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Proof of Samuelson's Theorem[†] Regarding the Ineffectiveness of Substitution in the Leontief Model

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Theorem: Let there be three closed convex sets C_1, C_2, C_3 of points in three-dimensional space (y_1, y_2, y_3) with the following properties.

$$\text{If } a_{(1)} \equiv (a_{11}, a_{21}, a_{31}) \in C_1, \text{ then } a_{11} > 0,\ a_{21} \leqq 0,\ a_{31} \leqq 0,$$

(1) $$\text{if } a_{(2)} \equiv (a_{12}, a_{22}, a_{32}) \in C_2, \text{ then } a_{21} \leqq 0,\ a_{22} > 0,\ a_{32} \leqq 0,$$

$$\text{if } a_{(3)} \equiv (a_{13}, a_{23}, a_{33}) \in C_3, \text{ then } a_{31} \leqq 0,\ a_{32} \leqq 0,\ a_{33} > 0.$$

There exist three points $a'_{(1)} \in C_1$, $a'_{(2)} \in C_2$, $a'_{(3)} \in C_3$ and three scalar weights $x'_1 \geqq 0$, $x'_2 \geqq 0$, $x'_3 \geqq 0$ such that

(2) $$\begin{aligned} y'_1 &\equiv a'_{11}x'_1 + a'_{12}x'_2 + a'_{13}x'_3 > 0, \\ y'_2 &\equiv a'_{21}x'_1 + a'_{22}x'_2 + a'_{23}x'_3 > 0, \\ y'_3 &\equiv a'_{31}x'_1 + a'_{32}x'_2 + a'_{33}x'_3 > 0, \\ 1 &= x'_1 + x'_2 + x'_3 . \end{aligned}$$

Then the boundary of the convex hull C of C_1, C_2, and C_3 intersects the positive octant $y_1 \geqq 0$, $y_2 \geqq 0$, $y_3 \geqq 0$ in a triangular segment of a plane such that the outward normal to that segment of the boundary has positive direction coefficients. This plane is a common plane of support

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[†] See P. Samuelson, "Abstract of Theorem Concerning Substitutability in Leontief Systems," LPC:804.

[*] I am indebted to Saunders MacLane for valuable discussions concerning this theorem.

of C_1, C_2, C_3.

The relation of this theorem to Leontief models is the following. Consider a matrix of input-output coefficients with respect to three products (and three industries) and one primary factor (labor), as follows:

	industry or activity		
	1	2	3
product 1	a_{11}	a_{12}	a_{13}
product 2	a_{21}	a_{22}	a_{23}
product 3	a_{31}	a_{32}	a_{33}
labor	-1	-1	-1

where a_{11}, a_{22}, a_{33} are positive, all other elements nonpositive. Samuelson has suggested to supplement column 1 by a number (finite, or discretely or continuously infinite) of columns all having -1 as fourth element and obeying the same inequalities governing the signs of the other elements. This corresponds to allowing choice in the proportions of inputs in industry 1. Column 1 plus the added columns of the same sign configuration can be made into the closed convex set C_1 satisfying (1) by adding all linear combinations with nonnegative weights that add up to unity. Similar additions of columns of the same sign configurations are made to columns 2 and 3, respectively, and convex sets C_2, C_3 satisfying (1) constructed thereon.

The point (y_1', y_2', y_3') in (2) then represents one achievable product combination from a total labor inflow of 1, obtained by rates of operation x_1', x_2', x_3' of the selected activities $a_{(1)}'$, $a_{(2)}'$, $a_{(3)}'$. The theorem states that, irrespective of the nonnegative quantities prescribed for any two of the three product flows y_1, y_2, y_3, the set of activities that maximizes the third of these product flows can be formed by linear combination, with positive weights, of the same three activities (or activity combinations) selected from C_1, C_2, C_3, respectively. After proper initial selection of the appropriate column to represent each industry, therefore, the possibility of substitution within each industry can be disregarded.

