

Secondary Market Liquidity and the Optimal Capital Structure*

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Abstract

We present a model in which endogenous liquidity generates a feedback loop between secondary market liquidity and firm's financing decisions in primary markets. The model features two key frictions: a costly state verification problem in primary markets, and search frictions in over-the-counter secondary markets. Our concept of liquidity depends endogenously on illiquid assets put up for sale by impatient investors relative to the resources available for buying those assets. The endogenous nature of liquidity creates a feedback loop as issuance in primary markets affects secondary market liquidity, and vice versa through liquidity premia. We show that the privately optimal allocations are inefficient because investors and firms fail to internalize how their respective behavior affects secondary market liquidity. These inefficiencies are established analytically through a set of wedge expressions for key efficiency margins. Our analysis provides a rationale for the effect of quantitative easing on secondary and primary capital markets and the real economy.

Keywords: Market liquidity, secondary markets, capital structure, quantitative easing.

JEL classification: E44, G18, G30.

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1 Introduction

Secondary market liquidity is an important consideration for investors buying long-term assets. At the same time, the issuance of long-term debt in primary markets affects market liquidity because it alters the maturity composition of investor's portfolios. The interaction between primary debt markets and secondary market liquidity is important for understanding the real effects of financial market imperfections. For example, how does debt issuance in primary markets affect the liquidity in secondary markets? How does investor's demand for compensation to bear liquidity risk affect the firm's incentive to issue debt in the primary market? Does the interaction between these two channels lead to an efficient capital structure of the firm? What is the effect of quantitative easing on the real economy through intervention in either the primary or secondary market?

This paper presents a model to formalize the interaction between primary and secondary capital markets in order to shed light on these questions. In particular, we are interested in imperfect secondary trading that gives rise to liquidity risk, as liquidity needs of investors can not be met by selling assets frictionlessly in secondary markets.

We make three main contributions. First, we uncover a novel feedback loop, illustrated in Figure 1, between secondary market liquidity and the firm's financing decision in primary capital markets. This feedback loop allows for liquidity risk associated with trade in the secondary market to influence firm's financing decisions through funding costs.¹ This direct channel has received considerable attention in the literature as it is closely related to the idea of transaction or information costs impeding trading, as well to the lending channel of monetary policy. But, what is different about our framework is that we also capture an additional channel whereby the firm's financing decisions in the primary market feed back into the determination of liquidity in the secondary market. This happens both directly through the supply of long-term assets as well as indirectly by altering the composition of investor portfolios. This link between primary issuance and secondary market liquidity has not been studied in the literature, but it is key to understand how the liability structure of firms matters for the optimal intermediation of liquidity risk and the real economy. We prove the existence and uniqueness of an equilibrium featuring this feedback loop, and fully characterize its properties.

The second main contribution is to show that this feedback loop distorts capital

¹In a seminal paper, Holmström and Tirole (1998) study a similar question to ours, but focus on the liquidity needs of firms to cover operational costs before their investment matures. In contrast, we focus on the liquidity demand of lenders. To this extent, we model the demand for liquidity as in the seminal paper of Diamond and Dybvig (1983), but bring re-trading of long-term assets, aggregate liquidity and the capital structure to the center of our analysis.

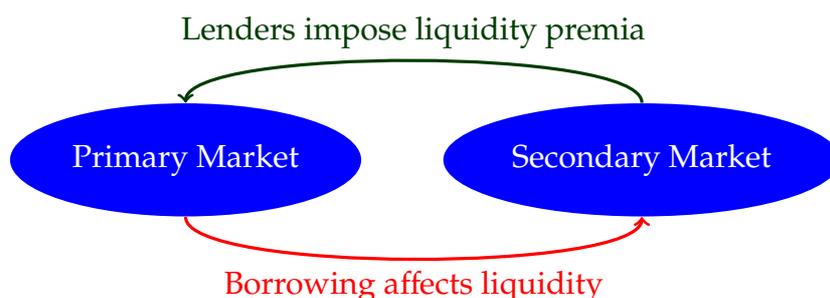


Figure 1: Feedback loop between primary and secondary market for corporate debt

markets. The interaction between the primary and secondary markets leads to two distortions: one in the capital structure of the firm and another in the allocation of investor portfolios. These distortions arise from the fact that neither firms nor investors internalize how their behavior affects liquidity in the secondary market. In equilibrium, market liquidity can be suboptimally low (or high) implying the firm is over-leveraged (under-leveraged), hence there is an undersupply (oversupply) of liquid assets for investors engaging in exchange on the secondary market. A social planner would like to implement the optimal level of liquidity in the secondary market by altering the financing decisions of firms and the portfolio allocations of investors. Such an outcome leads to higher firm profits while investors are no worse off. We derive a set of analytic wedge expressions that highlight two distorted margins and show how an appropriately designed tax system can decentralize the efficient equilibrium.

The third contribution is to provide a theoretical characterization for the effects of quantitative easing (QE) policies, like the ones observed following the Great Recession. Through the lens of our model, policies that affect the composition of investor portfolios, such as quantitative easing, affect the economy by compressing liquidity premia and, hence, directly influence savings and investment decisions in the real economy (see Stein, 2014, for a general discussion on the topic). Moreover, our analysis also highlights the benefits and limitations of these type of interventions. On the one hand, we show that QE can improve the intermediation capacity of the economy expanding its productive frontier. But, on the other hand, these policies are expected to be limited by their redistributive effects, the disadvantage of central banks monitoring borrowers, and the prospect for financial losses.

The model has three periods and is populated by firms that need external financing to invest in long-term projects and investors who want to transfer funds over time to consume in all periods. In the initial period, ex ante identical investors supply funds to firms in primary capital markets, while firms issue claims against their long-term revenues that materialize only in the final period. The contracting problem between the firm and investors in the primary debt market is subject to agency frictions, which we model using the costly state verification (CSV) framework (Townsend, 1979; Gale

and Hellwig, 1985; Bernanke and Gertler, 1989). The choice of the CSV framework is guided by the fact that it offers a convenient and well understood rationale for the firm's use of debt financing, which is central to our model. In addition, the CSV framework allows us to jointly study the effect of liquidity premia on the composition (leverage) and the riskiness of the capital structure of the firm. That said, the specific nature of the agency frictions in the primary market is not detrimental for the generality of our results.²

After the financial contract between the firm and investors has been written and investment decisions have been made, a subset of investors receive idiosyncratic (liquidity) shocks that make them want to consume before the firm's investments mature and proceeds are distributed. These shocks are private information and, thus, contingent contracts among patient and impatient investors cannot be written *ex ante*. Alternatively, investors can self-insure by investing part of their endowment in a storage technology and/or by holding corporate bonds and re-trading them in a secondary market once type has been revealed. Corporate bonds are thus not only a claim on real revenues, but also have a role in facilitating the process of exchange (see also Rocheteau and Wright, 2013).

In absence of frictions, the ability to trade long-term bonds in the secondary market would perfectly satisfy impatient investor's demand for liquidity. Indeed, in this special case we show that our model collapses to the benchmark CSV model of Bernanke and Gertler (1989) where liquidity concerns play no role. In practice, however, trading frictions may impinge on the ability of impatient investors to sell long-term assets. For corporate bonds, which are traded in over-the-counter (OTC) markets, empirical evidence by Edwards et al. (2007) and Bao et al. (2011) suggests that search frictions are an important driver of liquidity premia.³

²The reason is because we are able to disentangle the channel through which market liquidity affects liquidity premia in long-term assets from the choice of the optimal contract/capital structure of the firm. Hence, it does not matter how we introduce the financing frictions. For example, a situation where firms face collateral constraints as in Holmström and Tirole (1997) or Kiyotaki and Moore (1997) would yield the same qualitative results. That said, there is a fundamental difference between models featuring collateral constraints and our framework with respect to the concept of liquidity. In the language of Brunnermeier and Pedersen (2009), the former emphasizes funding liquidity (how much firms can raise by pledging assets as collateral), while our theory highlights the importance of market liquidity (the ease with which assets can be traded).

³ Bond financing has become one of the most important sources of external financing for U.S. corporations. Figure 3 shows that bond financing is the dominant source of credit liabilities for non-financial corporate firms (Flow of Funds data). This paper focuses on bond financing abstracting from the fact that firms enter into bank loans or other types of borrowing at the same time (see deFiore and Uhlig, 2011, Aoki and Nikolov, 2014, for models where bank and bond financing coexist). In principle, bank intermediation would be optimal to insure against idiosyncratic liquidity risk in the spirit of Diamond and Dybvig (1983) when bank runs are not very likely (see Cooper and Ross, 1998, and Goldstein and Pauzner, 2005) or bank credit is not sufficiently more expensive than bond financing as in deFiore and Uhlig (2011). However, Jacklin (1987) shows that the efficiency gains of bank intermediation for investors vanish when secondary capital markets are available and function frictionlessly. This should continue to be true when the associated frictions in secondary markets are not too severe, while bank

We follow Duffie et al. (2005), Lagos and Rocheteau (2007, 2009), and others by introducing illiquidity in the secondary market through search frictions between sellers and buyers that engage in OTC trade. In our framework, impatient investors submit sell orders that are matched with buy orders submitted by patient investors through a matching function. The probability that the two are matched, allowing the corporate bond to be exchanged for a quantity of the short-term asset, depends on market liquidity defined as the number of buy orders relative to sell orders. Hence, the initial period portfolio holdings of investors endogenously determines liquidity in the secondary market. This aspect of our model sets our approach apart from most of the existing literature studying search frictions in OTC markets, which tends to treat matching probabilities as exogenous.⁴ Moreover, most of this literature focuses on the implications of search frictions and illiquidity specifically on asset prices. Price effects are important in our framework as well, but our focus is broader in the sense that we are interested more generally in how primary markets for corporate assets interact with secondary market liquidity.

Endogenous liquidity is critical for linking the capital structure of the firm to outcomes in the secondary market. The capital structure influences secondary market liquidity through two channels. There is a direct effect whereby primary issuance determines the supply of the illiquid asset, and hence the number of possible sell orders. In addition, there is an indirect effect as the portfolio adjustment on the part of investors determines the supply of the liquid asset, or the number of possible buy orders. This interaction is what gives rise to the inefficiencies that we uncover. Firms do not internalize how their capital structure influences the liquidity position of investors and investors do not internalize how their liquidity needs affect the firm's capital structure.

Before turning to the details of the model, we should note that we have abstracted away from issues related to adverse selection arising from asymmetrically informed agents participating in the secondary market. In a seminal paper, Gorton and Pennacchi (1990) show how the information sensitivity of financial contracts affects their liquidity in secondary markets and study the capital structure of the firm and efficient intermediation.⁵ Although similar in spirit, our approach differs with respect to the

intermediation would dominate when markets are more imperfect (Diamond, 1997).

⁴Duffie et al. (2005) assume the holdings of agents participating in the OTC markets do not play an important role in equilibrium outcomes. Lagos and Rocheteau (2009) utilize the fact that agents can mitigate trading friction by adjusting their asset position to reduce their trading needs. Thus, they can study how liquidity premia affect the portfolio holdings of agents, but not the reverse linkage from portfolios to market liquidity. Finally, He and Milbradt (2014) present a model with search frictions in OTC markets for corporate bonds and show how default and liquidity premia, as well as the decision to default, are affected by market liquidity. However, they take the capital structure and investment of the firm as given, which in our model is endogenous and at the heart of our analysis. See Rogerson et al. (2005) for a detailed discussion of different concepts of equilibria in search models, albeit in the context of labor markets.

⁵There is an important literature following this tradition, such as Dang et al. (2011) and Gorton and Ordoñez (2014). Guerrieri and Shimer (2014) examine how adverse selection about the quality of

frictions resulting in illiquid liabilities of the firm. Gorton and Pennacchi (1990) consider informational asymmetries between buyers and sellers in the secondary market for corporate liabilities and show that uninformed investors respond by demanding informationally insensitive assets, notably riskless debt. Hence, their approach is important for understanding how investor's decisions to participate in these markets (the extensive margin of the investor's portfolio choice) affects the firm's capital structure. In contrast, our approach of introducing search frictions to limit trade in secondary markets allows us to examine how—given full participation in both asset markets—the intensive margin of the investor's portfolio choice affects the firm's financing decision and how the firm's financing decision, in turn, affects investor's portfolios.

We have also abstracted away from aggregate liquidity risk. When investors face aggregate liquidity risk which cannot be hedged due to market incompleteness, liquidity provision in the form of aggregate savings/reserves may be suboptimally low (Bhattacharya and Gale, 1987; Allen and Gale, 2004).⁶ In our paper, liquidity under-provision stems from trading frictions rather than aggregate shocks, which yields important implications for the liquidity premia of corporate bonds during periods that aggregate liquidity shocks are expected to occur rather infrequently. Consequently, our mechanism could potentially explain the fluctuations in liquidity and default risk premia, as well as firms' leverage even when aggregate liquidity shortages are unlikely or excluded due to the presence of unconventional policies, such as quantitative easing.

Moreover, our mechanism can also rationalize situations where there may be an over-provision of liquidity in the market economy. The reason is that our trading frictions do not only matter for the sellers of assets in the secondary market, who benefit from high liquidity, but also for the buyers, who extract rents when liquidity is low. Hart and Zingales (2015) show that the lack of a double coincidence of wants can result in a penuniary externality operating through the relative price of traded goods

assets affects their liquidity premia. They differ from the search microfoundations of illiquidity because the difficulty of finding a buyer depends primarily on the extent of private information rather than the availability of trading opportunities. Like us, but for different reasons, they suggest that unconventional policy interventions, such as asset purchase, can enhance the liquidity of assets not included in the purchase programs. Nevertheless, they do not study how illiquidity and policy interventions affect the equilibrium supply of assets, i.e. they abstract from corporate finance issues. Malherbe (2014), who builds on an adverse selection model of liquidity by Eisfeldt (2004), shows that, in contrast, excess cash-holdings impose a negative externality on others because they reduce the quality of assets put for sale in the secondary market. Although the direction of the externality in our model can result both in under- and over-supply of liquidity compared to a constrained optimum our mechanism is distinct from the literature studying adverse selection issues in secondary markets. See also Kurlat (2013) and Bigio (2015) for the interaction of business cycle dynamics and illiquidity induced by adverse selection in asset markets.

⁶Liquidity under-provision may also stems from hidden trades undoing the efficient sharing of liquidity risk across impatient and patient agents as in Farhi et al. (2009) or fire-sales externalities (Lorenzoni, 2008; Korinek, 2011; Acharya, Shin and Yorulmazer (2011)), which we abstract from in our paper. Perotti and Suarez (2011) show the desirability of Pigouvian taxation to address externalities from the under-provision of liquidity.

and services, and render private liquidity holdings inefficiently high. In our model, inefficient liquidity in general does not accrue from a relative price externality or a fire-sale, but rather from the relative easiness for buyers and sellers to trade.

The rest of the paper proceeds as follows. Section 2 presents the model and derives the equilibrium conditions. Section 3 shows how secondary market liquidity interacts with the optimal financing decisions of the firm. Section 4 present the social planner's problem, and identifies the externalities inherent in the private economy as well as the optimal policy mix. Section 5 analyses the effect of quantitative easing on secondary market liquidity and financing decisions. Finally, section 6 concludes.

2 Model

2.1 Physical Environment

There are three time periods $t = 0, 1, 2$, a single consumption good, and two type of agents: entrepreneurs and investors. Entrepreneurs have long-term investment projects and may fund these projects with internal funds or borrowing from investors. Ex ante identical investors lend funds to entrepreneurs, but once that lending has taken place and while production is underway, investors are subject to a preference (liquidity) shock which reveals whether they are impatient, and hence prefer to consume earlier rather than later, or patient. These two types of investors trade their assets in secondary asset markets with search frictions (see Figure 2).

There is a mass one of ex ante identical entrepreneurs, who are endowed with n_0 units of capital at $t = 0$. Entrepreneurs invest to maximize the return on their equity, i.e., to maximize profits per unit of endowment. The technology is linear and delivers $R^k\omega$ at $t = 2$, per unit invested at $t = 0$. The random variable ω is an idiosyncratic productivity shock that hits after the project starts, and is distributed according to the cumulative distribution function F , with unit mean. It is privately observed by the entrepreneur, but investors can learn about it when they seize entrepreneurs' assets and pay a monitoring costs μ as a fraction of assets. The (expected) gross return R^k is assumed to be known at $t = 0$, as there is no aggregate uncertainty in the model. In order to produce, the firm must finance investment, denoted k_0 , either through its own funds or by issuing financial contracts to investors. So profits equal total revenue in period 2, $R^k\omega k_0$, minus payment obligations from financial contracts. Entrepreneurs represent the corporate sector in our model, so we will talk about entrepreneurs' projects and firms interchangeably.

There is a mass one of ex ante identical investors, who are endowed with e_0 units of capital at $t = 0$. Investors have unknown preferences at $t = 0$, and learn their preferences at $t = 1$. At $t = 1$ investors realize if they are *patient* or *impatient consumers*,

a fraction $1 - \delta$ will turn out to be patient and a fraction δ impatient. Patient consumers have preferences only for consumption in $t = 2$, $u^P(c_1, c_2) = c_2$, whereas impatient consumers have preferences for both consumption in $t = 1$ and 2, but discount period 2 consumption at rate β , $u^I(c_1, c_2) = c_1 + \beta c_2$.

Investors in both period 0 and 1 have access to a storage technology with yield $r > 0$, i.e., every unit stored yields $1 + r$ units of consumption in the next period. The amount stored in period t is denoted s_t . In addition, at $t = 0$, they can invest in financial contracts issued by entrepreneurs in primary markets; and, at $t = 1$, they can buy and sell assets in secondary markets with search frictions (see Figure 2). When engaging in trade in the secondary market patient investors realize a return Δ . Both the primary and secondary markets are described in detail below.⁷

In what follows we make the following assumptions.

Assumption 1 (Relative Returns) *The long-term return of the productive technology is larger than the cumulative two-period storage return and the return on storage plus the return on secondary markets, i.e., $(1 + r)^2 < R^k$ and $(1 + r)\Delta < R^k$. In addition, monitoring costs are such that $R^k(1 - \mu) < (1 + r)^2$.*

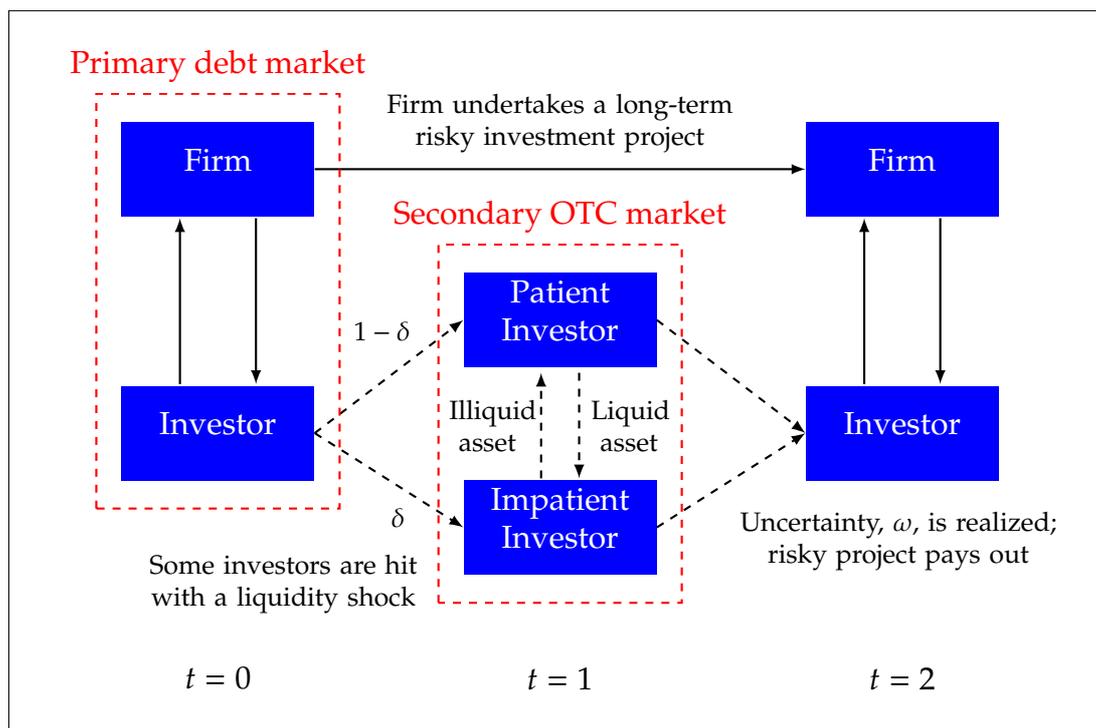


Figure 2: Timeline.

⁷Note that since $r > 0$ and since investors preferences have been assumed time separable and risk neutral, there was no loss of generality in abstracting away from consumption at $t = 0$ for investors, and consumption at $t = 1$ for patient investors.

