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SOME MODELS FOR THE STUDY OF THE ECONOMIC EFFECTS OF TECHNOLOGICAL CHANGE

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Some estimates of the magnitude of the direct effect upon national income of the introduction of technological changes--e.g., cheap atomic power--can be obtained by studying simple economic models. The technological change is introduced into the model as a change in one or more production functions.

In the first section, a rather general model of the Evans type will be examined. The second section will deal with the problem of aggregation. In the third and fourth sections, macromodels that are special cases of the general model will be studied. The fifth section will deal with the case of increasing returns.

These first five sections treat exclusively of the direct, short-run effect of technological change. Important indirect effects may also be produced through the impact of the change upon the rate of capital formation and the rate of growth of the labor force. These indirect effects will form the subject of the remaining sections of the ~~p~~aper.

I.

Consider an economy in which a consumption good (y) is produced by the use of certain factors of production $(x_\lambda : \lambda = 2, \dots, n)$. The supply of each of the $x_\rho; \rho = k+1, \dots, n$, will be considered fixed, while the supply of the other factors $(x_\mu : \mu = 2, \dots, k)$) will be determined by production functions for those factors. Thus we have:

(1.1) $$y = \phi(x_{21}, \dots, x_{n1}, \alpha_1),$$

(1.2) $$x_\mu = \psi^{(\mu)}(x_{2\mu}, \dots, x_{n\mu}, \alpha_\mu) \qquad (\mu = 2, \dots, k),$$

(1.3) $$\sum_{i=1}^{k} x_{\lambda i} = x_\lambda, \qquad (\lambda = 2, \dots, n),$$

(1.4) $x_\rho =$ constant, $\qquad (\rho = (k+1), \dots, n).$

The $x_{\lambda i}$ are components of the x_λ devoted to different types of production. The α_λ are parameters describing the state of technology in each industry. (It can easily be shown that the results which follow hold even if some of the x_{ij} are zero.)

This model may best be interpreted as a "short-run" model for an economy, since all the factors of production are consumed during the period of production. If this interpretation is adopted, the variable x_λ become semi-manufactured goods, while the constant X_λ are factors which, like land, labor, buildings and machinery, are fixed in the short run. The variable y is an index of the national income, including consumption plus net additions to fixed and working capital.

From the usual assumption of profit maximization under competition, we get additional relations:

(1.5) $$\phi_\lambda = p_\lambda / p_y \; ; \; \psi_\lambda^{(\mu)} = p_\lambda / p_\mu \quad (\lambda = 2, \ldots, m;\ \mu = 2, \ldots, k),$$

where the $p_y, p_\lambda (\lambda = 2, \ldots, m)$ are money prices.

Equations (1.1)-(1.5) give us $(m^2+2m-2+k-km)$ independent relations among the same number of variables; $y, x_{\lambda\mu}, X_\lambda, p_\lambda/p_y\ (\lambda = 2, \ldots, m;\ \mu = 2, \ldots, k)$.

(Note that only the price ratios are determined by the system.)

Hence we can consider these variables as functions of the α's, and in particular we can solve for dy as a function of the $d\alpha$'s. Taking total differentials of (1.1) and (1.2), we have:

(1.6) $$dy = \sum_{i=2}^{m} \phi_i \, dx_{i1} + \phi_\alpha \, d\alpha_1 = \sum_{i=2}^{m} \frac{p_i}{p_y} dx_{i1} + \phi_\alpha \, d\alpha_1$$

(1.7) $$dx_\mu = \sum_{i=2}^{m} \psi_i^{(\mu)} dx_{i\mu} + \psi_\alpha^{(\mu)} d\alpha_\mu , \qquad (\mu = 2, \ldots, k)$$

$$= \sum_{i=2}^{m} \frac{p_i}{p_\mu} dx_{i\mu} + \psi_\alpha^{(\mu)} d\alpha_\mu ,$$ where

(1.8) $$\sum_{i=1}^{k} dx_{\lambda i} = dX_\lambda , \qquad (\lambda = 2, \ldots, m),$$

(1.9) $$dx_\rho = 0, \qquad (\rho = (k+1), \ldots, m).$$

Adding and subtracting $\sum_{i=2}^{k} \phi_i dx_i$ from the right-hand side of (1.6), and expanding the terms by (1.7), we find:

(1.10) $$dy = \sum_{i=k+1}^{m} \phi_i \, dx_{i1} + \sum_{i=2}^{k} \phi_i \{ dx_{i1} - dx_i \} + \sum_{i=2}^{k} \phi_i \, dx_i + \phi_\alpha \, d\alpha,$$

$$= \sum_{i=k+1}^{m} \phi_i \, dx_{i1} + \sum_{i=2}^{k} \phi_i \{ dx_{i1} - dx_i \}$$

$$+ \sum_{i=2}^{k} \phi_i \left\{ \sum_{j=2}^{m} \psi_j^{(i)} dx_{ji} + \psi_\alpha^{(i)} d\alpha_i \right\} + \phi_\alpha \, d\alpha$$

Reversing the indices i and j of the third term, separating the two summations $\sum_{i=k+1}^{n}$ and $\sum_{i=2}^{k}$, and combining them, with the first and second terms, respectively, we have:

$$(1.11)\quad dy = \sum_{i=k+1}^{n}\left\{\phi_i\,dx_{i1} + \sum_{j=2}^{k}\phi_j\,\psi_i^{(j)}\,dx_{ij}\right\} + \sum_{i=2}^{k}\left\{\phi_i\,dx_{i1} + \sum_{j=2}^{k}\phi_j\,\psi_i^{(j)}\,dx_{ij} - \phi_i\,dx_i\right\} + \sum_{i=2}^{k}\phi_i\,\psi_\alpha^{(i)}\,d\alpha_i + \phi_\alpha\,d\alpha_1 .$$

Using (1.5), this reduces to:

$$(1.12)\quad dy = \sum_{i=k+1}^{n}\frac{p_i}{p_y}\,dx_{ij} + \sum_{i=2}^{k}\left(\sum_{j=1}^{k}\frac{p_i}{p_y}\,dx_{ij} - \frac{p_i}{p_y}\,dx_i\right) + \sum_{i=2}^{k}\phi_i\,\psi_\alpha^{(i)}\,d\alpha_i + \phi_\alpha\,d\alpha_1 .$$

But, by (1.8) and (1.9) the first two terms are zero. Hence,

$$(1.13)\quad dy = \sum_{i=2}^{k}\phi_i\,\psi_\alpha^{(i)}\,d\alpha_i + \phi_\alpha\,d\alpha_1 .$$

This very important result has previously been derived for similar models by Dresch and May. The fact that the dx_{ij}'s cancel out is due to the conditions (1.5) whereby, near the equilibrium, substitution of factors leads only to second-order changes in the quantities produced.