Proof: Define the convex hull C of C_1, C_2, C_3 as the set of all points obtainable by linear combination of three points selected from C_1, C_2, C_3, respectively, with weights adding up to unity. Consider the point $y'' = \lambda'' y'$ where λ'' is the algebraically largest value of λ for which

$$(3) \qquad \lambda y' = (\lambda y'_1, \lambda y'_2, \lambda y'_3)$$

is contained in C. Then y" is on the boundary of C, and can be made to satisfy

$$(4) \qquad \begin{cases} y''_1 = a''_{11}x''_1 + a''_{12}x''_2 + a''_{13}x''_3 > 0, & a''_{(1)} \in C_1, \\ y''_2 = a''_{21}x''_1 + a''_{22}x''_2 + a''_{23}x''_3 > 0, & a''_{(2)} \in C_2, \\ y''_3 = a''_{31}x''_1 + a''_{32}x''_2 + a''_{33}x''_3 > 0, & a''_{(3)} \in C_3, \end{cases}$$

for some x" satisfying

$$(5) \qquad \begin{cases} 1 = x_1 + x_2 + x_3 \\ x_1 \geqq 0, \quad x_2 \geqq 0, \quad x_3 \geqq 0 \end{cases}$$

if x" is substituted for x.

Consider the triangle spanned by $a''_{(1)}$, $a''_{(2)}$, $a''_{(3)}$, that is, the set of all points

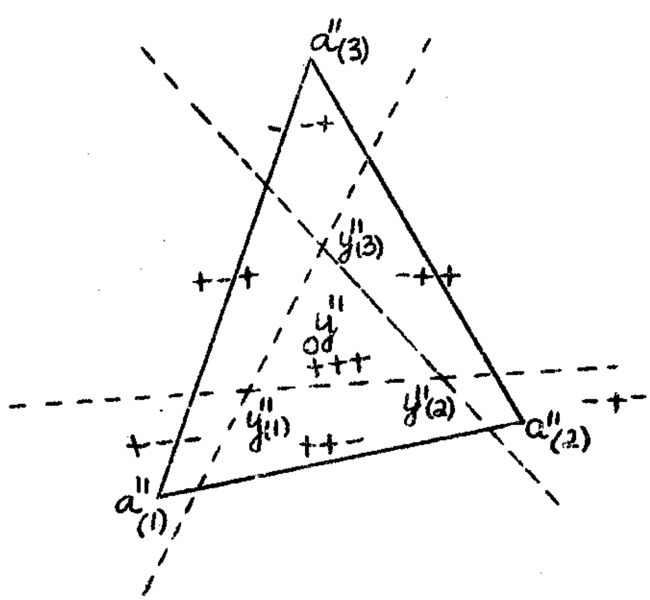
$$(6)\qquad \begin{cases} y_1 = a''_{11}x_1 + a''_{12}x_2 + a''_{13}x_3 \\ y_2 = a''_{21}x_1 + a''_{22}x_2 + a''_{23}x_3 \\ y_3 = a''_{31}x_1 + a''_{32}x_2 + a''_{33}x_3 \end{cases}$$

for which x satisfies (5). The following table indicates what can be said about the sign configurations of the coordinates of various points or point sets in it.

(7)

vertices			edges			internal point
$a''_{(1)}$	$a''_{(2)}$	$a''_{(3)}$	$\{a''_{(1)}, a''_{(2)}\}$	$\{a''_{(2)}, a''_{(3)}\}$	$\{a''_{(3)}, a''_{(1)}\}$	y''
+	−	−		−		+
−	+	−			−	+
−	−	+	−			+

Here + stands for >0, − for $\leqq 0$. Since y'' is contained in the triangle, the plane P of the triangle does not coincide with a coordinate plane. It follows from (7) that the intersections of this triangle with the coordinate side planes must run as follows (see dotted lines, drawn with

