(r)}$. It was shown in 23. above that

$$U^{(r)} = \sum_n u^{[n]} p_n^i ,$$

where $u^{[n]}$ $(n=1,\ldots,N)$ is the utility of having X_n with certainty; or briefly, $u^{[n]}$ is the utility of X_n. This result was derived from the fact that there exists choice between prospects, as described in axioms A and B. If the individual chooses the best achievable prospect--i.e., maximizes $U(P \text{ in } E_m)$ (see 13. above)--he also maximizes the expression $\sum_n u^{[n]} p_n^i$.*

34. Asymptotic properties of rational behavior? The statistician advises to "minimize the average loss", loss being defined by Wald as a weight attached to having made a particular error--so that the "weight function" is essentially the utility function of error. This advice is justified by the results of 23. and 33. only in the sense that this advice turns out to be a paraphrase of the statement "do make a choice between various achievable probability distributions of occurrences, according to your-presumably consistent-tastes." May one conjecture that the advice is not merely this formal one, but has some virtues of substance?

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*I have to correct an error in my review (Journal of Political Economy, 1946, pp.111-112). I wrongly concluded there that the result just stated would imply that "A plantation with wildly fluctuating crops will be considered an investment of equal value with a plantation with stable crops, provided the long-run value of crops is the same. The existence of investment trusts appears pointless." I see now that, in the notation of the review, "wild fluctuation of crops" (not of utilities!) must be measured by the variance.

$\alpha(\alpha u + \beta w - u)^2 + \beta(\alpha u + \beta w - w)^2$,--independent of the "valuation numbers" n(u), n(w)--and not by the expression on p.112.

Slightly changing the previous notation, let x_τ be a random point in the commodity-space (or more generally, a "situation") at time τ; and let ξ_τ be the observed value of x_τ. Define the following three sequences:

a sequence of "decision functions" $D_T = \{d_1, d_2, \ldots, d_T\}$;

a sequence of "utility functions" $U_T = \{u_1, u_2, \ldots, u_T\}$;

a sequence of random "situations" $X_T = \{x_1, x_2, \ldots, x_T\}$

a sequence of observed "situations" $\Xi_T = \{\xi_1, \xi_2, \ldots, \xi_T\}$,

as follows:

d_1 is a quantity; d_2 is a function, $d_2 = d_2(\xi_1)$; $d_3 = d_3(\xi_1, \xi_2)$;...; $d_T = d_T(\xi_1, \xi_2, \ldots, \xi_{T-1})$.

The distribution of X_T depends on d_1. The distribution of $\{x_2, \ldots, x_T | x_1\}$ depends on d_1 and $d_1(\xi_1)$. The distribution of $\{x_3, \ldots, x_T | x_1, x_2\}$ depends on d_1, $d_2(\xi_1)$, $d_3(\xi_1, \xi_2)$,... . The distribution of $\{x_T | x_1, \ldots, x_{T-1}\}$ depends on $d_1, d_2(\xi_1), \ldots d_T(\xi_1, \ldots, \xi_{T-1})$.

Thus the sequence Ξ depends on D_T, apart from random fluctuations.

u_τ, $\tau = 1, \ldots, T$ is a function of $\{x_1, \ldots, x_\tau\}$.

Define the "best strategy", D_T by the condition

$$\mathcal{E}\, U_T(X_T) \mid \hat{D}_T \geqq \mathcal{E}\, U_T(X_T) \mid D_T^*),$$

where D_T^* is any value of D_T.

It is conjectured that, for an important class* of utility functions U_T,

$$\text{Prob.} \left[U_T(X_T) \mid \hat{D}_T \geqq U_T(X_T) \mid D_T^* \right] \to 1,$$

as $T \to \infty$. That is, the rule of maximizing the average utility (of non-sure prospects) makes in the long run the utility of the actually realized sequence of events as high as possible. The environment, as known to the individual at a given time, was assumed constant above. More generally, $d_\tau(\tau = 1, \ldots, T)$ can be considered a function of $\{\xi_1, \ldots, \xi_{\tau-1}, E_\tau\}$, where E_τ is the man's judgment, at time τ, of all environmental conditions other than the events

*H. Chernoff points out that, for example, the function U_T must not be such as to be strongly influenced by early events, and not influenced by later ones.

$\xi_1 \ldots \xi_{\tau-1}$.

VIII. SIMPLIFICATION OF ECONOMIC THEORY

35. Theory of Games. This theory which promises to become the foundation of the theory of the market, as well as of the optimal allocation of social resources, has been worked out, so far, entirely on the basis of measurable utilities. Its difficulties--and therefore the chances for further development of economic theory--would increase enormously if one had to use non-measurable utilities.

36. Case of divisible commodities; the "utility of money". Neglecting all future time-points but one, and assuming the amounts of all commodities continuous, a prospect is given by a continuous function $p(x)$, where x is a point in the commodity space. If the utility of the (sure) prospect of having x with certainty is $u(x)$, the utility of the prospect is by 23. the average,

(36.1) $U(p) = \int_x u(x)\, p(x)\, dx$,

a quantity depending on the functions u and p. This quantity is maximized with respect to the function p, given the environment E_m. The optimal prospect p, as well as its utility, depends thus on E_m and on the utility function of sure prospects, u.

Let there be only one good (money), its quantity being x. If $p(x_0) = 1$, and $p(x') = 0$, for any $x' \neq x_0$, then (36.1) becomes

$$U(p) = u(x_0) \text{ ("utility function for money").}$$

Suppose $u'(x) > 0$, $u''(x) < 0$ ("decreasing marginal utility of money").

Consider the moments of the distribution $p(x)$:

$$\mu_1 = \int_{-\infty}^{\infty} xp(x)dx; \quad \mu_2 = \int_{-\infty}^{\infty} (x-\mu_1)^2\, p(x)dx; \text{ etc.}$$

Expand x about μ_1:

$$u(x) = u(\mu_1) + (x-\mu_1)\, u'(\mu_1) + \tfrac{1}{2}(x-\mu_1)^2\, u''(\mu_1) + \ldots$$

hence $U(p) = \int_{-\infty}^{\infty} u(x)p(x)dx = u(\mu_1) + u'(\mu_1) \cdot 0 + \frac{1}{2}u''(\mu_1) \cdot \mu_2 + ..$

We have thus

$$\frac{\partial U(p)}{\partial \mu_2} = u''(\mu_1);$$

that is, the variance is a "good" or a "bad" thing (all other moments being given) according to the sign of the second derivative of the utility function $u(x)$ at $x = \mu_1$. If $u''(\mu_1)$ is negative ("decreasing marginal utility of money"), the variance of a prospect is an undesirable feature. This result was essentially known to Marshall and goes back to Daniel Bernoulli, but was discarded while the theory of non-measurable utility held the field. Recently Friedman and Savage have made interesting attempts to find evidence for the sign of higher derivatives of $u(x)$ at various levels of income x, basing this evidence on the behavior of various social groups in their choice of prospects characterized by different probability distributions (gambling, insurance).

There are also useful applications of the results of this section to the theory of liquidity, the effect of profit taxes on enterprises, etc.

The concept of "complementarity" of goods is also greatly simplified. It becomes permissible to define two goods complementary if the cross derivative of utility with respect to the amounts of these goods is positive. (This definition is not possible if utility is determined up to monotone transformation, since the cross derivative of utility depends then on the second derivative of the transformation; but if the transformation is linear, its second derivative vanishes.