Introduction

- A theory of information acquisition in games
  - Endogenize assumptions on players’ information
  - Common extra layer of strategic interaction
- Flexible learning about state and what others know
  - Expose primitive incentives to acquire information
- Broad assumptions on cost of information
  - Costly to learn state and what others know
  - Example: Shannon mutual information
- Applications
  - Investment games: risk-dominance selection
  - Games on networks: Bonacich centrality
  - Large games: endogenous informational smallness
Introduction

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- Broad assumptions on cost of information
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  - Games on networks: Bonacich centrality
  - Large games: endogenous informational smallness
Today

Investment games: bunk runs, currency crises, ...

Exogenous information
- Common knowledge: multiplicity
  - Diamond & Dybvig 1983, Obstfeld 1996, ...
- Global games: risk-dominance selection
  - Carlsson & van Damme 1993, Morris & Shin 1998, ...
- Any perturbation: any selection
  - Weinstein & Yildiz 2007

Endogenous information w/ mutual information
- Flexible info acquisition about state: multiplicity
  - Yang 2015
- Unrestricted info acquisition: risk-dominance selection
  - Extend to potential games

Endogenous information w/out mutual information: next talk
Outline

- Investment game with incomplete information
  - Basic game: actions, states, utilities
  - Exogenous Information structure
- Recap: common knowledge, global games,…
- Investment game with information acquisition
  - Basic game: actions, states, utilities
  - Information acquisition technology
- Flexible info acquisition about state: multiplicity
- Unrestricted info acquisition: risk-dominance selection
Basic Game

- $N = \{1, \ldots, n\}$: finite set of players
- $A_i = \{\text{invest, not invest}\}$: set of $i$’s actions
- $\Theta \subseteq \mathbb{R}$: closed set of states
- $P_\Theta \in \Delta(\Theta)$: state distribution
- $\rho(\bar{a}_-, \theta) \in \mathbb{R}$: $i$’s non-decreasing return
  - $\bar{a}_-$: share of opponents who invest
  - $\rho$ integrable in $\theta$ w.r.t. $P_\Theta$
- $P_\Theta(\{\theta : \rho(1, \theta) < 0\}) > 0$: dominance region
- $P_\Theta(\{\theta : \rho(0, \theta) > 0\}) > 0$: dominance region

- $u_i(a, \theta) = 1\{\text{invest}\}(a_i)\rho(\bar{a}_-, \theta)$: $i$’s utility

Today:

<table>
<thead>
<tr>
<th></th>
<th>invest</th>
<th>not invest</th>
</tr>
</thead>
<tbody>
<tr>
<td>invest</td>
<td>$\theta, \theta$</td>
<td>$\theta - 1, 0$</td>
</tr>
<tr>
<td>not invest</td>
<td>$0, \theta - 1$</td>
<td>$0, 0$</td>
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Exogenous Information Structure

- $(\Omega, \mathcal{F}, P)$: underlying probability space
- $\theta : \Omega \rightarrow \Theta$: random variable with $\theta \sim P_\Theta$
- $X_i$: Polish space of $i$’s messages
- $x_i : \Omega \rightarrow X_i$: $i$’s signal, random variable
Recap

Common knowledge: multiplicity

- \( x_i = \theta \) for all \( i \)
- \( \theta \in [0, 1] \): equilibrium indeterminacy

Global games: risk-dominance selection

- \( x_i = \theta + \lambda \epsilon_i \) for all \( i \)
  - \( \lambda > 0 \): scale factor
  - \( \epsilon_i \): idiosyncratic noise
- \( \lambda \rightarrow 0 \): perturbation of complete information
- 1/2: risk-dominance threshold

Any perturbation: any selection

- Weinstein & Yildiz 2007
Information Acquisition Technology

- $(\Omega, \mathcal{F}, P)$: underlying probability space
- $\theta : \Omega \rightarrow \Theta$: random variable with $\theta \sim P_\Theta$
- $X_i$: Polish space of $i$’s messages
- $X_i$: nonempty set of $i$’s signals $x_i : \Omega \rightarrow X_i$
- $C_i : \Delta(X \times \Theta) \rightarrow [0, \infty]$: $i$’s cost of information
Game with Information Acquisition

Basic game + info technology = strategic form game:

- Set of players: \( N \)
- \( i \)'s strategy: signal \( x_i \in X_i \), contingency plan \( s_i \in S_i \)
  - \( S_i \): set of all measurable \( s_i : X_i \rightarrow A_i \)
- \( i \)'s payoff: \( E[u_i(s(x), \theta)] - \lambda C_i(P(x, \theta)) \)
  - \( \lambda > 0 \): scale factor
  - \( P(x, \theta) \in \Delta(X \times \Theta) \): joint distribution of \( x \) and \( \theta \)