Assumption 2 (Productivity Distribution) Let $h(\omega) = dF(\omega)/(1 - F(\omega))$ denote the hazard rate of the productivity distribution. It is assumed that $\omega h(\omega)$ is increasing.

Assumption 3 (Impatience) The rate of preference of impatient investors is such that $\beta \leq 1/(1 + r)$.

Assumption 4 (Investors Deep Pockets) It is assumed that investors' (total) endowment e_0 is significantly higher than entrepreneurs' (total) endowment n_0 , i.e., $e_0 \gg n_0$.

Assumption 1 is necessary for there to be a role for the entrepreneurial sector, $R^k > (1+r)^2$ and $R^k > (1+r)\Delta$, when the prospective return on secondary market is taken into account. Furthermore, this assumption rules out equilibria where entrepreneurs are always monitored, $(1+r)^2 > R^k(1-\mu)$. Assumption 2 ensures that there is no credit rationing in equilibrium, and together with Assumption 1 will ensure the existence and uniqueness of equilibrium, as we discuss below. Assumption 3 makes impatient investors behave as impatient agents when the interest rate is r . Finally, Assumption 4 ensures that investors can meet the credit demand of entrepreneurs.

2.2 The Financial Contract

Entrepreneurs finance their investments using either internal funds, n_0 , or by selling long-term financial contracts to investors in the primary corporate debt market. These contracts specify an amount, b_0 , borrowed from investors at $t = 0$ and a *promised* gross interest rate, Z , made upon completion of the project at $t = 2$. If entrepreneurs cannot make the promised interest payments, investors can take all firm's proceeds paying a monitoring cost, equal to a fraction μ of the value of assets.⁸

The $t = 0$ budget constraint for the entrepreneur is given by

$$k_0 \leq n_0 + b_0 . \tag{1}$$

For what follows it will be useful to define the entrepreneur's leverage, l_0 , as the ratio of assets to (internal) equity k_0/n_0 .

The entrepreneur is protected by limited liability, so its profits are always non-negative. Thus, the entrepreneur's expected profit in period $t = 2$ is given by

$$\mathbb{E}_0 \max \{0, R^k \omega k_0 - Z b_0\} .$$

Limited liability implies that the entrepreneur will default on the contract if the realization of ω is sufficiently low such that the payoff of the long-term project falls

⁸We consider deterministic monitoring rather than stochastic monitoring, which results in debt being the optimal contract. Krasa and Villamil (2000) derive the conditions under which deterministic monitoring occurs in equilibrium in costly enforcement models. Moreover, our model features perfect, but costly, ex-post enforcement. See Krasa et al. (2008) for a more elaborate enforcement process and its implications for firms' finance.

below the promised payout; that is, when $R^k \omega k_0 < Zb_0$. This condition defines a threshold productivity level, $\bar{\omega}$, such that the entrepreneur defaults when

$$\omega < \bar{\omega} = \frac{Z}{R^k} \frac{l_0 - 1}{l_0} . \quad (2)$$

The productivity threshold measures the credit risk of the financial contract; and is increasing in the spread between the promised return and the expected return on the entrepreneur investment, and increasing in firm's leverage.

For notational convenience, we define $G(\bar{\omega}) \equiv \int_0^{\bar{\omega}} \omega dF(\omega)$ and $\Gamma(\bar{\omega}) \equiv \bar{\omega}(1 - F(\bar{\omega})) + G(\bar{\omega})$. The function $G(\bar{\omega})$ equals the truncated expectation of entrepreneurs' productivity given default. The function $\Gamma(\bar{\omega})$ equals the expected value of a random variable equal to ω if there is default ($\omega < \bar{\omega}$) and equal to $\bar{\omega}$ when there is not ($\omega \geq \bar{\omega}$). It follows that $R^k k_0 \Gamma(\bar{\omega})$ corresponds to the expected transfers from entrepreneurs to investors.

Then, firms' objective, expected profits per unit of endowment, or return on equity, can be expressed using the previous notation as⁹

$$\frac{1}{n_0} \mathbb{E}_0 \max \{0, R^k \omega k_0 - Zb_0\} = [1 - \Gamma(\bar{\omega})] R^k l_0 . \quad (3)$$

Similarly, the *total* expected payoffs of bond contracts can be expressed as

$$\int_{\bar{\omega}}^{\infty} Zb_0 dF(\omega) + (1 - \mu) \int_0^{\bar{\omega}} R^k \omega k_0 dF(\omega) = k_0 R^k [\Gamma(\bar{\omega}) - \mu G(\bar{\omega})] .$$

Therefore, the expected gross return of holding a *single* bond to maturity R^b is given by

$$R^b = \frac{l_0}{l_0 - 1} R^k [\Gamma(\bar{\omega}) - \mu G(\bar{\omega})] , \quad (4)$$

which is a function of only leverage and the productivity threshold.

Clearly R^b is decreasing in l_0 as leverage dilutes lenders claim on the collateral. Moreover, in equilibrium it will be increasing in risk $\bar{\omega}$, as detailed below. Finally, note that the expected return is known in period 0 and 1, since there is no aggregate uncertainty or new information arriving after investors and the firm have agreed on the terms of lending. This means that idiosyncratic liquidity shocks in period 1 do not affect R^b and investors would trade bonds in a secondary market promising this expected payout.

⁹The objective of the firm in equation (3) is written in terms of return to equity rather than total profits. However, both formulations would yield the same equilibrium results as n_0 is positive and given.

2.3 The Secondary OTC Market

The ex post heterogeneity introduced by the preference shock generates potential gains from trading corporate debt in a secondary market. Impatient investors want to exchange long-term, imperfectly liquid, bonds for consumption, as they would rather consume at the end of period 1 than hold the bond to maturity until period 2 (Assumption 3). Patient investors are willing to exchange lower yielding storage for corporate debt with higher expected returns.

In order for such a trade to take place, buy and sell orders must be paired up according to a matching technology which aligns them. Impatient investors submit sale orders, one for each bond they are ready to sell at a given price q_1 . Patient investors submit buy orders, one for each package of q_1 units of storage they are ready to exchange for a bond. We model the OTC market such that matching is by *trade*, as opposed to by *investor*.¹⁰ Suppose, in aggregate, there are A sell (or ask) orders and B buy orders. The matching function is assumed to be constant returns to scale and is given by

$$m(A, B) = \nu A^\alpha B^{1-\alpha}, \quad (5)$$

with $0 < \nu$ a scaling constant and $0 < \alpha < 1$ the elasticity of the matching function with respect to sell orders.

We define a concept of market liquidity as the ratio of buy orders to sell orders, or $\theta = B/A$. When θ is large (small), the secondary market is relatively (il)liquid in the sense that the number of buy orders is large (small) relative to sell orders. This makes it easier (harder) for impatient investors to transform their corporate bonds into storage, which can then be consumed.

Indeed, the *probability that a sell order is executed* is expressed as

$$f(A, B) = \min \left[1, \frac{m(A, B)}{A} \right] \quad \text{or} \quad f(\theta) = \min [1, m(1, \theta)], \quad (6)$$

and the *probability that a buy order is executed* is expressed as

$$p(A, B) = \min \left[1, \frac{m(A, B)}{B} \right] \quad \text{or} \quad p(\theta) = \min [1, m(\theta^{-1}, 1)]. \quad (7)$$

Once a buy order and a sell order are matched, the terms of trade are determined via a simple surplus sharing rule known by all agents. From the seller's perspective, a trading match yields additional liquid wealth from unloading the incremental bond sold at price q_1 . If the seller walks away from the match she holds the bond, which matures in the final period, delivering an expected payout of R^b in $t = 2$, which is discounted at rate β . The value of a trading match to a buyer is the present value of the

¹⁰ This can be thought of as money chasing bonds, instead of investors chasing investors.

(expected) return on the bond, net of the price that needs to be paid for each bond in the secondary market. Then, the surplus that accrues to an impatient investor, $S^I(q_1)$, and the surplus that accrues to a patient investor, $S^P(q_1)$, respectively, are given by

$$S^I(q_1) = q_1 - \beta R^b \quad \text{and} \quad S^P(q_1) = \frac{R^b}{1+r} - q_1 .$$

The price of the debt contract on the secondary market is determined by a sharing rule that maximizes the Nash product of the respective surpluses,

$$\max_{q_1} \left(S^I(q_1) \right)^\psi \left(S^P(q_1) \right)^{1-\psi} ,$$

where $\psi \in [0, 1]$ is a parameter that determines the split of the surplus between patient and impatient investors.¹¹

The solution of the surplus splitting problem yields the following bond price in the secondary market

$$q_1 = R^b \left(\frac{\psi}{1+r} + (1-\psi)\beta \right) . \quad (8)$$

Note that $\psi = 1$ drives the price of the bond to the “bid” price, or the price that extracts full rent from the buyer, $q_1 = \frac{R^b}{1+r}$. By the same token, $\psi = 0$ drives the price of the bond to the “ask” price, or the price that extracts full rent from the seller, $q_1 = \beta R^b$. From equation (8) it follows that the return that patient investors make in the secondary market, per executed buy order, is given by

$$\Delta = \frac{R^b}{q_1} = \left(\frac{\psi}{1+r} + (1-\psi)\beta \right)^{-1} .$$

2.4 Investors

As described above, investors are ex ante identical and are endowed with e_0 units of capital. At $t = 0$ they can allocate their wealth across two assets: the storage technology and debt contracts. Thus, the budget constraint is given by¹²

$$s_0 + b_0 = e_0 , \quad (9)$$

where $s_0, b_0 \geq 0$, i.e. borrowing at the storage rate or short-selling corporate debt are not allowed.

The storage technology, denoted s_0 , pays a fixed rate of return $1 + r$ at $t = 1$ in units of consumption. The proceeds of this investment, if not consumed, can be reinvested

¹¹Our sharing rule is very close to Nash bargaining over the surplus. Under Nash bargaining the parameter ψ can be interpreted as the bargaining power of sellers.

¹² Since the mass of both entrepreneurs and investors equals one, and we focus on the symmetric equilibrium, we abuse notation and denote the individual supply and demand of debt by b_0 .

to earn an additional return of $1 + r$ between period 1 and 2, again paid in units of consumption. In this sense, storage is a liquid investment, as at any point in time it can be costlessly transformed into consumption. Alternatively, the corporate bond has an expected payoff of R^b , but only at the beginning of $t = 2$. Moreover, for an investor to turn her bond into consumption at $t = 1$, she will have to post an order in a secondary market characterized by search frictions. So the bond is illiquid, as it does not allow investors to transform their investment costlessly into consumption in period 1.

The relative illiquidity of corporate debt comes into play because at the beginning of $t = 1$, a fraction δ of investors receive a preference shock that makes them discount future consumption at rate β . Moreover, Assumption 3 implies that impatient investors strictly prefer to consume in period 1 relative to period 2. In contrast, the remaining fraction $1 - \delta$ are patient investors, who only enjoy consumption in $t = 2$.

Thus, impatient investors find themselves holding corporate debt contracts which cannot easily be transformed into period $t = 1$ consumption. Ideally, they would like to sell this asset to patient investors who are willing to give up units of liquid storage in exchange for the higher yielding corporate debt. This trading activity takes place in an OTC secondary market. As described above, impatient investors looking to unload corporate debt contracts will only get their orders executed with endogenous probability $f(\theta)$. Similarly, patient investors looking to purchase corporate debt will only get their orders executed with endogenous probability $p(\theta)$. If a buy and a sell order are lucky enough to be matched in the OTC market a bilateral trade takes place and units of bonds are exchanged for units of storage at the agreed upon price, q_1 .

To describe the portfolio choice problem of investors, it is useful to first consider the optimal behavior of impatient and patient investors in $t = 1$ when they arrive to that period with a generic portfolio of storage and bonds (s_0, b_0) .

2.4.1 Impatient Investors

By Assumption 3 at $t = 1$ impatient investors want to consume in the current period. They can consume the payout from investing in storage, $s_0(1 + r)$, plus the additional proceeds from placing b_0 sell orders in the OTC market. These orders are executed with probability f and each executed order yields q_1 units of consumption. Thus, the expected consumption of impatient investors in period 1 is given by

$$c_1^I = s_0(1 + r) + fq_1b_0 . \quad (10)$$

On the other hand, with probability $1 - f$ orders are not matched and impatient investors are forced to carry debt contracts into period 2. Therefore, expected con-

sumption in the final period is given by

$$c_2^I = (1 - f)R^b b_0, \quad (11)$$

and the utility derived from c_2^I is discounted by β .

2.4.2 Patient Investors

Patient investors only value consumption in the final period and will be willing to place buy orders in the OTC market if there is a surplus to be made, i.e., if $q_1 \leq R^b/(1 + r)$. The price determination in the OTC market guarantees that this is always the case ($1 + r \leq \Delta$), thus patient investor would ideally like to exchange all of the lower yielding units of storage for corporate debt with a higher expected returns. But their buy orders will be executed only with probability p .

Therefore, expected storage holdings at the end of $t = 1$, s_1^P , are equal to a fraction $1 - p$ of the available liquid funds $s_0(1 + r)$, i.e.,

$$s_1^P = (1 - p)s_0(1 + r).$$

On the other hand, patient investors place $s_0(1 + r)/q_1$ buy orders, of which a fraction p is expected to be executed. So patient investors expect to increase their bond holding by $ps_0(1 + r)/q_1$ units. It follows that expected consumption in the final period equals

$$c_2^P = (1 - p)s_0(1 + r)^2 + \left[b_0 + p \frac{s_0(1 + r)}{q_1} \right] R^b. \quad (12)$$

That is, the payout from units of storage that were not traded away in the secondary market plus the expected payout from corporate debt holdings.

2.4.3 Optimal Portfolio Allocation

In the initial period investors solve a portfolio allocation problem, choosing between storage and bonds to maximize their expected lifetime utility

$$\mathbb{U} = \delta(c_1^I + \beta c_2^I) + (1 - \delta)c_2^P,$$

subject to the period 0 budget constraint (9), and the expressions for expected consumption of impatient and patient investors (10)-(12).

Given the optimal behavior of patient and impatient investors described above, we can rewrite the expected lifetime utility as

$$\mathbb{U} = U_s s_0 + U_b b_0,$$

where U_s and U_b denote the expected utility from investing in storage and bonds in period 0, respectively, and are given by

$$U_s = \delta(1+r) + (1-\delta) \left[(1-p)(1+r)^2 + p(1+r) \frac{R^b}{q_1} \right], \quad (13)$$

$$\text{and} \quad U_b = \delta(fq_1 + \beta(1-f)R^b) + (1-\delta)R^b. \quad (14)$$

Note that both of these expressions depend on the characteristics of the financial contract, $(l_0, \bar{\omega})$, through the expected return on holding the bond to maturity R^b ; and on the characteristics of the secondary market, (q_1, θ) , through the secondary market price q_1 and matching probabilities $f(\theta)$ and $p(\theta)$.

Using these definitions, we can express the asset demand correspondence that maximizes the investors portfolio problem as

$$\begin{cases} s_0 = 0, & b_0 = e_0 & \text{if } U_s < U_b \\ s_0 \in [0, e_0], & b_0 = e_0 - s_0 & \text{if } U_s = U_b \\ s_0 = e_0, & b_0 = 0 & \text{if } U_s > U_b \end{cases}$$

That is, when the expected benefit of holding storage in period 0 is dominated by the benefit of holding the bond, then investors will demand only bonds in period 0. On the contrary, if the expected benefit of holding storage is greater than then expected utility of buying the bond in period 0, then investors will only hold storage in the initial period. Finally, if both expected utilities are equal, investors will be indifferent between investing in storage and bonds initially and their demands will be an element of the set of feasible portfolio allocations: $s_0, b_0 \in [0, e_0]$, such that the total value of assets equal the initial endowment (9). Given our assumptions in equilibrium the portfolio allocation will be interior (i.e., $U_s = U_b$ with $s_0, b_0 > 0$), thus we focus our analysis on them.

All told, in equilibrium it must be that the two assets in period 0 yield the same expected discounted utility, so the return to storage equals the return to lending to entrepreneurs,

$$U_s(l_0, \bar{\omega}, q_1, \theta) = U_b(l_0, \bar{\omega}, q_1, \theta).$$

For future reference we label the previous equation as the *investors' break-even condition*. Note that the discounted expected utility from investing in storage, U_s , is not smaller than the discounted expected utility in financial autarky: $U_a = \delta(1+r) + (1-\delta)(1+r)^2$, since the return of buying a bond in the secondary market $\Delta \geq 1+r$ (equation 8). So investors are always willing to provide liquidity in secondary markets.

2.5 Equilibrium

The equilibrium of the model is defined as follows.

Definition 1 (Competitive Equilibrium) *We say that $(l_0, \bar{\omega}, \theta, q_1)$ is a competitive equilibrium if and only if:*

1. *Given the characteristics of the secondary market (θ, q_1) , the debt contract is described by $(l_0, \bar{\omega})$ maximizes entrepreneurs' return on equity subject to investors' break-even condition.*
2. *Market liquidity corresponds to $\theta = (1 - \delta)(1 + r)s_0/q_1/(\delta b_0)$.*
3. *q_1 is determined via the surplus sharing rule.*
4. *All agents have rational expectations about q_1 and θ .*

The equilibrium of the model is described by the entrepreneur's choice of l_0 and $\bar{\omega}$ to maximize the payoff of the risky investment project. Entrepreneurs' profits are higher when leverage l_0 is higher and when the promised payout is lower, that is, when the risk $\bar{\omega}$ is lower. But entrepreneurs are constrained in their choices of l_0 and $\bar{\omega}$ as they need to offer terms that make financial contracts attractive to investors: investors' break-even condition.

Entrepreneurs are aware that when selling in the secondary market, investors obtain a price that depends on their individual contract characteristics. In fact, the price is determined via the sharing rule (equation 8). Substituting the secondary market price in the expressions for the discounted expected utilities of investing in storage and bonds (equations 13 and 14) we get

$$U_s(\theta) = \delta(1 + r) + (1 - \delta)(1 + r) [(1 - p(\theta))(1 + r) + p(\theta)\Delta] ,$$

and

$$U_b(l_0, \bar{\omega}, \theta) = \left\{ \delta \left[f(\theta)\Delta^{-1} + (1 - f(\theta))\beta \right] + (1 - \delta) \right\} R^b(l_0, \bar{\omega}) .$$

It follows that the entrepreneur's problem can be written as

$$\max_{l_0, \bar{\omega}} [1 - \Gamma(\bar{\omega})]R^k l_0$$

subject to:

$$U_s(\theta) = U_b(l_0, \bar{\omega}, \theta) . \tag{15}$$

Let λ be the multiplier on the break-even condition (15), then the entrepreneur's privately optimal choice of leverage is given by

$$[1 - \Gamma(\bar{\omega})]R^k = -\lambda \frac{\partial U_b(l_0, \bar{\omega}, \theta)}{\partial l_0} . \tag{16}$$

That is, the marginal increase in profits for entrepreneurs need to be proportional to the marginal reduction in expected utility of financial contracts for investors.

Similarly, the privately optimal choice for the risk profile of corporate debt is given by

$$\Gamma'(\bar{\omega})l_0 = \lambda \frac{\partial U_b(l_0, \bar{\omega}, \theta)}{\partial \bar{\omega}}. \quad (17)$$

That is, the marginal increase in profits for entrepreneurs need to be proportional to the marginal increase in expected utility of financial contracts for investors.

Taking a ratio of the equations (16) and (17) gives

$$\frac{1 - \Gamma(\bar{\omega})}{\Gamma'(\bar{\omega})l_0} = - \frac{\partial U_b(l_0, \bar{\omega}, \theta)/\partial l_0}{\partial U_b(l_0, \bar{\omega}, \theta)/\partial \bar{\omega}}. \quad (18)$$

This equation, which describes the privately optimal debt contract, taken together with the investors' break-even condition, given by equation (15), and the expressions that characterize the secondary market (θ, q_1) provide a complete description of the equilibrium of the model.

Finally, note that both the price in the secondary market q_1 and secondary market liquidity θ can be expressed as a function of the characteristics of the optimal financial contract $(l_0, \bar{\omega})$. In fact, the price is a function of the expected return on holding the bond to maturity R^b , which depends on $(l_0, \bar{\omega})$; so we can write market liquidity as

$$\theta = \frac{(1 - \delta)s_0(1 + r)}{\delta b_0 q_1} = \frac{(1 - \delta)(1 + r)\Delta(e_0 - n_0(l_0 - 1))}{\delta n_0(l_0 - 1)R^b(l_0, \bar{\omega})}. \quad (19)$$

The following theorem establishes the existence and uniqueness of equilibrium in our model.

