Bank Capital and Aggregate Credit

Nataliya Klimenko∗ Sebastian Pfeil † Jean-Charles Rochet ‡

April 17, 2015

PRELIMINARY AND INCOMPLETE

Abstract

This paper seeks to explain the role of bank capital in fluctuations of lending and output. We build a continuous time model of an economy in which commercial banks finance their loans by deposits and equity, while facing issuance costs when they raise new equity. The dynamics of the loan rate and the volume of lending in the economy are driven by aggregate bank capitalization. The model has a unique Markovian competitive equilibrium that can be solved in closed form. We show that the competitive equilibrium is constrained inefficient: banks lend too much and hold too little equity, since they do not internalize the impact of their lending decisions on the cost of credit and the social costs of bank distress. We also examine the impact of two regulatory tools: a standard capital ratio and a tax on dividends, showing that the latter is more efficient in dealing with the distortions inherent in the competitive equilibrium.

Keywords: macro-model with a banking sector, bank capital, pecuniary externalities

JEL: E21, E32, F44, G21, G28

Acknowledgements: we thank Philippe Bacchetta, Bruno Blais, Catherine Casamatta, Julien Daubanes, Jean-Paul Decamps, Hans Goerich, Hendrik Hakenes, Christian Hellwig, Eric Jondeau, Loriano Mancini, Semyoun Malamud, Erwan Morelec, Henri Pagès, Bruno Parigi, Jean Tirole and Stéphane Villeneuve, and seminar participants at ETHZ, TSE, SFI/EPFL, the University of Nanterre and Banque de France.

∗University of Zürich. E-mail: nataliya.klimenko@bf.uzh.ch.
†University of Bonn. E-mail: pfeil@uni-bonn.de.
‡University of Zürich, SFI and TSE-IDEI. E-mail: jean-charles.rochet@bf.uzh.ch.
1 Introduction

There is an ongoing debate among scholars and practitioners about the "right" level of bank capital. While proponents of higher capital ratios emphasize the stabilizing effect of bank capital, others argue that high leverage is a direct consequence of banks' intrinsic role as creators of information insensitive, liquid debt (e.g. deposits). The current paper brings together both aspects in a dynamic general equilibrium model where bank capital plays the role of a loss absorbing buffer that facilitates the creation of liquid, risk-free claims while providing (risky) loans to the real sector.

We consider an economy where firms borrow from banks that are financed by short term deposits and equity. The aggregate supply of bank loans is confronted with the firms' demand for credit, which is decreasing in the nominal loan rate. The firms' default probability depends on undiversifiable aggregate shocks, which ultimately translates into gains or losses for banks. Banks can continuously adjust their volumes of lending to firms. They also decide when to distribute dividends and when to issue new equity. Equity issuance is subject to deadweight costs, which constitutes the main financial friction in our economy. The supply of deposits is completely inelastic as long as banks can always guarantee full repayment. That is, as depositors are infinitely risk-averse, all losses that may be generated by the banks’ loan portfolio must be borne by the banks’ shareholders.

Note first that in a set-up without financial frictions (i.e., no issuance costs for bank equity) the equilibrium volume of lending and the nominal loan rate are constant. Furthermore, dividend payment and equity issuance policies are trivial in this case: Banks immediately distribute all profits as dividends and issue new shares to offset losses and honor obligations to depositors. This implies that, in a frictionless world, there is no need to build up a capital buffer: all loans are entirely financed by deposits.

In the model with financing frictions, banks' dividend and equity issuance strategy becomes more interesting. In the unique competitive equilibrium, aggregate bank equity,

\footnote{Most prominently, Admati et al. (2010), Admati and Hellwig (2013).}
\footnote{See DeAngelo and Stulz (2014) for an argument along these lines. More generally, the perception of financial intermediaries as creators of liquidity ("inside money") has a long tradition within the financial intermediation literature (see, e.g. Diamond and Dybvig (1983), Diamond and Rajan (2001), Gorton (2010), Gorton and Pennachi (1990), Holmstrom and Tirole (1998, 2011)).}
\footnote{As we abstract from incentive effects of bank capital ("skin in the game"), our model captures best the features of more traditional commercial banks whose business model makes them less prone to risk-shifting than investment banks. The incentive effects of bank capital in a setting allowing for risk-shifting are analyzed for instance in Martinez-Miera and Suarez (2014), DeNicolo et al. (2014), and van den Heuvel (2008).}
\footnote{Empirical studies report sizable costs of seasoned equity offerings (see e.g. Lee, Lochhead, Ritter, and Zhao (1996), Hennessy and Whited (2007)). Here we follow the literature (see e.g. Décamps et al. (2011) or Bolton et al. (2011)) by assuming that issuing new equity entails a deadweight cost proportional to the size of the issuance.}
\footnote{To focus on market forces, we do not consider deposit insurance.}
which serves as the single state variable, follows a Markov diffusion process reflected at two boundaries. Banks issue new shares at the lower boundary where book equity is depleted. When the book value of equity reaches its upper boundary, any further earnings are paid out to shareholders as dividends. Between the two boundaries, the level of equity changes only due to retained earnings or absorbed losses. That is, banks retain earnings in order to increase the loss-absorbing equity buffer and thereby reduce the frequency of costly recapitalizations.

The risk-adjusted spread for bank loans is strictly positive (except at the dividend payout boundary) and decreasing in the level of aggregate bank equity. To get an intuition for this result, note that loss absorbing equity is most valuable when it is scarce and least valuable when it is abundant. Therefore, the marginal (or market-to-book) value of equity decreases in aggregate bank equity. Negative shocks that reduce banks’ equity buffers are thus amplified by a simultaneous increase in the market-to-book value, whereas positive shocks are moderated by a simultaneous decrease in the market-to-book value. As a result, banks will lend to firms only if the loan rate incorporates an appropriate premium.

We study the long-run behavior of the economy, which is considerably facilitated by the fact that the loan rate dynamics can be obtained in closed form. As the long-run behavior of the economy is mainly driven by the (endogenous) volatility, it spends most of its time in states with low endogenous volatility. For high recapitalization costs and a low elasticity of demand for bank loans, this can lead to severe credit crunches with persistently high loan rates, low volumes of lending and low levels of bank equity. The occurrence of credit crunches is caused by the following simple mechanism: Assume that a series of adverse shocks has eroded bank capital. Since banks require a larger loan rate when capital is low, this drives down firms’ credit demand. As a result, banks’ exposure to macro shocks is reduced and thus also the endogenous volatility of aggregate equity, so that the economy may spend a long time in the credit crunch.

A welfare analysis of the competitive equilibrium shows that the latter is constrained inefficient. Specifically, banks lend too much and hold too little equity as compared to the Second Best allocation. The reason is that banks take into account neither the impact of their individual lending decisions on the cost of credit, nor the social costs of bank distress. We then investigate whether the distortions emerging in the competitive equilibrium can be corrected by regulation. To this end, we consider the effects of a standard capital ratio and a tax on bank dividends. While each of these tools help increase aggregate capitalization of the banking sector, a dividend tax does a better job in terms of asymptotically approaching the Second Best allocation.

\[6\] That is, the ergodic density function that characterizes the long run behavior of the economy can be obtained in closed form as well.
Related Literature. From a technical perspective, our paper is most closely related to the macroeconomic models with financial frictions that study the formation of asset prices in a dynamic endowment economy (see, e.g., Brunnermeier and Sannikov (2014), He and Krishnamurthy (2012, 2013)). As in the above-mentioned papers, our model lends itself to studying the full equilibrium dynamics of the economy, in contrast to traditional macroeconomic models that only allow for analyzing the equilibrium around a steady state. At the same time, the problem of individual banks with respect to dividend distribution and recapitalization in our model shares some similarities with the liquidity management models in Bolton et al. (2011, 2013), Décamps et al. (2011), Hugonnier and Morellec (2015).

The dynamic effects in our model are driven by an endogenous leverage constraint based on the loss absorbing capacity of book equity in the presence of external financing frictions. Via this transmission channel, temporary shocks can have persistent effects on loan rates, which in turn amplifies the initial shock on book equity. This amplification mechanism is similar in spirit to the collateral constraint in Kiyotaki and Moore (1997) or the limitation of pledgeable income as in Hohnstrom and Tirole (1997).

Since the reflection property of aggregate equity in our model generates quasi cyclical lending patterns, our paper is also related to the literature on credit cycles that has brought forward a number of alternative explanations for their occurrence. Fisher (1933) identified the famous debt deflation mechanism, that has been further formalized by Bernanke et al. (1996) and Kiyotaki and Moore (1997). It attributes the origin of credit cycles to the fluctuations of the prices of collateral. Several studies also place emphasis on the role of financial intermediaries, by pointing out the fact that credit expansion is often accompanied by a loosening of lending standards and "systemic" risk-taking, whereas materialization of risk accumulated on the balance sheets of financial intermediaries leads to the contraction of credit (see e.g. Aitken et al. (2013), Dell’Ariccia and Marquez (2006), Jimenez and Saurina (2006)). Suarez and Sussman (1997) consider instead risk-taking by borrowers in a dynamic version of the Stiglitz-Weiss model. They establish a credit reversion mechanism where high production in booms reduces prices, thereby, creating liquidity shortages and leading to more external financing next period. This entails more risk-taking and defaults, thus turning the economy into a bust.

Finally, our paper relates to the literature on pecuniary externalities. Recent contributions by Lorenzoni (2008), Bianchi (2011), Jeanne and Korinek (2011) show that collateral price fluctuations can be the source of welfare decreasing pecuniary externalities on credit markets, which could justify countercyclical public policies. Such pecuniary externalities can also be generated by agency problems (Gersbach and Rochet (2014)), market incompleteness (He and Kondor (2014)), or decreasing returns to aggregate lend-
ing (Malherbe (2015)). In our model the source of pecuniary externalities is rooted in the effect of competition. Namely, while making individual lending decisions, competitive banks do not internalize the feedback effect on their profits that works through the loan rate channel. As a result, the cost of credit in the competitive equilibrium (the volume of lending) turn out to be lower (higher) than in the set-up where the social planner could control lending so as to maximize social welfare.

The rest of the paper is structured as follows. Section 2 presents the discrete-time version of the model and discusses two useful benchmarks. In Section 3 we solve for the competitive equilibrium in the continuous-time set up and analyze its implications on financial stability and welfare. In Section 4 we study the impact of taxes on dividends and minimum capital requirements on bank policies. Section 5 concludes. All proofs and computational details are gathered in the Appendix.

2 The discrete-time model

To elucidate the main mechanisms at work in the continuous time set-up, we start by formulating our model in discrete time and then let the length $\Delta t$ of each period go to zero.

2.1 Model set-up

There is one physical good, taken as a numeraire, which can be consumed or invested. There are three types of agents: (i) depositors, who only play a passive role, (ii) bankers, who own and manage the banks, and (iii) entrepreneurs, who manage the productive sector. Depositors are infinitely risk averse and discount the future at rate $r$. Bankers and entrepreneurs discount the future at rate $\rho > r$ (i.e., they are more impatient) but they are risk neutral.

Banks behave competitively and finance loans to the productive sector by a combination of deposits and equity. Since we focus on credit, we do not introduce explicit liquidity provision activities associated with bank deposits. These deposits are modeled in a parsimonious fashion: depositors are infinitely risk averse and have a constant discount factor $r$. This implies two things: first, deposits must be absolutely riskless (all the risks will thus be borne by bank shareholders); second, depositors are indifferent as to the level of deposits and timing of withdrawal, provided that they receive an interest

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7 Our bankers are to be interpreted as the union of managers and shareholders. Likewise, dividends should be thought of as bonuses plus shareholder dividends.

8 The liquidity premium $\rho - r$ reflects the fact that depositors request perfect safety/liquidity and are thus ready to accept a lower rate of return. For an alternative justification of this liquidity premium, see Stein (2011).
rate \( r \) per unit of time. In sum, banks can collect any amount of deposits (i.e., deposits represent an infinitely inelastic source of funding), provided that they offer depositors the interest rate \( r \) and fully guarantee deposit value. As we will show in the following, the depositors’ preference for safety lies behind the role of bank capital in our economy.