Solution concept: pure-strategy Nash equilibrium

To ease notation: \( C_i(x, \theta) = C_i(P(x, \theta)) \)

\( \lambda \rightarrow 0 \): multiplicity/selection of equilibria?
Flexible Info Acquisition about State

Yang 2015: for all players $i$

Assumption 0. $|X_i| \geq |A_i|$. Moreover, if random variable $x'_i : \Omega \rightarrow X_i$ is measurable w.r.t. some $x_i \in X_i$, then $x'_i \in X_i$.

Assumption 1. Take any $x \in X$. Then $(x_i \perp x_{-i})|\theta$.

Assumption 2. Take any $P_{X_i \times \Theta} \in \Delta(X_i \times \Theta)$. If $\theta \sim \text{marg}_{\Theta}(P_{X_i \times \Theta})$, then $(x_i, \theta) \sim P_{X_i \times \Theta}$ for some $x_i \in X_i$.

Assumption 3. For all $x \in X$, $C_i(x, \theta) = l(x_i; x_{-i}, \theta)$.

Mutual information: for $X_i$ finite, $p$ p.m.f. of $x_i$,

$$l(x_i; x_{-i}, \theta) = E \left[ \log \frac{p(x_i|x_{-i}, \theta)}{p(x_i)} \right].$$
Flexible Info Acquisition about State

Yang 2015: for all players $i$

Assumption 0. $|X_i| \geq |A_i|$. Moreover, if random variable $x'_i : \Omega \rightarrow X_i$ is measurable w.r.t. some $x_i \in X_i$, then $x'_i \in X_i$.

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Assumption 3. For all $x \in X$, $C_i(x, \theta) = I(x_i; x_{-i}, \theta)$.

Mutual information: for $X_i$ finite, $p$ p.m.f. of $x_i$,

$$I(x_i; x_{-i}, \theta) = E \left[ \log \frac{p(x_i|x_{-i}, \theta)}{p(x_i)} \right].$$
Revelation Principle

Direct technology: \[ X_i = A_i \] for all \( i \)

Basic game + direct technology = strategic form game:

- Set of players: \( N \)
- \( i \)'s strategy: direct signal \( x_i \in X_i \)
- \( i \)'s payoff: \( E[u_i(x, \theta)] - \lambda C_i(x, \theta) \)

Solution concept: pure-strategy Nash equilibrium

Revelation principle (Yang 2015):
“A0-A3 \( \Rightarrow \) w.l.o.g. technology and signals are direct.”
Multiplicity

Theorem (Yang 2015)
Assume A0-A3.

Let $P_{\Theta}$ be abs. continuous w.r.t. Lebesgue measure.

Then $\forall \hat{\theta} \in [0, 1]$, $\exists$ equilibria $(x_\lambda : \lambda > 0)$: $\forall i \in N$

$$P(x_{i,\lambda} = \text{invest}|\theta) \xrightarrow{a.s.} \begin{cases} 1 & \text{if } \theta \geq \hat{\theta}, \\ 0 & \text{if } \theta < \hat{\theta}. \end{cases}$$

Remark. Also non-monotone equilibria
Monotone Equilibria

$\lambda > 0$:

$\ldots$

$P(x_1 = x_2 | \theta)$

$P(x_i = \text{invest} | \theta)$
Monotone Equilibria

$\lambda > 0$:

$\rho(\bar{x} - i, \theta) \geq 0$ since $\text{Var}(\bar{x} - i | \theta) \neq 0$
Monotone Equilibria

$\lambda > 0$:

$\lambda > 0$:

\[
\begin{align*}
\Pr(x_i = \text{invest} | \theta) &> \Pr(x_1 = x_2 | \theta) \\
\theta &> \hat{\theta} \\
\end{align*}
\]
Monotone Equilibria

\( \lambda > 0: \)

\[ \overset{\text{\(\lambda > 0:\)}}{\overset{\rightarrow}{i}} \text{\'s primitive incentive: learn} \{ \rho(\bar{x} - i, \theta) \geq 0 \} \]

\[ \overset{\text{\{}}{\overset{\rightarrow}{\rho(\bar{x} - i, \theta) \geq 0 \}} \neq \{ \theta \geq \hat{\theta} \} \text{since} \ Var(\bar{x} - i | \theta) \neq 0 \]
Monotone Equilibria

\[ \lambda \to 0: \]

\[ \Pr(x_i=\text{invest} | \theta) \]
Monotone Equilibria

$\lambda > 0$:

\[
\begin{align*}
\text{P}(x_1 = x_2 | \theta) & \quad \text{P}(x_i = \text{invest} | \theta)
\end{align*}
\]
Monotone Equilibria