The interpretation of equation (1.13) is simplified if we introduce $\delta\phi = \phi_\alpha\,d\alpha_1$, $\delta\psi^{(i)} = \psi_\alpha^{(i)}\,d\alpha_i$.

$$(1.14)\quad \frac{dy}{y} = \frac{\delta\phi}{\phi} + \sum_{i=2}^{k}\frac{p_i}{p_y}\,\frac{x_i}{y}\,\frac{\delta\psi^{(i)}}{\psi^{(i)}} = \frac{\delta\phi}{\phi} + \sum_{i=2}^{k}\frac{v_i}{v_y}\,\frac{\delta\psi^{(i)}}{\psi^{(i)}},$$

where $v_i = p_i x_i$; $v_y = p_y y$. For greater symmetry, we write $\psi^{(1)}$ for ϕ and v_1 for v_y, obtaining:

$$(1.15)\quad \frac{dy}{y} = \sum_{i=1}^{k}\frac{v_i}{v_1}\,\frac{\delta\psi^{(i)}}{\psi^{(i)}}$$

But $\frac{\delta\psi^{(i)}}{\psi^{(i)}}$ is the relative change in x_i (or y) brought about by the technological change in its production function for <u>given</u> quantities of the factors of production. Hence, from (1.15) we read that the relative change in y is the sum of the products v_i/v_y by the relative change in productivity of the ith industry. If we consider the special case where $\delta\psi^{(i)}/\psi^{(i)} = k$ for i in some set Ω, $\delta\psi^{(i)}/\psi^{(i)} = 0$ for i not in Ω we get:

$$(1.16)\quad \frac{dy}{y} = \frac{k}{v_y}\sum_{i \in \Omega} v_i .$$

It will be useful, at this point, to examine the economic significance of our results. We are considering, of course, only first-order effects, which will be dominant if the $\delta\psi^{(i)}$ are small (strictly, we are considering $\lim_{h \to 0} \frac{dy}{dh}$).[1]

[1] Closely similar results for the economic effects of a change in technology were obtained by Leontief more than ten years ago, using a somewhat different model than that employed here. Leontief assumed linear production functions with fixed coefficients (no factor substitution possible). Since our own result involves linear approximations, it is not surprising that the two methods should lead to about the same end. A careful examination of the relation between Leontief's model and the one used in this paper should be of considerable value to further investigations of this kind. See Wassily W. Leontief, "Interrelation of Prices, Output, Savings, and Investment," The Review of Economic Statistics, 19:109-132 (August, 1937).

1. From (1.9) we see that the increase in y is equal to the increase that would result if the quantities of factors used in producing each X_i were held constant, while the increment in X_i resulting from the change in technology were applied directly to the production of y. This does not mean that there will be in fact no substitution (see section III below). But because in equilibrium (equations (1.5)), the marginal productivities of the factors are equal in all uses, substitution causes only second-order effects which vanish in the limit. Hence, small technological changes produce an effect on y as if no substitution took place. This is the economic significance of the mutual cancellation of substitution terms that occured in reaching equation (1.13).

2. From (1.14) we see that dy/y is the sum of components corresponding to technological changes in individual industries—also a consequence of the fact that we are considering only first-order effects. From (1.15) we see that the relative change in y is proportional to the ratio of the value of the factor whose production is increased by the technological change (and not merely to the fraction of that factor used in producing y) to the value of y; and proportional also to the relative increase in the productivity of resources employed in producing that factor. From (1.16) we see further that if several factors are increased in productivity by the

same relative amount, they can be treated as a single factor.

3. An even simpler interpretation of the results is possible. The coefficient k may be interpreted as the percentage saving (due to the change in the production function) in the resources required to produce a given output. Writing $V_{\Omega} = \sum_{i \in \Omega} v_i$, we have from (1.16) $v_y \cdot dy/y = k \cdot V_{\Omega}$. That is, the increase, in money income is equal to the money value of the resources saved by the technological change. From this standpoint, the technological change leads to a "saving" of the fixed resources, the "saved" resources then being applied to the production of additional output. Since, in first approximation, the marginal productivity of the factors is constant and equal in all uses, the value of the increased output is exactly equal to the value of the saved resources.

II.

Point (2), above, leads us to consider the problem of aggregation. Is the estimate of dy/y invariant with respect to aggregation of the factors of production and can dy/y be estimated correctly from a model in which some of the industries are represented only as aggregates?

The k production functions may be considered the functions for individual firms (several of the x_λ may represent identical products). It would be highly unsatisfactory if the result obtained for a firm carrying on an integrated series of production processes would be altered by breaking up the production function for the firm into the several production functions for the separate processes.

The difficulty involved in vertical integration is that integration will in general reduce $\sum_{i \in \Omega} \sigma_i$ by eliminating some of the intermediate factors. Consider two firms (or industries), the product of the first being used entirely as a factor in the production of the second, then $\Sigma\sigma$ for these two firms will be $\sigma_a + \sigma_b$; while if the production function were written for a vertically integrated firm combining both production processes, we would have $\sigma = \sigma_b$.

The appearance of paradox is removed when it is recognized that substituting in our model the production function of the integrated firm for the two original production functions not only changes $\Sigma \upsilon$, but produces also a compensating change in the k of equation (1.16).

A more concrete understanding of these results may be obtained by applying them to the specific case of a technological change in the electric power industry. We may consider the production function for (a) electric energy at the central generating station and (b) the transmission of electric energy to the plant.

We distinguish two cases:

(A) There is a technological change of magnitude K in process (a), the other production functions remain unchanged.

(B) There is a technological change of magnitude k_a in process (a), and one of magnitude k_b in process (b).

We must first make clear what is meant by "a technological change of magnitude k." We estimate the product that will be produced in the industry with the original input of factors prior to the technological change (x_a). We estimate this product again after the technological change (x_a'). Then (2.1) $k = \dfrac{x_a' - x_a}{x_a} \longrightarrow \dfrac{\delta \psi^{(a)}}{\psi^{(a)}}$ ✓

✓ It may be objected that different proportions of the individual factors will be employed in producing x_a before and after the technological change, hence, that it is meaningless to speak of a fixed input of factors. Our justification in ignoring this lies in equations (1.5) which guarantee that in the neighborhood of the equilibrium substitution effects vanish.

