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Theory and Measurement of Production Functions

by

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This paper is concerned with an analysis of some aspects of the work of Professor Paul Douglas, associates and others who worked along the same line.

Statistical measurement of production functions is important as it may give us a tool to find out if and in how far the marginal productivity principle is working. In its most simple form this principle states that

$$p_0 \frac{\partial x_0}{\partial x_i} = p_i \qquad (1)$$

where

$$x_0 = a x_1^{\alpha_1} x_2^{\alpha_2} \qquad (2)$$

is the production function, giving the relation between the input of two factors of production-f.i. labor and capital- and output. Now when

$$\pi = y_0 - y_1 - y_2 \quad (3) \qquad \begin{array}{l} \pi = \text{profits} \\ y_j = p_j x_j \\ p_j = \text{price of } x_j \end{array}$$

then, under the assumption of profit maximization

$$\frac{\alpha_1}{\alpha_2} = \frac{p_1 x_1}{p_2 x_2} \qquad (4) \qquad \begin{array}{l} i = 1 \text{ or } 2 \\ j = 1,\ 2 \text{ or } 0. \end{array}$$

The marginal productivity principle, therefore, may be helpful in explaining:

a. the pricing process of the factors of production
b. the distribution of income.

I. We first analyze the assumptions that have to be made to find these two relations true when using the Cobb-Douglas function and measuring the variables, in physical units.

II. Secondly we analyze the assumptions when money-units are used to measure the variables.

III. Finally some remarks concerning the interpretation of results are made when using the single equation method.

I. Analyzing the assumptions that have to be made, to find the marginal productivity principle in its simple form true, we restate the principle as follows:

$$(1.1) \qquad \frac{\Delta y_0}{\Delta y_i} = 1$$

As will be clear from further analysis we have to make a distinction between two possible interpretations:

a. The increase in value of output is equal to the increase in outlay for the production factor generating the production-increase

b. The increase in revenue is equal to the increase of total costs.

Usually the increase in costs will be different from the increase in outlay for one of the production factors alone. The entrepreneur will act according to the last definition.

Case 1.1

$$(1.2) \qquad \frac{\Delta y_0}{\Delta y_i} = \frac{(p_0+\Delta p_0)(x_0+\Delta x_0)-x_0p_0}{(p_i+\Delta p_i)(x_i+\Delta x_i)-x_ip_i} = 1$$

or

$$(1.3) \qquad p_0 \cdot \frac{\beta_0}{\beta_i} \cdot \frac{x_0}{x_i} = p_i$$

where $\beta_i = \left(1+\frac{x_i}{p_i}\cdot\frac{\partial p_i}{\partial x_i}\right) = \left(1-\frac{1}{\varepsilon_i}\right)$ and $\beta_0 = \left(1-\frac{1}{\varepsilon_0}\right)$.

As β_0 and β_i approach 1 from different sides, $\frac{\Delta y_0}{\Delta y_i} = 1$ will only be true when $\beta_0 = \beta_i = 1$, or when free competition prevails.

The distribution of income is $\frac{\beta_2}{\beta_1}\cdot\frac{\alpha_1}{\alpha_2}$. So again only equal to $\frac{\alpha_1}{\alpha_2}$ under conditions of free competition.

Case 1.2

Only two factors of production were taken into account. Omitting the soil in testing a production function for the manufacturing section as a whole, or for parts of it, may be acceptable. Leaving the materials out of the analysis, however, has more influence. For now

$$\frac{\Delta y_0}{\Delta y_i} \quad \frac{\{(p_0+\Delta p_0)(x_0+\Delta x_0)-x_0p_0\}-\{(p_3+\Delta p_3)(x_3+\Delta x_3)-p_3x_3\}}{(x_i+\Delta x_i)(p_i+\Delta p_i)-x_ip_i} = 1$$

Adding the assumption that free competition prevails in the market of raw materials we find that

(1.4) $$p_i = p_0 \frac{\partial x_0}{\partial x_i}\left(1 - \frac{p_3}{p_0} \cdot \frac{\partial x_3}{\partial x_0}\right)$$

The assumption that x_0 and x_3 are proportional, which is not always true but in this case is the most favorable alternative, does not yet allow drawing the conclusion that $p_i = p_0 \frac{\partial x_0}{\partial x_i}$

Only when we add the assumption that also p_0 and p_3 change proportionally we are able to conclude that

(1.5) $$p_i = p_0 c_1 \frac{\partial x_0}{\partial x_i}$$ c_1 being a constant.