$\lambda > 0$:

- $i$’s primitive incentive: learn $\{\rho(\bar{x}_{-i}, \theta) \geq 0\}$
- $\{\rho(\bar{x}_{-i}, \theta) \geq 0\} \neq \{\theta \geq \hat{\theta}\}$ since $\text{Var}(\bar{x}_{-i}|\theta) \neq 0$
Unrestricted Info Acquisition

For all players $i$:

**Assumption 0.** $|X_i| \geq |A_i|$. Moreover, if random variable $x'_i : \Omega \rightarrow X_i$ is measurable w.r.t. some $x_i \in X_i$, then $x'_i \in X_i$.

**Assumption 3.** For all $x \in X$, $C_i(x, \theta) = l(x_i; x_{-i}, \theta)$.

**Assumption 4.** Take any $x_{-i} \in X_{-i}$ and $P_{X \times \Theta} \in \Delta(X \times \Theta)$. If $(x_{-i}, \theta) \sim \text{marg}_{X_{-i} \times \Theta}(P_{X \times \Theta})$, then $(x, \theta) \sim P_{X \times \Theta}$ for some $x_i \in X_i$.
Unrestricted Info Acquisition

For all players $i$:

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Finite-game Construction  General Construction
Unrestricted Info Acquisition

For all players $i$:

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Revelation principle:
“$A0, A3, A4 \Rightarrow \text{w.l.o.g. technology and signals are direct.}”
All Equilibria

$\lambda > 0$: 

$P(x_1 = x_2 | \theta)$

$P(x_i = \text{invest} | \theta)$
All Equilibria

$\lambda > 0$:

$$\hat{\theta} = \frac{1}{2} - \lambda \log \frac{P(x_i = \text{invest})}{P(x_i = \text{not invest})}$$
All Equilibria

$\lambda > 0$: 

$$\hat{\theta} = \frac{1}{2} - \lambda \log \frac{P(x_i = \text{invest})}{P(x_i = \text{not invest})}$$
All Equilibria

$\lambda > 0:$

\[
\hat{\theta} = \frac{1}{2} - \lambda \log \frac{P(x_i = \text{invest})}{P(x_i = \text{not invest})}
\]
All Equilibria

$\lambda > 0$:

$$\hat{\theta} = \frac{1}{2} - \lambda \log \frac{P(x_i = \text{invest})}{P(x_i = \text{not invest})}$$
All Equilibria

\( \lambda \to 0: \)

\[ P(x_i=\text{invest}|\theta) \]

\( \theta \)
Risk-dominance Selection

Theorem
Assume A0, A3, and A4.

Then \( \forall \) equilibria \( (x_\lambda : \lambda > 0) \): \( \forall i \in N \)

\[
P(x_{i,\lambda} = \text{invest}|\theta) \xrightarrow{a.s.} \begin{cases} 
1 & \text{if } \theta > \frac{1}{2}, \\
0 & \text{if } \theta < \frac{1}{2}.
\end{cases}
\]

Theorem extends to potential games.
Proof

- \((x, \lambda)\): family of equilibria

- \(v: A \times \Theta \rightarrow \mathbb{R}\): potential s.t. for all \(i\), \(a_i\), and \(a'_i\)

\[
u_i(a_i, \cdot) - u_i(a'_i, \cdot) = v(a_i, \cdot) - v(a'_i, \cdot)\]

- Investment games: for all \(a\) and \(\theta\)

\[
v(a, \theta) = \sum_{m=0}^{\lfloor |a| - 1 \rfloor} \rho \left( \frac{m}{n - 1}, \theta \right)
\]

with \(|a|\) number of players who invest

- \(a^*\) risk-dominant at \(\theta\): \(v(a^*, \theta) > v(a, \theta)\) for all \(a \neq a^*\)

- Info acquisition w/ mutual information, \(\lambda > 0\):
  - Static 1-player: Csiszar 1974, Matejka & McKay 2016
  - Dynamic 1-player: previous talk
  - Static \(n\)-player: next slide
Key Lemma

\( P_{(x_{\lambda},\theta)} \): joint distribution of \( x_\lambda \) and \( \theta \)