Theorem 1 (Existence and Uniqueness of Competitive Equilibrium) *Under the maintained assumptions there exists a unique competitive equilibrium of the model. Furthermore, in the unique equilibrium credit is not rationed, i.e., $\Gamma'(\bar{\omega}) - \mu G'(\bar{\omega}) > 0$.*

That is, $\exists!(l_0, \bar{\omega}, \theta, q_1)$ where the optimal contract in the primary market is described by (18), the investors' break-even condition (15) is satisfied, and both secondary market bond pricing and liquidity are consistent with the decisions in primary markets, i.e., they are given by equation (8) and (19), respectively.

The proof is relegated to the Appendix. As is the case in the canonical CSV model (e.g. Bernanke et al. 1999), the result on existence follows from our assumptions. That is, we have assumed that the return on the entrepreneurs' technology is better than the return on financial assets, including the possibility of secondary market retrading, so entrepreneurs will always be able to offer contractual terms that are attractive to investors. In contrast, while uniqueness is relatively straightforward to establish in the CSV model, our framework is complicated by the endogeneity of liquidity. Never-

theless, we are able to establish that even in our setup with feedback effects between outcomes in primary and secondary markets, multiple equilibria do not obtain.

3 Frictions and the (Ir)relevance of OTC Trade

We begin by describing the conditions on exogenous parameters under which there is no liquidity premium and, hence, trading frictions are irrelevant (Proposition 2). Next, we describe the feedback loop that arises due to frictional trade in the OTC market in Lemma 1 and Remark 2. We provide a comparative statics analysis of the effect of the parameters that determine the demand and supply of corporate bonds in the primary market (Propositions 3 and 4). Finally, our main theoretical findings are followed by a simple numerical exercise to make the results more concrete. All proofs are relegated to the Appendix.

First, it is useful to define a benchmark interest rate that is the return on a two-period bond that could *hypothetically* be traded in a perfectly liquid secondary market, *if such a market were available to investors*. Naturally, such a hypothetical contract needs to deliver the same return in expectation as a strategy of investing only in storage both in the initial and interim periods. This gives rise to the following definition.

Definition 2 (Liquid Two-period Rate) *The liquid two-period rate is defined as the gross interest rate on a perfectly liquid two-period bond.*¹³

$$R^\ell \equiv (1 + r)^2 .$$

The benchmark rate allows us to decompose the total gross return on the financial contract written by the firm into a default and a liquidity premium. In order to do this, express the total corporate bond premium as the gross return of the firm's contract relative to the benchmark rate, Z/R^ℓ . Then, this total premium is decomposed into a component owing to default risk, Z/R^b , and a component owing to liquidity risk, R^b/R^ℓ . With this decomposition, we have the following definitions for the default and liquidity premia, respectively.

Definition 3 (Default and Liquidity Premia) *The default premium Φ^d and the liquidity premium Φ^ℓ on the firm's debt contract are given by*

$$\Phi^d \equiv \frac{Z}{R^b} \quad \text{and} \quad \Phi^\ell \equiv \frac{R^b}{R^\ell} .$$

¹³No arbitrage under perfectly liquid markets implies that trading a two-period bond should yield the same expected return for investors to rolling over one period safe investments, i.e. $\delta \cdot R^\ell / (1+r) + (1-\delta) \cdot R^\ell = \delta \cdot (1+r) + (1-\delta) \cdot (1+r)^2$.

With these definitions the total corporate premium is given by

$$\Phi^t \equiv \frac{Z}{R^\ell} = \Phi^d \Phi^\ell .$$

These definitions provide sharp characterizations of both the default and liquidity premia, which are convenient to help trace out the underlying economic mechanisms in our model. The relationship between the liquidity premium and the investors' break-even condition, in equilibrium, is described in the next remark.

Remark 1 (Investors Break-even Condition and Liquidity Premium) *If investors correctly expect the period 1 bond price to be determined via the sharing rule, then the investors' break-even condition (15) can be expressed as*

$$(1 + r)^2 \Phi^\ell = R^b , \quad (20)$$

with the liquidity premium being only a function of secondary market liquidity given by

$$\Phi^\ell(\theta) = \frac{1}{1 + r} \frac{\delta + (1 - \delta) [(1 - p(\theta))(1 + r) + p(\theta)\Delta]}{\delta [f(\theta)\Delta^{-1} + (1 - f(\theta))\beta] + (1 - \delta)} . \quad (21)$$

On the other hand, the next proposition shows that the default premium, in equilibrium, is an increasing function of *only* the risk of the financial contract, $\bar{\omega}$.

Proposition 1 (Default Premium and Risk) *Under the maintained assumptions, the default premium, Φ^d , depends only on the risk of the financial contract, $\bar{\omega}$, and it is strictly increasing in this risk.*

Intuitively, investors demand a higher default premium for financial contracts that are more likely to default (i.e., contracts that are more risky or, specify a higher productivity threshold $\bar{\omega}$ for paying out the full value of the contract). The more subtle part of the argument is that leverage does not affect the default premium. This is due to the fact that, for a fixed threshold level $\bar{\omega}$, leverage affects both the face value of the contract, Z , and the hold-to-maturity return for investors, R^b , in the same way. So leverage is irrelevant for the default premium, as is the case in the classic CSV model, though leverage and risk are jointly determined in equilibrium.

We now turn to our main results.

3.1 A Frictionless Benchmark

Our first result, stated in Proposition 2, establishes the conditions under which trade in the secondary market is irrelevant, so that secondary OTC market liquidity has no bearing on the firm's optimal capital structure.

Proposition 2 (Irrelevance of OTC Trade) *Under the following conditions, there is no liquidity premium, i.e., $\Phi^l = 1$, implying that the model collapses to the benchmark costly state verification model:*

1. *All investors are patient, so that $\delta = 0$;*
2. *Impatient investors discount at rate $\beta = 1/(1 + r)$; or*
3. *Impatient investors extract their full value from all their sale orders in the secondary market, which is true for $\psi = 1$ and $\{e_0 \geq \bar{e}_0 : f(\theta) = 1\}$.*

The case in which $\delta = 0$ is straightforward. When all investors are patient, there is no need to trade in secondary markets; investors only care about the hold-to-maturity return. Liquidity is not priced in financial contracts and the model collapses to the standard costly state verification setup presented in, for example, Townsend (1979) and Bernanke and Gertler (1989).

The same result obtains for the second case, though for different reasons. When impatient investors discount future consumption at exactly the rate of return that comes from holding a unit of storage, so that $\beta = 1/(1 + r)$, they will be indifferent between consuming in the final or interim period. This indifference implies that there are no gains from OTC trade. In this case, the liquidity preference shock is immaterial and investors only consider the hold-to-maturity return when buying financial contracts in primary markets.

The final case considers the situation in which impatient investors can fully satisfy their liquidity needs in secondary markets. That is, the terms of trade are set such that impatient investors extract the entire surplus, such that $\psi = 1$, and all their sell orders will be executed, given that $f(\theta) = 1$. In this case, as before, liquidity considerations will not factor in the lending decision of investors in primary markets. In turn, $f(\theta) = 1$, requires that there is enough storage at $t = 1$ that all sell orders can be satisfied. Given that in this frictionless environment, entrepreneurs can only operate below a certain upper bound for leverage as an outcome of costly state verification, all the remaining endowment of investors not used for lending will be invested in the storage technology, which drives the probability that impatient investors find a match to 1. We derive this threshold for investors endowment in the proof of Proposition 2 in the Appendix.

3.2 OTC Trade in the Secondary Market

We now characterize the effects of frictional OTC trade. For the remainder of the paper, we consider only the cases in which trading frictions in the secondary market result in a non-negligible liquidity premium. That is, assume that (i) the probability of being an early consumer is positive, $\delta > 0$; (ii) impatient investors discount future consumption

strictly more than is implied by the storage rate, i.e., $\beta < 1/(1 + r)$; and (iii) impatient investors cannot fully satisfy their liquidity needs in secondary markets, $\psi < 1$ or $f(\theta) < 1$.

Under these assumptions we begin by establishing the link between imperfect liquidity in the secondary market and the associated liquidity premium.

Lemma 1 (Secondary Market Liquidity and the Liquidity Premium) *When secondary market liquidity, θ , is lower, investors require a higher liquidity premium, Φ^ℓ , or equivalently, a higher hold-to-maturity return, R^b . Moreover, the elasticity of the liquidity premium, Φ^ℓ , with respect to secondary market liquidity, θ , is lower than 1 in absolute terms.*

Lemma 1 formalizes the intuition that the price of liquidity risk (i.e., the liquidity premium) is inversely proportional to the amount of liquidity in secondary OTC markets. This relationship forms the basis for the direct link between primary and secondary markets shown by the upper arrow in Figure 1. In our model, market liquidity determines the likelihood that investors' orders will be executed in an OTC trade. In particular, as the market becomes less liquid sell orders will be more difficult to execute (i.e., $f(\theta)$ decreases), and impatient investors will have a harder time fulfilling their liquidity needs in secondary markets. By the same token, as liquidity declines buy orders are more likely to be executed (i.e., $p(\theta)$ increases) which provides an incentive for investors to shift their portfolios out of storage and into illiquid bonds. Both of these channels lead to a reduction in the demand for illiquid bonds in the primary market and an increase in the price of liquidity.

In equilibrium, the firm naturally responds to higher funding costs by altering the contract that it issues. A key contribution of this paper is to show that this, in turn, has knock-on effects for liquidity in the secondary market (the lower arrow in Figure 1). This transmission mechanism is summarized by the following remark.

Remark 2 (The Optimal Contract and Secondary Market Liquidity) *Secondary market liquidity, θ , is decreasing in leverage, l_0 , and the riskiness of the contract offered in the primary market, $\bar{\omega}$.*

Taken together with Lemma 1 this remark completes the feedback loop at the heart of this paper. Intuitively, when investors require additional compensation to bear liquidity risk, the firm has an incentive to alter the characteristics of the contract it offers in primary markets, reducing leverage and risk. By doing this, the firm's actions indirectly enhance liquidity in the secondary market, attenuating the initial increase in the liquidity premium. Similarly, an exogenous shock in the primary market will ripple through secondary market liquidity, affecting the liquidity premium, and thus, feeding back into the decisions in the primary market.

Now we describe the effect of the parameters that determine demand and supply

in the primary market in the equilibrium of the model. We begin by describing the effect on the demand for bonds.

Proposition 3 (Investors' Bond Demand) *Investors require a higher a higher liquidity premium, Φ^ℓ , and hence a higher hold-to-maturity return on the bond, R^b , when*

1. (Liquidity shock) *The probability of becoming impatient is higher, (i.e., δ is higher);*
2. (Impatience) *Impatient investors discount the future more heavily, (i.e., β is lower); and*
3. (Endowments) *Investors have fewer funds to invest in storage, (i.e., e_0 is lower).*

The proposition describes how the parameters that describe investors' preferences (δ and β) and endowments (e_0) affect demand in the primary market when the characteristics of the financial contract (leverage and risk) are held constant. As investors' preferences are more sensitive to liquidity risk (i.e., δ is higher or β is lower), the associated liquidity premium drives up the hold-to-maturity return that investors require to hold corporate debt. On the other hand, when investors are poorer (i.e., e_0 is smaller) they reduce their savings through storage one-for-one conditional on buying the same number of financial contracts. Less liquid savings reduces liquidity in secondary markets, and thus also drives up the required hold-to-maturity return through an increase in the liquidity premium (Lemma 1).

The equilibrium implications for the optimal capital structure, considering the feedback loop with secondary market liquidity, are summarized in the following proposition.

Proposition 4 (Equilibrium Comparative Statics) *In the private equilibrium, the firm's optimal leverage, l_0 , and the riskiness of the contracts it offers up in the primary market, $\bar{\omega}$, both decrease when*

1. (Liquidity shock) *The probability of becoming impatient is higher (i.e., δ is higher);*
2. (Impatience) *Impatient investors discount the future more heavily (i.e., β is lower);*
3. (Investors' Endowments) *Investors have fewer funds to invest in storage (i.e., e_0 is lower); and*
4. (Firms' Endowments) *Firms have more equity (i.e., n_0 is higher).*

This proposition presents the comparative statics in *equilibrium* for the parameters that describe preferences and endowments for investors and firms. For the first three cases, Proposition 3 establishes that an increase in δ or a decrease in β or e_0 will push up the firm's cost of funding through the liquidity premium. According to Proposition 4, entrepreneurs adjust to this increase in the cost of funding along two margins (recall

that the debt contract is two-dimensional). They offer fewer contracts in the primary market and the contracts that are offered are less risky relative to an equilibrium in which the firm's debt can be traded frictionlessly. A reduction in the number of bonds issued in the primary market lowers the number of possible sell orders in the secondary market, which attenuates the increase in the liquidity premium. That is, the adjustment of the firms' optimal capital structure mitigates the effect of trading frictions on the price of liquidity.

The fourth case of Proposition 4 deserves special attention. In the benchmark CSV model, altering the firm's endowment of equity has no impact on the characteristics of the optimal contract. The reason is because, given an increase in equity, the firm expands its size proportionally so that the optimal amount of leverage, $l_0 = k_0/n_0$, remains unchanged. This result does not carry through in our framework with endogenous secondary market liquidity. As in the benchmark model—indeed, for exactly the same reason—there is no *direct effect* of an increase in equity on the optimal contract. (Hence, absent imperfect liquidity in the OTC market we would obtain the same result as the benchmark model.) But, our framework is different in that an increase in equity raises the number of debt contracts issued in the primary market. To see this consider that the firm's budget constraint expressed in terms of leverage is given by, $b_0 = n_0(l_0 - 1)$. In order for l_0 to remain unchanged, the firm must increase primary issuance in proportion to the size of the equity injection. But, this alters liquidity because it raises the number of possible sell orders in the secondary market, which investors will price through the liquidity premium. Thus, in our framework equity influences the capital structure of the firm *indirectly* by altering secondary market liquidity.

Finally, we note that the link between the liquidity premium and the optimal capital structure of the firm has the following corollary.

Corollary 1 (Default Premium Comparative Statics) *In the private equilibrium, the default premium Φ^d decreases when*

1. (Liquidity shock) *The probability of becoming impatient is higher (i.e., δ is higher);*
2. (Impatience) *Impatient investors discount the future more heavily (i.e., β is lower);*
3. (Investors' Endowments) *Investors have fewer funds to invest in storage (i.e., e_0 is lower);*
and
4. (Firms' Endowments) *Firms have more equity (i.e., n_0 is higher).*

This corollary is a direct consequence of Propositions 1 and 4.

3.3 A Numerical Illustration

We present a simple numerical illustration using the following parameter values. We set the initial endowment of entrepreneurs so that $n_0 = 0.2$ and the endowment of investors at $e_0 = 1$. Investors' preferences are described by a discount factor for impatient investors $\beta = 0.85$, while δ will take different values in $[0, 1]$ to illustrate the results established above. Entrepreneurs' expected return is given by $R^k = 1.2$, whereas the return on storage is assumed to be $r = 0.01$. The parameters of the matching function in the OTC market are the scaling constant $\nu = 0.2$ and the elasticity of the matching function with respect to sell orders is $\alpha = 0.5$. The surplus that accrues to impatient investors in the sharing rule is $\psi = 1$. Idiosyncratic productivity shocks ω are distributed according to a log-normal distribution with mean equal 1 and variance equal to 0.25. Finally, monitoring costs are a share $\mu = 0.2$ of firms' revenue.

We begin with the frictionless benchmark, taking $\delta = 0$.¹⁴ The equilibrium of the model is described by entrepreneurs' choice of leverage, l_0 , and risk, $\bar{\omega}$, subject to the constraint imposed by investors' break-even condition and the consistency requirements for liquidity, θ , and price, q_1 , in the secondary market. The characteristics of the optimal contract $(l_0, \bar{\omega})$ determine the hold-to-maturity return, R^b , and thus the secondary market price q_1 . (Recall that the return on executed orders in secondary markets is pinned down by ψ , r , and β .) The optimal contract will determine the portfolio allocation of investors and thus secondary market liquidity θ (equation 19). Thus, we use the $(l_0, \bar{\omega})$ -space to describe the optimal contract and the equilibrium of the model. Figure 4 depicts the firm's isoprofit curves in green.¹⁵ Investors' break-even condition is shown by the red line. Firm's profits increase with leverage and decrease with risk, so isoprofit curves represent higher profits moving south-east in the figure. The private equilibrium in the frictionless benchmark economy is given by the tangency between the break-even condition and the isoprofit line shown by the solid black dot in Figure 4.

Figure 5 illustrates the case of an increase in the liquidity shock, δ , (i.e., the case 1 of Propositions 3 and 4). As the probability of becoming impatient increases, investors require a higher liquidity premium to compensate for liquidity risk (Proposition 3). In contrast, the firm's isoprofit lines for a given contract specified by $(l_0, \bar{\omega})$ are invariant to δ . Nevertheless, the firm adjusts the terms of the contract it offers in the primary market owing to the increase in the liquidity premium. In particular, the firm reduces its supply of primary debt, which partially compensates investors for the reduction in secondary market liquidity. The resulting equilibrium has a lower level of leverage

¹⁴From Proposition 2 the frictionless benchmark is obtained if alternatively we set $\beta = 1/1.01$, or if $\psi = 1$ (as in our example) and e_0 is sufficiently high so $f(\theta) = 1$.

¹⁵Note that the shape of the isoprofit curves (increasing and concave) holds in general, as follows from the properties of the $\Gamma(\bar{\omega})$ function, and does not depend on the particular values used in our example.

and a less risky debt contract, as shown in Figure 5 (Proposition 4).

Finally, Figure 6 presents a decomposition of the total corporate premium Φ^t paid on the primary debt contract in terms of the default premium Φ^d and the liquidity premium Φ^ℓ . The figure shows that lower levels of leverage and risk due to increased liquidity demand result in lower total corporate bond premia. Naturally, the liquidity premium goes up, but the default premium decreases since the firm is offering a lower $\bar{\omega}$ (Corollary 1), and the latter effect dominates in this case.

4 The Efficient Structure of Corporate Debt

We analyze the efficient structure of corporate debt by considering a social planner that is constrained by the presence of matching frictions and the structure of trade in the secondary market. To this extent, our concept of efficiency is one of constrained efficiency, or second best.

The planner chooses the optimal contract to maximize the profits of the firm while internalizing the effect of the capital structure on secondary markets through liquidity and bond prices. To formalize the planner's problem let $(l_0, \bar{\omega}, \theta, q_1)$ be allocations that describe the socially efficient outcome and let $(l_0^{ce}, \bar{\omega}^{ce}, \theta^{ce}, q_1^{ce})$ be the allocations in the competitive equilibrium described in section 3. Then, the planner's problem can be written as

$$\max_{\bar{\omega}, l_0, \theta, q_1} [1 - \Gamma(\bar{\omega})] R^k l_0 \quad (22)$$

subject to:

$$\mathbb{U}(l_0, \bar{\omega}, \theta, q_1) \geq \mathbb{U}(l_0^{ce}, \bar{\omega}^{ce}, \theta^{ce}, q_1^{ce}) \quad (23)$$

and equations (8) and (19).

Condition (23) says that the planner cannot choose equilibrium allocations that result in lower welfare for investors compared to the competitive equilibrium, whereas equations (8) and (19) force the planner to respect the determination of prices and liquidity, respectively, in secondary markets.¹⁶ The social planning problem differs from the competitive equilibrium in two respects: (1) the planner need not respect the investor's break-even condition (15), but may want to influence it to satisfy (23); and (2) the planner internalizes how period 0 choices affect liquidity in the secondary market by explicitly considering (19) as a constraint, which, in contrast, is an equilibrium condition in the competitive economy.¹⁷

¹⁶In an Online Appendix we present a more general problem, where the planner can additionally determine the terms of trade in the secondary market and assigns Pareto weights on the two agents to maximize a social welfare function.

¹⁷Recall that investors, and thus firms, explicitly considered (8) in the competitive equilibrium as well, thus its explicit consideration does not modify the planner's problem relative to the competitive

We substitute equations (8) and (19) in the planner's problem, and let λ be the multiplier on constraint (23), to obtain that the socially optimal choice of leverage is given by

$$[1 - \Gamma(\bar{\omega})]R^k = -\lambda \left[n_0(U_b - U_s) + b_0 \frac{\partial U_b}{\partial l_0} + \frac{\partial \mathbb{U}}{\partial \theta} \frac{\partial \theta}{\partial l_0} \right]. \quad (24)$$

That is, the marginal increase in the firm's profits needs to be proportional to the marginal reduction in total expected utility for investors. The latter has three components: (i) the portfolio composition effect: as leverage increases investors need to re-allocate n_0 units from storage to bonds; (ii) the effect on the expected utility of bond holdings U_b ; and (iii) the effect through secondary market liquidity: as liquidity increases it becomes easier for impatient investors to sell their bonds, but it becomes more difficult for patient investors to buy bonds and earn the return Δ in the secondary market.

