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Second note on the response of entrepreneurs to taxes,

by J. Marschak, September 1, 1947

The problem: Is it possible for the government to manipulate the profit-tax schedule (in order to increase the revenue, or to achieve greater equality, or greater demand for consumption goods, or for other purposes) in such a way as to maintain entrepreneurial decisions unaffected?

1. Assumption I: each entrepreneur maximizes what he considers the mathematical expectation of his net profit;

2. Assumption II: the government controls only one parameter of the tax schedule; we shall denote this parameter by a and write

   $$N=N(G,a),$$

   where G is the gross profit and N the net profit, i.e., the profit after taxes.

3. Assumption III: the entrepreneur has only one degree of freedom. We shall denote his unique decision variable by z. The distribution density of gross profit (as imagined by the entrepreneur) will be denoted by

   $$f(G,z),$$

   since z can be regarded as a parameter of the distribution function.

4. Remark. Assumptions II and III are introduced for simplicity and are probably unimportant for the gist of the argument. Assumption I is probably essential.

5. Statement of the problem. Write for the expected value of net profit

   $$(1)\quad M \equiv \int_P N(G,a)\, f(G,z)dG \equiv M(a,z),$$

   where the integral $\int_P$ is taken over the range of all possible gross profits, i.e.,

   $$(2)\quad \int_P f(G,z)dG = 1.$$

   Write $\mathrm{Max}_z M \equiv M(a,z')$, so that z' denotes the optimal decision. We have to find the restrictions upon the form of the function $N(G,a)$, necessary and sufficient to make z' independent of a:

   $$(3)\quad \frac{dz'}{da} = 0.$$

6. <u>Solution.</u>

(4) $0 = \left(\frac{\partial M}{\partial z}\right)_{z=z'} = \int N(G,a)\ f_z(G,z')dG.$

Differentiate with respect to a:

(5) $0 = \left(\frac{\partial^2 M}{\partial z\,\partial a}\right)_{z=z'} = \int_p N_a(G,a)f_z(G,z')dG + \frac{dz'}{da}\int_p N(G,a)f_{zz}(G,z')dG;$

where $N_z = \frac{\partial N(G,a)}{\partial a}$. We now introduce the plausible [In view of (2)]

7. <u>Assumption IV</u>: the second integral in (5) has finite value. We shall now prove the following

Proposition: If Assumption IV is fulfilled, then a sufficient and necessary condition for dz'/da to vanish, is

(6) $N(G,a) = A(a) + B(a) \cdot C(G)$

where each of the functions A, B, C is a function of one variable.

To prove this proposition we shall write

$f_z\ (G,a\ ) = g(G)$, so that, by (2)

(7) $\int_p g(G)dG = 0$; and by (4)

(8) $\int_p Ng(G)dG = 0$; finally (3) jointly with (5) and with Assumption IV is equivalent to

(9) $\int_p N_a g(G)dG = 0.$

We have thus to show that (6) is a sufficient and necessary condition for (9) whenever (7) and (8) are satisfied.

The <u>sufficiency</u> is seen immediately by substituting from (6) into (8) and (9) and remembering (7).

To prove the <u>necessity</u> of (9), express the integrals in (7), (8), (9) as limits of the following sums:

(10) $\begin{cases} g^{(1)} + g^{(2)} + \ldots = 0 \\ N^{(1)}g^{(1)} + N^{(2)}g^{(2)} + \ldots = 0 \\ N_a^{(1)}g^{(1)} + N_a^{(2)}g^{(2)} + \ldots = 0, \end{cases}$

where $g^{(i)} = g(G_i)\,\Delta G_i$; $N^{(i)} = N(G_i,a)\,\Delta G_i$;

$N_a^{(i)} = \frac{\partial N(G_i,\ a)}{\partial a}\,.\Delta G_i$; $i = 1,\ldots$ It follows from (10) that the determinant

(11) $\begin{vmatrix} 1 & 1 & 1 \\ N^{(i)}g^{(i)} & N^{(j)}g^{(j)} & N^{(k)}g^{(k)} \\ N_a^{(i)}g^{(i)} & N_a^{(j)}g^{(j)} & N_a^{(k)}\ {}^{(k)} \end{vmatrix} = 0$; $i,\ j,\ k = 1,\ldots$

Passing now to the limit we conclude that there exist two quantities P and Q, both independent of G (but possibly dependent on a) and such that

$N_a + PN - Q = 0$; or in more detail,

$$(12) \quad \frac{\partial N(G,a)}{\partial a} + P(a)\, N(G,a) = Q(a).$$

To solve this differential equation, with N as the dependent and a as the independent variable (Piaggio, p.17), multiply it by

$$R \equiv \exp \int Pda \equiv R(a),$$

$\int$ denoting an indefinite integral; we obtain

$$\frac{d}{da}(NR) = QR$$

$$(13) \quad NR = C + \int QRda;$$

in general, the integration constant C depends on G, so that (13) results in

$$(6) \quad N(G,a) = A(a) + C(G)\, B(a),$$

where $A(a) = (\int Q(a)\, R(a) da)/R(a)$

$B(a) = 1/R(a)$

This completes the proof.

8. Special cases. The linear tax schedule, with losses refunded by the government,

$$N = \ell_1 + \ell_2 G$$

is a special case of (6) with either ℓ_1 or ℓ_2 playing the role of the controlled parameter a. A curvilinear schedule may also satisfy (6), for example the schedule

$$N = aG^{\frac{1}{2}}.$$

9. Actual tax schedules. Cases like those just mentioned do not have the usual properties of progressive taxes. These properties can be stated as follows:

$$(14) \quad \begin{array}{ll} N = G & ; G \leq 0 \\ 0 < N < G, \quad 0 < dN/dG < 1, \quad d^2N/dG^2 < 0 & ; G > 0. \end{array}$$

In the inequalities on the extreme right of each of the two lines above, zero is sometimes replaced by a positive "exemption limit". Modifications are also required to represent the progressive schedule as consisting not of two smooth sections (as in (14)) but of three or more smooth sections. These modifications of the conditions (14) add, however, little that is not trivial. It can be shown that (14) and (3) are inconsistent (on the lines of the First Note on this subject, July 24).

However, it should be possible to construct functions N(G,a) which would satisfy (6) and at the same time approximate the conditions (14). One could then devise families of progressive tax schedules which would permit, say, to increase or decrease the public revenue, the income equality etc., without affecting the decisions of entrepreneurs by more than a preassigned small extent.