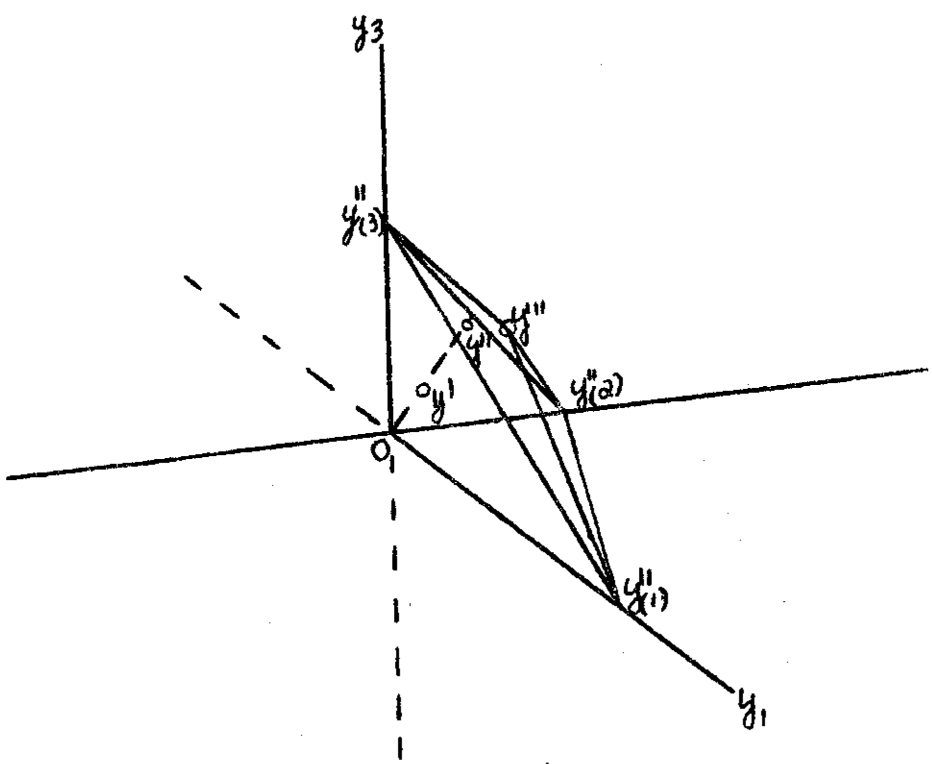


- meaning < 0): Intersection with

$$(8)\qquad \begin{cases} y_1 = 0 \text{ separates } y'' \text{ from } \{a''_{(2)}, a''_{(3)}\}, \\ y_2 = 0 \text{ separates } y'' \text{ from } \{a''_{(3)}, a''_{(1)}\}, \\ y_3 = 0 \text{ separates } y'' \text{ from } \{a''_{(1)}, a''_{(2)}\}, \end{cases}$$

separation meaning strictly that y'' is not on the line $y_1 = 0$ while no point of $\{a''_{(2)}, a''_{(3)}\}$ is on the same side of $y_1 = 0$ as y'' is, etc.

Denote by $y''_{(1)} \equiv (y''_{11}, 0, 0)$, $y''_{(2)} \equiv (0, y''_{22}, 0)$, $y''_{(3)} \equiv (0, 0, y''_{33})$ the three points at which the lines of separation intersect. Now suppose that C contains any point y''' of the positive octant separated from the origin by P. Then, since C is convex, the entire tetrahedron constructed on the vertices $y''_{(1)}$, $y''_{(2)}$, $y''_{(3)}$, y''' belongs to C, has y'' on its boundary



facing the origin, and contains a point (3) with $\lambda > \lambda''$, in contradiction with the definition of λ''. Hence the triangle $\{y''_{(1)}, y''_{(2)}, y''_{(3)}\}$ is a boundary of the intersection of C with the positive octant. It represents the entire production possibility locus whenever negative net outputs are excluded. It follows from the positive signs of y''_{11}, y''_{22}, y''_{33} that the direction coefficients of its outward normal are positive.

It is believed that this proof can (a) be purged of its visual elements by more careful formulation and (b) be generalized to n dimensions without essential new difficulties.

The statement proved here ceases to be true when either (i) one of the three industries has joint production (e.g., $a_{11} > 0$, $a_{21} > 0$) or (ii) more than one scarce primary input is required. Any proof should therefore make essential use of the restrictions (1) on the signs of the coefficients a_{nk}. I have not found the place where Samuelson's proof (LPC:804) does.