Similarly, the socially optimal choice for the risk profile of corporate debt is given by

$$l_0 \Gamma'(\bar{\omega}) R^k = \lambda \left[b_0 \frac{\partial U_b}{\partial \bar{\omega}} + \frac{\partial \mathbb{U}}{\partial \theta} \frac{\partial \theta}{\partial \bar{\omega}} \right]. \quad (25)$$

That is, the marginal increase in the firm's profits need to be proportional to the marginal reduction in total expected utility for investors, which has two components: the effect on the hold-to-maturity return R^b and the effect through secondary market liquidity.

Taking a ratio of equations (24) and (25) gives

$$\frac{1 - \Gamma(\bar{\omega})}{\Gamma'(\bar{\omega}) l_0} = - \frac{n_0(U_b - U_s) + b_0 \frac{\partial U_b}{\partial l_0} + \frac{\partial \mathbb{U}}{\partial \theta} \frac{\partial \theta}{\partial l_0}}{b_0 \frac{\partial U_b}{\partial \bar{\omega}} + \frac{\partial \mathbb{U}}{\partial \theta} \frac{\partial \theta}{\partial \bar{\omega}}}. \quad (26)$$

This equation, together with the constraint on investors total expected utility (23), describes the socially optimal debt contract.¹⁸ We are ready to establish the generic inefficiency of the debt contract in competitive markets.¹⁹

Proposition 5 (Generic Constrained Inefficiency of the Debt Contract) *Consider a planner that designs an optimal debt contract, as described in (23), (26), (8) and (19). If the parameters (α, ψ, r) belong to a generic set \mathcal{P} , the planner will set a level of secondary market liquidity that is different from the competitive equilibrium. That is, the competitive equilibrium is generically constrained inefficient.*

equilibrium, unless the planner can affect the terms of secondary trade.

¹⁸The constraint will always be binding since the planner cares only about the firm, but this need not be the case if the planner maximizes aggregate social welfare. In that case the planner may want to split the aggregate gains according to some set of Pareto weights.

¹⁹See also Geanakoplos and Polemarchakis (1986) for a general characterization of constrained inefficiency.

Given Proposition 5, we can identify two distorted margins that drive a set of wedges between the private and socially efficient outcomes. Comparing the equilibrium conditions (15) and (18) to the social planner's counterparts (23) and (26), the first distortion is evident from the $\partial\mathbb{U}/\partial\theta$ term in equation (26) that does not appear in equation (18). This term captures the liquidity externality. It arises because neither the firm nor investors internalize the effect that their decisions in the primary market have on liquidity in the secondary market. This additional term changes the trade-off between risk and leverage for the planner relative to the firm.

To understand the role of the term $\partial\mathbb{U}/\partial\theta$, which measures the externality of market liquidity on investors ex ante welfare, consider the following reinterpretation of the conditions that determine the optimal contract. Let the negative of risk measure the safety of the financial contract. Then, firms profits are increasing in both leverage and safety, and the optimality conditions can be reinterpreted as equating the marginal benefit with the marginal cost, in terms of investors compensation, of increasing leverage or safety. Using this interpretation, the planner finds that a positive externality increases the compensation required to increase leverage and reduces the compensation required to increase safety. Consequently, a planner that internalizes this externality would reduce leverage and risk (increase safety), leading to higher secondary market liquidity and firm's profits.

The second distortion appears in the optimal portfolio composition of investors. It can be easily seen by comparing the weak Pareto improvement constraint (23) that the planner faces to the break-even condition (15) in the competitive equilibrium, i.e., $U_b = U_s$. Since $U_s = U_a + (1 - \delta)(1 + r)(\Delta - (1 + r))p(\theta)$, we can rewrite equation (23) as $n_0(l_0 - 1)(U_b - U_s) = e_0(1 - \delta)(1 + r)(\Delta - (1 + r))[p(\theta^{ce}) - p(\theta)]$. Written this way, the equation tells us that as long as $\partial\mathbb{U}/\partial\theta \neq 0$ the planner chooses a different level of market liquidity, so that $\theta^{ce} \neq \theta$. In this case, the expected return on holding bonds will not be equated with the return to storage, as must be the case in the competitive equilibrium.

The following proposition summarizes the linkages between these two distortions.

Proposition 6 (Constrained Efficient Equilibrium) *The constrained efficient allocations can be characterized conditional on the model parameters (α, r, ψ) as follows:*

- *If $\psi(1 + \alpha r) > \alpha(1 + r)$ then secondary market liquidity generates a positive externality on investors ($\partial\mathbb{U}/\partial\theta > 0$); the planner implements a higher level of secondary market liquidity ($\theta > \theta^{ce}$); and the optimal capital structure of the firm is characterized by lower leverage, $l_0 < l_0^{ce}$, and less risk, $\bar{\omega} < \bar{\omega}^{ce}$.*
- *If $\psi(1 + \alpha r) < \alpha(1 + r)$ then secondary market liquidity generates a negative externality on investors ($\partial\mathbb{U}/\partial\theta < 0$); the planner implements a lower level of secondary market*

liquidity ($\theta < \theta^{ce}$); and the optimal capital structure of the firm is characterized by higher leverage, $l_0 > l_0^{ce}$, and more risk, $\bar{\omega} > \bar{\omega}^{ce}$.

- If $\psi(1 + \alpha r) = \alpha(1 + r)$ then there is no externality ($\partial \mathbb{U} / \partial \theta = 0$) constrained efficient, i.e., $(l_0, \bar{\omega}, \theta) = (l_0^{ce}, \bar{\omega}^{ce}, \theta^{ce})$.

To understand the intuition behind the proposition, consider first the role that liquidity has on investors' welfare. On the one hand, an increase in liquidity generates ex ante welfare gains for impatient investors simply because they will have an easier time selling unwanted corporate debt in secondary markets. On the other hand, patient investors suffer welfare losses as it becomes more difficult to earn a higher return by purchasing bonds at a discounted price in the secondary market. Whether investors are ex ante better off with higher liquidity depends on the parameterization of (α, r, ψ) .

In particular, the gains to impatient investors outweigh the losses to patient investors, making ex ante investors better off, when $\psi(1 + \alpha r) > \alpha(1 + r)$. This occurs when the trade surplus that accrues to impatient investors is sufficiently large relative to the elasticity of the matching function, $\frac{\psi}{\alpha} > (1 + r)/(1 + \alpha r)$.²⁰ Or, alternatively, when the return to storage is sufficiently low such that $r < (\psi - \alpha)/(\alpha - \alpha\psi)$. In either case, we say the liquidity externality is positive because ex ante investors benefit from an increase in market liquidity.

How can the planner implement a higher level of liquidity in a way that increases the profitability of firms? Recall from equation (19) that liquidity can be expressed as a function of the characteristics of the firm's debt contract, $\theta(l_0, \bar{\omega})$. Furthermore, we know $\partial \theta / \partial l_0 < 0$ and $\partial \theta / \partial \bar{\omega} < 0$. So, from the firm's perspective, the planner can increase secondary market liquidity by directing the firm to take on less leverage, $l_0 < l_0^{ce}$, and write debt contracts that are less risky, $\bar{\omega} < \bar{\omega}^{ce}$. Profitability increases because, despite the reduction in scope owing to lower leverage, the firm reduces its overall cost of funding. Higher liquidity lowers the liquidity premium and the less risky nature of the debt contracts lowers the default premium.²¹

Increasing liquidity in this way has, by design, implications for the portfolio composition of investors. Specifically, it requires that investors shift out of corporate bonds and into storage. At the same time, increasing secondary market liquidity depresses the return to storage (given that $\partial p(\theta) / \partial \theta < 0$) and increases the return on bond holdings (given that $\partial f(\theta) / \partial \theta > 0$). So, investors are being asked to shift their portfolios out of higher return corporate bonds and into storage, which offers a lower return. The

²⁰The parameter restriction is analogous to the Hosios (1990) rule that determines the efficient surplus split in search and matching models of the labor market. Arseneau and Chugh (2012) study the implications of inefficient surplus sharing for optimal labor taxation in a dynamic general equilibrium economy.

²¹It is interesting to note that by implementing higher secondary market liquidity, the planner in essence increases *funding* liquidity in the primary market by implementing a reduction in total bond premia.

only way such an outcome can be supported without making investors worse off is if the expected return for holding bonds in the more liquid portfolio outweighs the expected return from holding storage, so that ($U_b > U_s$). In other words, the only way to support allocations that deliver higher liquidity is to violate the breakeven condition.

The opposite intuition holds when $\psi(1+ar) < \alpha(1+r)$, so that the liquidity externality is negative and the planner desires less liquidity relative to the competitive equilibrium. In this case it will implement higher bond premia, but make the firm better off by increasing firm's leverage. Finally, in the knife-edge case where $\psi(1+ar) = \alpha(1+r)$ private liquidity is efficient so that at the margin an increase in liquidity generates gains for impatient investors that are perfectly offset by losses to patient investors and the planner cannot exploit the externality to improve upon the competitive equilibrium.

4.1 Decentralizing the Efficient Equilibrium

A complete set of tax instruments allows us to decentralize the efficient equilibrium. We introduce a marginal tax τ^s on the return from storage U_s ($\tau^s < 0$ corresponds to a subsidy) and a marginal tax τ^l on leverage ($\tau^l < 0$ corresponds to a subsidy). With these tax instruments, the objective of investors becomes $\mathbb{U} = b_0 U_b + s_0 U_s (1 - \tau^s) + T^s$ and the objective of the firm changes to $[1 - \Gamma(\bar{\omega})] R^k l_0 - \tau^l \lambda l_0 + T^l$. The taxes are funded in a lump-sum fashion on the same agents, thus $T^l = \tau^l \lambda l_0$ and $T^s = \tau^s s_0 U_s$ in equilibrium. Also, in order to simplify the exposition note that we have normalized the tax on leverage by the Lagrange multiplier, $\lambda > 0$, on the constraint faced by firms in the competitive economy (i.e., the investor's break-even condition).

Proposition 7 provides a general characterization of the optimal tax policy.

Proposition 7 (Optimal Policy) *The planner's solution can be decentralized by levying distortionary taxes on the portfolio allocation decision of investors and the capital financing decision of firms. The resulting optimal taxes on investors storage, τ^s , and firms' leverage, τ^l , are given by:*

$$\tau^s = \frac{e_0}{b_0} \left(1 - \frac{U_s(\theta^{ce})}{U_s(\theta)} \right), \quad (27)$$

$$\tau^l = \frac{n_0 U_s \frac{\partial U_b}{\partial \bar{\omega}} \tau^s + \left[\frac{\partial U_b}{\partial l_0} \frac{\partial \theta}{\partial \bar{\omega}} - \frac{\partial U_b}{\partial \bar{\omega}} \frac{\partial \theta}{\partial l_0} \right] \frac{\partial \mathbb{U}}{\partial \theta}}{b_0 \frac{\partial U_b}{\partial \bar{\omega}} + \frac{\partial \theta}{\partial \bar{\omega}} \frac{\partial \mathbb{U}}{\partial \theta}} \quad (28)$$

where the term in square brackets and the denominator in (28) are strictly positive.

Combining the insights of Proposition 7 with Proposition 6 above, it is easy to characterize the optimal tax system more specifically. When $\psi(1+ar) > \alpha(1+r)$, the liquidity externality is positive so that the planner wants to implement higher liquidity relative to the competitive equilibrium, $\theta > \theta^{ce}$. Accordingly, the optimal tax system needs to be designed in a way that results in investors holding a more liquid portfolio.

This can be achieved through a storage subsidy, so that $\tau^s < 0$. Moreover, the optimal tax system needs to be designed in a way that results in firms issuing fewer debt contracts in the primary market, which can be achieved through a tax on leverage, so that $\tau^l > 0$. By the same logic, when $\psi(1 + \alpha r) < \alpha(1 + r)$, the liquidity externality is negative, $\theta < \theta^{ce}$. The optimal tax system calls for a tax on storage, $\tau^s > 0$, and leverage subsidy, $\tau^l = 0$. Only in the knife-edge case of constrained efficiency, where $\psi(1 + \alpha r) = \alpha(1 + r)$, do we have that $\tau^l = \tau^s = 0$.

4.2 A Numerical Illustration

We continue the numerical example in section 3.3. Recall that in this illustration, $\psi = 1$. Moreover, because the planner has the same objective as the competitive firm, the isoprofits lines are the same in both problems. Figure 7 shows the planner's solution and the private equilibrium for two cases: $\delta = 0$ and $\delta = 0.1$. In a frictionless environment ($\delta = 0$), the planner's solution coincides with the private equilibrium (as we proved in Proposition 2). However, when there is a positive demand for liquidity, $\delta > 0$ and $\beta < (1 + r)^{-1}$, and secondary market liquidity is not sufficiently high to guarantee $f(\theta) = 1$, the planner chooses lower leverage and a less risky capital structure, i.e. lower l_0 and $\bar{\omega}$. The reason is because the planner internalizes the affect of the leverage decision on liquidity in the secondary market. This induces the planner to consider a steeper constraint compared to the breakeven condition considered by competitive firms (where market liquidity is taken as given). As a result, the planner understands how lower leverage and risk improves borrowing terms on the margin, when the total social costs are taken into account.

Table 1 shows the change in equilibrium allocations between the competitive and planner's solutions for $\delta = 0.1$ as ψ moves from 1 to 0. Consistent with the analysis above, the planner's allocations can be replicated by imposing a tax (subsidy) on leverage and storage. For $\psi = 1$, the liquidity externality is positive, implying that liquidity is suboptimally low in the competitive equilibrium. The planner would like to implement a tax on leverage to generate more liquidity in the secondary market. However, as the share of the gains from trade that accrues to impatient investors declines, the size of the liquidity externality shrinks. Hence, the planner is less aggressive in choosing the optimal combination of leverage tax and storage subsidy (i.e., both τ^l and τ^s shrink in absolute value as θ^{CE} increases to a value that is closer to θ). When the parameterization of ψ satisfies $\psi(1 + \alpha r) = \alpha(1 + r)$, the externality zeros out and the optimal tax system implies $\tau^l = \tau^s = 0$. For values of ψ below that point, the liquidity externality becomes negative, so that liquidity is over-provided in the competitive equilibrium. Accordingly, the sign of the optimal tax system flips so that leverage is subsidized, $\tau^l < 0$, and storage is taxed, $\tau^s > 0$, in order to reduce θ to a level that is closer to θ^{CE} .

5 Quantitative Easing as part of the Optimal Policy Mix

Many central banks following the Great Recession of 2007-09 have turned to unconventional monetary policies, such as quantitative easing (QE), to provide further monetary accomodation after they reduced standard policy rates to its minimum feasible levels. Ultimately, the goal of QE is for the central bank to influence the real economy through direct intervention in the markets for certain assets. Our model provides a stylized framework to analyze the effect of these policies.

5.1 Quantitative Easing Policy

We model QE through direct purchases by the central bank of long-term illiquid assets (i.e., the financial contracts issued by firms and which are retraded by investors in OTC markets, much like Treasuries and MBS). These purchases are financed by the issuance of short-term liquid liabilities (referred to as reserves) that offer a return that is at least as large as that offered by the storage technology. This seems a reasonable approximation for the policies implemented by the Federal Reserve during the Great Recession, where lending facilities and asset purchases were financed primarily with redeemable liabilities in the form of reserves (see Carpenter et al. 2013)..

In period $t = 0$, and before markets open, the central bank announces the quantity of bonds, \bar{b}_0 , it will purchase in period 0 and commits to holding those bonds to maturity. These bond purchases are financed through the issuance of \bar{s}_0 units of reserves that pay interest $\bar{r} \geq r$. Thus, the central bank budget constraint in period 0 is simply

$$\bar{b}_0 = \bar{s}_0 . \tag{29}$$

We assume the central bank finances itself in period 1 with reserves only. This assumption prevents the central bank from injecting additional resources into the economy in the interim period. Thus, in period 1 the central bank keeps its bond holdings, so it needs to roll over its outstanding reserves \bar{s}_0 and pay interest on them. The central bank will have to borrow an amount equal to $(1 + \bar{r})\bar{s}_0$.²² Finally, in period 2 the central bank receives the debt payout from the financial contract and expends $(1 + \bar{r})^2\bar{s}_0$ in interest and principal on outstanding reserves. It is assumed that the central bank allocates reserves evenly across investors who demand reserves in a given time period.

The central bank faces three constraints that, taken together, serve to limit the size of its QE program. First, we assume that the central bank is at a disadvantage

²²In practice, the long-term assets held by central banks pay interest in the interim period, and in an environment of low short-term interest rates these holdings will generate a positive net-interest income for the central bank. But for simplicity we abstract from these considerations. See, for instance, Carpenter et al (2013) for estimates of net-interest income for the Federal Reserve.

relative to the private sector in monitoring investment projects. It thus needs to pay a higher monitoring cost relative to investors, denoted by $\bar{\mu} > \mu$.²³ This implies that in expectation the central bank anticipates receiving $\bar{R}^b \bar{b}_0$ for its asset holdings, with \bar{R}^b the expected hold-to-maturity return on financial contracts for the central bank, given by

$$\bar{R}^b(l_0, \bar{\omega}) = \frac{l_0}{l_0 - 1} R^k [\Gamma(\bar{\omega}) - \bar{\mu} G(\bar{\omega})] = R^b(l_0, \bar{\omega}) - \frac{l_0}{l_0 - 1} R^k (\bar{\mu} - \mu) G(\bar{\omega}) .$$

Second, the central bank needs to fully finance its funding cost, i.e., the total interest on reserves, with its expected return on assets. That is,

$$\bar{R}^b \geq (1 + \bar{r})^2 . \quad (30)$$

Finally, we assume that investors cannot be made worse off by QE, as we describe in section 5.3.

5.2 Investors' Problem and Liquidity with QE

In period 0 investors allocate their wealth across three assets: the storage technology, debt contracts, and reserves. So the budget constraint at $t = 0$ is given by

$$s_0 + \bar{s}_0 + b_0 = e_0 ,$$

with $s_0, \bar{s}_0, b_0 \geq 0$. Following the approach of Section 2, we consider the optimal behavior of impatient and patient investors in $t = 1$ when they arrive with a generic portfolio of storage, reserves, and bonds (s_0, \bar{s}_0, b_0) .

Impatient Investors. By Assumption 3 impatient investors want to consume all their wealth at $t = 1$. They can consume the payouts of their liquid assets: $(1 + r)s_0 + (1 + \bar{r})\bar{s}_0$; in addition, they can consume the proceeds from their sell orders in the OTC market: q_1 units of consumption for each order executed. Thus, the expected consumption of impatient investors in periods 1 and 2, respectively, is given by

$$c_1^I = (1 + r)s_0 + (1 + \bar{r})\bar{s}_0 + f(\theta)q_1 b_0 , \quad (31)$$

$$\text{and} \quad c_2^I = (1 - f(\theta))R^b b_0 . \quad (32)$$

Patient Investors. Patient investors only value consumption in the final period and, as a result, are willing to place buy orders in the OTC market because the return

²³Consequently, any positive effects of QE would not accrue from enhanced monitoring, as in the delegated monitoring models of Diamond (1984) and Krasa and Villamil (1992), but from its effect on liquidity premia.

from doing so, Δ , is strictly greater than the return on storage, $1 + r$. Moreover, it is also the case that the return on reserves, $1 + \bar{r}$, is at least as large as that on storage, so patient investors are willing to allocate liquid wealth to reserves. Accordingly, liquidity provision in the secondary market will depend on the return on OTC trade, Δ , relative to the return on reserves, $1 + \bar{r}$. Specifically, if $1 + \bar{r} < \Delta$ patient investors will pledge *all* their liquid wealth to place buy orders in the OTC market. On the other hand, if $1 + \bar{r} > \Delta$ patient investors will use their liquid wealth to buy higher yielding reserves first and then allocate the remainder of their liquid wealth to placing buy orders in the OTC market. For expositional purposes, we assume throughout the remainder of the paper that $1 + \bar{r} < \Delta$ (although for the main results of this section—stated below in Propositions 8 and 9—we trace out the proofs over the entire parameter space of the model, where appropriate).