The productive sector consists of a continuum of entrepreneurs controlling investment projects that are parametrized by a productivity parameter \( x \). The productivity parameter \( x \) is privately observed by each entrepreneur\(^9\) and is distributed according to a continuous distribution with density function \( f(x) \) defined on a bounded support \([0, R]\).\(^{10}\)

Entrepreneurs’ projects are short lived and each of them requires an investment of one unit of good. If successful, a project yields \((1 + x \Delta t)\) units of good in the next period and zero otherwise. Entrepreneurs have no own funds and finance themselves via bank loans.\(^{11}\) Thus, the entire volume of investment in the economy is determined by the volume of bank credit. Entrepreneurs are protected by limited liability and default when their projects are not successful. Given a nominal loan rate \( R \Delta t \) (for a loan of duration \( \Delta t \)), only the projects such that \( x > R \) will demand financing. Thus, the total demand for bank credit in the economy is:

\[
D(R) = \int_{R}^{R} f(x) dx.
\]

We focus on the simple case where all projects have the same default probability: \(^{12}\)

\[
p \Delta t + \sigma_0 \sqrt{\Delta t} \varepsilon_t,
\]

where \( p \) is the unconditional probability of default per unit of time, \( \varepsilon_t \) represents an aggregate shock faced simultaneously by all firms and \( \sigma_0 \) reflects the change in the default probability caused by the aggregate shock. For simplicity, \( \varepsilon_t \) is supposed to take only two values \(+1\) (bad state) and \(-1\) (good state) with equal probabilities. Then, up to the first

\(^9\)This assumption is made mainly to facilitate exposition as it prevents contracts in which the loan rate depends on \( x \). Even if productivity were publicly observable, competition between banks would lead to identical loan rates for all borrowers.

\(^{10}\)Parameter \( R \) can be thought of as the maximum productivity among all the firms. This is also the maximum loan rate: when \( R > \bar{R} \), the demand for loans is nil.

\(^{11}\)Firms in our economy should be thought of as small and medium-sized enterprises (SMEs). SMEs represent the major pillar of the real economy and typically highly rely on bank financing. We recognize, however, that the importance of bank financing varies across countries. For example, according to the TheCityUK research report (October 2013), in EU area, bank loans account for 81% of the long term debt in the real sector, whereas in US the same ratio amounts to 19%.

\(^{12}\)The extension to heterogeneous default probability is straightforward and would not change the results of our analysis.
order terms, the net expected return per loan for a bank after an aggregate shock $\varepsilon_t$ is

\[(R - r - p)\Delta t - \sigma_0 \sqrt{\Delta t}\varepsilon_t, \quad (1)\]

where the first term reflects the expected earnings per unit of time and the second term captures the exposure to aggregate shocks.

At the equilibrium of the credit market, the net aggregate output per period in the economy is

\[\left( F[D(R)] - pD(R) \right)\Delta t - \sigma_0 D(R)\sqrt{\Delta t}\varepsilon_t, \quad (2)\]

where

\[F[D(R)] = \int_R^\infty x f(x)dx \quad (3)\]

is the aggregate production function.

Note that $F'[D(R)] \equiv R$, so that the total expected surplus per unit of time, $F[D(R)] - pD(R) - rD(R)$, is maximized for $R_{fb} = r + p$. Thus, in the first best allocation, the loan rate is the sum of two components: the riskless rate and the unconditional probability of default. This implies that banks make zero expected profit, and the total volume of credit in the economy is given by $D(R_{fb})$.

### 2.2 One-period benchmark

To introduce the main tools of our analysis and to provide the basic intuitions behind our results, we start with a simple static set-up in which banks only live one period. At the beginning of the period, an individual bank’s shareholders have an exogenous amount of equity $e_t$. Given this equity level, they choose the volume of deposits $d_t$ to collect and lend $k_t = e_t + d_t$ to the productive sector. At the end of the period, the aggregate shock is realized, and the profit of an individual bank is:

\[k[(R - r - p)\Delta t - \sigma_0 \sqrt{\Delta t}\varepsilon_t].\]

For $\Delta t$ small enough, this profit is negative (i.e., the bank incurs losses) when $\varepsilon_t = 1$. Since depositors are infinitely risk-averse, they will deposit their money into the bank only if they are certain to get it back at the end of the period. This implies that any potential losses must be absorbed by the bank’s equity, i.e.,

\[e \geq k[\sigma_0 \sqrt{\Delta t} - (R - r - p)\Delta t], \quad (4)\]

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13 The neglected term $-R(\Delta t)^2[p + \sigma_0/\sqrt{\Delta t}]$ is, indeed, negligible when $\Delta t$ is small enough.

14 Differentiating Equation (3) with respect to $R$ yields $F'[D(R)]D'(R) = -Rf(R).$ Since $D'(R) = -f(R)$, this implies $F'[D(R)] \equiv R.$
where the term in brackets can be interpreted as the maximum loss that can be incurred per unit of investment. Thus, the depositors’ need for safety imposes a leverage constraint on each bank, which must also be satisfied at the aggregate level:

\[ E \geq K[\sigma_0\sqrt{\Delta t} - (R - r - p)\Delta t]. \tag{5} \]

The competitive equilibrium is characterized by a loan rate \( R(E) \) and a lending volume \( K(E) \) that are compatible with expected profit maximization by each individual bank:

\[ v(e, E) = \max_k e + k(R(E) - r - p)\Delta t \quad \text{s.t.} \quad (4) \]

and the loan market clearing condition

\[ K(E) = D[R(E)]. \]

Note that the shareholder value \( v(e, E) \) of each bank depends both on its own equity level \( e \) and on aggregate bank capitalization \( E \), as the latter affects the loan rate \( R(E) \). It is easy to see that, depending on the level of banks’ capitalization, two cases are possible.

**Case 1: well-capitalized banking sector.** When aggregate bank capitalization is sufficiently high, the leverage constraint does not bind and thus the equilibrium loan rate is given by \( R^*(E) = r + p \equiv R_{fb} \), which corresponds to the First-Best allocation of credit. Specifically, this case is feasible when

\[ E \geq E^* \equiv D[R_{fb})\sigma_0\sqrt{\Delta t}. \]

**Case 2: undercapitalized banking sector.** When aggregate bank capitalization is low, the leverage constraint binds and the equilibrium loan rate is defined implicitly by the value \( R = R^*(E) \) such that

\[ D(R)[\sigma_0\sqrt{\Delta t} - (R - r - p)\Delta t] = E. \tag{6} \]

Since the left-hand side of (6) is decreasing in \( R \), the loan rate \( R^*(E) \) resulting from (6) is decreasing in \( E \) (for \( E < E^* \)), which implies that \( R^*(E) > R^*(E^*) = r + p \). Thus, this parsimonious model illustrates an important idea: it is the limited loss absorbing capacity combined with the need to guarantee riskless investment to depositors that drives the loan rate away from its First-Best level.

Note that the shareholder value of any individual bank is proportional to its book

\[ \text{An equivalent interpretation is that banks finance themselves by repos, and the lender applies a hair cut equal to the maximum possible value of the asset (the loan portfolio) that is used as collateral.} \]
value $e$ and depends on aggregate capitalization. This implies that the market-to-book ratio is the same for all banks:

$$\frac{v(e, E)}{e} \equiv u(E) = 1 + \frac{(R(E) - r - p)\Delta t}{\sigma_0 \sqrt{\Delta t} - (R(E) - r - p)\Delta t}.$$

It is easy to see that, when the banking sector is well capitalized (i.e., $E \geq E^*$), the market-to-book ratio equals one. In the alternative case ($E < E^*$), it is strictly higher than one and a decreasing function of aggregate bank capitalization: \[ u(E) = \frac{\sigma_0 \sqrt{\Delta t}}{\sigma_0 \sqrt{\Delta t} - (R(E) - r - p)\Delta t} > 1, \quad E < E^*. \]

As will become apparent below, these properties of the market-to-book value also hold in the continuous time set-up.

Finalizing the discussion of this one-period benchmark, it is worthwhile to note that, in this static set-up, the competitive equilibrium is constrained efficient. Indeed, a social planner would choose a volume of lending $K(E)$ that maximizes social welfare under the aggregate leverage constraint:

$$W(E) = \max_K E + [F(K) - (r + p)K]\Delta t \quad \text{s.t.} \quad E \geq K[\sigma_0 \sqrt{\Delta t} - (F'(K) - r - p)\Delta t].$$ \tag{7}

$$E \geq K[\sigma_0 \sqrt{\Delta t} - (F'(K) - r - p)\Delta t].$$ \tag{8}

Note that the social welfare function $W(E)$ is increasing and concave. Moreover, $W'(E) \geq 1$, with strict inequality when $E < E^*$. Given these observations, it is easy to see that the solution of the social planner’s problem coincides with the competitive equilibrium allocation, which is therefore constrained efficient. However, it is no longer true in a dynamic setting. To show this, we start by a simple two-period benchmark.

### 2.3 Two-period benchmark

In this section, banks live two periods (0 and 1) and, for simplicity, there is no aggregate shock in period 0 (i.e., $\varepsilon_0 = 0$). In that case, the banks are not subject to a leverage constraint in period 0 but only face it in period 1. Assuming no dividend nor recapitalization in period 0, shareholder value is

$$v_0(e_0, E_0) = \max_{k_0} \left[ e_0 + k_0(R_0 - r - p) \right] u(E_0 + K_0(R_0 - p - r)).$$

The only loan rate $R_0$ that is compatible with shareholder value maximization is

\[16\text{Recall that } R'(E) < 0 \text{ when } E < E^*.\]
\[ R_0 = r + p \] and the volume of credit is then given by \( K_0 = D(R_0) \). Thus, in the initial period, banks make zero profit and aggregate equity remains constant \((E_0 = E_1 \equiv E)\). As a result, the equilibrium loan rate at date \( t = 1 \) equals \( R^*(E) \), as in the one-period benchmark. However, we show below that, in contrast to the one-period benchmark where a social planner’s choice of the volume of lending coincides with the competitive outcome, in the two-period setting a social planner would choose a higher volume of lending in period 0.

A social planner would choose the volume of lending so as to maximize intertemporal welfare \( W_0 \):\(^{17}\)

\[ W_0(E_0) = \max_K \left[ F(K) - KF'(K) \right] \Delta t + W(E_0 + K[F'(K) - r - p] \Delta t), \]  

where the first term captures the firms’ profit in period 0.

The profit of the banks in period 0 is retained and appears in the argument \( E_1 \) of the welfare function for period 1. The first-order condition of this problem is

\[ 0 = -KF''(K) + W'(E_1)[F'(K) - r - p + KF''(K)], \]

which enables us to express the corresponding loan rate as follows:

\[ R_{sb} \equiv F'(K_{sb}) = r + p - K_{sb}F''(K_{sb}) \left[ 1 - \frac{1}{W'(E_1)} \right]. \]

The above expression suggests that, when \( W'(E_1) > 1 \) (undercapitalized banking sector) and \( F''(K) < 0 \) (elastic loan rate), \( R_{sb} > R_{fb} \equiv r + p \). The reason is that the reduction of lending below \( K^* = D(R_{fb}) \) allows the banks to increase their capitalization in period 1. When there are frictions on financial markets, and banks cannot recapitalize costlessly, a reduction in lending in period 0 increases banks’ profit and thus relaxes their leverage constraint in period 1.

Two important remarks are in order at this stage. First, the leverage constraint in our model comes from the depositors’ need for a safe investment. In such a context, bank equity plays the role of a loss absorbing buffer, very much in the spirit of the "loss absorbing capacity" concept that is put forward by regulators. This justification of leverage constraints differs from the other ones that are put forward by academics. One of them relates to the limitation on pledgeable income by bank insiders (Holmstrom and Tirole (1997)) and backs the concept of "inside" equity that plays the role of "the skin in the game" for banks. The other academic justification for leverage constraints stems

\(^{17}\)For simplicity, we do not allow for discounting and set \( \rho \) to zero for this subsection only.
on the limited resalability of collateral (Kiyotaki and Moore (1997)), thereby, placing emphasis on the asset side of borrowers’ balance sheets, whereas we focus here on the banks’ liabilities.\footnote{These three justifications for leverage are different, but not independent.}

Second, it is worthwhile to note that, even though the leverage constraint only binds in period 1, it also impacts the banks’ choices in period 0. If we also allow for a shock in period 0 (we ruled it out only for a simpler exposition), then a leverage constraint also appears in period 0, but it is not always binding, in contrast to period 1. The reason is that the market-to-book value of banks $u(E)$ fluctuates with $E$, and bankers are aware that they are exposed to the same aggregate shocks. Therefore, each bank’s loss is magnified by an increase in $u(E)$ and, vice-versa, each bank’s gain is reduced because of the decrease in $u(E)$. As we will see in the continuous-time setting, there exists a particular value of the loan spread $R(E) - p$ that exactly compensates for the fluctuation of the market-to-book value, and allows for an interior choice of lending by banks.