For the explanation of the income distribution no assumptions have to be made in excess of those in case 1.1.

Case 1.3a.

When prices of capital goods do not change, the above analysis will lead to the same results when measuring x_2 in physical or money units. When price changes do occur, however, capital has to be measured in physical units, or the value has to be deflated. The value of capital is $x_2' = p_2' x_2$ (p_2' being the price of capital goods). We now find

$$p_2^* = p_0 \frac{\partial x_0}{\partial x_2}$$

where

(1.6) $$p_2^* = p_2' p_2$$

The rate of interest therefore is the computed p_2^* divided by the index of the prices of capital goods.

The distribution of income will be, under conditions of free competition,

(1.7) $$\frac{\alpha_1}{\alpha_2} = \frac{p_1 x_1}{p_2^* x_2}$$

but p_2^* is no longer identifiable with the rate of interest. Using the rate of interest in this formula, we will have to substitute x_2' for x_2, and write

(1.8) $$\frac{\alpha_1}{\alpha_2} = \frac{p_1 x_1}{p_2 x_2'}$$

In the denominator we find either the value of capital multiplied by the actual rate of interest, or the physical quantity of capital multiplied by p_2^*.

Case 1.3b

The analysis of 1.3a is only true when the changes in the price of capital goods is a result of shifts in a horizontal supply curve for these goods. The assumption of free competition in the market of capital goods has to be added to allow the old conclusion concerning the pricing process. Only with this assumption we will find

$$(1.9)\quad \rho_2 \rho_2' = \rho_2^* = c_1 \rho_0 \frac{\partial x_0}{\partial x_2}$$

For the distribution of income we find

$$(1.10)\quad \frac{\alpha_1}{\alpha_2} \cdot \frac{\beta_2 \beta_2'}{\beta_1}$$

where $\beta_2' = (1 - \frac{1}{\varepsilon_2})$

so a form expressing the elasticity of supply in the market of capital goods. Also here the assumption of free competition is necessary.

Case 1.4

Using more capital goods does not only enlarge the outlay for interest, but will also be of influence on the costs of the capital-goods itself. Not only interest enters into the cost of production; depreciation also has to be taken into account. We assume

a. adding labor does not have influence on depreciation

b. depreciation is a fixed percentage of the value of the capital goods.

Analyzing $\frac{\Delta y_0}{\Delta y_2}$ we now find, making all the assumptions of the foregoing cases, that

$$(1.11)\quad \rho_2 \rho_2' = \rho_2^* = \rho_0 c_1 \frac{\partial x_0}{\partial x_2} - a_1 \rho_2'$$

a_1 being the % of depreciation.

Therefore, when we want to compare the actual rate of interest with a "rate of interest" computed from the derivative of the production function, we will have to deflate this computed rate by the prices of capital goods. And only after that we could find a linearity between the computed and actual rate.

The distribution of income is now:

$$(1.12)\quad \frac{y_1}{y_2} = \frac{\beta x_1}{\rho_2^* x_2} = \frac{\alpha_1}{\alpha_2 - \frac{a_1 x_2}{\rho_0 x_0 c_1}}$$

and the simplification of this expression to $\frac{\alpha_1}{\alpha_2}$ is no longer possible. The greater a_1 the greater the deviations in the distribution of income will be from $\frac{\alpha_1}{\alpha_2}$.

Case 1.5

We finally have to add one more complication. Employing more labor does not only increase production, the costs of labor and at the same time the quantity of materials used, but also other cost items will increase. The capital invested in the goods in process will increase, as a result of the fact that wages are paid while the receipts will come in at a later moment; for materials the same is true in the case of a larger production resulting either from an increase in labor or capital input.

Assuming that the capital invested in the goods in process is the sum of

a. a fixed % of the total wage-bill

b. a fixed % of the total outlay for materials

c. a fixed % of the total value of capital goods,

we find, under conditions of free competition, that

$$(1.13) \quad p_1 = p_0 \frac{\partial x_0}{\partial x_1} \left\{ \frac{c_4 p_2 + c_2}{c_3 p_2 + 1} \right\}$$

C_2, C_3 and C_4 are constants including the c_1 from formula (1.5) and the percentages mentioned above.