When the anticipated return to OTC trade exceeds the return on reserves, patient investors use $(1 + r)s_0 + (1 + \bar{r})\bar{s}_0$ units of liquid wealth to place buy orders. A fraction $p(\theta)$ are matched allowing patient investors to exchange liquid wealth for corporate debt, while the $1 - p(\theta)$ unmatched portion needs to be reinvested in liquid assets in period $t = 1$. Because the central bank needs to finance itself in the interim period, it removes a total of $(1 + \bar{r})\bar{s}_0$ reserves from a mass $1 - \delta$ of patient investors. Individual reserve holdings in the interim period for patient investors, \bar{s}_1^P , totals $(1 + \bar{r})\bar{s}_0/(1 - \delta)$. All remaining liquid funds are placed into the lower yielding storage technology, so expected storage holdings at the end of $t = 1$, s_1^P , equal

$$s_1^P = (1 - p(\theta)) [(1 + r)s_0 + (1 + \bar{r})\bar{s}_0] - \frac{(1 + \bar{r})\bar{s}_0}{1 - \delta},$$

which is strictly positive from Assumption 4. It follows that expected consumption of patient investors equals

$$c_2^P = s_1^P(1 + r) + \frac{(1 + \bar{r})^2\bar{s}_0}{1 - \delta} + \left\{ b_0 + p(\theta) \frac{(1 + r)s_0 + (1 + \bar{r})\bar{s}_0}{q_1} \right\} R^b. \quad (33)$$

Using the optimal behavior of investors in period 1, summarized in equations (31)-(33), we can rewrite the expected lifetime utility as the portfolio weighted average of the utilities of the three assets available in the initial period:

$$\mathbb{U} = U_s s_0 + U_{\bar{s}} \bar{s}_0 + U_b b_0.$$

As before, the expected utility of investing in storage and bonds, U_s and U_b , are given by equations (13) and (14), respectively. On the other hand, the expected utility of

reserves is given by

$$U_{\bar{s}} = \delta(1 + \bar{r}) + (1 - \delta)(1 + \bar{r}) \left[(1 - p(\theta))(1 + r) + \frac{\bar{r} - r}{1 - \delta} + p(\theta)\Delta \right]. \quad (34)$$

Reserves yield $1 + \bar{r}$ for patient investors. For impatient investors, there is additional compensation that comes from the expected return from buy orders in the secondary market, plus the spread between reserves and storage, $\bar{r} - r$, for the additional reserves bought in period 1.²⁴

We are now ready to establish the link between QE and secondary market liquidity.

Proposition 8 (The Real Effects of Quantitative Easing) *Quantitative easing, i.e., the size of the bond buying program, \bar{b}_0 , increases secondary market liquidity θ and, hence, has implications for the firm's optimal capital structure.*

The intuition behind this result is straightforward, each bond bought by the central bank will be held to maturity, reducing the number of sell orders in the secondary market. At the same time, these bonds need to be financed with reserves, which patient investors can use to submit additional buy orders in the secondary market. So, a bond buying program affects secondary market liquidity *directly* through the purchase of bonds as well as *indirectly* through the liquidity created by issuing central bank reserves. Moreover, Remark 2 establishes an equilibrium link between liquidity and the optimal capital structure of the firm.

Now we turn to main results regarding QE.

5.3 QE and Optimal Policy

To understand the role of QE in the optimal policy mix, we assume the planner to maximize firm profits, but is restricted by the central bank budget constraint, equation (29), and the financing constraint, equation (30). In addition, as with the planner in Section 4, we assume the QE program cannot make investors worse off. To write this later constraint, let $\mathbb{U}(l_0, \bar{\omega}, \theta, \bar{b}_0, \bar{r})$ be the expected lifetime utility of investors when the equilibrium is described by $(l_0, \bar{\omega}, \theta)$, with the secondary market price given by (8), and the QE program described by (\bar{b}_0, \bar{r}) . Similarly, let $\mathbb{U}(l_0^{ce}, \bar{\omega}^{ce}, \theta^{ce})$ be the expected lifetime utility of investors in the competitive equilibrium, when the secondary market price is given by (8). We refer to this planner that have access to QE policies as the central bank. Then, the central bank's problem can be written as

$$\max_{l_0, \bar{\omega}, \theta, \bar{b}_0, \bar{r}} [1 - \Gamma(\bar{\omega})] R^k l_0 \quad (35)$$

²⁴If, $1 + \bar{r} > \Delta$, patient investors will use their liquid wealth first to buy reserves, and then will use their remaining liquid wealth to place buy orders in the OTC market. Proceeding as above we can derive for patient investors s_1^P , c_2^P , and $U_{\bar{s}}$.

subject to:

$$\mathbb{U}(l_0, \bar{\omega}, \theta, \bar{b}_0, \bar{r}) \geq \mathbb{U}(l_0^{ce}, \bar{\omega}^{ce}, \theta^{ce}) \quad (36)$$

and equations (19), (29) and (30).

The following proposition characterizes the role of QE as part of the optimal policy mix.

Proposition 9 (Quantitative Easing as Part of the Optimal Policy Mix) *The optimal design of QE is conditional on the model parameters (α, r, ψ) as follows:*

- *If $\psi(1 + \alpha r) > \alpha(1 + r)$, then QE improves upon the constrained efficient allocation; the optimal QE program consists of a positive bond buying program, $\bar{b}_0 > 0$, and paying interest on reserves that are strictly greater than the return on storage, $\bar{r} > r$.*
- *If $\psi(1 + \alpha r) \leq \alpha(1 + r)$, then QE does not improve upon the constrained efficient allocation; the optimal QE program is just $\bar{b}_0 = 0$ and $\bar{r} = r$.*

As long as the liquidity externality is positive (i.e., liquidity is suboptimally low in the private equilibrium), a QE program can lead to a Pareto improvement over the constrained efficient allocations studied in Section 4. The reason this is possible is because the central bank can finance its purchases of long-term illiquid corporate debt by issuing liquid liabilities to investors subject to liquidity risk, much like a deposit contract offered by banks. The central bank has an advantage over a typical bank, however, in that it is not subject to runs by investors. In this sense, a central bank that is not subject to liquidity risk effectively enhances the intermediation technology of the economy. This technological improvement can only be realized when there are social gains from raising liquidity. When the liquidity externality is negative (i.e., liquidity is suboptimally high in the competitive equilibrium), QE is ineffective because the central bank cannot take a short position in the primary corporate debt market.

It is useful to point out that the proposition suggests QE is effective when the interest rate on storage is sufficiently low, $r < \psi/(1 + \alpha - \alpha\psi)$.²⁵ Although it is beyond the scope of this model, these conditions indicate that QE may be an effective policy response in a protracted low interest rate environment.

The other issue worth mentioning is that when QE is effective, the absence of constraints that limit the size of the program could lead to an extreme outcome in which the central bank disintermediates the bond market. That is, if there is nothing holding back the size of the program, as long as QE is effective, the optimal policy is for the central bank to buy all the bonds offered by the firm and offer the corresponding amount of reserves to investors, paying $\bar{r} = r$. Doing so allows the central bank to replicate the frictionless benchmark of section 3.1. However, as mentioned above, the

²⁵Alternatively, $\psi > \alpha(1 + r)/(1 + \alpha r)$ or $\alpha < \psi/(1 + r - r\psi)$.

size of the QE program is limited in our model by: (1) the higher monitoring cost that the central bank pays relative to investors; (2) the fact that the expected return on assets cannot be lower than the total cost of reserves; and (3) investors cannot be made worse off.

5.4 A Numerical Illustration

Table 2 extends our numerical example to study QE. The table shows the changes in allocations relative to the competitive equilibrium for three different economies. The first column shows the decentralization of the socially efficient outcomes (through the leverage tax, τ^l , and storage subsidy, τ^s) without QE, the second column shows the effects of QE by itself, and, finally, the third column shows QE in conjunction with optimal tax policy. All cases assume the parameterization $\alpha = 0.5$, $\psi = 0.9$, and $r = 0.01$, unless otherwise noted. We choose this parameterization because it puts the model in a region of the parameter space where QE is effective, as per proposition 9. In addition, we assume the $\bar{\mu} = 0.3$, which is 50% higher than the baseline value of $\mu = 0.3$.

The first column (which, for reference, corresponds to a point half way between the results shown in the first and second columns of table 1) shows that in absence of QE, the efficient allocation is decentralized with a leverage tax, $\tau^l = 0.21$, and a modest subsidy for storage, $\tau^s = -0.04$. By raising liquidity in the secondary market, and hence depressing the liquidity premium, the resulting reduction in funding costs raises profits by 0.14% relative to the competitive equilibrium, leaving the utility of investors unchanged. The second column presents results where we shut down the tax system, but allow the planner access to a QE program. Even when we shut down the tax system, so that $\tau^l = \tau^s = 0$, the planner can use QE to achieve an even greater increase in firm profitability without harming investors. The central bank is able to improve the intermediation technology in the economy by directly intervening in the primary debt market, financing its bond purchases through the issuance of reserves (upon which the central bank must pay investors a premium above the return on storage). With QE the planner can achieve a similar outcome in terms of liquidity, without tax instruments. Finally, the last column of the table shows that QE, by itself, is not a panacea. A planner can do even better by implementing QE in conjunction with tax policy. The way to interpret this last result is that although QE improves the intermediation technology in the economy, it does nothing to remove the underlying distortions.

Figure 8 shows how the gains to the firm vary with ψ for different levels of the efficiency of the central bank monitoring technology. The thin lines show the case for $\mu = 0.3$ assuming QE in conjunction with the optimal tax system (the thick solid line) and, alternatively, assuming QE alone with no supporting tax system (the thick dashed

line). The thin solid and dashed lines correspond to the same information when the monitoring cost is higher, so that $\mu = 0.35$. Finally, the thick blue line shows the gains to the firm from optimal tax policy alone in absence of QE. There are four things to take from the figure. First, QE is always more effective when combined with the optimal tax policy (the solid lines are always above the dashed line for the same monitoring cost assumption). Second, the effectiveness of QE is limited by the parameterization of ψ (the dashed lines are downward sloping, so that as the gains from trade that accrue to impatient investors declines, QE becomes less effective). Third, the effectiveness of QE depends importantly on the quality of the central bank's monitoring technology (the thin lines are below the thick ones, so the worse the technology, the less effective is QE). Finally, there are parts of the parameter space in which QE is ineffective to the point at which a planner would strictly prefer optimal taxation to QE (the regions in which the thick and thin dashed lines lie below the solid blue line).

6 Conclusion

We presented a model to study the feedback loop between secondary market liquidity and firm's financing decisions in primary capital markets. We showed that imperfect secondary market liquidity accruing from search frictions results in positive liquidity premia, lower levels of leverage—or equivalently lower debt issuance,—and less credit risk in primary markets. Lower issuance in primary markets enhances liquidity in secondary markets, but this effect is not enough to offset the rise of liquidity premia.

Furthermore, this feedback loop creates externalities operating via secondary market liquidity, as private agents do not internalize how their borrowing and liquidity provision decisions affect secondary market liquidity. This externality changes the trade-off between risk and leverage and *generically* makes the competitive equilibrium constrained inefficient. We showed how efficiency can be restored by correcting two distorted margins: one on firms and one on investors. Finally, we showed how unconventional policies like quantitative easing are expected to affect both secondary market liquidity and debt issuance in primary capital markets. By substituting illiquid assets for liquid short-term securities, these policies increase the intermediation capacity of the economy, and under some circumstances may lead to an improvement on the productive capacity of the economy. Our analysis suggests that these type of policies ought to be implemented in conjunction with other policies, in particular policies to limit corporate borrowing.

In our model liquidity holdings by investors can be either too low or too high relative to the efficient level, with borrowing by firms being too high or too low, respectively. This inefficiency arises as both type of agents fail to internalize how they affect secondary market liquidity. The result is similar to other results in the literature

of overborrowing and liquidity under supply. However, our result has different policy prescriptions as two policy tools are needed to restore efficiency. This contrasts with previous results, which have just focussed on one of these inefficiencies (Fostel and Geanakoplos, 2008; Farhi, Golosov and Tsyvinski, 2009); or where borrowers are also liquidity providers and one policy instrument is enough to restore efficiency (Holmström and Tirole, 1998; Caballero and Krishnamurthy, 2001; Lorenzoni, 2008; Jeanne and Korinek, 2010; Bianchi, 2011).

Our model suggest a set of testable predictions for the relationship between the availability of short-term liquid assets and liquidity premia. In our model there is only one set of investors who participate in OTC markets, but in practice there are many, potentially segmented OTC markets. In this context, the intuition of our model will predict that liquidity premia for a given asset, should be inversely related to the liquidity of the portfolio of the participants in the OTC market for that asset. Along these lines, our model predicts that quantitative easing financed with bank reserves should have an effect on the liquidity premia of *all* the securities traded in OTC markets where banks are relevant participants, not only affect the liquidity premia of the illiquid assets purchased by central banks.

Finally, this paper leaves open questions that we are taking on future work. First, we would like to explore the quantitative relevance of the mechanisms described herein. For that we have deliberately stayed very close to the quantitative model of Bernanke et al. (1999), and we are planning to explore the quantitative prescription of our model. Second, in practice many different assets are traded in OTC markets, a dimension that we have abstracted from in our analysis but seems important in practice. Future work should explore the relationship between market segmentation in OTC trade and secondary market liquidity. Two important considerations that we abstracted from will have to be accounted for in this work: what are the strategic incentives in such an environment?, and, how is liquidity allocated across these markets?

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Appendix

A Proofs

Proof of Theorem 1: We need to show that there is a unique equilibrium, and that in this equilibrium credit is not rationed. For that, first, we rule out credit rationing equilibria (Part 1). Then, we establish existence of a non-rationed equilibria (Parts 2-3). Finally, we establish the uniqueness (Part 4).

Part 1. Rule out credit rationing equilibrium.

First of all, note that from Assumption 2, $\bar{\omega}dF(\bar{\omega})/(1 - F(\bar{\omega}))$, is increasing so

$$1 = \mu \frac{\bar{\omega}dF(\bar{\omega})}{1 - F(\bar{\omega})}$$

has only one root, which is strictly positive and we denote by $\bar{\bar{\omega}} > 0$.

Note that from the definition of $\Gamma(\omega)$ and $G(\omega)$ it follows that for $\bar{\omega} > 0$

$$\begin{aligned} \Gamma(\bar{\omega}) &> 0, & 1 - \Gamma(\bar{\omega}) &= \mathbb{P}(\omega \geq \bar{\omega})\mathbb{E}[\omega - \bar{\omega} | \omega \geq \bar{\omega}] > 0 \\ 1 > \Gamma'(\bar{\omega}) &= 1 - F(\bar{\omega}) > 0, & \Gamma''(\bar{\omega}) &= -dF(\bar{\omega}) < 0 \\ 0 < G(\bar{\omega}) &< 1, & \mu G(\bar{\omega}) &< G(\bar{\omega}) < \Gamma(\bar{\omega}) \\ G'(\bar{\omega}) &= \bar{\omega}dF(\bar{\omega}) > 0, & G''(\bar{\omega}) &= dF(\bar{\omega}) + \bar{\omega} \frac{d(dF(\bar{\omega}))}{d\bar{\omega}} \end{aligned} \quad (\text{A.1})$$

$$\lim_{\bar{\omega} \rightarrow 0} \Gamma(\bar{\omega}) = 0, \quad \lim_{\bar{\omega} \rightarrow \infty} \Gamma(\bar{\omega}) = \bar{\omega}\mathbb{P}(\omega \geq \bar{\omega}) + \mathbb{P}(\omega < \bar{\omega})\mathbb{E}[\omega | \omega < \bar{\omega}] = 1$$

$$\lim_{\bar{\omega} \rightarrow 0} G(\bar{\omega}) = 0 \quad \text{and} \quad \lim_{\bar{\omega} \rightarrow \infty} G(\bar{\omega}) = 1$$

Then

$$\Gamma'(\bar{\omega}) - \mu G'(\bar{\omega}) = (1 - F(\bar{\omega})) \left(1 - \mu \frac{\bar{\omega}dF(\bar{\omega})}{1 - F(\bar{\omega})} \right) \begin{cases} > 0 & \text{if } \bar{\omega} < \bar{\bar{\omega}} \\ = 0 & \text{if } \bar{\omega} = \bar{\bar{\omega}} \\ < 0 & \text{if } \bar{\omega} > \bar{\bar{\omega}} \end{cases}.$$

On the other hand, the investors break-even condition (15) defines a relationship between risk $\bar{\omega}$ and leverage l_0 that we can characterize as follows. Let $\bar{\omega}^{ibec}(l_0)$ be the correspondence that gives the values of risk compatible with the break-even condition for a given value of leverage, then these values of risk are implicitly defined by

$$U_s(\theta) = u_b(\theta)R^b[l_0, \bar{\omega}^{ibec}(l_0)],$$

where

$$u_b(\theta) \equiv \delta [f(\theta)\Delta^{-1} + (1 - f(\theta))\beta] + (1 - \delta).$$

Since investors and the firm take secondary market liquidity, θ , as given for any $\bar{\omega} \neq \bar{\bar{\omega}}$, applying the Implicit Function Theorem we have that

$$\frac{d\bar{\omega}^{ibec}}{dl_0} = - \frac{\frac{\partial U_b}{\partial l_0}}{\frac{\partial U_b}{\partial \bar{\omega}}} = - \frac{\frac{\partial R^b}{\partial l_0}}{\frac{\partial R^b}{\partial \bar{\omega}}} \quad (\text{A.2})$$

In fact, from equation (4) we have that

$$\frac{\partial R^b}{\partial l_0} = -\frac{R^b}{l_0(l_0 - 1)} < 0 \quad \text{and} \quad \frac{\partial R^b}{\partial \bar{\omega}} = \frac{R^b[\Gamma'(\bar{\omega}) - \mu G'(\bar{\omega})]}{\Gamma(\bar{\omega}) - \mu G(\bar{\omega})} \quad (\text{A.3})$$

so we can apply the Implicit Function Theorem for $\bar{\omega} \neq \bar{\bar{\omega}}$. It follows that the firm will never choose a contract with risk $\bar{\omega} > \bar{\bar{\omega}}$, as firm profits are decreasing in $\bar{\omega}$, and for $\bar{\omega} > \bar{\bar{\omega}}$ additional risk will reduce the return to investors and they will not be willing to extend additional credit (higher leverage) at these higher risk levels.

An equilibria with $\bar{\omega} = \bar{\bar{\omega}}$ constitute a credit rationing equilibrium as credit is rationed and is not at the margin trading off the benefits to the firm with the costs to investors. We want to rule out that such an equilibrium exists. Note that using a similar argument as above we have that there exists a function $l_0^{ibec}(\bar{\omega})$ that gives the single value of leverage consistent with the break-even condition, and let $\bar{l}_0 = l_0^{ibec}(\bar{\bar{\omega}}) > 0$. Then, there are three potential types of credit rationing equilibria: (i) $l_0 < \bar{l}_0$; (ii) $l_0 = \bar{l}_0$; and (iii) $l_0 > \bar{l}_0$.

Suppose in equilibrium the firm chooses $(l_0, \bar{\omega})$ with $l_0 < \bar{l}_0$. Then, from equation (A.3) it must that $u_b(\theta)R^b(l_0, \bar{\omega}) > U_s(\theta)$. But then the firm can do better by lowering the risk of the contract, while still offering enough compensation to investors to hold bonds, so this cannot be an equilibrium. On the other hand, if equilibrium is described by $(l_0, \bar{\omega})$ with $l_0 > \bar{l}_0$, then $u_b(\theta)R^b(l_0, \bar{\omega}) < U_s(\theta)$. Then, investors at $t = 0$ will allocate all their wealth to storage, which is a contradiction with $\bar{l}_0 > 0$.

Finally, consider the case where the equilibrium is given by $(\bar{l}_0, \bar{\bar{\omega}})$. This contract is suboptimal for the firm as

$$\Gamma'(\bar{\bar{\omega}}) = (1 - F(\bar{\bar{\omega}})) > 0 ,$$

which is incompatible with the optimality conditions (16) and (17). Intuitively, the firm can give up an infinitesimal amount of leverage for an infinite reduction of risk, so it will never choose these contract terms in equilibrium.

Part 2. Rewrite the equilibrium conditions as a single-valued equation $\mathcal{H}(\bar{\omega})$.

Note that from equation (18), which characterizes the optimal contract in a non-rationing equilibrium, we can rearrange to get

$$l_0(\bar{\omega}) = 1 + \frac{\Gamma(\bar{\omega}) - \mu G(\bar{\omega})}{1 - \Gamma(\bar{\omega})} \frac{\Gamma'(\bar{\omega})}{\Gamma'(\bar{\omega}) - \mu G'(\bar{\omega})} . \quad (\text{A.4})$$

In addition, note that from equation (19) we have

$$\theta(\bar{\omega}) = \theta(l_0(\bar{\omega}), \bar{\omega}) = \frac{(1 - \delta)(1 + r)\Delta(e_0 - n_0(l_0(\bar{\omega}) - 1))}{\delta R^b(l_0(\bar{\omega}), \bar{\omega})n_0(l_0(\bar{\omega}) - 1)} . \quad (\text{A.5})$$

Finally, using equations (A.4) and (A.5) we can express the break-even condition as the zero of the function $\mathcal{H}(\bar{\omega})$, defined by

$$\mathcal{H}(\bar{\omega}) = U_s(\theta(\bar{\omega})) - u_b(\theta(\bar{\omega}))R^b(l_0(\bar{\omega}), \bar{\omega}) , \quad (\text{A.6})$$

Part 3. Existence of a non-credit rationing equilibrium.