3 The continuous-time model

We now turn to the full-fledged continuous-time model. For the rest of the paper we assume that $r = 0$ and solve the model for the case $r > 0$ in Appendix D.

The main financial friction in our model is that banks face a proportional issuance cost $\gamma$ when they want to issue new equity.\footnote{On top of direct costs of equity issuance, $\gamma$ may also capture inefficiencies caused by asymmetric information, which are not modeled here.} For simplicity, we neglect other external frictions such as adjustment costs for loans or fixed costs of issuing equity.\footnote{We also disregard any frictions caused by governance problems inside the banks or government explicit/implicit guarantees.} This implies that our economy exhibits an homotheticity property: all banks’ decisions (lending, dividends, recapitalization) are proportional to their equity levels. In other words, all banks make the same decisions at the same moment, up to a scaling factor equal to their equity level. This entails an important simplification: only the aggregate size of the banking sector, reflected by aggregate bank capitalization, matters for our analysis, whereas the number of banks and their individual sizes do not play any role.

Taking the continuous time limit of the return on assets for banks (see expression (1)), we obtain:

$$ (R_t - p)dt - \sigma_0 dZ_t, $$

where $\{Z_t, t \geq 0\}$ is a standard Brownian motion.

We investigate the existence of a Markovian competitive equilibrium, where all aggregate variables are deterministic functions of the single state variable, namely, aggregate
bank equity, $E_t$. When $r = 0$, the dynamics of aggregate equity $E_t$ satisfies

$$dE_t = K(E_t)[(R(E_t) - p)dt - \sigma_0 dZ_t] - d\Delta_t + dI_t,$$

(11)

where $K(E_t)$ denotes aggregate lending, $d\Delta_t \geq 0$ reflects aggregate dividend payments and $dI_t \geq 0$ captures aggregate equity injections. Aggregate deposits are determined by the residual $K(E_t) - E_t$.21

Definition 1 A Markovian competitive equilibrium consists of an aggregate bank capital process $E_t$, a loan rate function $R(E)$ and a credit volume $K(E)$ that are compatible with individual banks’ profit maximization and the credit market clearing condition $K(E) = D[R(E)]$.

In the following subsections we show the existence of a unique Markovian equilibrium and study its implications on financial stability and social welfare.

3.1 The competitive equilibrium

To characterize the competitive equilibrium, we have to determine the optimal recapitalization and financing decisions of individual banks as well as a functional relation between the aggregate level of bank equity $E_t$ and the loan rate $R_t$. Consider first the optimal decision problem of an individual bank that takes the loan rate function $R_t = R(E_t)$ as given and makes its decisions based on the level of its own equity $e_t$ and aggregate equity $E_t$. Bank shareholders choose lending $k_t \geq 0$, dividend $d\delta_t \geq 0$ and recapitalization $di_t \geq 0$ policies so as to maximize the market value of equity:22

$$v(e, E) = \max_{k_t, d\delta_t, di_t} \mathbb{E} \left[ \int_0^{+\infty} e^{-\rho t} (d\delta_t - (1 + \gamma)di_t)e_t = e, E_t = E \right],$$

(12)

where aggregate equity $E_t$ evolves according to (11) and

$$de_t = k_t[(R(E_t) - p)dt - \sigma_0 dZ_t] - d\delta_t + di_t.$$

(13)

A fundamental property of the individual decision problem of a bank is that the feasible set, in terms of trajectories of $(k_t, d\delta_t, di_t)$, and the objective function are homogeneous of degree one in the individual equity level $e_t$. Therefore, the value function itself must

21Note also that we do not restrict the sign of $K(E) - E$, so that in principle, we also allow for liquid reserves (cash) in the form of negative deposits. However, in the competitive equilibrium, holding liquid reserves turns out to be suboptimal for banks.

22Throughout the paper, we use lower case letters for individual variables and upper case letters for aggregate variables.
satisfy: 

$$v(e, E) = eu(E),$$

where $e$ reflects the book value of equity and $u(E)$ can be thought of as the market-to-book value of equity for banks.

Using the above property and applying standard dynamic programming methods (see Appendix A), it can be shown that the market-to-book value of equity drives all bank policies in our framework. The optimal dividend and recapitalization policies turn out to be of the "barrier type". In particular, dividends are distributed only when $E_t = E_{\text{max}}$, where $E_{\text{max}}$ is such that $u(E_{\text{max}}) = 1$. In other words, distribution of dividends only takes place when the marginal value of equity capital equals the shareholders' marginal value of consumption. Recapitalizations occur only when $E_t = E_{\text{min}}$, where $E_{\text{min}}$ satisfies $u(E_{\text{min}}) = 1 + \gamma$, i.e., when the marginal value of equity equals the total marginal cost of equity issuance. As long as aggregate bank equity $E_t$ remains in between $E_{\text{min}}$ and $E_{\text{max}}$, fluctuations of the individual bank’s equity are only caused by retained earnings or absorbed losses. Given that the market-to-book value is the same for all banks, bank recapitalizations and dividend payments in our economy are perfectly synchronized in time.

Maximization with respect to the level of lending $k_t$ shows that the optimal lending policy of the bank is indeterminate, i.e., bank shareholders are indifferent with respect to the volume of lending. Instead, the latter is entirely determined by the firms’ demand for credit. We show in Appendix A that the maximization problem (12) has a non-degenerate solution if and only if the market-to-book ratio simultaneously satisfies two equations:

$$[R(E) - p]u(E) = -K(E)\sigma_0^2 u'(E),$$

and

$$\rho u(E) = K(E)[R(E) - p]u'(E) + \frac{\sigma_0^2 K^2(E)}{2} u''(E),$$

where $K(E) = D[R(E)]$.\footnote{This useful property of the value function is a natural consequence of the scale invariance property of our model.}

\footnote{The barrier-type recapitalization and payout policies have been extensively studied by the corporate liquidity management literature (see e.g. Jeanblanc and Shiryaev (1996), Milne and Robertson (1996), Décamps et al. (2011), Bolton et al. (2012, 2013), Hugonnier et Morellec (2015) among others) that places emphasis on the loss-absorbing role of corporate liquid reserves in the presence of financial frictions. In our model, the role of book equity is very similar to the role of liquidity buffers in those models. However, we differentiate from this literature by allowing for the feedback loop between the individual decisions and the dynamics of individual book equity via the general equilibrium mechanism that determines the loan rate and thus affects the expected earnings of a bank.}

\footnote{This situation is analogous to the case of an economy with constant returns to scale, in which the equilibrium price of any output is only determined by technology (constant marginal cost), whereas the volume of activity is determined by the demand side.}
Combining these two equations, we find that \( R(E) \) satisfies a first-order differential equation:

\[
R'(E) = -\frac{1}{\sigma_0^2} \left( \frac{2\rho \sigma_0^2 + (R(E) - p)^2}{D[R(E)] - [R(E) - p]D'[R(E)]} \right) = -\frac{1}{H[R(E)]}.
\] (16)

Given that \( D'(R) < 0 \), it is easy to see that \( R'(E) < 0 \): In the states with higher aggregate capital, banks charge lower loan rates, which leads to higher volume of credit and output in the economy. In contrast, when aggregate bank capital gets scarce after a long series of negative aggregate shocks, the loan rate increases, which entails a reduction in credit and output.\(^{26}\) Thus, in line with empirical evidence (see, e.g., Becker and Ivashina (2014)), our model generates a procyclical pattern of lending.

It is important to emphasize that, for any level of bank capitalization \( E \in [E_{\text{min}}, E_{\text{max}}) \), the risk-adjusted credit spread \( R(E) - p \) remains strictly positive. Indeed, expressing the loan rate \( R(E) \) from equation (14) immediately shows that, for any \( E > E_{\text{max}} \), bank shareholders require a strictly positive premium for accepting to lend:

\[
R(E) = p + \sigma_0^2 K(E) \left[ -\frac{u'(E)}{u(E)} \right].
\] (17)

To understand the “raison d’être” for this lending premium, consider the impact of the marginal unit of lending on shareholder value \( eu(E) \). A marginal increase in the volume of lending increases the bank’s exposure to aggregate shocks. However, note that the aggregate shock not only affects the individual bank’s equity \( e_t \) but also aggregate equity \( E_t \) and thus the market-to-book ratio \( u(E) \) that is decreasing in \( E \).\(^{27}\) Thus, if there is a negative aggregate shock \( dZ_t > 0 \) that depletes the individual bank’s equity, the effect of this loss on shareholder value gets amplified via the market-to-book ratio. Symmetrically, a positive aggregate shock \( (dZ_t < 0) \), while increasing book equity, translates into a reduction of \( u(E_t) \), which reduces the impact of positive profits on shareholder value. This mechanism gives rise to effective risk aversion with respect to variation in aggregate capital, which explains why risk-neutral bankers require a positive spread for accepting to lend.

The following proposition summarizes the characterization of the competitive equilibrium.

\(^{26}\) Another remark to be made in light of the negative relation between the loan rate and aggregate equity is that recapitalizations occur when the bank makes a strictly positive profit in expectation, whereas dividends are distributed when the bank makes a zero expected profit.

\(^{27}\) Intuitively, having an additional unit of equity reduces the probability of facing costly recapitalizations in the short-run, so that the marginal value of equity, \( u(E) \), is decreasing with bank capitalization.
Proposition 1 There exists a unique Markovian equilibrium, in which aggregate bank capital evolves according to:

\[ dE_t = D[R(E_t)][(R(E_t) - p)dt - \sigma_0 dZ_t], \quad E_t \in (0, E_{\text{max}}). \] (18)

Banks recapitalize when \( E_t = 0 \) and distribute dividends when \( E_t = E_{\text{max}} \).

The loan rate function \( R(E) \) is defined in the interval \([R_{\text{min}}, R_{\text{max}}]\), such that \( R_{\text{min}} \equiv R(E_{\text{max}}) = p \) and \( R_{\text{max}} \equiv R(0) \), and is implicitly given by the equation

\[ E = \int_{R_{\text{max}}}^{R_{\text{min}}} H(s) ds, \quad \text{where} \quad H(s) = \frac{\sigma_0^2 D(s) - (s - p)D'(s)}{2\rho\sigma_0^2 + (s - p)^2}. \] (19)

The number \( R_{\text{max}} \) is uniquely determined by the equation

\[ \int_{p}^{R_{\text{max}}} \frac{(R - p)H(R)}{\sigma_0^2 D(R)} dR = \log(1 + \gamma). \] (20)

Proposition 1 shows in particular that \( E_{\text{min}} = 0 \), which means that unregulated banks would wait until the last moment before recapitalizing. We interpret these “last moment recapitalizations” of the entire banking sector as systemic crises. In our welfare analysis in Section 3.3 we introduce a social costs of the systemic crisis, which is not internalized by private banks.

![Figure 1: Loan rate and mark et-to-book ratio in the competitive equilibrium](image)

**Notes:** This figure reports the typical patterns of the loan rate \( R(E) \) (left panel) and market-to-book ratio \( u(E) \) (right panel) in the competitive equilibrium.