The conclusion must be that p_1 will only be proportional with $p_0 \dfrac{\partial x_0}{\partial x_1}$ when p_2 is a constant. For p_2 we find

$$p_2^* = p_2 p_2' = c_5 p_0 \frac{\partial x_0}{\partial x_2}$$

so that the conclusion here can be the same as in case 1.4.

That also the distribution of income will not simply be $\frac{\alpha_1}{\alpha_2}$, needs no proof. An expression, more complicated than in 1.4 will be found.

The case where depreciation and the complication of this case are taken into account together, was not analyzed.

Conclusion:

Measuring a production-function in physical units may be useful for computing physical product at different combinations of x_1 and x_2. For the explanation of the pricing process of the factors of production and also of the distribution of income, the function will be helpful only under very restricting conditions, or when we have much more information than only the production-function itself.

II. In statistical investigations the variables were often not measured in their respective units but in value units. In this part we will analyze if, and in how far, the assumptions, which we found to be necessary when measuring in physical units, can be dropped when different units of measurement are used, and if the conclusions concerning the pricing process and the distribution of income hold.

Case 2.1

As long as the prices of each of the three variablesare the same in in each observation, the results will not change. When, however, this is not the case, different results will be found. Measuring output in value, the factors of production in their respective units, the production function therefore being $y_0 = a x_1^{\alpha_1} x_2^{\alpha_2}$, we can derive from the profit equation, under the assumption of profit-maximization, that $p_i = \frac{\partial y_0}{\partial x_i}$.

Multiplication of the right side with p_0 is no longer necessary. Analyzing the derivative of the production function we find that $p_i = p_0/\beta_0 \frac{\partial x_0}{\partial x_i}$ and in comparison with the cases in Chapter I the assumption of free competition in the market of the product can be dropped.

The distribution of income is $\frac{p_1 x_1}{p_2 x_2} = \frac{\alpha_1}{\alpha_2}$ and also here the assumption of free competition in the product market is no longer necessary.

Case 2.2

When measuring production and labor-input in physical quantities, the quantity of capital however in value units, then

$$p_1 = p_0 \frac{\partial x_0}{\partial x_1}$$

$$p_2 = p_0 \frac{\partial x_0}{\partial x_2'}$$

Analyzing the derivative of the production function we find that

$$p_0 \frac{\partial x_0}{\partial x_2'} = \frac{p_0}{p_2'/\beta_2'} \cdot \frac{\partial x_0}{\partial x_2}$$

The assumption of free competition in the market of capital goods is no longer necessary, and also the deflation with p_2', as mentioned in case 1.3 need not take place.

For the distribution of income we still find $\frac{\alpha_1}{\alpha_2}$.

Case 2.3

When production as well as capital is measured in money units, we find a combination of the results of cases 2.1 and 2.2. The price of the factors of production will be equal to the derivative of the production function, therefore not multiplied by p_0, and the assumptions of free competition in the markets of the product and the capital goods can be dropped.

The distribution of income still simply is $\frac{\alpha_1}{\alpha_2}$.

Case 2.4

To avoid the criticisms mentioned before, concerning the cost of materials and others, production has been measured as value added (y'). Then $y_0' = a x_1^{\alpha_1} x_2'^{\alpha_2}$ is the production function, and from the profit equation follows that

$$p_1 = \frac{\partial y_0'}{\partial x_1} \qquad and \qquad p_2 = \frac{\partial y_0'}{\partial x_2'}$$

With which case from Chapter I we can compare the results of this approach depends on what is included in y_0'. When the value added is the revenue minus costs of materials, depreciation, interest on capital invested in the goods in process, then we find a result about the same as in Case 1.5 as far as the assumptions are concerned. All the assumptions are included except the one of free competition in the market of capital.

Case 2.5

Measuring production as value added, labor input as the wagebill and capital input as the interest outlay, then analyzing the derivative of the "production function" will show that all assumptions found necessary in Chapter I are included in this kind of production function. However, from the profit equation now follows that $\frac{\partial y_0'}{\partial y_i} = 1$.

No conclusions can be drawn regarding the prices of the factors of production, and also concerning the distribution of income the conclusion is very simple; namely the shares of each of the factors will be in the proportion y_1 / y_2.