In case (A), if processes (a) and (b) are represented by separate production functions, we have (2.2) $\dfrac{dy}{y} = \dfrac{\upsilon_a}{\upsilon_y} K = \dfrac{\upsilon_a}{\upsilon_y} \dfrac{\delta \psi^{(a)}}{\psi^{(a)}}$. Consider now an integrated production function representing processes (a) and (b) together. Then we estimate:

(2.3) $\quad \bar{k} = \dfrac{\delta \Psi_b}{x_b} = \dfrac{\partial \psi_b / \partial x_a \cdot \delta \psi_a}{x_b} = \dfrac{p_a \, \delta \psi_a}{p_b \, x_b} = \dfrac{\upsilon_a}{\upsilon_b} k$; Hence,

(2.4) $\quad \dfrac{dy}{y} = \dfrac{\upsilon_a}{\upsilon_y} k = \dfrac{\upsilon_b}{\upsilon_y} \bar{k}$.

In case (B), if processes (a) and (b) are represented by separate production functions, we have:

(2.5) $$\frac{dy}{y} = \frac{v_a}{v_y} k_a + \frac{v_b}{v_y} k_b,$$

(2.6) If the functions are combined, we have:

(2.6) $$\bar{k} = \frac{\delta \bar{\psi}_b}{x_b} = \frac{\delta \psi_b + \frac{\partial \psi_b}{\partial x_a} \delta \psi_a}{x_b} = k_b + \frac{v_a}{v_b} k_a; \text{ Hence}$$

(2.7) $$\frac{dy}{y} = \frac{v_b}{v_y} k_b + \frac{v_a}{v_y} k_a = \frac{v_b}{v_y} \bar{k}.$$

It follows that if v and k are estimated for corresponding production processes, or combinations of processes, it does not matter whether we consider the individual process in which ~~the~~ each technological change has taken place, or an integrated industry of which these processes are only parts. It should be noted, however, that if (a) and (b) each undergo a change of magnitude k, the change in the integrated industry will not be k, but $\bar{k} = (1 + \frac{v_a}{v_b})\, k > k$; while if only (a) has a change of magnitude k, the change in the integrated industry will be $\bar{k} = \frac{v_a}{v_b} k < k$. In either case, accounting estimates of the changed cost of production will provide correct estimates of k or $\bar{k}$, as the case may be.

1. This result was first obtained for a particular model by Kenneth May, "Technological Change and Aggregation," Econometrica 15:51-63 (January, 1947)

III

From these results it will be interesting to derive actual estimates of the increase in production of consumption goods that might result from technological improvements in the electric power industry. At an average price of 1.5 cents per KWH, the value of the 280 billion KWH electric power generated in the United States in 1944 (including power generated in industrial plants for their own use) was about 4.2 billion dollars. The total national income in the same year was 160 billion dollars. Hence, $v_\Omega / v_y = .026$. An increase in productivity of five per cent in the power industry (a cost reduction of about .75 mills per KWH) would therefore lead to a percentage increase in the national income of 5 x .026 = .13. Stated in other terms, each one mill reduction in power costs would increase the national income by about one-sixth of one per cent. This result holds strictly, of course, only for cost reductions of small magnitude.

The same estimates can also be obtained from a two-industry macromodel[3] — a result that is not surprising in view of our previous discussion of integration. Let:

(3.1) $\quad Y = \eta(X_a, E)$,

(3.2) $\quad E = \xi(X_b, \alpha)$,

(3.3) $\quad X_a + X_b = X = \text{constant}$,

(3.4) $\quad \eta_X = \frac{\partial Y}{\partial X}, \; \eta_E = \frac{\partial Y}{\partial E}, \; \xi_X = \frac{\partial E}{\partial X}$

Taking total differentials, we find that

(3.5) $\quad dY = \eta_E \, \xi_\alpha \, d\alpha$.

This equation is of the same form as (1.14) when all the α's except one are taken as constant. Hence, our macromodel is a special case of the micromodel of section I.

Defining again $\delta\xi = \xi_\alpha d\alpha$ ~~dd~~, we get

(3.6) $\quad \frac{dY}{Y} = \frac{\partial Y}{\partial E} \frac{E}{Y} \frac{\delta\xi}{\xi} = \frac{v_E}{v_Y} \frac{\delta\xi}{\xi}$.

In applying equation (3.6) for the statistical estimation of the effects of technological change, E must be considered the quantity of power employed in the entire economy, not simply the quantity employed directly in the production of consumption goods, i.e., E corresponds to one of the X_λ, and not to $X_{\lambda 1}$ in the micromodel of section I.

We may also solve equations (3.1) - (3.5) for dE. We obtain:

(3.7) $\quad dE = \xi_\alpha d\alpha + \frac{\xi_X}{\Delta}\left\{\eta_E \xi_{X\alpha} - (\eta_{XE} - \xi_X \eta_{EE})\xi_\alpha\right\} d\alpha$,

where $\Delta = \eta_E \xi_{XX} + 2\xi_X \eta_{EX} - \xi_X^2 \eta_{EE} - \eta_{XX}$.

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3. This is essentially the model employed by May, op. cit. in footnote 2, pp. 54-58.

From second order stability conditions on (3.1)-(3.5), we can show that $\Delta > 0$.

Hence, substitution effects do not vanish in the limit for dE as they do for dY. The terms (i) $\eta_E \xi_{x\alpha}$, (ii) $(-\eta_{xE}\xi_{\alpha})$, (iii) $\xi_x \eta_{EE} \xi_{\alpha}$ have the following economic interpretation.

(i) The factor X will be shifted from the production of Y to E (from E to Y) as the technological change increases (decreases) the marginal productivity of X in ξ.

(ii) The factor X will be shifted to Y as the increased consumption of E increases the marginal productivity of X in η.

(iii) The factor X will be shifted to Y as the increased consumption of E decreases the marginal productivity of E in η.

If the second partial derivatives in (3.7) are small enough (nearly constant returns), we have:

(3.8) $\quad dE \doteq \xi_{\alpha}\, d\alpha = \delta \xi_{\alpha}$.

Hence, the same quantity of resources will be used in producing E after the technological change as were used before if the effects of decreasing returns are sufficiently small to be ignored. But our previous estimate of dY is free from this limitation.

IV

For large values of k we may expect the percentage increment in y to be smaller than that given by our estimate. To obtain some idea of the effect of decreasing returns, consider a two-industry macromodel for the special case of the Cobb-Douglas function.