The typical patterns of the loan rate \( R(E) \) and the market-to-book value \( u(E) \) that emerge in the competitive equilibrium are illustrated in Figure 1. Note that the loan rate function \( R(E) \) cannot generally be obtained in closed form. However, it turns out that the dynamics of the loan rate \( R_t = R(E_t) \) is explicit. Indeed, applying Itô’s lemma to \( R_t = R(E_t) \) yields:
After some computations involving the use of (16), one can obtain the drift and the volatility of \( R_t = R(E_t) \) in closed form. This yields the following proposition:

**Proposition 2** The loan rate \( R_t = R(E_t) \) has explicit dynamics

\[
dR_t = \tilde{\mu}(R_t) \, dt + \sigma(R_t) \, dZ_t, \quad p \leq R_t \leq R_{\text{max}},
\]

where

\[
\tilde{\mu}(R_t) = \sigma(R_t) (R_t - p),
\]

and

\[
\sigma(R) = \frac{2\rho \sigma_0^2 + (R - p)^2}{\sigma_0 \left( 1 - (R - p) \frac{D'(R)}{D(R)} \right)}. \tag{23}
\]

The drift function is

\[
\tilde{\mu}(R) = \sigma(R)(R - p) \frac{h(R)}{2\sigma_0}, \tag{24}
\]

where

\[
h(R) = \frac{2(R - p)D'(R)}{D(R) - (R - p)D'(R)} + \frac{(2\rho \sigma_0^2 + (R - p)^2)D(R)D''(R)}{[D(R) - (R - p)D'(R)]^2}. \tag{25}
\]

The dynamics of \( E_t \) and \( R_t = R(E_t) \) in the competitive equilibrium depends on the credit demand function \( D(R) \) and four parameters: the exposure to aggregate shocks (or fundamental volatility) \( \sigma_0 \), the unconditional probability of default \( p \), the discount factor \( \rho \) and financial frictions \( \gamma \). In equilibrium, the loan rate \( R_t \) fluctuates between its first-best level \( p \) and \( R_{\text{max}} \). Moreover, it follows immediately from the expression (20) that the maximum lending premium, \( R_{\text{max}} - p \), is increasing with the magnitude of financial frictions, \( \gamma \). Thus, our model predicts that loan rates, lending and, thereby, output will be more volatile in the economies with stronger financial frictions. At the same time, the expression (19) shows that the target level of bank capitalization, \( E_{\text{max}} \), is increasing with \( R_{\text{max}} \). Thus, the loss absorbing capacity of equity becomes more important under stronger financing frictions.\(^{28}\) By contrast, in the absence of financial frictions, i.e., when \( \gamma = 0 \), one would have \( R_{\text{max}} = p \) and \( E = 0 \), so that there would no role for bank equity and no fluctuations of credit.

\(^{28}\)Note that the cost of recapitalization \( \gamma \) only affects \( R_{\text{max}} \) and \( E_{\text{max}} \), without intervening in the expressions of \( \tilde{\mu}(R), \sigma(R) \) and \( u(E) \).
3.2 Long run behavior of the economy

We now study the long-run behavior of the economy in the competitive equilibrium. To this end, we look at the long-run behavior of the loan rate, whose dynamics is explicitly defined in (22).\footnote{Working with $R_t$ instead of $E_t$ enables us to provide an analytic characterization of the system’s behavior, because the drift and volatility of $R_t$ are closed-form expressions. By contrast, the drift and volatility of the process $E_t$ cannot in general be obtained in closed form, since $R(E)$ has an explicit expression only for particular specifications of the credit demand function.} It turns out that this behavior is ergodic and thus can be described by the ergodic density function. The ergodic density measures the average time spent by the economy in the neighborhood of each possible loan rate $R$: the states with lower $R$ (equivalently, high aggregate capital $E$) can be interpreted as "boom" states and the states with higher $R$ (equivalently, low aggregate capital $E$) can be thought of as "bust" states. The ergodic density function can be computed by solving the Kolmogorov forward equation (see details in Appendix A).

**Proposition 3** The competitive loan rate process $R_t$ is ergodic. Its asymptotic distribution is characterized by the probability density function

$$g(R) = \frac{C_0}{\sigma^2(R)} \exp \left( \int_p^R \frac{2\mu(s)}{\sigma^2(s)} ds \right),$$

where the constant $C_0$ is such that $\int_p^{R_{\text{max}}} g(R) dR = 1$.

By differentiating the logarithm of the ergodic density defined in (26), we obtain:

$$\frac{g'(R)}{g(R)} = \frac{2\mu(R)}{\sigma^2(R)} - \frac{2\sigma'(R)}{\sigma(R)}.$$  \hspace{1cm} (27)

Using the general formulas for $\sigma(R)$ and $\mu(R)$, it can be shown that $\sigma(p) = 2\rho \sigma_0$, $\sigma'(p) = 2\rho \sigma_0 \frac{D'(p)}{D(p)} < 0$ and $\mu(p) = 0$. Hence, $g'(p) > 0$, which means that the state $R = p$ that would correspond to the deterministic steady state is definitely not the one at which the economy spends most of the time in the stochastic set up.\footnote{See Appendix E for the analysis of the properties of the deterministic steady-state.} To get a deeper understanding of the determinants of the system behavior in the long run, we resort to the particular demand specification:

$$D(R) = (\bar{R} - R)^\beta,$$  \hspace{1cm} (28)

where $\beta > 0$ and $p < \bar{R}$.

Figure 2 reports the typical patterns of the endogenous volatility $\sigma(R)$ (left panel) and the ergodic density $g(R)$ (right panel) for the above loan demand specification. It
Figure 2: Volatility and ergodic density of $R$

Notes: this figure reports the typical patterns of the loan rate volatility (left panel) and the ergodic density (right panel).

shows that the extrema of the ergodic density almost coincide with those of the volatility function, i.e., the economy spends most of the time in the states with the lowest loan rate volatility. Intuitively, the economy can get "trapped" in the states with low loan rate volatility because the endogenous drift is generally too small to move it away from these states. In fact, $\sigma(R)$ turns out to be much larger than $\mu(R)$ for any level of $R$, so that the volatility impact always dominates the drift impact. In this light, relying on the results of the impulse response analysis (see Appendix E) in order to infer the long-run behavior of the economy would be misleading.

Note that functions $\sigma(.)$ and $f(.)$ must be truncated (and, in the case of the ergodic density, rescaled) on $[p, R_{max}]$, where $R_{max}$ depends on the magnitude of issuing costs $\gamma$. For the chosen specification of the loan demand function, $D(R) = (R - p)^{\beta}$, we always have $R_{max} < \overline{R}$. However, $R_{max}$ can be arbitrary close to $\overline{R}$, which typically happens with very strong financial frictions and low elasticity of credit demand. In that case the economy will spend quite some time in the region where the loan rate is close to $R_{max}$.

We interpret this situation as a persistent "credit crunch": it manifests itself via scarce bank equity capital, high loan rates, low volumes of lending and output.

This "credit crunch" scenario is reminiscent to the "net worth trap" documented by Brunnermeier and Sannikov (2014). In their model, the economy may fall into a recession because of the inefficient allocation of productive capital between more and less productive agents, which they call “experts” and “households” respectively. This allocation is driven by the dynamics of the equilibrium price of capital, which depends on the fraction of the total net worth in the economy that is held by experts. After

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31 The reason is that the factor $h(R)$ in the expression of $\mu(R)$ is very small.

32 To prove this property, it is sufficient to show that the integral $\int_{\overline{R}}^{\overline{R}} \frac{(R-p)^{H(R)}}{\sigma^2(R_{max})} dR$ diverges.

33 In a partial equilibrium set-up, the similar phenomenon is found by Bohatíl, Milne and Roberston (2014).
experiencing a series of negative shocks on their net worth, experts have to sell capital to less productive households, so that the average productivity in the economy declines. Under a reduced scale of operation, experts may struggle for a long time to rebuild net worth, so that the economy may be stuck in a low output region. In our model, the output in the economy is driven by the volume of credit that entrepreneurs can get from banks, whereas the cost of credit depends on the level of aggregate bank capitalization. When the banking sector suffers from a series of adverse aggregate shocks, its loss absorbing capacity deteriorates. As a result, the amplification mechanism via the market-to-book value becomes more pronounced and bankers thus require a larger lending premium. The productive sector reacts by reducing its demand for credit and the banks have to shrink their scale of operations, which makes it even more difficult to rebuild equity capital.

3.3 Welfare analysis

Proposition 1 has shown that, in the absence of regulation, competition between banks tends to generate recurrent systemic crises, which are likely to entail social costs. One simple way to introduce these costs into the model is to consider that there is a social costs $\varphi$ of recapitalizing banks, which adds to the private cost $\gamma$. Taking into account the existence of the social costs of bank distress, we now turn to the welfare analysis.

In our simple set-up where deposit taking does not generate any surplus, social welfare can be computed as the sum of the market value of the firms (i.e., the expected discounted profit of the productive sector) and the aggregate value of banks net of the social costs of banking distress:\footnote{We assume that the social planner has the same discount factor as banks’ shareholders.}

$$
W(E) = \mathbb{E} \left[ \int_0^{+\infty} e^{-\rho t} \left( \pi_F(K) + d\Delta_t - (1 + \gamma + \varphi) dI_t \right) | E_t = E \right],
$$

(29)

where $\pi_F(K) \equiv F(K) - KF'(K)$ denotes the aggregate instantaneous production of firms net of credit costs (recall that the market for bank credit must clear, i.e., $K(E) = D(R(E))$ and $F'(K(E)) = R(K(E)))$.

By using aggregate bank equity $E$ as a state variable, we can apply standard pricing methods to compute the social welfare function. Recall that, in the region $(0, E_{max})$, banks neither distribute dividends nor recapitalize, so that the available cash flow consists uniquely of the firms’ profit. Therefore, for $E \in (0, E_{max})$, the social welfare function at the competitive equilibrium, $W(E)$, must satisfy the following differential equation:

$$
\rho W(E) = \pi_F[K(E)] + K(E)[F'(K(E)) - p]W'(E) + \frac{\sigma^2}{2}K^2(E)W''(E).
$$

(30)
Note that dividend distribution and bank recapitalizations only affect the market value of banks, without producing any immediate impact on the firms’ profit. However, bank recapitalizations generate social costs that are taken into account by the social planner. This observation yields two boundary conditions, \( W'(E_{max}) = 1 \) and \( W'(0) = 1 + \gamma + \varphi \). Thus, the welfare function corresponding to the competitive allocation of credit can be computed numerically (see Appendix B for the computation details).

Our first result is that, since banks do not take into account the social costs of banking distress, \( \varphi \), the target equity level \( E_{max} \) where dividends are distributed in the competitive equilibrium is too low from a social perspective unless social costs \( \varphi \) are negligible.

**Proposition 4** There exists a critical level of the social costs of banking distress \( \varphi^* \), such that for \( \varphi > \varphi^* \), it holds that \( W''(E_{max}) < 0 \).

We next investigate whether, in the competitive equilibrium, banks lend too little or too much from a social perspective. Taking the first derivative of the right-hand side of equation (30) with respect to \( K \) yields:

\[
L(E) := [F'(K) - p + F''(K)K]W'(E) - F''(K)K + \sigma_0^2KW''(E). \tag{31}
\]

The above expression measures the impact of a marginal change in lending on social welfare. The first term in \( L(E) \) denotes the change in social welfare due to the change in the instantaneous expected profits of banks that are caused by a marginal increase in \( K \). The second term reflects the analogous change in firms’ profit and the last term captures the change in social welfare caused by larger exposure of the banking system to credit risk. To study the sign of \( L(E) \), it is more convenient to work with the expression \( L(E)/W'(E) \), which, after taking into account Equation (14), can be rewritten as follows:

\[
F''(K)K\left(1 - \frac{1}{W''(E)} \right) + \sigma_0^2K\left(\frac{W''(E)}{W'(E)} - \frac{u'(E)}{u(E)} \right). \tag{32}
\]

The first term in the above expression captures the *price-depressing* effect of an increase in lending, which affects banks’ and firms’ profits with a different sign. In particular, a marginal increase in lending drives down the loan rate, which increases the firms’ profits but bites into the expected profit of banks, thereby, reducing their loss absorbing capacity.\(^3\) It is important to note that competitive banks, which take the loan rate as given, do not internalize this adverse effect of their individual lending decisions. Overall, the negative impact of a lower loan rate on banks’ expected profits outweighs the

\(^{35}\)For the sake of space, we abstain of writing the argument of \( K(E) \).

\(^{36}\)This "margin effect" of competition is also acknowledged in Martinez-Miera and Repullo (2010) albeit in an entirely different application.
positive impact on firms' profits, so that the first term in (32) is weakly negative (recall that $W'(E) \geq 1$ and $F''(K) < 0$). The second term in (32) captures the difference in the effective risk aversion of the social planner and banks, and can, in principle, have either sign.