Want to show that $\mathcal{H}(\bar{\omega})$ as a zero in $(0, \bar{\bar{\omega}})$. But since $\mathcal{H}(\bar{\omega})$ is continuous, it suffices to show that $\mathcal{H}(0) < 0$ and $\mathcal{H}(\bar{\bar{\omega}}) > 0$.

Consider first the case $\bar{\omega} = 0$, in this case from equation (A.4) we have that $l_0(0) = 1$, and from equation (4) $\lim_{\bar{\omega} \rightarrow 0} R^b(\bar{\omega})$ equals

$$\lim_{\bar{\omega} \rightarrow 0} \frac{l_0(\bar{\omega})}{l_0(\bar{\omega}) - 1} R^k [\Gamma(\bar{\omega}) - \mu G(\bar{\omega})] = \lim_{\bar{\omega} \rightarrow 0} R^k \frac{l_0'(\bar{\omega})[\Gamma(\bar{\omega}) - \mu G(\bar{\omega})] + l_0(\bar{\omega})[\Gamma'(\bar{\omega}) - \mu G'(\bar{\omega})]}{l_0'(\bar{\omega})} = R^k ,$$

where we used that from differentiating equation (A.4) we get

$$\frac{dl_0}{d\bar{\omega}} = \frac{\Gamma'(\bar{\omega})}{1 - \Gamma(\bar{\omega})} + \frac{\Gamma(\bar{\omega}) - \mu G(\bar{\omega})}{[1 - \Gamma(\bar{\omega})][\Gamma'(\bar{\omega}) - \mu G'(\bar{\omega})]} \left\{ \frac{[\Gamma'(\bar{\omega})]^2}{1 - \Gamma(\bar{\omega})} + \frac{\mu[\Gamma'(\bar{\omega})G''(\bar{\omega}) - \Gamma''(\bar{\omega})G'(\bar{\omega})]}{\Gamma'(\bar{\omega}) - \mu G'(\bar{\omega})} \right\}, \quad (\text{A.7})$$

so $l'_0(0) = 1$. In addition, from equation (A.5) we have

$$\lim_{\bar{\omega} \rightarrow 0} \theta(\bar{\omega}) = \lim_{\bar{\omega} \rightarrow 0} \frac{(1 - \delta)(1 + r)\Delta(e_0 - n_0(l_0(\bar{\omega}) - 1))}{\delta R^b(\bar{\omega})n_0(l_0(\bar{\omega}) - 1)} = \infty.$$

This imply that $p(\theta) = 0$ and $f(\theta) = 1$, and thus

$$\begin{aligned} \mathcal{H}(0) &= \delta(1 + r) + (1 - \delta)(1 + r)^2 - R^k [\delta\Delta^{-1} + 1 - \delta] \\ &= \delta [(1 + r) - R^k\Delta^{-1}] + (1 - \delta) [(1 + r)^2 - R^k] < 0, \end{aligned}$$

where the inequality follows from Assumption 1.

Consider now the case $\bar{\omega} = \bar{\bar{\omega}}$, in this case from equation (A.4) we have that

$$\lim_{\bar{\omega} \rightarrow \bar{\bar{\omega}}} l_0(\bar{\omega}) = \lim_{\bar{\omega} \rightarrow \bar{\bar{\omega}}} 1 + \frac{\Gamma(\bar{\omega}) - \mu G(\bar{\omega})}{1 - \Gamma(\bar{\omega})} \frac{\Gamma'(\bar{\omega})}{\Gamma'(\bar{\omega}) - \mu G'(\bar{\omega})} = \infty.$$

In addition, from equation (4)

$$\lim_{\bar{\omega} \rightarrow \bar{\bar{\omega}}} R^b(\bar{\omega}) = \lim_{\bar{\omega} \rightarrow \bar{\bar{\omega}}} \frac{1}{1 - 1/l_0(\bar{\omega})} R^k [\Gamma(\bar{\omega}) - \mu G(\bar{\omega})] = R^k [\Gamma(\bar{\bar{\omega}}) - \mu G(\bar{\bar{\omega}})].$$

Furthermore, if leverage diverges then investors are allocating all their wealth to bonds and none to storage, so $s_0(\bar{\omega}) = 0$, then it follows from equation (A.5) that

$$\lim_{\bar{\omega} \rightarrow \bar{\bar{\omega}}} \theta(\bar{\omega}) = \lim_{\bar{\omega} \rightarrow \bar{\bar{\omega}}} \frac{(1 - \delta)(1 + r)\Delta s_0(\bar{\omega})}{\delta R^b(\bar{\omega})n_0(l_0(\bar{\omega}) - 1)} = 0.$$

This imply that $p(\theta) = 1$ and $f(\theta) = 0$, and thus

$$\mathcal{H}(\bar{\bar{\omega}}) = \delta(1 + r) + (1 - \delta)(1 + r)\Delta - [\delta\beta + 1 - \delta] R^k [\Gamma(\bar{\bar{\omega}}) - \mu G(\bar{\bar{\omega}})].$$

To show that $\mathcal{H}(\bar{\bar{\omega}}) > 0$ we proceed by contradiction. Suppose that

$$\delta(1 + r) + (1 - \delta)(1 + r)\Delta < [\delta\beta + 1 - \delta] R^k [\Gamma(\bar{\bar{\omega}}) - \mu G(\bar{\bar{\omega}})].$$

Then, at $\bar{\omega} = \bar{\bar{\omega}}$ a portfolio with $s_0 = 0$ and $b_0 = e_0$ is optimal for investors, since in this case the hold-to-maturity return of bonds, $R^b = (e_0 + n_0)/e_0 R^k [\Gamma(\bar{\bar{\omega}}) - \mu G(\bar{\bar{\omega}})] > R^k [\Gamma(\bar{\bar{\omega}}) - \mu G(\bar{\bar{\omega}})]$. Moreover, in with this portfolio allocation liquidity equals zero, so the previous inequalities capture the return on storage and bond investments. So we have found an equilibrium with $\bar{\omega} = \bar{\bar{\omega}}$, which is a contradiction. Thus, we conclude that

$$\delta(1 + r) + (1 - \delta)(1 + r)\Delta < [\delta\beta + 1 - \delta] R^k [\Gamma(\bar{\bar{\omega}}) - \mu G(\bar{\bar{\omega}})],$$

and $\mathcal{H}(\bar{\bar{\omega}}) > 0$.

Part 4. Uniqueness: Show that $\mathcal{H}(\bar{\omega})$ is strictly increasing in $(0, \bar{\bar{\omega}})$.

Differentiating equation (A.6) we obtain

$$\frac{d\mathcal{H}(\bar{\omega})}{d\bar{\omega}} = \frac{dU_s(\theta(\bar{\omega}))}{d\theta} \frac{d\theta(\bar{\omega})}{d\bar{\omega}} - \frac{du_b(\theta(\bar{\omega}))}{d\theta} \frac{d\theta(\bar{\omega})}{d\bar{\omega}} R^b(l_0(\bar{\omega}), \bar{\omega}) - u_b(\theta(\bar{\omega})) \left[\frac{\partial R^b(l_0(\bar{\omega}), \bar{\omega})}{\partial l_0} \frac{dl_0(\bar{\omega})}{d\bar{\omega}} + \frac{\partial R^b(l_0(\bar{\omega}), \bar{\omega})}{\partial \bar{\omega}} \right],$$

To sign this derivative note that

$$\frac{dU_s}{d\theta} = (1 - \delta)(1 + r)p'(\theta) [\Delta - (1 + r)] < 0,$$

and $\frac{du_b}{d\theta} = \delta f'(\theta) [\Delta^{-1} - \beta] > 0.$

where the inequalities follow from $p'(\theta) < 0$, $f'(\theta) > 0$, and $(1 + r) < \Delta < \beta^{-1}$.

Note that we can express

$$\frac{d\theta}{d\bar{\omega}} = \frac{\partial \theta}{\partial l_0} \frac{dl_0}{d\bar{\omega}} + \frac{\partial \theta}{\partial \bar{\omega}} < 0. \quad (\text{A.8})$$

In fact, from the definition of θ , equation (19), and equation (A.3) we have that

$$\frac{\partial \theta}{\partial l_0} = -\frac{\theta(e_0 + n_0)}{l_0(e_0 - n_0(l_0 - 1))} < 0 \quad \text{and} \quad \frac{\partial \theta}{\partial \bar{\omega}} = -\frac{\theta[\Gamma'(\bar{\omega}) - \mu G'(\bar{\omega})]}{\Gamma(\bar{\omega}) - \mu G(\bar{\omega})} < 0, \quad (\text{A.9})$$

where the first inequality follows from Assumption 4, whereas the second inequality follows from $\bar{\omega} < \bar{\omega}$. Moreover, from Assumption 2 for $\bar{\omega} < \bar{\omega}$, $dl_0/d\bar{\omega} > 0$. In fact, evaluating equation (A.7) and using that $\Gamma'(\bar{\omega}) - \mu G'(\bar{\omega}) > 0$ for $\bar{\omega} < \bar{\omega}$; $\Gamma'(\bar{\omega})G''(\bar{\omega}) - \Gamma''(\bar{\omega})G'(\bar{\omega}) = \frac{d(\bar{\omega}h(\bar{\omega}))}{d\bar{\omega}}(1 - F(\bar{\omega}))^2 > 0$ from Assumption 2; and $\Gamma'(\bar{\omega}) > 0$ and $\Gamma(\bar{\omega}) < 1$ from (A.1) we obtain that $dl_0/d\bar{\omega} > 0$. Therefore, $d\theta/d\bar{\omega} < 0$.

It is just left to show that

$$\frac{\partial R^b}{\partial l_0} \frac{dl_0}{d\bar{\omega}} + \frac{\partial R^b}{\partial \bar{\omega}} < 0, \quad (\text{A.10})$$

which is the case iff

$$\frac{1}{l_0} \frac{dl_0}{d\bar{\omega}} > (l_0 - 1) \frac{\Gamma'(\bar{\omega}) - \mu G'(\bar{\omega})}{\Gamma(\bar{\omega}) - \mu G(\bar{\omega})} = \frac{\Gamma'(\bar{\omega})}{1 - \Gamma(\bar{\omega})} \Leftrightarrow \frac{1 - \Gamma(\bar{\omega})}{\Gamma'(\bar{\omega})} \frac{dl_0}{d\bar{\omega}} - l_0 > 0.$$

Substituting in the expressions for $l_0(\bar{\omega})$ and $\frac{dl_0}{d\bar{\omega}}$ from equations (A.4) and (A.7) we get

$$1 + \frac{\Gamma(\bar{\omega}) - \mu G(\bar{\omega})}{\Gamma'(\bar{\omega})[\Gamma'(\bar{\omega}) - \mu G'(\bar{\omega})]} \left\{ \frac{[\Gamma'(\bar{\omega})]^2}{1 - \Gamma(\bar{\omega})} + \frac{\mu[\Gamma'(\bar{\omega})G''(\bar{\omega}) - \Gamma''(\bar{\omega})G'(\bar{\omega})]}{\Gamma'(\bar{\omega}) - \mu G'(\bar{\omega})} \right\} - 1 - \frac{\Gamma(\bar{\omega}) - \mu G(\bar{\omega})}{1 - \Gamma(\bar{\omega})} \frac{\Gamma'(\bar{\omega})}{\Gamma'(\bar{\omega}) - \mu G'(\bar{\omega})} = \frac{[\Gamma(\bar{\omega}) - \mu G(\bar{\omega})]\mu[\Gamma'(\bar{\omega})G''(\bar{\omega}) - \Gamma''(\bar{\omega})G'(\bar{\omega})]}{\Gamma'(\bar{\omega})[\Gamma'(\bar{\omega}) - \mu G'(\bar{\omega})]^2} > 0 \blacksquare$$

Proof of Proposition 1: From the optimal default decision (2) we have that $Z = l_0/(l_0 - 1)R^k\bar{\omega}$. On the other hand, from equation (4) we have $R^b = l_0/(l_0 - 1)R^k[\Gamma(\bar{\omega}) - \mu G(\bar{\omega})]$. Then, the default premium is given by

$$\Phi^d(\bar{\omega}) = \frac{\bar{\omega}}{\Gamma(\bar{\omega}) - \mu G(\bar{\omega})} \quad (\text{A.11})$$

Taking the derivative:

$$\frac{d\Phi^d(\bar{\omega})}{d\bar{\omega}} = \frac{1}{\Gamma(\bar{\omega}) - \mu G(\bar{\omega})} - \frac{\bar{\omega}[\Gamma'(\bar{\omega}) - \mu G'(\bar{\omega})]}{[\Gamma(\bar{\omega}) - \mu G(\bar{\omega})]^2}$$

which is larger than zero iff

$$\Gamma(\bar{\omega}) - \mu G(\bar{\omega}) > \bar{\omega}[\Gamma'(\bar{\omega}) - \mu G'(\bar{\omega})]$$

Using the definition of $\Gamma(\bar{\omega}) = \bar{\omega}(1 - F(\bar{\omega})) + G(\bar{\omega})$ and that $\Gamma'(\bar{\omega}) = 1 - F(\bar{\omega})$, the previous inequality is equivalent to

$$(1 - \mu)G(\bar{\omega}) > -\bar{\omega}\mu G'(\bar{\omega})$$

But the previous inequality follows from $1 - \mu > 0$, $G(\bar{\omega}) \geq 0$ and $G'(\bar{\omega}) = \bar{\omega}dF(\bar{\omega}) > 0$, for any $\bar{\omega} > 0$. ■

Proof of Proposition 2: When there is no need to compensate investors for liquidity risk, the expected return from lending to entrepreneurs is equal to the outside option of holding storage. In other words, the liquidity premium is zero, i.e. $\Phi^\ell(\theta) = 1$, and $R^b = (1 + r)^2$ or $k_0 R^k [\Gamma(\bar{\omega}) - \mu G(\bar{\omega})] = (k_0 - n_0)(1 + r)^2$. This is equivalent to the benchmark costly state verification model where investors are only compensated for credit risk. Note that entrepreneurs' profits do not depend directly on secondary market liquidity. We proceed by showing that $\Phi^\ell(\theta) = 1$ under the three alternative condition stated in Proposition 2.

Condition 1: $\delta = 0$. This implies that secondary market liquidity $\theta \rightarrow \infty$, hence $p(\theta) = 0$. Setting $\delta = 0$ and $p(\theta) = 0$ yields $\Phi^\ell(\theta) = 1$.

Condition 2: $\beta = (1 + r)^{-1}$. Simple substitution yields $\Phi^\ell(\theta) = 1$.

Condition 3: $\psi = 1$ and $f(\theta) = 1$. Simple substitution yields $\Phi^\ell(\theta) = 1$.

For any given distribution for the idiosyncratic productivity shock ω , the upper threshold $\bar{\omega}$ that entrepreneurs can promise to investors under perfect secondary markets is given by $\Gamma'(\bar{\omega}) - \mu G'(\bar{\omega}) = 0$, this also gives an upper bound on leverage \bar{l}_0 (see the proof of Theorem 1). This implies a maximum amount of borrowing, \bar{b}_0 , which is given by the break-even condition $(\bar{b}_0 + n_0)R^k [\Gamma(\bar{\omega}) - \mu G(\bar{\omega})] = \bar{b}_0(1 + r)^2$. In turn this implies a lower bound for investors' endowment, \bar{e}_0 , such that $m(1, \theta) > 1$ for $e_0 > \bar{e}_0$ given that $\theta = (1 - \delta)(1 + r) \left(\frac{e_0}{n_0} + 1 - \bar{l}_0 \right) / (\delta q_1 (\bar{l}_0 - 1))$ is increasing in e_0 and the highest value for q_1 is $\bar{l}_0 / (\bar{l}_0 - 1) R^k [\Gamma(\bar{\omega}) - \mu G(\bar{\omega})]$. ■

Proof of Lemma 1: We want to show that the derivative of the liquidity premium *wrt* liquidity is negative. Denote by C and D the numerator and denominator in $\Phi^\ell(\theta)$ given by 21. Then,

$$C = \delta + (1 - \delta) [(1 - p)(1 + r) + p\Delta] > 0 \tag{A.12}$$

$$D = \delta [f\Delta^{-1} + \beta(1 - f)] + (1 - \delta) > 0$$

where the inequalities follow from the fact that probabilities and returns are non-negative. In addition, denote C_θ and D_θ the derivatives of C and D , respectively, *wrt* θ . Then

$$C_\theta = \frac{\partial C}{\partial \theta} = (1 - \delta) [\Delta - (1 + r)] \frac{dp(\theta)}{d\theta} \leq 0$$

$$D_\theta = \frac{\partial D}{\partial \theta} = \delta \psi \left(\frac{1}{1+r} - \beta \right) \frac{df(\theta)}{d\theta} \geq 0$$

where the inequalities follow from $\beta < 1/(1+r)$, equations (6) and (7), and that the matching function $m(A, B)$ is increasing in both arguments. From equation (21) we have that

$$\frac{d\Phi^\ell}{d\theta} = \frac{1}{1+r} \left[\frac{C_\theta}{D} - \frac{CD_\theta}{D^2} \right] \leq 0 \quad (\text{A.13})$$

where the inequality follows from the previously established inequalities: $D, C > 0$, $D_\theta \geq 0$ and $C_\theta \leq 0$. In the case where $f(\theta) < 1$, then $D_\theta > 0$, so $d\Phi^\ell/d\theta < 0$. Alternatively, if $f(\theta) = 1$, our assumptions require that $\psi < 1$. In addition, if $f(\theta) = 1$ it must be that $p(\theta) < 1$ so $dp(\theta)/d\theta < 0$. With $\psi < 1$ and $dp(\theta)/d\theta < 0$ then $C_\theta < 0$, so $d\Phi^\ell/d\theta < 0$. Therefore, we conclude that $d\Phi^\ell/d\theta < 0$ when OTC trade is relevant.

Regarding the second part of the Lemma, the elasticity of the liquidity premium, Φ^ℓ , with respect to the secondary market liquidity, θ , is written, using equation (A.13), as:

$$\zeta_{\Phi^\ell, \theta} = \frac{\theta}{\Phi^\ell} \frac{d\Phi^\ell}{d\theta} = \frac{\theta}{1+r} \frac{C_\theta D - CD_\theta}{CD}, \quad (\text{A.14})$$

Then $|\zeta_{\Phi^\ell, \theta}| < 1$ requires:

$$-\frac{\theta}{1+r} \frac{C_\theta D - CD_\theta}{D^2} < \frac{1}{1+r} \frac{C}{D} \quad \Leftrightarrow \quad CD + \theta C_\theta D - \theta CD_\theta > 0$$

First, let's consider the case where $f(\theta), p(\theta) < 1$. In this case, $f(\theta) = \nu\theta^{1-\alpha}$ and $p(\theta) = \nu\theta^{-1}$. Thus, $\theta \frac{df(\theta)}{d\theta} = (1-\alpha)f(\theta)$ and $\theta \frac{dp(\theta)}{d\theta} = -\alpha p(\theta)$. Then,

$$C_\theta \theta = -\alpha C + \alpha[\delta + (1-\delta)(1+r)] \leq 0 \quad (\text{A.15})$$

$$D_\theta \theta = (1-\alpha)D - (1-\alpha)[\beta\delta + (1-\delta)] \geq 0 \quad (\text{A.16})$$

Then,

$$\begin{aligned} CD + \theta C_\theta D - \theta CD_\theta &= CD + D \{-\alpha C + \alpha[\delta + (1-\delta)(1+r)]\} - C \{(1-\alpha)D - (1-\alpha)[\beta\delta + (1-\delta)]\} \\ &= \alpha D[\delta + (1-\delta)(1+r)] + C(1-\alpha)[\beta\delta + (1-\delta)] > 0. \end{aligned}$$

Second, consider the case where $f(\theta) < 1$, but $p(\theta) = 1$. In this case, $f(\theta) = \nu\theta^{1-\alpha}$ and $\theta \frac{df(\theta)}{d\theta} = (1-\alpha)f(\theta)$, so equation (A.16) still holds. But $\theta \frac{dp(\theta)}{d\theta} = \theta C_\theta = 0$. Then,

$$CD + \theta C_\theta D - \theta CD_\theta = C[D - \theta D_\theta] = C[\alpha D + (1-\alpha)(\beta\delta + 1 - \delta)] > 0.$$

Finally, consider the case where $f(\theta) = 1$ and $\psi < 1$ and $p(\theta) < 1$. In this case, $\theta \frac{df(\theta)}{d\theta} = \theta D_\theta = 0$, but equation (A.15) still holds. Then,

$$CD + \theta C_\theta D - \theta CD_\theta = D[C + \theta C_\theta] = D\{(1-\alpha)C + \alpha[\delta + (1-\delta)(1+r)]\} > 0. \quad \blacksquare$$

Proof of Proposition 3: From the investors' break-even condition (20), we see that an increase in the liquidity premium, Φ^ℓ induces investors to require a higher expected return R^b to invest in corporate bonds. Hence, the liquidity premium Φ^ℓ and the hold-to-maturity bond return R^b are proportional to one another. In fact,

$$(1+r)^2 \Phi^\ell = R^b \quad \Rightarrow \quad \frac{dR^b}{d\Phi^\ell} = \frac{R^b}{\Phi^\ell} > 0$$

For this proof we consider the liquidity premium a function of both secondary market

liquidity, θ , and model parameters δ and β . That is, we can write the liquidity premium as $\Phi^\ell(\theta, \delta, \beta)$.