(4.1) $\quad Y = AX_a^{s}\; E^{m}$

(4.2) $\quad E = \alpha X_b^{n}$

(4.3) $\quad X_a + X_b = X = \text{Constant}$

(4.4) $\quad \frac{p\hat{x}}{p\hat{y}} = \frac{\partial Y}{\partial X} = \frac{SY}{Xa}; \quad \frac{p\hat{E}}{p\hat{Y}} = \frac{mY}{E}; \quad \frac{p\hat{x}}{p\hat{E}} = \frac{nE}{X_b}$

From (4.4), we derive

(4.5) $\quad \frac{Xb}{Xa} = \frac{n m}{s}$

If we consider a change, $\Delta\alpha$ in the parameter we will obtain new equilibrium values, $\bar{Y}$, $\bar{X}_a$, $\bar{X}_b$, $\bar{E}$, for our variables. But, since

(4.6) $\dfrac{\bar{X}_b}{\bar{X}_a} = \dfrac{rm}{s}$ and $\bar{X}_a + \bar{X}_b = X = X_a + X_b$, we have:

(4.7) [6] $\bar{X}_a = X_a$, $\bar{X}_b = X_b$ Hence,

(4.8) $\bar{Y} = Y + \Delta Y = AX_a^s X_b^{rm} (\alpha + \Delta\alpha)^m$;

(4.9) $\dfrac{Y + \Delta Y}{Y} = \dfrac{(\alpha + \Delta\alpha)^m}{\alpha^m} = \left(1 + \dfrac{\Delta\alpha}{\alpha}\right)^m$.

If as previously, we set $\dfrac{\Delta\alpha}{\alpha} = k$, we find

(4.10) $\log \dfrac{Y + \Delta Y}{Y} = m \log (1 + k)$.

But $m = \dfrac{\sigma_E}{\sigma_Y}$, and we have:

(4.11) $\log \dfrac{Y + \Delta Y}{Y} = \dfrac{\sigma_E}{\sigma_Y} \log (1 + k)$

Hence, if the ratio of total energy costs to total value of consumption goods is known, $\dfrac{\Delta Y}{Y}$ can be calculated for various values of k, as shown in the following table:

Table I

Values of $\dfrac{\Delta Y}{Y}$

$\frac{\sigma_E}{\sigma_Y}$ \ k	1	9	19	49	99
.02	.014	.05	.06	.08	.10
.05	.035	.12	.16	.22	.26
.10	.07	.26	.35	.48	.59
.20	.15	.59	.82	1.19	1.51
.30	.23	1.00	1.45	2.24	2.98

It can be seen from the table that if the value of energy produced in the economy is only two percent of the value of consumption goods, a 100% increase in productivity of the power industry ($k = 1$) will increase production of consumption goods 1.4%; a 900% increase ($k = 9$) will increase production 5%.

In computing the increased productivity of the power industry our previous analysis of aggregation shows that it does not matter whether we consider cost of energy at the generating plant, cost at the factory, or cost including power appliances. In each case we will get a different value for V_E, but a correspondingly different value for k; hence, our estimates of ΔY will be identical.

Finally, it is interesting to compare the results obtained in this section with the first-order effects of the model of section VI. For $\frac{V_E}{V_X} = .05$ and $k = 1$, $\frac{\Delta Y}{Y} = .035$ in the present model, but .05 in the previous model. For $k = 9$, $\frac{\Delta Y}{Y} = .12$, while $\frac{dy}{Y} = .45$. Hence, our earlier model gives an estimate only fifty per cent too large for $k = 1$, but four times too large for $k = 9$.

V

By examining the second-order conditions, it can be shown that the models which have been considered are stable only in case of decreasing returns. In particular, in section IV, we must have $S + m < 1$, $\rho < 1$. This is a reflection of the well-known economic condition that increasing returns are incompatible with the stability of competitive equilibrium.

By ignoring the stability conditions, we can use the model of section IV, formally, to examine the effects of increasing returns under conditions of monopoly, imperfect competition, or external economies. In the first two cases—monopoly and imperfect competition, we generally have:

(5.1) $$\frac{Px}{Py} < \frac{\partial y}{\partial x}$$

Hence, in (4.10), $m > \frac{V_E}{V_Y}$, and we have to enter Table II with $\rho \frac{V_E}{V_Y}$ rather than $\frac{V_E}{V_Y}$, where $\rho = \frac{m}{V_E/V_Y}$ is a measure of the departure from competitive conditions.

External economies can be treated within the same general framework. Let us consider production functions for n firms manufacturing the same product, using a single factor (with total quantity fixed), and subject to external economies that depend on the total output of the industry.

(5.2) $$y_i = f_i(x_i, y_i, \alpha_i)\,;\; y = \sum_{i=1}^{n} y_i\,;\; x = \sum_{i=1}^{n} x_i = \text{constant}\,;$$

If each entrepreneur attempts to maximize his profit, taking y and prices as given, we find:

(5.3) $$\frac{p_x}{p_y} = \frac{\partial f_i}{\partial x_i} \qquad (i = 1, \ldots, n)$$

Suppose now a technological change affects each of the firms, as represented by changes in the α_i's. We have:

(5.4) $$dy = \sum_{i=1}^{n} \left(\frac{\partial f_i}{\partial x_i} dx_i + \frac{\partial f_i}{\partial y} dy + \frac{\partial f_i}{\partial \alpha_i} d\alpha_i \right)$$

$$= \frac{p_x}{p_y} \sum_{i=1}^{n} dx_i + dy \sum_{i=1}^{n} \frac{\partial f_i}{\partial y} + \sum_{i=1}^{n} \frac{\partial f_i}{\partial \alpha_i} d\alpha_i \,.$$

Introducing for simplicity the notations: $f = \sum_{i=1}^{n} f_i$, $\sum_{i=1}^{n} \frac{\partial f_i}{\partial \alpha_i} d\alpha_i = \sum_{i=1}^{n} \delta f_i = \delta f$, this reduces to:

(5.5) $$dy = dy \frac{\partial f}{\partial y} + \delta f$$, or

(5.6) $$dy = \frac{\delta f}{(1 - \partial f / \partial y)}.$$

In this case, to correctly estimate the effect of technological change, our k must be divided by the factor $(1 - \frac{\partial f}{\partial y})$ which measures the additional output accruing from external economies.