Our numerical analysis shows that, as long as the social cost of banking distress $\varphi$ is not too small, i.e., $\varphi > \varphi^*$, the overall marginal impact of lending on welfare in the competitive equilibrium is always negative, i.e., $L(E) < 0$ for any level of aggregate equity $E$.

Put differently, this result suggests that social welfare in the competitive equilibrium would be increased by a marginal reduction in lending.

To sum up, in the competitive equilibrium, banks lend too much and hold too little equity as compared with what would be optimal from the social perspective. This is because they do not internalize both the impact of their lending decisions on the cost of credit and the social costs of bank distress. In such a situation, one may expect that introducing some regulatory measures may improve welfare. We investigate this question in the following section.

4 Impact of regulation

So far our analysis has been focused on the "laissez-faire" environment in which banks face no regulation. Our objective in this section is to understand the impact of two regulatory tools on bank policies and welfare: namely, we study the consequences of the implementation of a minimum capital ratio and a tax on dividend.

The main challenge of our analysis is to find an acceptable way of assessment of welfare implications of different regulatory tools. To this end, we first need to establish a benchmark set-up that will serve for comparison. We assume that in such a benchmark set-up, further referred to as the Second Best, the social planner enforces both the aggregate lending volumes (or, equivalently the loan rate) and bank dividend policies, so as to maximize social welfare.

Our approach to the assessment of welfare implications of regulation relies on the comparison of the probabilistic behavior of the regulated equilibrium with the probabilistic behavior of the Second Best. The probabilistic behavior of the economy in our model can be well captured by the cumulative distribution of the loan rate. Thus, comparing the cumulative distribution of the loan rate emerging under different regulatory parameters to the Second-Best benchmark may bring an insight into the sign of the effect generated by regulation. With such an approach in mind, we start

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37 This result is robust for different specifications of the loan demand function.

38 The formal statement of the Second Best maximization problem and its solution for the linear credit demand can be found in the Appendix B.
by considering the welfare implications of a standard minimum capital ratio and then investigate the effects induced by a dividend tax.

4.1 Capital regulation

Let us assume that public authorities enforce a minimum capital requirement, under which each bank must maintain equity capital above a certain fraction of loans, i.e.,

\[ e_t \geq \Lambda k_t, \]

where \( \Lambda \) is the minimum capital ratio.\(^{39}\)

Note that, under such a formulation, banks have two options to comply with minimum capital requirements. The first option is to immediately recapitalize as soon as the regulatory constraint starts binding. The second option consists in cutting on lending and reducing deposit taking. We show below that, in our model, banks use the first option when \( E_t \) is small and the latter when \( E_t \) is large. In other words, a capital ratio does two things: it forces banks to recapitalize earlier (i.e., \( E_{\text{min}} > 0 \)) and reduce lending as compared to the unregulated case.

To solve for the regulated equilibrium, we again start by looking at the maximization problem of an individual bank. As in the unregulated set-up, bank shareholders maximize the market value of their claim by choosing their lending, recapitalization and dividend policies subject to the regulatory restriction on the volume of lending:

\[
v_A(e, E) = \max_{k_t \leq \frac{e}{\Lambda}, d\delta_t, d\iota_t} \mathbb{E} \left[ \int_0^{+\infty} e^{-\rho_t} (d\delta_t - (1 + \gamma)d\iota_t) | e_t = e, E_t = E \right]. \quad (33)
\]

To have the intuition of the solution to the above problem, recall that, in the unregulated case, bank recapitalizations take place only when equity is completely depleted. Thus, it is natural to expect that the regulatory constraint will be binding for relatively low levels of equity. Indeed, in the general case, the bank may find itself in one of two cases: (i) when its level of equity is relatively high, the regulatory constraint is not binding and the volume of lending is still determined by the firms’ demand for credit; (ii) in the states with low equity, the regulatory constraint binds and the volume of lending is determined by \( k_t = e_t / \Lambda \). Due to the homotheticity property, at each point in time, all banks have the same leverage ratio. Thus, it is legitimate to anticipate the existence of the critical level of bank capital \( E^\Lambda_c \), such that the regulatory constraint binds (for all banks) for any \( E \in [E^\Lambda_{\text{min}}, E^\Lambda_c] \) and is slack for any \( E \in (E^\Lambda_c, E^\Lambda_{\text{max}}] \). This critical

\(^{39}\)Since our model only considers one type of bank assets (loans), we cannot discuss the issue of risk weights or distinguish a leverage ratio from a risk-weighted capital ratio.
threshold $E_c^\Lambda$ must satisfy

$$\frac{K(E_c^\Lambda)}{E_c^\Lambda} = \frac{1}{\Lambda}.$$  

For $\Lambda$ high enough, $E_c^\Lambda$ tends to $E_{\text{max}}^\Lambda$ and the unconstrained region disappears entirely.

**Proposition 5** For all $\Lambda$, there exists a unique regulated equilibrium, where the support of $E_t$ is $[E_{\text{min}}^\Lambda, E_{\text{max}}^\Lambda]$. This equilibrium is characterized by one of two regimes:

a) for $\Lambda$ high enough, the regulatory constraint binds over the entire interval $[E_{\text{min}}^\Lambda, E_{\text{max}}^\Lambda]$. The loan rate is explicitly given by

$$R(E_t) = D^{-1}[E_t/\Lambda],$$

where $D^{-1}$ is the inverse function of the loan demand. The dynamics of the aggregate bank capital is given by:

$$\frac{dE_t}{E_t} = \frac{1}{\Lambda} \left( D^{-1}[E_t/\Lambda] - p \right) dt - \sigma_0 dZ_t, \quad E \in (E_{\text{min}}^\Lambda, E_{\text{max}}^\Lambda).$$

b) for relatively low $\Lambda$, capital constraint only binds for $E \in (E_c^\Lambda, E_{\text{max}}^\Lambda]$ and is slack for $E \in (E_{\text{min}}^\Lambda, E_c^\Lambda]$, where $E_c^\Lambda$ is a critical capitalization level. When $E \in (E_{\text{min}}^\Lambda, E_c^\Lambda]$, the dynamics of aggregate equity and the loan rate function are defined as in the regime a). When $E \in (E_c^\Lambda, E_{\text{max}}^\Lambda]$, the loan rate satisfies the first-order differential equation

$$R'(E) = -1/H(R(E))$$

with the boundary condition $R(E_c^\Lambda) = D^{-1}[E_c^\Lambda/\Lambda]$.

In either regime, banks distribute dividends when $E_t = E_{\text{max}}^\Lambda$ and recapitalize when $E_t = E_{\text{min}}^\Lambda$.

We show in the Appendix C.1 that, in the unconstrained region $(E_{\text{min}}^\Lambda, E_{\text{max}}^\Lambda)$, the market-to-book value still simultaneously satisfies (14) and (15), whereas in the constrained region $(E_{\text{min}}^\Lambda, E_c^\Lambda]$ it satisfies instead

$$\rho = \frac{E(D^{-1}[E/\Lambda] - p)}{\Lambda} \frac{u'_\Lambda(E)}{u_\Lambda(E)} + \frac{\sigma_0^2 E^2 u''_\Lambda(E)}{2\Lambda^2 u_\Lambda(E)},$$

under the condition

$$\frac{u'_\Lambda(E)}{u_\Lambda(E)} \geq - \frac{D^{-1}[E/\Lambda] - p}{E/\Lambda \sigma_0^2},$$

with strict equality at $E = E_c^\Lambda$.

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\(^{40}\) The function $H(.)$ is defined in (16). Thus, in the region $E \in (E_c^\Lambda, E_{\text{max}}^\Lambda)$, the dynamics of the aggregate equity is described by the same differential equation as in the unregulated set-up.
The optimal recapitalization and payout decisions are characterized by two boundaries, $E_{\min}^\Lambda$ (recapitalizations) and $E_{\max}^\Lambda$ (dividend payments), such that $u_\Lambda(E_{\min}^\Lambda) = 1 + \gamma$ and $u_\Lambda(E_{\max}^\Lambda) = 1$. In Appendix C.1 we provide the detailed description of the computational procedure that enables us to numerically solve for the regulated equilibrium.

To get general insight into the impact of capital regulation on the cost of credit and the bank’s policies, we perform a comparative static analysis by computing the equilibrium characteristics of bank policies for all values of $\Lambda \in (0, 1]$. The left panel of Figure 3 reports the values of $E_{\min}^\Lambda$, $E_{\max}^\Lambda$ and $E_c^\Lambda$ (solid lines), contrasting them to the values $E_{\min}$ and $E_{\max}$ computed in the unregulated setting (dashed lines). The minimum and the maximum boundaries of the loan rate are reported in the right panel of Figure 3. This figure shows that, as long as the capital ratio is not too high, the bank may find itself in either constrained or unconstrained region, but above some critical level $\Lambda^*$ of a capital ratio, the regulatory constraint is always binding. A higher capital ratio induces banks to operate with a higher target level of equity and drives up the maximum cost of credit. Moreover, in contrast with the unregulated set-up, shareholders recapitalize the bank before completely exhausting bank capital.\textsuperscript{41} The optimal patterns of the loan rate and aggregate lending that emerge under different levels of the minimum capital ratio are depicted in Figure 4.

\textbf{Figure 3: Minimum capital ratio and bank policies}

*Notes: this figure illustrates the effect of minimum capital requirements on banks’ policies. Solid lines in the left panel depict the optimal recapitalization ($E_{\min}^\Lambda$) and payout ($E_{\max}^\Lambda$) barriers, as well as the critical barrier $E_c^\Lambda$ such that the regulatory constraint is binding for $E \leq E_c^\Lambda$. Solid lines in the right panel depicts the minimum ($R_{\min}^\Lambda \equiv R(E_{\min}^\Lambda)$) and maximum ($R_{\max}^\Lambda \equiv R(E_{\max}^\Lambda)$) boundaries for the loan rate, as well as the critical loan rate $R_c^\Lambda \equiv R(E_c^\Lambda)$ such that the regulatory constraint is binding for any $R \geq R_c^\Lambda$. Dashed lines in both panels illustrate the outcomes of the competitive equilibrium in the unregulated set-up. For $\Lambda > \Lambda^*$, a critical level $E_c^\Lambda$ does not exist, i.e., the regulatory constraint is binding for any $E \in [E_{\min}^\Lambda, E_{\max}^\Lambda]$. Parameter values used: $\rho = 0.05$, $\sigma_0 = 0.05$, $p = 0.02$, $\bar{R} = 0.07$, $\gamma = 0.2$, $D(R) = R - R_\text{max}$.\textsuperscript{41}*

To investigate the impact of a minimum capital ratio on welfare, we compute the cumulative distribution function of the loan rate, $G(R)$, under different levels of minimum

\textsuperscript{41}The last feature is due to the binding regulatory constraint: for any positive level of lending, book equity is strictly positive at a recapitalization boundary.
Notes: this figure depicts the loan rate and aggregate credit as the functions of aggregate bank capitalization for two levels of minimum capital requirements (black dashed curves). They are contrasted to the loan rate and aggregate credit that arise in the unregulated (blue solid line) and the Second Best (red solid line) set-ups. Under the lower level of minimum capital ratio (5% in our example), the regulatory constraint binds for the lower levels of bank capitalization, which dramatically affects the slope of the loan rate and lending curves. For the higher level of capital ratio (25% in our example), the slope of the loan rate and lending curves is regular, as it is entirely determined by the binding regulatory constraint, but the support of the function drastically shifts to the right, which reflects substantial changes in the recapitalization and dividend policies. Parameter values used: $\rho = 0.05, \sigma_0 = 0.05, p = 0.02, \bar{R} = 0.07, \gamma = 0.2, \varphi = 0.3, D(R) = \bar{R} - R$.

capital requirements and make a comparison with the cumulative distribution function of $R$ that emerges in the Second Best to see, whether the implementation of capital regulation changes the probabilistic behavior of the system in the direction of the Second Best outcomes. A snapshot of our numerical results is reported in Figure 5. It turns out that, as long as $\Lambda < \Lambda^*$, an increase in the minimum capital ratio shifts the cumulative distribution of $R$ to the left, whereas for $\Lambda \geq \Lambda^*$, the cumulative distribution of $R$ moves away from the Second Best benchmark to the right. Overall, it turns out that no any single level of capital ratio is able to push the cumulative distribution of $R$ towards the Second Best benchmark, which suggests that a minimum capital ratio does a poor job in terms of approaching the Second Best outcomes.