Case 1: Effect of δ . Want to show that

$$\frac{d\Phi^\ell}{d\delta} = \frac{\partial\Phi^\ell}{\partial\delta} + \frac{\partial\Phi^\ell}{\partial\theta} \frac{\partial\theta}{\partial\delta} > 0$$

From the definition of secondary market liquidity, given in equation (19), and considering the dependence of secondary market pricing on liquidity premia, we have that

$$\frac{\partial\theta}{\partial\delta} = -\frac{\theta}{\delta(1-\delta)} - \frac{\theta}{q_1} \frac{dq_1}{dR^b} \frac{dR^b}{d\Phi^\ell} \frac{d\Phi^\ell}{d\delta} = -\frac{\theta}{\delta(1-\delta)} - \frac{\theta}{\Phi^\ell} \frac{d\Phi^\ell}{d\delta}$$

Using this expression we get

$$\frac{d\Phi^\ell}{d\delta} = \frac{\frac{\partial\Phi^\ell}{\partial\delta} - \frac{1}{\delta(1-\delta)}\zeta_{\Phi^\ell,\theta}}{1 + \zeta_{\Phi^\ell,\theta}}$$

where $\zeta_{\Phi^\ell,\theta}$ is the elasticity of the liquidity premium with respect to secondary market liquidity, which is negative and strictly greater than -1 (Lemma 1). Therefore, $1 + \zeta_{\Phi^\ell,\theta} > 0$.

It is left to show that $\partial\Phi^\ell/\partial\delta > 0$. For that we use the notation introduced in equation (A.12). In addition, denote C_δ and D_δ the derivatives of C and D , respectively, wrt δ . Then

$$C_\delta = \frac{\partial C}{\partial\delta} = 1 - [(1-p)(1+r) + p\Delta]$$

$$D_\delta = \frac{\partial D}{\partial\delta} = [f\Delta^{-1} + (1-f)\beta] - 1$$

Then, from equation (21) we have that

$$\frac{\partial\Phi^\ell}{\partial\delta} = \frac{1}{1+r} \left[\frac{C_\delta}{D} - \frac{CD_\delta}{D^2} \right]$$

which is strictly greater than zero if and only if

$$C_\delta D > CD_\delta$$

$$C_\delta [\delta D_\delta + 1] > [\delta C_\delta + 1 - C_\delta] D_\delta$$

$$C_\delta > [1 - C_\delta] D_\delta$$

or

$$1 - [(1-p)(1+r) + p\Delta] > [(1-p)(1+r) + p\Delta] \left\{ [f\Delta^{-1} + (1-f)\beta] - 1 \right\}$$

$$\Leftrightarrow 1 > [(1-p)(1+r) + p\Delta] [f\Delta^{-1} + (1-f)\beta]$$

It is easy to check that after distributing terms in the previous expression the four remaining terms, are a weighted average of terms strictly smaller than 1, with the weights given by the product of probabilities f and p adding up to 1. In fact, $\beta < 1/(1+r)$ imply that

$$\beta(1+r) < 1, \quad \Delta^{-1}(1+r) < 1, \quad \text{and} \quad \Delta\beta < 1.$$

Case 2: Effect of β . Want to show that

$$\frac{d\Phi^\ell}{d\beta} = \frac{\partial\Phi^\ell}{\partial\beta} + \frac{\partial\Phi^\ell}{\partial\theta} \frac{\partial\theta}{\partial\beta} < 0.$$

For that we use the notation introduced in equation (A.12). In addition, denote C_β and D_β the derivatives of C and D , respectively, wrt β . Then

$$C_\beta = \frac{\partial C}{\partial \beta} = -(1-\delta)p\Delta^2(1-\psi) < 0,$$

$$\text{and } D_\beta = \frac{\partial D}{\partial \beta} = \delta[f(1-\psi) + 1 - f] = \delta(1-\psi f) > 0.$$

where the inequalities follow from our assumption about δ , ψ , and $f(\theta)$. Then,

$$\frac{\partial \Phi^\ell}{\partial \beta} = \frac{1}{1+r} \left[\frac{C_\beta}{D} - \frac{CD_\beta}{D^2} \right] < 0,$$

as $C_\beta < 0$ and $D_\beta, C, D > 0$.

From the definition of secondary market liquidity, given in equation (19), and considering the dependence of the secondary market price on liquidity premia, we have that

$$\frac{\partial \theta}{\partial \beta} = -\frac{\theta}{q_1} \left[\frac{\partial q_1}{\partial \beta} + \frac{\partial q_1}{\partial R^b} \frac{dR^b}{d\Phi^\ell} \frac{d\Phi^\ell}{d\beta} \right] = -\theta \left[(1-\psi)\Delta + \frac{1}{\Phi^\ell} \frac{d\Phi^\ell}{d\beta} \right]$$

Thus,

$$\frac{d\Phi^\ell}{d\beta} = \frac{\frac{\partial \Phi^\ell}{\partial \beta} - (1-\psi)\Delta\theta \frac{\partial \Phi^\ell}{\partial \theta}}{1 + \zeta_{\Phi^\ell, \theta}}$$

where $\zeta_{\Phi^\ell, \theta}$ is the elasticity of the liquidity premium with respect to secondary market liquidity. From Lemma 1 the denominator, $1 + \zeta_{\Phi^\ell, \theta}$, is strictly positive. But the sign of the numerator is ambiguous. The reason is that a higher β on one hand reduces the preference for liquidity by impatient households, i.e., $\partial \Phi^\ell / \partial \beta < 0$. But on the other hand it increases the secondary market price, q_1 , which pushes market liquidity θ down and liquidity premia up. This second force, represented by the second term in the numerator, depends crucially on the bargaining power of impatient investors: the lower their bargaining power the more important the effect of their valuation, i.e., β , will be on the price.

The numerator is negative if and only if

$$(1-\psi)\Delta\theta[C_\beta D - CD_\beta] - C_\beta D + CD_\beta > 0$$

Using the expressions derived above for C , D , C_β , D_β , C_θ , D_θ , C_β , and D_β , we have

$$\begin{aligned} & (1-\psi)\Delta\theta[C_\beta D - CD_\beta] - C_\beta D + CD_\beta \\ &= -(1-\psi)\Delta p(1-\delta)[\Delta - (1+r)][\delta(f\Delta^{-1} + (1-f)\beta) + (1-\delta)] \\ & \quad - (1-\psi)\Delta(1-\alpha)f\delta[\Delta^{-1} - \beta][\delta + (1-\delta)[(1-p)(1+r) + p\Delta]] \\ & \quad + (1-\psi)p(1-\delta)\Delta^2[\delta[f\Delta^{-1} + (1-f)\beta] + (1-\delta)] \\ & \quad + \delta[f(1-\psi) + 1 - f][\delta + (1-\delta)[(1-p)(1+r) + p\Delta]] \\ &= (1-\psi)[\delta(f\Delta^{-1} + (1-f)\beta) + (1-\delta)] \left\{ (1-\alpha)p(1-\delta)\Delta^2 + p(1-\delta)(1+r) \right\} \\ & \quad + (1-\psi)[\delta + (1-\delta)[(1-p)(1+r) + p\Delta]] \left\{ \alpha f\delta + (1-\alpha)f\delta\Delta\beta \right\} \\ & \quad + (1-f)\delta[\delta + (1-\delta)[(1-p)(1+r) + p\Delta]] > 0 \end{aligned}$$

Case 3: Effect of e_0 . Want to show that

$$\frac{d\Phi^\ell}{de_0} < 0. \quad (\text{A.17})$$

Note that investors' endowment e_0 affects liquidity premium Φ^ℓ only through its effect on secondary market liquidity θ . In particular, it has an effect only through $s_0 = e_0 - b_0$ given that we have fixed leverage in this exercise. Thus,

$$\frac{d\Phi^\ell}{de_0} = \frac{\partial\Phi^\ell}{\partial\theta} \frac{\partial\theta}{\partial s_0} \frac{ds_0}{de_0} = \frac{\partial\Phi^\ell}{\partial\theta} \frac{\theta}{s_0} < 0$$

where the inequality follows from Lemma 1. ■

Proof of Proposition 4:

Case 1: *Comparative Statics on δ* . Recall that from equation (18) we can rearrange terms to get leverage as a function of risk, $l_0(\bar{\omega})$, equation (A.4). In addition, from equation (19) we can express θ as a function of $l_0(\bar{\omega})$, $\bar{\omega}$, and δ , i.e., $\theta(l_0(\bar{\omega}), \bar{\omega}, \delta)$. Using these expressions, equilibrium conditions boil down to the investors' break-even condition, which can be expressed as

$$(1+r)^2 \Phi^\ell(\theta(l_0(\bar{\omega}), \bar{\omega}, \delta), \delta) = R^b(l_0(\bar{\omega}), \bar{\omega})$$

By the Implicit Function Theorem, if the derivative of the previous expression *wrt* $\bar{\omega}$ is different than 0, then we can define $\bar{\omega}(\delta)$ and calculate its derivative from the previous expression. We want to show that $\frac{d\bar{\omega}}{d\delta} < 0$.

Fully differentiating *wrt* to $\bar{\omega}$ we obtain

$$(1+r)^2 \left\{ \frac{\partial\Phi^\ell}{\partial\theta} \left[\frac{\partial\theta}{\partial l_0} \frac{dl_0}{d\bar{\omega}} \frac{d\bar{\omega}}{d\delta} + \frac{\partial\theta}{\partial \bar{\omega}} \frac{d\bar{\omega}}{d\delta} + \frac{\partial\theta}{\partial \delta} \right] + \frac{\partial\Phi^\ell}{\partial \delta} \right\} = \frac{\partial R^b}{\partial l_0} \frac{dl_0}{d\bar{\omega}} \frac{d\bar{\omega}}{d\delta} + \frac{\partial R^b}{\partial \bar{\omega}} \frac{d\bar{\omega}}{d\delta}$$

Thus,

$$\frac{d\bar{\omega}}{d\delta} = \frac{H_\delta}{J}$$

with

$$\begin{aligned} H_\delta &= -(1+r)^2 \left\{ \frac{\partial\Phi^\ell}{\partial\theta} \frac{\partial\theta}{\partial \delta} + \frac{\partial\Phi^\ell}{\partial \delta} \right\} \\ J &= (1+r)^2 \left\{ \frac{\partial\Phi^\ell}{\partial\theta} \left[\frac{\partial\theta}{\partial l_0} \frac{dl_0}{d\bar{\omega}} + \frac{\partial\theta}{\partial \bar{\omega}} \right] \right\} - \frac{\partial R^b}{\partial l_0} \frac{dl_0}{d\bar{\omega}} - \frac{\partial R^b}{\partial \bar{\omega}} \end{aligned} \quad (\text{A.18})$$

From Proposition 3 $\frac{\partial\Phi^\ell}{\partial\delta} > 0$. In addition,

$$\frac{\partial\theta}{\partial \delta} = -\frac{\theta}{\delta(1-\delta)} < 0$$

and $\partial\Phi^\ell/\partial\theta < 0$ from Lemma 1. Thus, $H_\delta < 0$.

Next we want to show that $J > 0$. For that first recall that from equation (A.8) we have that $(\partial\theta/\partial l_0)(dl_0/d\bar{\omega}) + (\partial\theta/\partial \bar{\omega}) < 0$. Second, note that from equation (A.10) we have that $(\partial R^b/\partial l_0)(dl_0/d\bar{\omega}) + \partial R^b/\partial \bar{\omega} < 0$.

Therefore, we conclude that $J > 0$ and $d\bar{\omega}/d\delta < 0$. It follows from $dl_0/d\bar{\omega} > 0$, equation (A.7), that $dl_0/d\delta < 0$. ■

Proof of Corollary 1: The effect of any parameter ρ on the default premium is described by

$$\frac{d\Phi^d}{d\rho} = \frac{d\Phi^d}{d\bar{\omega}} \frac{\partial \bar{\omega}}{\partial \rho}.$$

Since $\frac{d\Phi^d}{d\bar{\omega}} > 0$ from Proposition 1, the result on the default premium follows from Proposition 4. ■

Proof of Proposition 5: We want to show that if the competitive equilibrium is constrained efficient, then $(\alpha, \psi, r) \in \emptyset$, a set of measure zero.

Suppose $(l_0^{ce}, \bar{\omega}^{ce}, \theta^{ce}, q_1^{ce})$, the competitive equilibrium, is constrained efficient. Since $(l_0^{ce}, \bar{\omega}^{ce}, \theta^{ce}, q_1^{ce})$ is a competitive equilibrium the investor break-even condition (15) holds, i.e., $U_s = U_b$, and from equation (18) it must be that

$$\frac{1 - \Gamma(\bar{\omega}^{ce})}{l_0^{ce} \Gamma'(\bar{\omega}^{ce})} = - \frac{\partial U_b / \partial l_0}{\partial U_b / \partial \bar{\omega}}.$$

On the other hand, since $(l_0^{ce}, \bar{\omega}^{ce}, \theta^{ce}, q_1^{ce})$ is constrained efficient, from equation (26) it must be that

$$\frac{[1 - \Gamma(\bar{\omega}^{ce})]}{l_0^{ce} \Gamma'(\bar{\omega}^{ce})} = - \frac{n_0(U_b - U_s) + b_0^{ce} \frac{\partial U_b}{\partial l_0} + \frac{\partial \mathbb{U}}{\partial \theta} \frac{\partial \theta}{\partial l_0}}{b_0^{ce} \frac{\partial U_b}{\partial \bar{\omega}} + \frac{\partial \mathbb{U}}{\partial \theta} \frac{\partial \theta}{\partial \bar{\omega}}}.$$

Using that $U_s = U_b$, then

$$\frac{b_0^{ce} \frac{\partial U_b}{\partial l_0} + \frac{\partial \mathbb{U}}{\partial \theta} \frac{\partial \theta}{\partial l_0}}{b_0^{ce} \frac{\partial U_b}{\partial \bar{\omega}} + \frac{\partial \mathbb{U}}{\partial \theta} \frac{\partial \theta}{\partial \bar{\omega}}} = \frac{\frac{\partial U_b}{\partial l_0}}{\frac{\partial U_b}{\partial \bar{\omega}}},$$

which is the case iff

$$\frac{\partial \mathbb{U}}{\partial \theta} \left[\frac{\partial U_b}{\partial \bar{\omega}} \frac{\partial \theta}{\partial l_0} - \frac{\partial U_b}{\partial l_0} \frac{\partial \theta}{\partial \bar{\omega}} \right] = 0 \quad (\text{A.19})$$

Note that,

$$\frac{\partial U_b}{\partial \bar{\omega}} \frac{\partial \theta}{\partial l_0} - \frac{\partial U_b}{\partial l_0} \frac{\partial \theta}{\partial \bar{\omega}} < 0, \quad (\text{A.20})$$

since

$$\frac{\partial U_b}{\partial l_0} = - \frac{U_b}{l_0(l_0 - 1)} < 0 \quad \text{and} \quad \frac{\partial U_b}{\partial \bar{\omega}} = \frac{U_b[\Gamma'(\bar{\omega}) - \mu G'(\bar{\omega})]}{\Gamma(\bar{\omega}) - \mu G(\bar{\omega})} > 0 \quad (\text{A.21})$$

where the last inequality follows from Theorem 1; and $\partial \theta / \partial l_0, \partial \theta / \partial \bar{\omega} < 0$ from equation (A.9). Then, A.19 holds iff $\partial \mathbb{U} / \partial \theta = 0$, which is the case iff

$$s_0^{ce} \frac{\partial U_s}{\partial \theta} + b_0^{ce} \frac{\partial U_b}{\partial \theta} = 0$$

$$s_0^{ce} (1 - \delta)(1 + r)[\Delta - (1 + r)]p'(\theta^{ce}) + b_0^{ce} \delta[\Delta^{-1} - \beta]f'(\theta^{ce})R^b = 0$$

$$p(\theta^{ce}) \frac{\alpha}{\theta^{ce}} s_0^{ce} (1 - \delta)(1 + r)[\Delta - (1 + r)] = f(\theta^{ce}) \frac{1 - \alpha}{\theta^{ce}} b_0^{ce} \delta[\delta^{-1} - \beta]R^b$$

$$\theta^{ce} = \frac{\alpha s_0^{ce} (1 - \delta)(1 + r)[\Delta - (1 + r)]}{(1 - \alpha) b_0^{ce} \delta[\Delta^{-1} - \beta]R^b}$$

But from equation (19) $\theta^{ce} = (1 - \delta)(1 + r)\Delta s_0^{ce} / (\delta b_0^{ce} R^b)$, then

$$\begin{aligned}
& \frac{\alpha[\Delta - (1+r)]}{(1-\alpha)[\Delta^{-1} - \beta]} = \Delta \quad \Leftrightarrow \quad \Delta[\alpha + (1-\alpha)\beta] = 1 + \alpha r \\
\Leftrightarrow & \quad \frac{\psi}{1+r} + (1-\psi)\beta = \frac{\alpha + (1-\alpha)\beta}{1 + \alpha r} \quad \Leftrightarrow \quad \psi = \frac{\alpha(1 - \beta(1+r))}{1 + \alpha r} \frac{1+r}{(1 - \beta(1+r))} \\
& \quad \Leftrightarrow \quad \psi(1 + \alpha r) = \alpha(1 + r)
\end{aligned}$$

The set of (α, ψ, r) satisfying (A.22) is, thus, of measure zero. ■

Proof of Proposition 6:

Part 1. The sign of the externality determines the socially optimal level of secondary market liquidity.

Let \mathcal{L} be the Lagrangian of the planner's problem, which is given

$$\mathcal{L} = [1 - \Gamma(\bar{\omega})]R^k l_0 - \lambda[U^{ce} - s_0 U_s - b_0 U_b],$$

Fully differentiating and evaluating at the competitive equilibrium allocation $(l_0^{ce}, \bar{\omega}^{ce}, \theta^{ce})$ we have

$$d\mathcal{L}(l_0^{ce}, \bar{\omega}^{ce}, \theta^{ce}) = \lambda \frac{\partial \mathcal{U}}{\partial \theta} d\theta,$$

where we have substituted the optimality conditions in the competitive equilibrium. Thus, the planner, who internalizes the effect of liquidity on the investor's utility, would like to increase liquidity in secondary markets when the externality is positive, i.e., $\partial \mathcal{U} / \partial \theta > 0$, and decrease liquidity if the externality is negative, i.e., $\partial \mathcal{U} / \partial \theta < 0$.

Part 2. Show that the sign of the externality depends on the relationship between the parameters (α, r, ψ) .

Want to show that

$$\psi(1 + \alpha r) > \alpha(1 + r) \quad \Leftrightarrow \quad \frac{\partial \mathcal{U}}{\partial \theta} > 0.$$

In fact,

$$\begin{aligned}
\psi(1 + \alpha r) > \alpha(1 + r) & \quad \Leftrightarrow \quad \Delta > \frac{\alpha[\Delta - (1+r)]}{(1-\alpha)[\Delta^{-1} - \beta]} \\
& \quad \Leftrightarrow \quad \theta > \frac{\alpha s_0 (1-\delta)(1+r)[\Delta - (1+r)]}{(1-\alpha)b_0 \delta [\Delta^{-1} - \beta] R^b} \\
& \quad \Leftrightarrow \quad b_0 \frac{\partial U_b}{\partial \theta} + s_0 \frac{\partial U_s}{\partial \theta} > 0 \quad \Leftrightarrow \quad \frac{\partial \mathcal{U}}{\partial \theta} > 0.
\end{aligned}$$

Part 3. Characterization of the efficient contract.