Suppose, in particular, that f is of the form:

(5.7) $$y = \alpha x^r y^s \quad ; \quad 0 < s < 1.$$ Then,

(5.8) $$dy = x^r y^s d\alpha + s \alpha x^r y^{s-1} dy,$$

(5.9) $$dy = \frac{x^r y^s d\alpha}{1 - s \alpha x^r y^{s-1}} = \frac{y \, d\alpha}{\alpha(1-s)}$$

This same result may be obtained by solving (5.7) explicitly for y, and differentiating

(5.10) $$y^{1-s} = \alpha x^r$$

(5.11) $$(1-s) y^{-s} dy = x^r d\alpha$$
$$dy = \frac{x^r y^s d\alpha}{1-s} = \frac{y \, d\alpha}{\alpha(1-s)}$$

Equation (5.10), in turn may be transformed into an explicit production function for y:

(5.12) $$y = \beta x^{r'} \quad ; \quad \beta = \alpha^{\frac{1}{1-s}}, \; r' = \frac{r}{1-s} > r$$

We see that stability from the standpoint of the individual entrepreneur ($r < 1$ in equation (5.7)) is entirely compatible with increasing returns from the standpoint of the entire economy ($r' > 1$ in equation (5.12)).

VI

Our analysis thus far has been a static one. The quantities of the "ultimate" factors of production - labor, capital- have been taken as fixed. The problem to be solved next is that of evaluating the dynamic, long-term effects of technological change. Let us start again, with a simple macromodel in the variables: real income (y), capital (c), labor (x), and population (p), and the following relations connecting them:

$$(6.1)\qquad y = \phi(x, c, \alpha(t)),$$

$$(6.2)\qquad \frac{dc}{dt} = y\,\psi\left(\frac{y}{p}\right),$$

$$(6.3)\qquad \frac{dp}{dt} = p\,\xi\left(\frac{y}{p}\right),$$

$$(6.4)\qquad x = p\,\eta\left(\frac{y}{p}\right).$$

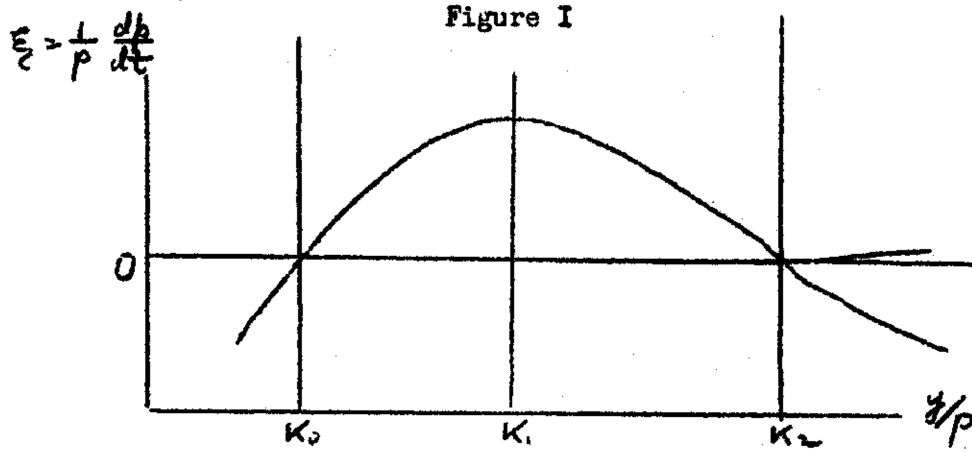
Equation (6.1) is the production function, which includes a parameter of technology, $\alpha(t)$, that is itself a function of time. Equation (6.2) is a saving (and investment) function. It is assumed that a fraction of the annual income is devoted to capital formation—the fraction depending on per capita real income. Equation (6.3) states the dependence of birth and death rates upon per capita real income. Equation (6.4) makes the size of the working force dependent upon the population and the per capita real income. In fact, the variables we have made dependent on real income undoubtedly depend also upon income distribution and other economic and non-economic factors. Nevertheless, our model represents in general form some of the salient features of the dynamic growth of an economy, in particular the relationships of growth in per capita income, capital formation, changes in technology, and population trends. As before, full employment of resources is assumed.

Provided our functions satisfy appropriate continuity and Lipschitz conditions, the system (6.1) - (6.4) determines x, y, c, and p as functions of t. Let us first make some qualitative assumptions about these functions, in order to determine how the economy will develop.

First, y is a monotonic increasing function of x, c, and α. In most cases we will want to assume decreasing returns for x and c separately, but we may also be interested in the case of increasing returns. The rate of saving is a monotonic increasing function of real income per capita, but in first approximation we may also consider the case where ψ is a constant.

The actual relation between population increase and real income is extremely complicated, and probably involves important lag relations. For certain purposes we may assume that the population is static for $y/p = K_0$ where K_0 is the "subsistence" level of real income in the Malthusian sense. For $K_0 < y/p < K_1$ birth rates will be assumed to remain at their "natural" levels (i.e. no preventitive measures, or change in marriage ages or fertility), while death rates will decline. Hence, in this interval, ξ is an increasing function of y/p. For $y/p \geq K_1$, we may assume a continued, but smaller decline in death rates, but a large drop in birth rates so that for $y/p > K_2 > K_1$, ξ may actually be negative. The hypothetical graph of $\frac{1}{p}\frac{dp}{dt}$ against $\frac{y}{p}$ is shown in Figure I

Figure I



It is difficult to make a defensible hypothesis about the function η. High per capita real incomes may lead to a substitution either of goods for leisure or of leisure for goods, depending on a variety of circumstances. The early stages of industrialization in most countries have seen an increase in the proportion of the population that is employed, but in "mature" economies increases in real income appear usually to be associated with the withdrawal of women, children and the aged from the labor market and a shortening of hours. In first approximation, η may be assumed to be a constant, and x eliminated as a separate variable.

Even with these assumptions, our system of equations is rather general and unmanageable. It will be helpful to start with a simpler special case, and gradually add complications.

VII

A. In the simplest case we take α, γ, p and ψ as constants, and write:

(7.1) $$y = \alpha x_0^n C^s, \qquad (s < 1);$$

(7.2) $$\frac{dC}{dt} = \psi_0 y.$$

Solving for C and y as explicit functions of time, we get:

$$(7.3)\qquad C = \left[\psi_0 \frac{y_0}{C_0^s}(1-s)t + C_0^{1-s}\right]^{\frac{1}{1-s}}$$

$$(7.4)\qquad y = \alpha \chi_0^r \left[\psi_0 \frac{y_0}{C_0^s}(1-s)t + C_0^{1-s}\right]^{\frac{s}{1-s}}$$

where $y_0 = y(t_0),\ C_0 = C(t_0),\ \chi_0 = \chi(t_0)$.

We consider now the development of another economy identical to the first except that it enjoys an increase in productivity at time to.