4.2 A tax on dividends

We now consider the setting in which the social planner imposes a tax $\tau$ on bank dividends. The only change induced by the implementation of a dividend tax compared to the unregulated set-up considered in Section 3.1 is that the marginal value of dividends reduces to $1 - \tau$. As a result, at the point where it is optimal to distribute dividends, the market-to-book value of aggregate bank capital must satisfy

$$u(E_{max}^\tau) = 1 - \tau.$$  

---

42 The details on the computation of the ergodic density in the regulated equilibrium can be found in the Appendix B.2.

43 Alternatively, $\tau$ can be thought of as a tax on the managers’ bonuses.
Figure 5: Minimum capital ratio and cumulative distribution of $R$

Notes: this figure illustrates the impact of minimum capital ratios on the cumulative distribution of the loan rate $R$ (black dashed lines). The Second Best cumulative distribution of $R$ is represented by the red solid line, whereas the blue solid line refers to the case $\Lambda = 0$. Parameter values used: $\rho = 0.05$, $\sigma_0 = 0.05$, $p = 0.02$, $R = 0.07$, $\gamma = 0.2$, $\varphi = 0.3$, $D(R) = R - R$.

An immediate adaptation of the results stated in Proposition 1 shows that the above condition leads to the change of the support for both the loan rate and aggregate capitalization, since now the numbers $R_{\tau}^{\tau_{\tau}}$ and $E_{\tau_{\tau}}^{\tau_{\tau}}$ are determined by

$$\int_{p}^{R_{\tau}^{\tau_{\tau}}} \frac{(R - p)H(R)}{\sigma^2 D(R)} dR = \log \left( \frac{1 + \gamma}{1 - \tau} \right) \quad \text{and} \quad E_{\tau_{\tau}}^{\tau_{\tau}} = \int_{p}^{R_{\tau}^{\tau_{\tau}}} H(R) dR.$$ \hspace{1cm} (37)

It is easy to see from (37) that increasing the tax rate $\tau$ would induce banks to postpone dividend payments, by building larger target capital buffers.\footnote{All other properties of the competitive equilibrium that we discussed in Section 1 (namely, a lending premium satisfying Equation (17), recapitalizations at zero equity level) remain intact.} To get intuition for this effect, note that the optimal choice of the target capital buffer $E_{\tau_{\tau}}^{\tau_{\tau}}$ is driven by the trade-off between the expected consumption and the expected recapitalization costs. On the one hand, bank shareholders want to consume more, which can be achieved by lowering the target capital buffer. On the other hand, they want to reduce the frequency of costly recapitalizations, which can be achieved by building the higher capital buffer. A higher dividend tax reduces the marginal utility from consumption, thereby, weakening the first effect. As a result, bankers’ effective risk-aversion increases in the tax rate (cf. the right panel of Figure 6), inducing banks to build higher capital buffers. At the same time, higher effective risk aversion drives up the lending premium, which ultimately leads to higher loan rates and lower volumes of lending (cf. Figure 7). This mechanism suggests that taxing dividends could potentially address both inefficiencies identified in Section 3.3.

To illustrate the welfare effect of a dividend tax, we plot the cumulative densities
Figure 6: Taxes on dividends, market-to-book value and effective risk aversion

Notes: this figure reports the patterns of the market-to-book values $u(E)$ and the effective risk aversion $-u'(E)/u(E)$ emerging under different levels of dividend taxes (black dashed curves), contrasting them to those arising in the unregulated (blue solid line) set-up. Parameter values: $\rho = 0.05, \sigma_0 = 0.05, p = 0.02, \bar{R} = 0.07, \gamma = 0.2, \varphi = 0.3, D(R) = \bar{R} - R$.

Figure 7: Taxes on dividends and lending

Notes: this figure reports the levels of aggregate lending and the loan rate emerging under different levels of a dividend tax (black dashed curves). They are contrasted to those arising in the unregulated (blue solid line) and the Second Best (red solid line) set-ups. Parameter values: $\rho = 0.05, \sigma_0 = 0.05, p = 0.02, \bar{R} = 0.07, \gamma = 0.2, \varphi = 0.3, D(R) = \bar{R} - R$.

of the equilibrium lending rate for different levels of $\tau$ in Figure 8, contrasting them to the Second Best outcome. This shows that raising the dividend tax rate leads to higher lending rates in the sense of first-order stochastic dominance, which suggests that the Second Best asymptotic behavior can be approached by implementing the intermediary levels of a dividend tax.

5 Conclusion

This paper develops a general equilibrium model of commercial banking, in which banks satisfy households’ needs for safe deposits and channel funds to the productive sector. Bank capital plays the role of a loss-absorbing buffer that insulates banks from the need to undertake costly recapitalizations too often. In our model, the aggregate level of bank capitalization drives the cost and the volume of lending. Specifically, we
Figure 8: Taxes on dividends and cumulative distribution of $R$

Notes: this figure depicts the cumulative distribution of the loan rate for different levels of a dividend tax (black dashed lines). The Second Best cumulative distribution of $R$ is represented by the red solid line, whereas the blue solid line refers to the case $\tau = 0$. Parameter values used: $\rho = 0.05$, $\sigma_0 = 0.05$, $p = 0.02$, $\bar{R} = 0.07$, $\gamma = 0.2$, $\varphi = 0.3$, $D(R) = \bar{R} - R$.

establish a negative relation between the equilibrium loan rate and the level of aggregate bank capital. The closed-form characterization of the equilibrium dynamics of loan rates enables us to study analytically the long-run behavior of the economy. We show that this behavior is ergodic and is essentially determined by the volatility of the loan rate and the magnitude of financing frictions. The economy never spends a lot of time at the deterministic steady state and, under severe financing frictions, may spend quite a lot of time in a credit crunch regime.

We show that competitive equilibrium is constrained inefficient: competitive banks do not internalize the impact of their lending decisions on the loan rate and, as a result, lend too much as compared to the social optimum. Moreover, since bank shareholders do not internalize the social costs of bank distress, they hold too little equity as compared to the social planner's choice. We then investigate the effects of two regulatory tools, a minimum capital ratio and a dividend tax, and show that the latter is more efficient in dealing with the distortions inherent in the competitive equilibrium.

Our model suffers from two important limitations. First, it only considers commercial banking activities (deposit taking and lending), while neglecting market activities such as securities and derivatives trading. Second, it only considers diffusion risks that do not lead to actual bank defaults, but merely fluctuations in the size of the banking sector. A consequence of these limitations is that we cannot address the important questions of banks' excessive risk-taking and the role of capital regulation in the mitigation of this behavior, which have already been the subject of a large academic literature.
Appendix A. Proofs

Proof of Proposition 1. By the standard dynamic programming arguments, shareholder value \( v(e, E) \) must satisfy the Bellman equation:\(^{45}\)

\[
\rho v = \max_{k \geq 0, d \delta \geq 0, d_i \geq 0} \left\{ \delta \delta (1 - v_e) - d_i (1 + \gamma - v_e) + \right. \\
+ k [(R(E) - p) v_e + \sigma_0^2 K(E) v_e E] + \frac{k^2 \sigma_0^2}{2} v_{ee} \\
+ K(E) (R(E) - p) v_E + \frac{\sigma_0^2 K^2(E)}{2} v_{EE} \right\}. \\
\]

(A1)

Using the fact that \( v(e, E) = e u(E) \), one can rewrite the Bellman equation (A1) as follows:

\[
\rho u(E) = \max_{k \geq 0, d \delta \geq 0, d_i \geq 0} \left\{ \frac{d \delta}{e} [1 - u(E)] - \frac{d_i}{e} [1 + \gamma - u(E)] + \frac{k}{e} [(R(E) - p) u(E) + \sigma_0^2 K(E) u'(E)] + \\
+ K(E) (R(E) - p) u'(E) + \frac{\sigma_0^2 K^2(E)}{2} u''(E) \right\}. \\
\]

(A2)

A solution to the maximization problem in \( k \) only exists when

\[
\frac{u'(E)}{u(E)} \leq - \frac{R(E) - p}{\sigma_0^2 K(E)}, \\
\]

with strict equality when \( k > 0 \).

Under conjecture that \( R(E) \geq p \) (which will be verified ex-post), it follows from the above expression that \( u(E) \) is a decreasing function of \( E \). Then, the optimal payout policy maximizing the right-hand side of (A2) is characterized by a critical barrier \( E_{\text{max}} \) satisfying

\[
u(E_{\text{max}}) = 1, \quad \text{(A4)}
\]

and the optimal recapitalization policy is characterized by a barrier \( E_{\text{min}} \) such that

\[
u(E_{\text{min}}) = 1 + \gamma. \quad \text{(A5)}
\]

In other words, dividends are only distributed when \( E_t \) reaches \( E_{\text{max}} \), whereas recapitalization occurs only when \( E_t \) reaches \( E_{\text{min}} \). Given (A3), (A4), (A5) and \( k > 0 \), it is easy to see that, in the region \( E \in (E_{\text{min}}, E_{\text{max}}) \), market-to-book value \( u(E) \) satisfies:

\[
\rho u(E) = K(E) (R(E) - p) u'(E) + \frac{\sigma_0^2 K^2(E)}{2} u''(E). \quad \text{(A6)}
\]

Note that, at equilibrium, \( K(E) = D[R(E)] \). Taking the first derivative of (A3), we

\begin{footnote}
For the sake of space, we omit the arguments of function \( v(e, E) \).
\end{footnote}
can compute $u''(E)$. Inserting $u''(E)$ and $u'(E)$ into (A6) and rearranging terms yields:

$$R'(E) = -\frac{1}{\sigma_0^2} \frac{2\rho \alpha^2 + (R(E) - p)^2}{D[R(E)] - [R(E) - p]D'[R(E)]}. \tag{A7}$$

Since $D'(R(E)) < 0$, it is clear that $R'(E) < 0$ if $R(E) > p$. To verify that $R(E) > p$ for any $E \in [E_{\min}, E_{\max})$, it is sufficient to show that $R_{\min} \equiv R(E_{\max}) \geq p$.

To obtain $R_{\min}$, let $V(E) \equiv Eu(E)$ denote the market value of the entire banking sector. At equilibrium, one must have $V'(E_{\max}) = 1$ and $V'(E_{\min}) = 1 + \gamma$. Given that $V'(E) = u(E) + Eu'(E)$, it must hold that $Eu'(E) = 0$. Hence, $u'(E_{\max}) = 0$ and $E_{\min} = 0$. Inserting $u'(E_{\max}) = 0$ into the binding condition (A3) immediately shows that $R_{\min} = p$, so that $R(E) > p$ for any $E \in [E_{\min}, E_{\max})$.

Hence, the loan rate $R(E)$ can be computed as a solution to the differential equation (A7), which yields:

$$\int_0^E R'(s)ds = R(E) - R_{\max}, \tag{A8}$$

where $R_{\max} \equiv R(0)$.

To obtain $E_{\max}$, we use the fact that individual banks’ optimization with respect to the recapitalization policy implies $u(E_{\min}) = 1 + \gamma$. Integrating equation (A3) in between $E_{\min} = 0$ and $E_{\max}$, while using $u(E_{\max}) = 1$, yields an equation that implicitly determines $E_{\max}$:

$$u(E_{\max}) \exp \left( \int_0^{E_{\max}} \frac{R(E) - p}{\sigma_0^2 D[R(E)]} dE \right) = 1 + \gamma. \tag{A9}$$

The system of equations (19) and (20) immediately follows from the change of variable of integration in equations (A8) and (A9), i.e., $dR = R'(E)dE$.

**Proof of Proposition 2.** Omitted.

**Proof of Proposition 3.** Consider the process $R_t$ that evolves according to

$$dR_t = \mu(R_t)dt + \sigma(R_t)dZ_t, \quad p \leq R_t \leq R_{\max}, \tag{A10}$$

with reflections at both ends of the support.