1. **Quality.** Almost surely,

\[
\frac{dP_{(x_{\lambda},\theta)}}{d(P_\theta \times \prod_{i \in N} P_{x_i,\lambda})}(a, \theta) = \frac{e^{\frac{\nu(a, \theta)}{\lambda}}}{\int_{A} e^{\frac{\nu(a', \theta)}{\lambda}}(\prod_{i \in N} P_{x_i,\lambda})(da')}.
\]

2. **Quantity.** \( P_{x_1,\lambda}, \ldots, P_{x_n,\lambda} \) eq. of potential game \( V \):

\[
V(P_{A_1}, \ldots, P_{A_n}) = \int_{\Theta} \log \left( \int_{A} e^{\frac{\nu(a, \theta)}{\lambda}}(\prod_{i \in N} P_{A_i})(da) \right) P_\Theta(d\theta).
\]

\( P_{A_1}, \ldots, P_{A_n} \): equilibrium of \( V \)

- **There is equilibrium** \( x_\lambda \) s.t. \( x_{i,\lambda} \sim P_{A_i} \) for all \( i \in N \).
Key lemma: for all $\theta$ and $a \neq a^*$

\[
\frac{dP_{(x_\lambda, \theta)}}{d(P_\theta \times_{i \in N} P_{x_i, \lambda})}(a, \theta) = \frac{e^{\frac{\nu(a, \theta)}{\lambda}}}{\int_A e^{\frac{\nu(a', \theta)}{\lambda}} \left( \times_{i \in N} P_{x_i, \lambda} \right)(da')} \\
\lesssim \frac{e^{\frac{\nu(a^*, \theta)}{\lambda}}}{e^{\frac{\nu(a^*, \theta)}{\lambda}} \left( \times_{i \in N} P_{x_i, \lambda} \right)(\{a^*\})} \cdot \frac{1}{e^{\frac{\nu(a^*, \theta) - \nu(a, \theta)}{\lambda}} \left( \times_{i \in N} P_{x_i, \lambda} \right)(\{a^*\})} \to 0.
\]

- Recall: $\nu(a^*, \theta) > \nu(a, \theta)$ for all $a \neq a^*$
- Dominance regions: $\liminf_{\lambda \to 0} P_{x_i, \lambda}(\{a_i^*\}) > 0$ for all $i$

$\Rightarrow P(x_\lambda = a^*|\theta = \theta) \to 1$.  

Extension  
Infinite $N$
Conclusion

Today: multiplicity/selection of eq. in coordination games

▶ Exogenous: info players have about others’ info
▶ Endogenous: info players want about others’ info

Unrestricted info acquisition

▶ Broad assumptions on cost of information: A5, A6
▶ Games on networks, Bonacich centrality
▶ Large games, endogenous informational smallness

⇒ Rich yet tractable language for info acquisition in games
Appendix
Finite-game Construction

- Θ and X finite
- (Ω, 𝒇, 𝑃): nonatomic probability space
- \(X_i\): all measurable functions \(x_i : \Omega \rightarrow X_i\)
Finite-game Construction

- Θ and X finite
- (Ω, ℱ, P): nonatomic probability space
- X_i: all measurable functions x_i : Ω → X_i

Diagram:

- Ω: 0 → 1
- θ: θ_1 → θ_2
- x_j: x_j_1 → x_j_2
- x_i: x_i_1 → x_i_2

Back to text.
Finite-game Construction

- $\Theta$ and $X$ finite
- $(\Omega, \mathcal{F}, P)$: nonatomic probability space
- $X_i$: all measurable functions $x_i : \Omega \to X_i$

\[ x_i \quad x_{i2} \quad x_{i1} \quad x_{i2} \quad x_{i1} \quad x_{i2} \]

\[ x_j \quad x_{j1} \quad x_{j2} \quad x_{j1} \quad x_{j2} \]

\[ \Theta \quad \theta_1 \quad \theta_2 \]

\[ \Omega \quad 0 \quad 1 \]
General Construction

$T$: uncountable set.

$(\Omega, \mathcal{F}, P)$:

- $\exists \theta : \Omega \to \Theta: \theta \sim P_\Theta$.
- $\exists z_t : \Omega \to [0, 1], t \in T$:
  - $\theta$ and $z_t, t \in T$, independent
  - $z_t, t \in T$, uniformly distributed

$\forall i \in \mathbb{N}, x_i \in X_i$ iff $\exists$ countable $Q \subset T$:

- $x_i$ measurable w.r.t. $\theta$ and $z_t, t \in Q$. 