Conclusion:

It may be possible that measuring in value units will give us the possibility of dropping one or more of the assumptions which seemed to be necessary in Chapter I. This method, however, will never enable us to find a solution for all assumptions. It is possible to measure the variables such that all assumptions can be dropped, but then no conclusions can be drawn regarding the pricing process of the factors of production.

At the same time we can conclude that, when measuring in value units, the distribution of income will still be in the proportion $\frac{\alpha_1}{\alpha_2}$ but no assumptions concerning free competition are necessary, and therefore no conclusions can be drawn from the results concerning the degree of free competition, monopoly or monopsony in the different markets.

From the analysis in the first two parts we may conclude that measuring the production function in physical units will give us the best possibilities to draw conclusions concerning the working of the marginal productivity principle and concerning the degree of free competition in the respective markets, although either many assumptions will be necessary or much information must be at hand.

III. Not only for the reasons mentioned in the last chapter must we have doubt concerning the significance of the results of some of Professor Douglas' investigations, when he uses value units in measuring one or more of the variables. The following analysis will show that also in other respects the use of value units may lead to false conclusions.

Case 3.1

The most favorable case to measure a production function seems to be the one where we have as observations a number of firms in the same industry, all working under conditions of free competition. In this case, however, using the single equation method, it will not be possible to find α_1 and α_2 separately as, for theoretical reasons, x_1 and x_2 will be highly correlated. We have to assume that the production function is the same for each firm, and for each firm it is true that $\frac{\alpha_1}{\alpha_2} = \frac{p_1 x_1}{p_2 x_2}$. Where now the α's are the same for each firm and also the prices of the factors of production are the same, each firm will use the production factors in the same proportion. To find significant results we will have to add the assumption that $\alpha_1 + \alpha_2 = 1$.

When measuring in value units we have the same difficulty in the case of free competition, and we will be able to measure the α's separately only when the proportion of the factors used varies from firm to firm. These variations may, e.g., result from differences in transportation costs or other differences in the market situation. As soon as we need these differences, however, measurement of the production function, using value units for the variables, and using the single equation method, will be without any meaning. From the results no conclusions can be drawn concerning for example the degree of free competition or the meaning of the exponents in the function.

This can easily be shown with the help of the graphs in Figure 1. In each graph the two points give the situation of two firms in the market of the product, of labor, and of capital, respectively.

Computing a production function from these observations will not lead us to any information either concerning the exponents in the (physical) production function or concerning the degree of monopoly or monopsony. The same production function will be obtained in this case for a number of possible and different situations in the market of the product (A), of labor (B) and of capital (C).

A. 1. one horizontal demand curve and different supply curves for each of the firms

2. both different supply and demand curves

B. 1. horizontal but not identical supply curves of labor, together with different demand curves

2. both the demand and supply curves are different

3. the demand curve is the same for each firm

C. 1. horizontal but not identical supply curves for capital together with different demand curves

2. different demand and supply curves

3. both the same vertical demand or supply curve and different supply and demand curves respectively.

Figure II gives two other sets of possible situations.

From these graphs can be concluded at the same time that the coefficients in this "production function" do not give us any information about the elasticities of the different demand and supply curves.

Conclusion:

A cross section study, using value units for measuring one or more of the variables, is theoretically only acceptable (disregarding statistical implications) when free competition exists. Then, however, the exponents in the production function cannot be computed separately unless linearity is assumed.

Case 3.2

A cross section study for the whole manufacturing section is only possible when measuring production and capital in the different observations (industries) in money units. Under free competition multicollinearity must exist again. In the case where deviations from free competition occur the same kind of difficulties as mentioned in the last case will arise.

Case 3.3

When using time-series in the analysis of one firm, one industry or the manufacturing section as a whole, none of the foregoing difficulties disappears. We will find multicollinearity as long as p_1 and p_2 fluctuate proportionally. As soon as no multicollinearity exists we will find different market situations in the different years and instead of having the firms as observations in our graphs, now the years appear as such, and again no conclusions can be drawn when the variables are measured in money units.

Measuring in physical units may still be possible but the problem of technical changes (which appeared in the cross section studies as different techniques in the separate firms) will perhaps be of more significance and will cause more troubles, while also many other criticisms have been mentioned in literature.