Let $g(t, R)$ denote the probability density function of $R_t$. It must satisfy the forward Kolmogorov equation:

$$\frac{\partial g(t, R)}{\partial t} = -\frac{\partial}{\partial R} \left\{ \mu(R)g(t, R) - \frac{1}{2} \frac{\partial}{\partial R} \left[ \sigma^2(R)g(t, R) \right] \right\}. \tag{A11}$$

Since the process $R_t$ is stationary, we have $\frac{\partial g(t, R)}{\partial t} = 0$. Integrating Equation (A11) over $R$ yields:

$$\mu(R)g(R) = \frac{1}{2} \frac{\partial}{\partial R} \left[ \sigma^2(R)g(t, R) \right],$$

where the constant of integration is set to zero because of reflection properties of the
process. Solving the above equation by using the change of variable \( \hat{g}(R) = \sigma^2(R)g(R) \) ultimately yields:

\[
g(R) = \frac{C_0}{\sigma^2(R)} \exp\left(\int_p^{R_{\text{max}}} \frac{2\mu(s)}{\sigma^2(s)} ds\right), \tag{A12}
\]

where the constant \( C_0 \) is chosen so as to normalize the solution to 1 over the region \([p, R_{\text{max}}]\), i.e., \( \int_p^{R_{\text{max}}} g(R) dR = 1 \).

To ensure that the distribution of \( R \) is non-degenerate, it is sufficient to check that \( \sigma(R) > 0 \) for any \( R \in [p, R_{\text{max}}] \). From the expression of \( \sigma(R) \), it is easy to see that this condition holds for any loan demand specifications such that \( D'(R) < 0 \) and \( D(R) > 0 \).

**Proof of Proposition 4.** Note first that at \( E_{\text{max}} \) we have \( R(E_{\text{max}}) = p \). Hence, for \( E = E_{\text{max}} \), the ODE (30) can be rewritten as follows:

\[
\rho W(E_{\text{max}}) = \pi_F(D(p)) + \frac{\sigma_0^2[D(p)]^2}{2} W''(E_{\text{max}}).
\]

Observe that welfare can not exceed the value of the first best output plus the value of the banking sector net of the social costs of banking distress, i.e.,

\[
W(E_{\text{max}}) \leq \frac{\pi_F(D(p))}{\rho} + E_{\text{max}} - \varphi E\left[\int_0^{+\infty} e^{-\rho t} dI_t\right].
\]

It is immediate that \( \partial W(E_{\text{max}})/\varphi < 0 \). Therefore, for sufficiently large \( \varphi \), it holds that

\[
W(E_{\text{max}}) \leq \frac{\pi_F(D(p))}{\rho}
\]

and thus \( W''(E_{\text{max}}) < 0 \).

**Appendix B. Computing social welfare**

Consider the simple case where the credit demand is linear, i.e., \( D(R) = \overline{R} - R \).

**B.1. Social welfare in the competitive equilibrium**

Given the linear specification of the credit demand function, the loan rate \( R(E) \) can be computed in a closed form:

\[
R(E) = p + \sqrt{2\rho} \sigma_0 \tan \left(\frac{\sqrt{2\rho}}{\sigma_0(\overline{R} - p)}(E_{\text{max}} - E)\right), \tag{A13}
\]

and thus

\[
K(E) = \overline{R} - p - \sqrt{2\rho} \sigma_0 \tan \left(\frac{\sqrt{2\rho}}{\sigma_0(\overline{R} - p)}(E_{\text{max}} - E)\right). \tag{A14}
\]
To recover the production function, $F(D(R))$, recall that $F'(D(R)) = R$. Using the
fact that $R = \bar{R} - D$, we obtain $F'(D) = (\bar{R} - D)$ and, thereby,

$$F(D(R)) = \bar{R}D(R) - \frac{|D(R)|^2}{2}. \quad (A15)$$

At equilibrium, we have $D(R(E)) = K(E)$, so the that firm’s expected profit is given by

$$\pi_F(K(E)) = F(K(E)) - K(E)F'(K(E)) = \frac{|K(E)|^2}{2}. \quad (A15)$$

Then, social welfare follows ODE:

$$\rho W(E) = \frac{|K(E)|^2}{2} + K(E)(\bar{R} - K(E) - p)W'(E) + \frac{\sigma_0^2}{2}|K(E)|^2 W''(E), \quad (A16)$$
given that $W'(0) = 1 + \gamma + \varphi$ and $W'(E_{\max}) = 1$.

Differentiating the above expression with respect to $E$ and solving the obtained equation numerically with respect to $W'(E)$ ultimately enables us to uncover $W(E)$.

### B.2. Social welfare in the Second Best

Assume now that a social planner can optimally choose the target equity level $E_{\max}$ and lending volumes $K(E)$ so as to maximize social welfare. Note that searching for the socially optimal $K(E)$ is equivalent to searching for the socially optimal $R(E)$. Inserting $K(E) = \bar{R} - R(E)$ into (A16) yields:

$$\rho W(E) = \frac{|R(E)|^2}{2} + K(E)(\bar{R} - K(E) - p)W'(E) + \frac{\sigma_0^2}{2}|K(E)|^2 W''(E). \quad (A17)$$

The first-order condition with respect to $R(E)$ implies:

$$R^{sp}(E) = \bar{R} - \frac{(\bar{R} - p)W'(E)}{2W'(E) - \sigma_0^2 W''(E) - 1}. \quad (A18)$$

Substituting $R(E)$ in Equation (A17), by $R^{sp}(E)$ yields a simple second-order differential equation:

$$W''(E) = \frac{2\rho W(E)[2W'(E) - 1] - (\bar{R} - p)^2[W'(E)]^2}{2\rho \sigma_0^2 W(E)}, \quad (A19)$$

that can be solved numerically under the boundary conditions $W'(E_{\max}^{sp}) = 1$ and $W'(0) = 1 + \gamma + \varphi$. The free boundary $E_{\max}^{sp}$ satisfies a super-contact condition $W''(E_{\max}^{sp}) = 0$.

\footnote{It is easy to see from (A18), at the target level of aggregate bank equity, we have $R^{sp}(E_{\max}) = p$.}
Appendix C. Solving for the regulated equilibrium

C.1. Solving for the regulated equilibrium

Consider the shareholders’ maximization problem stated in (33). By the standard dynamic programming arguments and the fact that \( v(\nu, E) = \nu u(\nu, E) \), the optimal bank’s policies must satisfy the following Bellman equation:

\[
\rho u(\nu, E) = \max_{\nu \geq 0, \nu \geq 0} \left\{ \frac{d \delta}{e} [1 - u(\nu, E)] - \frac{d \delta}{e} [1 + \gamma - u(\nu, E)] \right\} + \max_{0 < k \leq \epsilon / \Lambda} \left\{ \frac{k}{e} \left[ (R(E) - p) u(\nu, E) + \sigma_0^2 K(E) u'(\nu, E) \right] \right\} + K(E) [R(E) - p] u'(\nu, E) + \frac{\sigma_0^2 K^2(E)}{2} u''(\nu, E). \tag{A20}
\]

The solution to (A20) exists only if

\[
B(E) := \frac{R(E) - p}{\sigma_0^2 K(E)} \geq -\frac{u'(\nu, E)}{u(\nu, E)}, \tag{A21}
\]

with strict equality for \( 0 < k < E/\Lambda \).

The optimal dividend and recapitalization policies are characterized by the barriers \( E_{\text{max}}^\Lambda \) and \( E_{\text{min}}^\Lambda \) such that \( u(\nu, E_{\text{max}}^\Lambda) = 1 \) and \( u(\nu, E_{\text{min}}^\Lambda) = 1 + \gamma \). Moreover, by the same reason than in the unregulated equilibrium, it must hold that \( E_{\text{max}}^\Lambda u'(\nu, E_{\text{min}}^\Lambda) = 0 \) and \( E_{\text{min}} u'(\nu, E_{\text{min}}^\Lambda) = 0 \). This implies \( u'(\nu, E_{\text{max}}^\Lambda) = 0 \) and \( u'(\nu, E_{\text{min}}^\Lambda) = 0 \).

Under the conjecture that there exists some critical threshold \( E_e^\Lambda \) such that the regulatory constraint is binding for \( E \in [E_{\text{min}}^\Lambda, E_{\text{max}}^\Lambda] \) and is slack for \( E \in [E_e^\Lambda, E_{\text{max}}^\Lambda] \), let

\[
\alpha_i(E) = -\frac{u'(\nu, E)}{u(\nu, E)},
\]

where \( i = 1 \) for \( E \in (E_e^\Lambda, E_{\text{max}}^\Lambda] \) and \( i = 2 \) for \( E \in [E_{\text{min}}^\Lambda, E_e^\Lambda] \).

Then, in the region \( (E_{\text{min}}^\Lambda, E_{\text{max}}^\Lambda) \), equation (A20) can be rewritten as a system of two first-order equations:

\[
\begin{align*}
\rho &= -\pi_B(E) \alpha_2(E) + \frac{\sigma_0^2 K^2(E)}{2} [\alpha_2^2(E) - \alpha_2'(E)] + \frac{[R(E) - p - \sigma_0^2 K(E) \alpha_2(E)]}{\Lambda}, \quad E \in [E_{\text{min}}^\Lambda, E_e^\Lambda], \tag{A22} \\
\rho &= -\pi_B(E) \alpha_1(E) + \frac{\sigma_0^2 K^2(E)}{2} [\alpha_1^2(E) - \alpha_1'(E)], \quad E \in (E_e^\Lambda, E_{\text{max}}^\Lambda], \tag{A23}
\end{align*}
\]

where \( \pi_B(E) \) denotes the aggregate expected profit of banks:

\[
\pi_B(E) = K(E) [R(E) - p].
\]

\[\text{Note that in the unregulated equilibrium, we had } E_{\text{min}} = 0. \text{ Under capital regulation, this is no longer possible, since } E_{\text{min}} = \Lambda K(E_{\text{min}}).\]

\[\text{Note that } u(\nu, E) \text{ must be continuous, but not necessarily its derivative. Hence, we do not impose continuity of } \alpha_i(E) \text{ at } E_e^\Lambda.\]
the volume of credit $K(E)$ satisfies
\[
K(E) = \begin{cases} 
\frac{E}{\Lambda}, & E \in [E_{\text{min}}^{\Lambda}, E_{c}^{\Lambda}] \\
D[R(E)], & E \in (E_{c}^{\Lambda}, E_{\text{max}}^{\Lambda}], 
\end{cases}
\]
and the loan rate $R(E)$ is given by:
\[
R(E) = \begin{cases} 
D^{-1}[E/\Lambda], & E \in [E_{\text{min}}^{\Lambda}, E_{c}^{\Lambda}] \\
R'(E) = -1/H(R(E)), & R(E_{c}^{\Lambda}) = D^{-1}[E_{c}^{\Lambda}/\Lambda], & E \in (E_{c}^{\Lambda}, E_{\text{max}}^{\Lambda}], 
\end{cases}
\]
where $D^{-1}$ is the inverse function of the demand for credit and $H(.)$ is defined in Proposition 1.

The critical threshold $E_{c}^{\Lambda}$ must satisfy equation $\alpha_{2}(E_{c}^{\Lambda}) = B(E_{c}^{\Lambda})$.

If $\alpha_{2}(E) < B(E)$ for any $E \in (E_{\text{min}}^{\Lambda}, E_{\text{max}}^{\Lambda})$, then the regulatory constraint is always binding and the ODE (A20) translates into
\[
\rho = -\pi_{B}(E)\alpha_{2}(E) + \frac{\sigma_{2}^{2}K^{2}(E)}{2} \left[ \alpha_{2}^{2}(E) - \alpha_{2}'(E) \right] + \frac{[R(E) - p - \sigma_{0}^{2}K(E)\alpha_{2}(E)]}{\Lambda}, & E \in [E_{\text{min}}^{\Lambda}, E_{\text{max}}^{\Lambda}]. 
\]
(A24)

The condition $u'_{\Lambda}(E_{\text{min}}^{\Lambda}) = 0$ yields the boundary condition $\alpha_{2}(E_{\text{min}}^{\Lambda}) = 0$. Similarly, the condition $u'_{\Lambda}(E_{\text{max}}^{\Lambda}) = 0$ translates into the boundary condition $\alpha_{1}(E_{\text{max}}^{\Lambda}) = 0$, when $E_{c}^{\Lambda} \in (E_{\text{min}}^{\Lambda}, E_{\text{max}}^{\Lambda})$, or $\alpha_{2}(E_{\text{max}}^{\Lambda}) = 0$, when $E_{c}^{\Lambda}$ does not exist (i.e., the regulatory constraint is always binding).