\( \text{Back} \)
Existence

- \( A_i, \Theta \): Polish spaces
- \( v : A \times \Theta \to \mathbb{R} \): measurable function s.t.
  \[
  -\infty < \int_{\Theta} \inf_{a \in A} v(a, \theta) P_{\Theta}(d\theta) < \int_{\Theta} \sup_{a \in A} v(a, \theta) P_{\Theta}(d\theta) < \infty
  \]

- \( V : \Delta(A_1) \times \cdots \times \Delta(A_n) \to \mathbb{R} \):
  \[
  V(P_{A_1}, \ldots, P_{A_n}) = \int_{\Theta} \log \left( \int_{A} e^{v(a,\theta)} (\times_{i \in \mathbb{N}} P_{A_i})(da) \right) P_{\Theta}(d\theta)
  \]

Fact

*Function \( V \) has a maximum if:*

- \( A_i \) is compact for all \( i \in \mathbb{N} \).
- \( v \) is upper semi-continuous in \( a \).
Unique Selection for Potential Games

- $N = \{1, \ldots, n\}$: finite set of players
- $A_i$: Polish space of $i$’s actions
- $\Theta$: Polish space of states
- $P_\Theta \in \Delta(\Theta)$: state distribution
- $v : A \times \Theta \to \mathbb{R}$: potential

Integrability: $\int_\Theta \sup_{a \in A} v(a, \theta) P_\Theta(d\theta) < \infty$
Theorem
Assume A0, A3, and A4.

Take any $a \in A$ s.t. $\forall i \in N \exists \Theta_{a_i} \subseteq \Theta$:

- $P_{\Theta}(\Theta_{a_i}) > 0$
- $\inf_{\theta \in \Theta_{a_i}} \inf_{a_i \neq a_i} \inf_{a_i \in A_i} v(a, \theta) - v(a_i, a_i, \theta) > 0$

Then $\forall$ equilibria $(x_\lambda : \lambda > 0)$: almost surely

$$v(a, \theta) > \sup_{a' \neq a} v(a', \theta) \implies \lim_{\lambda \to 0} P(x_\lambda = a | \theta = \theta) = 1.$$
Infinite $N$: Multiplicity

**Theorem**

Assume $A0$, $A3$, and $A4$.

Let $N = \{1, 2, \ldots\}$.

Let $P_\Theta$ be abs. continuous w.r.t. Lebesgue measure.

Then $\forall \hat{\theta} \in [0, r]$, $\exists$ equilibria $(x_\lambda : \lambda > 0)$: $\forall i \in N$

\[ (x_i \perp x_{-i})|\theta \]

\[ P(x_{i,\lambda} = \text{invest}|\theta) \xrightarrow{\text{a.s.}} \begin{cases} 1 & \text{if } \theta \geq \hat{\theta} \\ 0 & \text{if } \theta < \hat{\theta} \end{cases} \]
Proof

- Assume $(x_i \perp x_{-i})|\theta$ for all $i \in N$.

- Law of large numbers: $Var(\bar{x}|\theta) = 0$.

- Hence, $Var(\rho(\bar{x}, \theta)|\theta) = 0$.

- Independent information acquisition is optimal.

- Multiplicity as in Yang 2015.
Assumption 5

Take any \( i \in \mathbb{N}, \ x_{-i} \in X_{-i} \), and \( x_i, x'_i \in X_i \).

Assume that \( (x'_i \perp (x_{-i}, \theta))|x_i) \).

Then \( C_i(x_i, x_{-i}, \theta) \geq C_i(x'_i, x_{-i}, \theta) \).

Equality holds only if \( (x_i \perp (x_{-i}, \theta))|x'_i) \).
Assumption 6

Take any $i \in \mathbb{N}$, $x_{-i} \in X_{-i}$, and $x_i, x'_i \in X_i$.
Assume there is measurable $f : X_{-i} \times \Theta \rightarrow Z$:

$\triangleright (x_i, f(x_{-i}, \theta)) \sim (x'_i, f(x_{-i}, \theta))$

$\triangleright (x'_i \perp (x_{-i}, \theta))|f(x_{-i}, \theta)$

Then $C_i(x_i, x_{-i}, \theta) \geq C_i(x'_i, x_{-i}, \theta)$.
Equality holds only if $(x_i \perp (x_{-i}, \theta))|f(x_{-i}, \theta)$.

Proof
Set $z = f(x_{-i}, \theta)$:

$I(x'_i; x_{-i}, \theta) = I(x'_i; z) = I(x_i; z) \leq I(x_i; x_{-i}, \theta)$.