For this economy, we write:

$$(7.5)\qquad y' = \alpha' \chi_0^r C'^s \qquad \text{where } \alpha' > \alpha,\ C_0' = C_0$$

$$(7.6)\qquad \frac{dC'}{dt} = \psi_0 y'$$

From the solution of this system, it follows that:

$$(7.7)\qquad \frac{y'(t)}{y(t)} = \frac{\alpha'}{\alpha}\left\{\frac{\psi_0\alpha'\chi_0^r(1-s)t + C_0^{1-s}}{\psi_0\alpha\chi_0^r(1-s)t + C_0^{1-s}}\right\}^{\frac{s}{1-s}}$$

As t increases, we get, in the limit:

$$(7.8)\qquad \lim_{t\to\infty}\frac{y'}{y} = \frac{\alpha'}{\alpha}\left(\frac{\alpha'}{\alpha}\right)^{\frac{s}{1-s}} = \left(\frac{\alpha'}{\alpha}\right)^{\frac{1}{1-s}} = \left(\frac{y_0'}{y_0}\right)^{\frac{1}{1-s}}$$

Hence, the short-term effect of the technological change, corresponding to the effect measured in the Evans model, is magnified in the long run by the exponent $1/(1-s)$ due to the increased rate of capital accumulation that results.

B. For the case of constant returns to capital (S=1), (7.4) becomes:

$$(7.9)\qquad y = \alpha \chi_0^r C_0 e^{\psi_0\alpha\chi_1^r t},$$

and we get:

$$(7.10)\qquad \frac{y'}{y} = \frac{\alpha'}{\alpha} e^{\psi_0\chi_0^r(\alpha'-\alpha)t}.$$

C. Returning to the case of diminishing returns, we consider next two economies both experiencing a linear advance in technology, but with one advancing at a more rapid rate than the other. That is, instead of α and α' constant, we write:

$$(7.11)\qquad y = (\alpha_0 + kt)\chi_0^r C^s, \qquad (s < 1)$$

$$(7.12)\qquad y' = (\alpha_0 + k't)\chi_0^r C'^s.$$

The solution of the system comprised of

(7.11) and (7.2) is:

$$(7.13)\qquad y = (\alpha_0 + kt)\chi_0^r\left\{\frac{\psi_0 y_0}{C_0^s}(1-s)\left(t + \frac{k}{2}t^2\right) + C_0^{1-s}\right\}^{\frac{s}{1-s}}$$

From this, we readily deduce:

$$(7.14) \qquad \lim_{t\to\infty} \frac{y'}{y} = \frac{k'}{k}\left(\frac{k'}{k}\right)^{\frac{s}{1-s}} = \left(\frac{k'}{k}\right)^{\frac{1}{1-s}}$$

These results are illustrative of the course of growth of real income, and the effects upon this growth of technological change, in an economy with stationary population, and having an aggregate production function of the Cobb-Douglas type. We see that the economy with the more advanced technology has a constant (Models A and C), or constantly growing (Model B), advantage in productivity over the economy that is less advanced. The technological advantage is magnified by the more rapid accumulation of capital that is permitted by the superior technology; and it is in this fact that the principal distinction lies between the results of the dynamic analysis of the present section and the static analyses of previous sections.

VIII

Next, we introduce the assumption of a growing population. The most interesting special case is that in which the rate of growth of population is constant. Such a situation is likely to arise-approximately at least-when a backward economy, entering the first stages of industrialization, finds its per capita income rising above the subsistence level, so that the "positive" checks on population growth are at least temporarily diminished, while the "preventitive" checks have not yet come into play. Evidence from the industrialization of India and Japan shows that such an economy can pass rather rapidly from a static population balance to a condition of rapid and almost constant rate of population increase—amounting at the maximum to one or two per cent per year. We consider, first of all, the case where there is no change in technology apart from that produced by accumulation of capital (i.e. no change in the production function itself.)

A. The simplest model representing these conditions is embodied in the following equations:

$$(8.1) \qquad y = \alpha_0 x^r c^s \qquad (s<1),$$

$$(8.2) \qquad \frac{dc}{dt} = \varphi_0 y,$$

$$(8.3) \qquad \frac{1}{x}\frac{dx}{dt} = \rho_0 = \text{constant}.$$

Solving (8.3) explicitly for χ as a function of time, substituting the result in (8.1), and (8.1) in (8.2), and integrating (8.2), we get:

$$(8.4) \qquad C = \left\{ (1-s)\frac{\psi_0 \alpha_0 x_0^{\lambda}}{\xi_0 \lambda}\left[e^{\xi_0 \lambda t} - 1\right] + C_0^{1-s} \right\}^{\frac{1}{1-s}},$$

where the subscript o indicates initial values at $t = t_0$. We are chiefly interested in per capita real income, y/x. From (8.1) and (8.4), we get:

$$(8.5) \qquad \frac{y}{x} = \alpha_0 x_0^{\lambda-1} e^{\xi_0(\lambda-1)t} \left\{ (1-s)\frac{\psi_0 \alpha_0 x_0^{\lambda}}{\xi_0 \lambda}\left[e^{\xi_0 \lambda t} - 1\right] + C_0^{1-s} \right\}^{\frac{s}{1-s}}$$

$$(8.6) \qquad \frac{x}{y}\frac{d(y/x)}{dt} = \left\{ \xi_0(\lambda-1) + \frac{s\,\psi_0 \alpha_0 x_0^{\lambda} e^{\xi_0 \lambda t}\xi_0 \lambda}{(1-s)\psi_0 \alpha_0 x_0^{\lambda}\left[e^{\xi_0 \lambda t} - 1\right] + \xi_0 \lambda C_0^{1-s}} \right\}$$

The time paths of these rather complicated expressions are not immediately obvious. Equation (8.6) can be transformed, however, into a simpler expression that gives the rate of change of per capita income as a function of y/C. We get:

$$(8.7) \qquad \frac{x}{y}\left(\frac{d(y/x)}{dt}\right) = \xi_0(\lambda-1) + s\left(\frac{y}{C}\right)\psi_0 .$$

We can also find the limiting value of (8.6) as t increases:

$$(8.8) \qquad \lim_{t \to \infty} \frac{x}{y}\left(\frac{d(y/x)}{dt}\right) = \xi_0(\lambda-1) + \frac{s\,\xi_0 \lambda}{(1-s)} = \xi_0\frac{(\lambda+s-1)}{(1-s)}$$

From equation (8.7), we see that per capita income at any time will be increasing, constant, or decreasing as:

$$(8.9) \qquad s\frac{y}{C}\psi_0 \gtreqless \xi_0(1-\lambda).$$

That is, the conditions favorable to increasing per capita income are: nearly constant returns for capital and labor, a high percentage of income saved, a low rate of population increase, and a high ratio of income to capital (which, under decreasing returns, implies a low ratio of capital to labor.) None of these conclusions is particularly surprising except, perhaps, the second which shows that the scarcity of capital in the backward county is, initially, an advantage to it in securing an increase in per capita incomes. Since all the expressions in (8.9) except y/C are constant, accumulation of capital at a rate exceeding that of population increase may in time, by reducing y/C, bring the rise in per capita incomes to a halt. To see under what conditions this will happen, we turn to equation (8.8).