**Numerical procedure to solve for the regulated equilibrium.** This numerical algorithm solving for the regulated equilibrium can easily be implemented with the Mathematica software. $\Lambda$ is taken as a parameter.

- Pick a candidate value $\hat{E}_{\text{min}}^{\Lambda}$.
- Solve ODE (A24) for $\alpha_{2}(E)$ under the boundary condition $\alpha_{2}(\hat{E}_{\text{min}}^{\Lambda}) = 0$.
- Assume that the regulatory constraint always binds and compute a candidate value $\hat{E}_{\text{max}}^{\Lambda}$ such that satisfies equation $\alpha_{2}(\hat{E}_{\text{max}}^{\Lambda}) = 0$.
- Check whether $\alpha_{2}(\hat{E}_{\text{max}}^{\Lambda}) \leq B(\hat{E}_{\text{max}}^{\Lambda})$, which is equivalent to
\[
\frac{D^{-1}[\hat{E}_{\text{max}}^{\Lambda}/\Lambda] - p}{\sigma_{0}^{2}\hat{E}_{\text{max}}^{\Lambda}/\Lambda} \geq 0
\]
(A25)
- Conditional on the results of the above verification, one of the two scenarios is possible:
  a) if (A25) holds, then the regulatory constraint is always binding for a given $\Lambda$ and the market-to-book value $u_{\Lambda}(\hat{E}_{\text{min}}^{\Lambda})$ can be computed according to
\[
u_{\Lambda}(\hat{E}_{\text{min}}^{\Lambda}) = u_{\Lambda}(\hat{E}_{\text{max}}^{\Lambda}) \exp \left( \int_{\hat{E}_{\text{min}}^{\Lambda}}^{\hat{E}_{\text{max}}^{\Lambda}} \alpha_{2}(E)dE \right) = \exp \left( \int_{\hat{E}_{\text{min}}^{\Lambda}}^{\hat{E}_{\text{max}}^{\Lambda}} \alpha_{2}(E)dE \right).
\]
b) If (A25) is violated, solve equation $\alpha_2(\hat{E}_c^\Lambda) = B(\hat{E}_c^\Lambda)$ for the critical level of equity $\hat{E}_c^\Lambda$ above which the regulatory constraint is slack. Compute the market-to-book value $u_{\Lambda}(E_{\text{min}}^\Lambda)$ according to:

$$u_{\Lambda}(E_{\text{min}}^\Lambda) = A_0 \exp\left(\int_{E_{\text{min}}^\Lambda}^{E_{\text{max}}^\Lambda} \alpha_1(E)dE\right),$$

where

$$A_0 = \exp\left(\int_{E_{\text{min}}^\Lambda}^{E_{\text{max}}^\Lambda} \alpha_1(E)dE\right), \quad \text{where} \quad \alpha_1(E) = \frac{R(E) - p}{\sigma_0^2 K(E)}.$$ 

- If $u_{\Lambda}(\hat{E}_{\text{min}}^\Lambda) = 1 + \gamma$, then $E_{\text{min}}^\Lambda = \hat{E}_{\text{min}}^\Lambda$, $E_{\text{max}}^\Lambda = \hat{E}_{\text{max}}^\Lambda$ and $E_c^\Lambda = \hat{E}_c^\Lambda$ (if exists). Otherwise, pick a different $\hat{E}_{\text{min}}^\Lambda$, repeat the procedure from the beginning.

C.2. Ergodic density function in the regulated equilibrium

Consider first the case when $\Lambda$ is sufficiently high, so that the regulatory constraint is binding for any level of $E$. It is easy to show that, in this case, the dynamics of the loan rate is characterized by the process\(^{19}\)

$$dR_t = \mu_{\Lambda}(R_t)dt + \sigma_{\Lambda}(R_t)dZ_t, \quad R_{\text{min}}^\Lambda \leq R_t \leq R_{\text{max}}^\Lambda, \quad (A26)$$

where

$$\sigma_{\Lambda}(R) = -\frac{\sigma_0 D(R)}{\Lambda D'(R)}, \quad (A27)$$

$$\mu_{\Lambda}(R) = -\sigma_{\Lambda}(R) \left( \frac{R - p}{\sigma_0} + \frac{\sigma_{\Lambda}(R) D''(R)}{2 D'(R)} \right). \quad (A28)$$

In our numerical simulations, we stick to the following specification of the credit demand function: $D(R) = (\bar{R} - R)^\beta$, where $\beta > 0$ and $\bar{R} > p$. Under this specification, we obtain

$$\sigma_{\Lambda}(R) = \frac{(\bar{R} - R)\sigma_0}{\beta \Lambda}, \quad (A29)$$

$$\mu_{\Lambda}(R) = -\sigma_{\Lambda}(R) \left( \frac{R - p}{\sigma_0} + \frac{(1 - \beta)\sigma_0}{2\beta \Lambda} \right). \quad (A30)$$

Let

$$\phi_0 := \frac{\beta^2 \Lambda^2}{\sigma_0^2}, \quad \phi_1 := \frac{1}{\sigma_0^2} \left[ (1 - \beta)\sigma_0^2 + 2\beta \Lambda(\bar{R} - p) \right] - 2, \quad \phi_2 := \frac{2\beta \Lambda}{\sigma_0^2}. \quad (34)$$

\(^{19}\)Note that, one can alternatively use $R$ as a state variable, while looking for the unique mapping $E(R)$. In a Markov Equilibrium, one must have $-\sigma_0 K(R) = \sigma_{\Lambda}(R) E'(R)$ and $(R - p)K(R) = \mu_{\Lambda}(R) E'(R) + \sigma_{\Lambda}^2(R)/2E''(R)$. Using the fact that, under the binding regulatory constraint, $E(R) = \Lambda K(R)$ and, at equilibrium, $K(R) = D(R)$, we immediately obtain Expressions (A27) and (A28).
The ergodic density function of $R$ is then given by\footnote{The general formula for the ergodic density function in the regulated set-up is similar to the one stated in Proposition 3. After simplification, it can be expressed as in (A31).}

$$g_{\Lambda}(R) = C_{\Lambda} \phi_0(R - R)^{\phi_1} \exp\left(\phi_2 R\right), \quad (A31)$$

where the constant $C_{\Lambda}$ is such that $\int_{R_{min}^{\Lambda}}^{R_{max}^{\Lambda}} g_{\Lambda}(R) dR = 1$.

For the relatively low levels of $\Lambda$, the ergodic density function is discontinuous at $R_c$ and can be computed according to the following formula:

$$g_{\Lambda}(R) = \begin{cases} \frac{1}{\sigma(R)} \exp\left(\int \frac{\mu(R)}{\sigma(R)} dR\right), & R \in [R_{min}^{\Lambda}, R_c], \\ \phi_0(R - R)^{\phi_1} \exp\left(\phi_2 R\right), & R \in [R_{max}^{\Lambda}, R_c], \end{cases}$$

where $C_{\Lambda}$ is such that $\int_{R_{min}^{\Lambda}}^{R_{max}^{\Lambda}} g_{\Lambda}(R) dR = 1$, and $\mu(R)$ and $\sigma(R)$ are given in (23) and (24), respectively.

**Appendix D. Competitive equilibrium with $r > 0$**

In this appendix, we solve for the competitive equilibrium in the set up where $r > 0$. In this case, the dynamics of equity value of an individual bank follows:

$$de_t = re_t dt + k_t [(R(E_t) - p - r) dt - \sigma_0 dZ_t] - \delta_t + d\bar{i}_t. \quad (A32)$$

The aggregate equity of the banking sector evolves according to:

$$dE_t = [K(E_t)(R(E_t) - p - r) + rE_t] dt - \sigma_0 K(E_t) dZ_t - d\Delta_t + dI_t. \quad (A33)$$

Solving the shareholders’ maximization problem in the same way as we did in the proof of Proposition 1 and allowing for $k > 0$ yields us two equations:

$$\frac{u'(E)}{u(E)} = -\frac{R(E) - p - r}{\sigma_0^2 K(E)}, \quad (A34)$$

$$(\rho - r)u(E) = K(E)(R(E) - p - r)u'(E) + \frac{\sigma_0^2 K^2(E)}{2} u''(E). \quad (A35)$$

Substituting $u'(E)$ and $u''(E)$ into (A35), while allowing for $K(E) = D[R(E)]$, enables us to express $R'(E)$:

$$R'(E) = -\frac{1}{\sigma_0^2} \frac{2(\rho - r)\sigma_0^2 + (R(E) - p - r)^2 + 2(R(E) - p - r)E/[D[R(E)]]}{\left(D[R(E)] - [R(E) - p - r][D'[R(E)]]\right)} \quad (A36)$$

Applying the same arguments as in the setting with $r = 0$, we can show that $E_{min} = 0$.
and \( R_{\min} = r + p \). The boundary \( R_{\max} \) must be computed numerically by solving equation

\[
\int_{p}^{R_{\max}} E'(R) \frac{(R - p - r)}{\sigma_0^2 D(R)} dR = \log(1 + \gamma), \tag{A37}
\]

where \( E'(R) = 1/R' E \). Note that the left-hand side of the above expression is increasing in \( R_{\max} \). Hence, there exists a unique solution to (A37), which guarantees the uniqueness of the equilibrium.

### Appendix E. Impulse response analysis

In this Appendix we apply the impulse response methodology to study the stability of the deterministic steady state. Such an exercise enables us to show that the system behavior in a stochastic environment can be in sharp contrast to the behavior predicted by the impulse response analysis.

The usual methodology to analyze the long-term behavior of macro-variables in a DSGE model is to linearize it around the deterministic steady-state and perturb the system by a single unanticipated shock. The equivalent here would be to look at the case where \( dZ_t = 0 \) for \( t > 0 \). The dynamics of the system then becomes deterministic and can be described by the ordinary differential equation (linearization is not needed here):

\[
dR_t = \mu(R_t) dt,
\]

where the initial shock determines \( R_0 > p \).

It is easy to see from expression (24) that \( \mu(p) = 0 \). Hence, the frictionless loan rate \( R_t = p \) is an equilibrium of the deterministic system that is further referred to as the deterministic steady-state (DSS). It is locally stable when \( \mu'(p) < 0 \) and is globally stable when \( \mu(R) < 0 \) for all \( R \). After some computations, it can be shown that

\[
\mu'(p) = 2\rho^2 \sigma_0^2 D''(p) D(p).
\]

Hence, the DSS is locally stable when \( D''(p) < 0 \). Moreover, it also follows from (25), that condition \( D''(R) < 0 \) ensures global stability.

**Illustrative example.** To illustrate the properties of the equilibrium, consider the following specification of the loan demand function:

\[
D(R) = (\bar{R} - R)^\beta, \tag{A38}
\]

where \( \beta > 0 \) and \( p < \bar{R} \).

Under the above specifications, the volatility of the loan rate is

\[
\sigma(R) = \frac{[2\rho \sigma_0^2 + (R-p)^2] (\bar{R}-R)}{\sigma_0 \bar{R} + (\beta-1)R - \beta p}. \tag{A39}
\]
The drift of the loan rate is given by

\[ \mu(R) = \sigma(R) \frac{\beta(R - p)Q(R)}{2\sigma_0[\bar{R} + (\beta - 1)R - \beta p]^2}, \]  

where \( Q(R) \) is a quadratic polynomial:

\[ Q(R) = (1 - \beta)((R - p)^2 - 2\rho \sigma_0^2) - 2(R - p)(\bar{R} - p). \]

Given the above specification, it can be easily shown that, when \( \beta < 1 \) (which is equivalent to \( D''(R) < 0 \)), \( \mu'(p) < 0 \) and \( \mu(R) < 0 \) in the entire interval \([p, \bar{R}]\). Thus, the DSS is locally and globally stable. By contrast, when \( \beta > 1 \) (which is equivalent to \( D''(R) > 0 \)), the DSS is locally unstable, i.e., \( \mu'(p) > 0 \), and there exists a unique \( R^* \in (p, \bar{R}) \) such that \( \mu(R) \) is positive in the region \((0, R^*)\) and negative in the region \((R^*, \bar{R})\) (left panel of Figure 9).

![Figure 9: The loan rate drift and volatility, \( \beta > 1 \).](image)
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