By equation (8.8) we see that a continued increase in real income in the long-run does not depend at all upon the initial quantities of labor and capital, nor upon their increase (provided this be in the manner specified in the original system of equations), but only upon the properties of the production function. Per capita real income will increase, remain constant, or decrease in the long run as $(n+s) \gtreqless 1$, that is as labor and capital jointly have increasing, constant, or decreasing returns. This is again the familiar Malthusian conclusion. What is interesting, however, is that under these assumptions the mere accumulation of capital does not permit a permanent escape from population pressure unless production is carried out under conditions of increasing returns.

It may be pointed out that the essential features of equation (8.7) do not depend upon the special form of the production function, provided only that we assume y to be some function of x, and c. For we have in general:

$$(8.10)\quad \frac{x}{y}\frac{d(y/x)}{dt} = \frac{1}{y}\left\{\frac{\partial y}{\partial x}\frac{dx}{dt} + \frac{\partial y}{\partial c}\frac{dc}{dt}\right\} - \frac{1}{x}\frac{dx}{dt}$$

$$= \left\{\frac{x}{y}\frac{\partial y}{\partial x} - 1\right\}\xi_0 + \left\{\frac{c}{y}\frac{\partial y}{\partial c}\right\}\frac{y}{c}\psi_0$$

Finally, we compare two economies, both represented by the system (8.1)-(8.3), except that in the second economy we replace α_0 with α_0' where $\alpha_0' > \alpha_0$.

Corresponding to (7.8), we get:

$$(8.11)\quad \lim_{t\to\infty}\frac{(y'/x')}{(y/x)} = \left(\frac{\alpha_0'}{\alpha_0}\right)^{\frac{1}{1-s}} = \left(\frac{\psi_0'}{\psi_0}\right)^{\frac{1}{1-s}}.$$

B. We consider now an economy in which technology is changing at a constant rate. That is, we replace (8.1) with:

$$(8.12)\quad y = \alpha x^n c^s, \quad (s<1), \quad \text{where}$$

$$(8.13)\quad \alpha = \alpha_0 e^{\theta t}.$$

In this system, we readily find:

$$(8.14)\quad \frac{y}{x} = \alpha_0 x_0^{n-1} e^{[\xi_0(n-1)+\theta]t}\left\{\frac{(1-s)\psi_0\alpha_0 x_0^n}{\xi_0 n+\theta}\left[e^{(\xi_0 n+\theta)t}-1\right] + c_0^{1-s}\right\}^{\frac{s}{1-s}}$$

$$(8.15)\quad \frac{x}{y}\frac{d(y/x)}{dt} = \left\{\xi_0(n-1)+\theta + \frac{s\psi_0\alpha_0 x_0^n(\xi_0 n+\theta)e^{(\xi_0 n+\theta)t}}{(1-s)\psi_0\alpha_0 x_0^n\left[e^{(\xi_0 n+\theta)t}-1\right]+(\xi_0 n+\theta)c_0^{1-s}}\right.$$

$$(8.16)\quad = \xi_0(n-1)+\theta+s\frac{y}{c}\psi_0$$

$$(8.17)\quad \lim_{t\to\infty}\frac{x}{y}\frac{d(y/x)}{dt} = \frac{\xi_0(n+s-1)+\theta}{(1-s)}$$

From (8.16) and (8.17) we conclude that if the rate of improvement of the technology is sufficiently great, population pressure can be avoided, even under conditions of decreasing returns. That this conclusion depends on the compounded rate of technological progress is shown by another model in which the progress is linear.

C. We now replace (8.1) with

$$(8.18)\qquad y = (\alpha_0 + kt)\, x^r c^s$$

From (8.18), together with (8.2) and (8.3), we derive:

$$(8.19)\qquad \frac{y}{x} = (\alpha_0 + kt)\, x_0^{r-1} e^{\xi_0 (r-1)t} \left[\frac{(1-s)\psi_0 x_0^r}{\xi_0 r} \left\{ kt e^{\xi_0 r t} + \left(\alpha_0 - \frac{k}{\xi_0 r}\right)\left(e^{\xi_0 r t} - 1\right) \right\} + c_0^{1-s} \right]^{\frac{s}{1-s}},$$

$$(8.20)\qquad \frac{x}{y}\frac{d(y/x)}{dt} = \xi_0 (r-1) + \frac{k}{\alpha + kt} + \frac{s \psi_0 x_0^r \xi_0 r \{kt + \alpha_0\} e^{\xi_0 r t}}{(1-s)\psi_0 x_0^r \left\{ kt e^{\xi_0 r t} + \left(\alpha_0 - \frac{k}{\xi_0 r}\right)\left(e^{\xi_0 r t} - 1\right) \right\} + \xi_0 r c_0^{1-s}}$$

$$(8.21)\qquad \frac{x}{y}\frac{d(y/x)}{dt} = \xi_0 (r-1) + \frac{k}{\alpha} + s \frac{y}{c} \psi_0 \qquad (\alpha = \alpha_0 + kt)$$

$$(8.22)\qquad \lim_{t \to \infty} \frac{x}{y}\frac{d(y/x)}{dt} = \frac{\xi_0 (r+s-1)}{(1-s)}$$

In this case, while the advancing technology improves the initial chances for rising incomes (by the addition of the term $\frac{k}{\alpha}$), in the long run, population pressure will reassert itself so long as $(r+s) < 1$.

For models (B) and (C) we may again compare two economies with different rates of technological progress. Replacing θ with θ' $(\theta' > \theta)$ in model B, we get:

$$(8.23)\qquad \lim_{t \to \infty} \frac{(y'/x')}{(y/x)} = \left(\frac{\xi_0 r + \theta'}{\xi_0 r + \theta} \right)^{\frac{s}{1-s}} e^{\frac{\theta' - \theta}{1-s} t}$$

Replacing k with k' $(k' > k)$ in model C, we get:

$$(8.24)\qquad \lim_{t \to \infty} \frac{(y'/x')}{(y/x)} = \left(\frac{k'}{k} \right)^{\frac{1}{1-s}}$$

This result may be compared with (7.14).

IX

An attempt to find y/p as an explicit function of time in the general case represented by equations (6.1)-(6.4) would lead to serious complications, but it is easy to derive a relation comparable to (8.10). We have:

$$(9.1)\qquad \frac{p}{y}\frac{d(y/p)}{dt} = \frac{1}{y}\left\{ \phi_x \frac{dx}{dt} + \phi_c \frac{dc}{dt} + \phi_\alpha \frac{d\alpha}{dt} \right\} - \frac{1}{p}\frac{dp}{dt}$$

$$(9.2)\qquad = \left\{ \frac{p}{y}\phi_x \eta - 1 \right\}\xi + \frac{p}{y}\phi_x \eta' \frac{d(y/p)}{dt} + \phi_c \psi + \frac{1}{y}\phi_\alpha \frac{d\alpha}{dt}$$

Collecting terms we finally get:

$$(9.3)\qquad \frac{p}{y}\frac{d(y/p)}{dt} = \frac{\left(\frac{p}{y}\phi_x \eta - 1\right)\xi + \phi_c \psi + \frac{1}{y}\phi_\alpha \frac{d\alpha}{dt}}{(1 - \phi_x \eta')}$$

This may also be rewritten as

$$(9.4)\qquad \frac{p}{y}\frac{d(y/p)}{dt} = \frac{\left(\frac{x}{y}\phi_x - 1\right)\frac{1}{p}\frac{dp}{dt} + \frac{c}{y}\phi_c \frac{1}{c}\frac{dc}{dt} + \frac{\alpha}{y}\phi_\alpha \frac{1}{\alpha}\frac{d\alpha}{dt}}{(1 - \phi_x \eta')}$$

For the case of the Cobb-Douglas function, this reduces further to:

$$(9.5)\qquad \frac{p}{y}\frac{d(y/p)}{dt} = \frac{(r-1)\frac{1}{p}\frac{dp}{dt} + s\frac{1}{c}\frac{dc}{dt} + \frac{1}{\alpha}\frac{d\alpha}{dt}}{\left(1 - r\frac{y}{x}\eta'\right)}$$

If η' is small the sign of the left-hand side depends only upon the numerator of the right-hand side and we will have increasing real per capito income whenever:

$$s\frac{1}{c}\frac{dc}{dt}+\frac{1}{\alpha}\frac{d\alpha}{dt} > (1-r)\frac{1}{p}\frac{dp}{dt} \tag{9.6}$$

From the denominator of (9.5), we see that the rate of increase in income will be greater if an increase in income brings a larger proportion of the population into employment $(\eta' > 0)$, and less if the reverse is true $(\eta' < 0)$. From (9.6) we see that if $(r+s) = 1$ and if technology is static, the percentage rate of capital increase must be greater than the percentage rate of population increase in order for per capita incomes to rise. Likewise, so long as $s < 1$, a given percentage rate of improvement in technology is more effective in raising real incomes than an equal percentage rate of capital accumulation. Backward countries in a period of rising productivity commonly experience population increases in the neighborhood of one to two percent per year. Assuming r to be in the neighborhood of .75, we see that to escape population pressure, these same countries must achieve rates of capital accumulation of one or two percent per year, rates of technological change of one-quarter to one-half of one percent a year, or some combination of these.

Maintenance of a constant percentage rate of capital increase requires a larger and larger proportion of income to be saved as the ratio y/c becomes smaller, since:

$$\frac{1}{c}\frac{dc}{dt} = \frac{y}{c}\psi. \tag{9.7}$$

Since, under decreasing returns, y/c will decrease in the long run if capital accumulation is more rapid than population growth, a constant rate of growth of capital will become impossible in the long run, and economic progress will continue only if there is a constant (and sufficiently great) rate of technological progress, or if population growth levels off sufficiently soon.

We may ask, how rapidly the rate of population growth must change in order that an economy that has reached a balance between increasing population and increasing income continue to maintain this balance, under the assumption that there is no technological change, that $\eta \equiv 1$ and that the percentage of income saved is constant.

We have:

(9.8) $$s \frac{y}{c} \psi_0 = (1-r)\frac{1}{p}\frac{dp}{dt}$$

Hence

(9.9) $$\frac{d}{dt}\left(\frac{1}{p}\frac{dp}{dt}\right) = \frac{s}{(1-r)}\psi_0 \frac{d(y/c)}{dt}$$

From (9.8) and (9.9), we obtain

(9.10) $$\frac{1}{\xi}\frac{d\xi}{dt} = \xi\left\{\frac{r+s-1}{s}\right\}$$

If, for example, ξ is two per cent, per year, S= .25 and = .50, then ξ must decrease by .02 x .04 = .0008 per year. A rather moderate drop in the net reproduction rate would permit the maintenance of incomes at the level achieved even without technological progress.

Finally, we determine the time interval required for y/x to reach its maximum value, for the special case of model A in Section VIII. Setting the right-hand side of (8.6) equal to zero, and simplifying, we get:

(9.11) $$e^{\xi_0 rt} = \frac{(r-1)}{(r+s-1)}\left\{(1-s) - \frac{\xi_0 r}{\psi_0}\frac{c_0}{y_0}\right\}$$

If $\left(\frac{d(y/x)}{dt}\right)_0 > 0$ then, from (8.9), $s\frac{y_0}{c_0}\psi_0 > \xi_0(1-r)$

If further $(r+s)<1$, it follows that $\frac{y_0}{c_0}\psi_0 > \xi_0$ and hence $(1-s)\frac{y_0}{c_0}\psi_0 > r\xi_0$. Therefore, the right-hand side of (9.11) is positive. Within these limits, the larger is $\frac{y_0}{c_0}$ or ψ_0, the larger is $e^{\xi_0 rt}$, hence t. Hence, initial scarcity of capital and a high percentage of income saved lengthen the period during which per capita income rises. On the other hand, the larger is ξ_0, the smaller is $e^{\xi_0 rt}$ But for a given value of $e^{\xi_0 rt}$ $\xi_0 rt$ = constant. Hence, a larger value of ξ_0 implies a smaller value of t, for given values of $e^{\xi_0 rt}$ and r and *a fortiori* for a small value of $e^{\xi_0 rt}$.

X

There is no basis for a claim that any of the models that have been examined here describe the real world with any precision. Nevertheless, we have been able to represent in these models certain important features of an economy undergoing population growth, capital accumulation, and technological change. If such models have no other value, they are of use in permitting a precise statement of the Malthusian hypothesis, and an examination of the conditions under which that hypothesis holds.

Moreover, the conclusions we have been able to draw from our dynamic models give credence to the commonly advanced argument that by a sufficiently rapid industrialization, backward countries can gain a breathing spell from population pressure. We have established a method that permits an estimate of the necessary initial rate of progress, of the length of the consequent breathing spell, and of the effect upon that length, of technological change.