Let $\bar{\omega}^{pi}(l_0)$ be the function implicitly defined by the Pareto improvement constraint in the planner's problem (23). Using the Implicit Function Theorem and equation (A.20) we have that

$$\frac{d\bar{\omega}^{pi}}{dl_0} = - \frac{\frac{\partial \mathcal{U}}{\partial l_0} + \frac{\partial \mathcal{U}}{\partial \theta} \frac{\partial \theta}{\partial l_0}}{\frac{\partial \mathcal{U}}{\partial \bar{\omega}} + \frac{\partial \mathcal{U}}{\partial \theta} \frac{\partial \theta}{\partial \bar{\omega}}}$$

Similarly, using the notation introduced in the proof of Theorem 1, where $\bar{\omega}^{ibec}(l_0)$ denotes the function implicitly defined by the investors' break-even condition in the competitive

economy for $\bar{\omega} < \bar{\omega}$. From equation (A.2) we had that

$$\frac{d\bar{\omega}^{ibec}}{dl_0} = -\frac{\frac{\partial U_b}{\partial l_0}}{\frac{\partial U_b}{\partial \bar{\omega}}}$$

Note that the competitive equilibrium is a feasible point of the pareto improvement constraint, so $\bar{\omega}^{pi}(l_0^{ce}) = \bar{\omega}^{ibec}(l_0^{ce})$. Moreover, note that

$$\frac{d\bar{\omega}^{pi}(l_0^{ce})}{dl_0} - \frac{d\bar{\omega}^{ibec}(l_0^{ce})}{dl_0} = \frac{\frac{\partial \mathbb{U}}{\partial \theta} \left[\frac{\partial \theta}{\partial \bar{\omega}} \frac{\partial U_b}{\partial l_0} - \frac{\partial \theta}{\partial l_0} \frac{\partial U_b}{\partial \bar{\omega}} \right]}{\frac{\partial U_b}{\partial \bar{\omega}} \left[b_0 \frac{\partial U_b}{\partial \bar{\omega}} + \frac{\partial \mathbb{U}}{\partial \theta} \frac{\partial \theta}{\partial \bar{\omega}} \right]}$$

where all the derivatives on the RHS are evaluated at $(l_0^{ce}, \bar{\omega}^{ce}, \theta^{ce})$, and we used that

$$\frac{\partial \mathbb{U}(l_0^{ce}, \bar{\omega}^{ce}, \theta^{ce})}{\partial l_0} = n_0(U_b(l_0^{ce}, \bar{\omega}^{ce}, \theta^{ce}) - U_s(\theta^{ce})) + b_0^{ce} \frac{\partial U_b(l_0^{ce}, \bar{\omega}^{ce}, \theta^{ce})}{\partial l_0} = b_0^{ce} \frac{\partial U_b(l_0^{ce}, \bar{\omega}^{ce}, \theta^{ce})}{\partial l_0}$$

It follows from above and equation (A.20) that

$$\frac{d\bar{\omega}^{pi}(l_0^{ce})}{dl_0} - \frac{d\bar{\omega}^{ibec}(l_0^{ce})}{dl_0} > 0 \quad \Leftrightarrow \quad \frac{\partial \mathbb{U}}{\partial \theta} > 0.$$

Then, if $\psi(1 + \alpha r) > \alpha(1 + r)$, from Part 2, $\partial \mathbb{U} / \partial \theta > 0$, and, thus,

$$\frac{d\bar{\omega}^{pi}(l_0^{ce})}{dl_0} > \frac{d\bar{\omega}^{ibec}(l_0^{ce})}{dl_0} > 0$$

where the last inequality follows from equation (A.21). That means there are points that are feasible for the planner where $(l_0, \bar{\omega}) \ll (l_0^{ce}, \bar{\omega}^{ce})$ that achieve higher profits for the firm, so the planner will choose an allocation with lower leverage and risk. (Note that by equation (A.9) this imply that the planer will set a higher secondary market liquidity: $\theta > \theta^{ce}$.)

Similarly, if $\psi(1 + \alpha r) < \alpha(1 + r)$, from Part 2, $\partial \mathbb{U} / \partial \theta < 0$, so

$$0 < \frac{d\bar{\omega}^{pi}(l_0^{ce})}{dl_0} < \frac{d\bar{\omega}^{ibec}(l_0^{ce})}{dl_0}$$

That means there are points that are feasible for the planner where $(l_0, \bar{\omega}) \gg (l_0^{ce}, \bar{\omega}^{ce})$ that achieve higher profits for the firm, so the planner will choose an allocation with higher leverage and risk. ■

Proof of Proposition 7:

Part 1. Deriving the tax instruments.

The firm's problem with taxes on storage and leverage can be written as

$$[1 - \Gamma(\bar{\omega})]R^k l_0 - \tau^l \lambda l_0 + T^l \tag{A.22}$$

subject to

$$U_b = (1 - \tau^s)U_s \tag{A.23}$$

We write the Lagrangian for this problem as

$$\mathcal{L} = [1 - \Gamma(\bar{\omega})]R^k l_0 - \tau^l \lambda l_0 + T^l - \lambda[(1 - \tau^s)U_s - U_b] \tag{A.24}$$

Then, the optimality conditions are

$$[1 - \Gamma(\bar{\omega})]R^k = \tau^l \lambda - \lambda \frac{\partial U_b}{\partial l_0} \quad (\text{A.25})$$

$$\Gamma'(\bar{\omega})R^k l_0 = \lambda \frac{\partial U_b}{\partial \bar{\omega}} \quad (\text{A.26})$$

Note that the FOC for $(\bar{\omega})$, equation (A.27), together with equation (A.21) ensures that $\lambda > 0$, which is not necessarily the case with equality constraints. And the optimal contract is described by

$$\frac{1 - \Gamma(\bar{\omega})}{l_0 \Gamma'(\bar{\omega})} = - \frac{\frac{\partial U_b}{\partial l_0} - \tau^l}{\frac{\partial U_b}{\partial \bar{\omega}}}. \quad (\text{A.27})$$

Equating the previous expression and equation (26), and using that $U_b - U_s = -\tau^s U_s$, we derive the tax on leverage:

$$\tau^l = \frac{n_0 U_s \frac{\partial U_b}{\partial \bar{\omega}} \tau^s + \left[\frac{\partial U_b}{\partial l_0} \frac{\partial \theta}{\partial \bar{\omega}} - \frac{\partial U_b}{\partial \bar{\omega}} \frac{\partial \theta}{\partial l_0} \right] \frac{\partial \mathbb{U}}{\partial \theta}}{b_0 \frac{\partial U_b}{\partial \bar{\omega}} + \frac{\partial \theta}{\partial \bar{\omega}} \frac{\partial \mathbb{U}}{\partial \theta}}$$

The term in square brackets is positive from equation (A.20). On the other hand, using equations (A.9) and (A.21) the denominator is positive iff

$$b_0 \left\{ \delta \left[f(\theta) \Delta^{-1} + (1 - f(\theta)) \beta \right] + 1 - \delta \right\} R^b - s_0 (1 - \delta) (1 + r) [\Delta - (1 + r)] p'(\theta) \theta - b_0 \delta \left[\Delta^{-1} - \beta \right] f'(\theta) \theta R^b > 0.$$

Using that $p'(\theta) \theta = -\alpha p(\theta)$ and $f'(\theta) \theta = (1 - \alpha) f(\theta)$ the previous expression equals

$$b_0 \{ \delta \beta + 1 - \delta \} R^b + \alpha s_0 (1 - \delta) (1 + r) [\Delta - (1 + r)] p(\theta) + \alpha b_0 \delta \left[\Delta^{-1} - \beta \right] f(\theta) R^b > 0,$$

where the inequality follows from $\Delta > 1 + r$ and $\Delta^{-1} > \beta$, since $\beta < 1/(1 + r)$.

On the other hand, the break-even condition of investors with a tax on storage was given by equation (A.24). Combining it with constraint (23) we derive the tax on storage:

$$\tau^s = \frac{e_0}{b_0} \left(1 - \frac{U_s(\theta^{ce})}{U_s(\theta)} \right)$$

Part 2. Signing the tax on storage.

If $\psi(1 + ar) > \alpha(1 + r)$ then from Proposition 6 the planner wants to increase secondary market liquidity so $\theta > \theta^{ce}$. Thus, the storage technology is subsidized: $\tau^s \leq 0$. In fact, the tax on storage is negative from equation (27) if $\psi < 1$ and is zero if $\psi = 0$.

On the contrary, if $\psi(1 + ar) < \alpha(1 + r)$, then the externality is negative, the planner wants to reduce secondary market liquidity, and, therefore, $\tau^s > 0$.

Part 3. Signing the tax on leverage.

We start by describing the feasible allocations for a firm that chooses the optimal contract and faces the the optimal tax on storage, and the efficient level of secondary market liquidity. That is, τ^s is given by equation (27) and θ is the one that the planner would choose optimally. In this case we have

$$(1 - \tau^s) U_s(\theta) = \left(1 - \frac{e_0}{b_0} \frac{U_s(\theta) - U_s(\theta^{ce})}{U_s(\theta)} \right) U_s(\theta) = \frac{b_0 U_b(l_0^{ce}, \bar{\omega}^{ce}, \theta^{ce}) + s_0 U_s(\theta^{ce}) - s_0 U_s(\theta)}{b_0}$$

where we used that in the competitive equilibrium $U_s(\theta^{ce}) = U_b(l_0^{ce}, \bar{\omega}^{ce}, \theta^{ce})$, and $b_0^{ce} + s_0^{ce} = e_0$.

Lets consider first the case when $\psi(1 + ar) > \alpha(1 + r)$. In this case $\partial U/\partial \theta > 0$ and $\theta > \theta^{ce}$, then

$$b_0 U_b(l_0^{ce}, \bar{\omega}^{ce}, \theta^{ce}) + s_0 U_s(\theta^{ce}) < b_0 U_b(l_0^{ce}, \bar{\omega}^{ce}, \theta) + s_0 U_s(\theta)$$

So we conclude that

$$(1 - \tau^s) U_s(\theta) < U_b(l_0^{ce}, \bar{\omega}^{ce}, \theta)$$

Since $\partial U_b/\partial \bar{\omega} > 0$, for the leverage of the competitive equilibrium l_0^{ce} a feasible level of risk lies below the risk in the competitive equilibrium. So the investor's break-even condition with the optimal tax and the efficient level of liquidity will lie below the investor's break-even condition in the competitive problem. Moreover, from equation (A.2) the slope of this constraint at l_0^{ce} , which has the same expression regardless of the tax, will be flatter.

The firm, then, if it were to face this constraint without a tax on leverage will choose a higher leverage, at odds with the planner optimal prescriptions. The planner then will distort the firm's decision to disincentivize the use of leverage by levying a tax on leverage. One way to see this is that the planner will introduce a distortion such that the distorted isoprofit lines are flatter in the $(l_0, \bar{\omega})$ -space.

Let $\Pi^\tau = [1 - \Gamma(\bar{\omega})]R^k l_0 - \tau^l \lambda l_0 + T^l$, and denote by $\bar{\omega}^{\Pi^\tau}(l_0)$ the function that for any l_0 gives the associated risk level $\bar{\omega}$ along the taxed firm isoprofit line. Then, the Implicit Function Theorem implies that

$$\frac{d\bar{\omega}^{\Pi^\tau}}{dl_0} = \frac{[1 - \Gamma(\bar{\omega})]R^k - \tau^l \lambda}{\Gamma'(\bar{\omega})R^k l_0}$$

so a flatter slope requires a positive τ^l .

Using the same reasoning we conclude that if $\psi(1 + ar) < \alpha(1 + r)$, then $\tau^l < 0$. ■

Proof of Proposition 8: In the presence of quantitative easing, firms' borrowing is given by $b_0 + \bar{b}_0$, whereas investors' lending is given by b_0 . Then from the budget constraint of entrepreneurs we have that $k_0 = n_0 + b_0 + \bar{b}_0$, so investors' lending can be written in terms of entrepreneurs leverage and QE as $b_0 = n_0(l_0 - 1 - \bar{b}_0/n_0)$. On the other hand, from the investors' budget constraint, $b_0 + s_0 + \bar{s}_0 = e_0$, so we can express the amount invested in the storage technology in terms of entrepreneurs leverage as $s_0 = n_0(e_0/n_0 - (l_0 - 1))$. Note that the size of the QE program does not affect the amount ultimately invested in storage, as the bonds the central bank purchases are offset with the reserves it takes from investors. Finally, from the central bank's budget constraint we have that $\bar{s}_0 = \bar{b}_0$.

Using the previous expressions we can express secondary market liquidity in terms of entrepreneurs leverage and QE, conditional on the interest on reserves relative to the return on the OTC market. Note that the number of sell orders is always equal to $A = \delta b_0$, as impatient investors will put all their bond holdings for sale in the OTC market.

If $\Delta > 1 + \bar{r}$ patient investors pledge all their liquid assets to place buy orders in the OTC market so the number of buy orders $B = (1 - \delta)[(1 + r)s_0 + (1 + \bar{r})\bar{s}_0]/q_1$ and market liquidity is given by

$$\theta = \frac{(1 - \delta)[(1 + r)s_0 + (1 + \bar{r})\bar{s}_0]}{\delta b_0 q_1} = \frac{(1 - \delta)\Delta [(1 + r)(e_0 - n_0(l_0 - 1)) + (1 + \bar{r})\bar{b}_0]}{\delta R^b (n_0(l_0 - 1) - \bar{b}_0)} \quad (\text{A.28})$$

Then,

$$\frac{\partial \theta}{\partial \bar{b}_0} = \frac{(1 - \delta)\Delta(1 + \bar{r})}{\delta R^b (n_0(l_0 - 1) - \bar{b}_0)} + \frac{(1 - \delta)\Delta [(1 + r)(e_0 - n_0(l_0 - 1)) + (1 + \bar{r})\bar{b}_0]}{\delta R^b (n_0(l_0 - 1) - \bar{b}_0)^2} > 0 \quad (\text{A.29})$$

On the other hand, when $1 + \bar{r} > \Delta$ patient investors place buy orders in the OTC market only using the liquid assets they hold after funding the reserves liquidated by impatient investors, so the number of buy orders $B = (1 - \delta)[(1 + r)s_0 - \delta/(1 - \delta)(1 + \bar{r})\bar{s}_0]/q_1$ and market liquidity is given by

$$\theta = \frac{(1 - \delta)[(1 + r)s_0 - \frac{\delta}{1 - \delta}(1 + \bar{r})\bar{s}_0]}{\delta b_0 q_1} = \frac{(1 - \delta)\Delta \left[(1 + r)(e_0 - n_0(l_0 - 1)) - \frac{\delta}{1 - \delta}(1 + \bar{r})\bar{b}_0 \right]}{\delta R^b (n_0(l_0 - 1) - \bar{b}_0)}$$

Then,

$$\begin{aligned} \frac{\partial \theta}{\partial \bar{b}_0} &= - \frac{\Delta(1 + \bar{r})}{R^b (n_0(l_0 - 1) - \bar{b}_0)} + \frac{(1 - \delta)\Delta \left[(1 + r)(e_0 - n_0(l_0 - 1)) - \frac{\delta}{1 - \delta}(1 + \bar{r})\bar{b}_0 \right]}{\delta R^b (n_0(l_0 - 1) - \bar{b}_0)^2} \\ &= \frac{(1 - \delta)\Delta \left[(1 + r)e_0 - (1 + \bar{r})n_0(l_0 - 1) + (\bar{r} - r)n_0(l_0 - 1) \right]}{\delta R^b (n_0(l_0 - 1) - \bar{b}_0)^2} > 0 \end{aligned}$$

where the inequality follows from Assumption 4. Then, $\partial \theta / \partial \bar{b}_0 > 0$. ■

Proof of Proposition 9: We want to show that a planner that has access to QE as an additional policy tool will only use it when the return on storage r is strictly lower than $(\psi - \alpha)/(\alpha - \alpha\psi)$, or equivalently, when $\psi(1 + \alpha r) > \alpha(1 + r)$. Let $(l_0^{sp}, \bar{\omega}^{sp}, \theta^{sp})$ be the allocations chosen by the social planner studied in section 4 and denote by λ^{sp} the lagrange multiplier on the constraint of this planner (23).

Let \mathcal{L} be the Lagrangian of the central bank, which can be written as

$$\mathcal{L} = [1 - \Gamma(\bar{\omega})] R^k l_0 - \lambda \left[\mathbb{U}^{ce} - \mathbb{U}(l_0, \bar{\omega}, \theta(l_0, \bar{\omega}, \bar{b}_0, \bar{r}), \bar{b}_0, \bar{r}) \right] - \gamma \left[(1 + \bar{r})^2 - \bar{R}^b \right] - \nu [r - \bar{r}] + \eta \bar{b}_0$$

where we are considering the constraint imposed by the definition of secondary market liquidity (19) writing $\theta(l_0, \bar{\omega}, \bar{b}_0, \bar{r})$ and where we have already substituted in $\bar{s}_0 = \bar{b}_0$. An optimal allocation for this planner needs to satisfy the following FOCs:

$$\begin{aligned} (l_0) \quad 0 &= \frac{\partial \mathcal{L}}{\partial l_0} = [1 - \Gamma(\bar{\omega})] R^k + \lambda \left[\frac{\partial \mathbb{U}}{\partial l_0} + \frac{\partial \mathbb{U}}{\partial \theta} \frac{\partial \theta}{\partial l_0} \right] + \gamma \frac{\partial \bar{R}^b}{\partial l_0} \\ (\bar{\omega}) \quad 0 &= \frac{\partial \mathcal{L}}{\partial \bar{\omega}} = -\Gamma'(\bar{\omega}) R^k l_0 + \lambda \left[\frac{\partial \mathbb{U}}{\partial \bar{\omega}} + \frac{\partial \mathbb{U}}{\partial \theta} \frac{\partial \theta}{\partial \bar{\omega}} \right] + \gamma \frac{\partial \bar{R}^b}{\partial \bar{\omega}} \\ (\bar{b}_0) \quad 0 &= \frac{\partial \mathcal{L}}{\partial \bar{b}_0} = \lambda \left[\frac{\partial \mathbb{U}}{\partial \bar{b}_0} + \frac{\partial \mathbb{U}}{\partial \theta} \frac{\partial \theta}{\partial \bar{b}_0} \right] + \eta \\ (\bar{r}) \quad 0 &= \frac{\partial \mathcal{L}}{\partial \bar{r}} = \lambda \left[\frac{\partial \mathbb{U}}{\partial \bar{r}} + \frac{\partial \mathbb{U}}{\partial \theta} \frac{\partial \theta}{\partial \bar{r}} \right] - 2\gamma(1 + \bar{r}) + \nu \end{aligned}$$

Note that the size of the bond buying program \bar{b}_0 does not affect firm's profits directly, as the additional funds that the firm receives from the central bank, \bar{b}_0 , are perfectly offset by the reduction in the amount of funds received from investors, $b_0 = n_0(l_0 - 1) - \bar{b}_0$, as long as firm leverage is unchanged.

The next step is to evaluate the FOCs at the constrained efficient allocation (without QE), i.e., $(l_0^{sp}, \bar{\omega}^{sp}, \theta^{sp}, 0, r)$. If $\bar{R}^b(l_0^{sp}, \bar{\omega}^{sp}) \leq (1 + r)^2$ the central bank cannot implement QE without violating its funding constraint (30). So we consider that we are in the interesting case where

$\bar{R}^b(l_0^{sp}, \bar{\omega}^{sp}) > (1+r)^2$ and the central bank has some scope to offer a higher return on reserves relative to the storage technology. In this case the multiplier of this constraint at $(l_0^{sp}, \bar{\omega}^{sp}, \theta^{sp}, 0, r)$ equals zero, i.e., $\gamma = 0$. Moreover, note that at $\bar{b}_0 = 0$, investors' expected utility \mathbb{U} has the same functional form as in the case of the planner studied in section 4. Similarly, at $\bar{b}_0 = 0$ secondary market liquidity θ , equation (A.29), is the same function of choice variables as in the case without QE, equation (19). So we conclude that the FOCs *wrt* leverage l_0 and risk $\bar{\omega}$ are satisfied at $(l_0^{sp}, \bar{\omega}^{sp}, \theta^{sp}, 0, r)$. (In fact, we can use either FOC to obtain that $\lambda = \lambda^{sp}$, from where the other FOC follows.)

Next, note that

$$\frac{\partial \mathbb{U}}{\partial \bar{r}} = \bar{s}_0 \frac{\partial U_{\bar{s}}}{\partial \bar{r}} = \bar{b}_0 \frac{\partial U_{\bar{s}}}{\partial \bar{r}} \quad \Rightarrow \quad \frac{\partial \mathbb{U}(l_0^{sp}, \bar{\omega}^{sp}, \theta^{sp}, 0, r)}{\partial \bar{r}} = 0$$

And given that $(1 + \bar{r}) = (1 + r) < \Delta$ from equation (A.29) we have that

$$\frac{\partial \theta}{\partial \bar{r}} = \frac{(1 - \delta)\Delta \bar{b}_0}{\delta R^b(n_0(l_0 - 1) - \bar{b}_0)} \quad \Rightarrow \quad \frac{\partial \theta(l_0^{sp}, \bar{\omega}^{sp}, \theta^{sp}, 0, r)}{\partial \bar{r}} = 0$$

So the FOC *wrt* on interest on reserves \bar{r} is trivially satisfied, with $\nu = 0$.

Finally, we need to evaluate the FOC *wrt* \bar{b}_0 at $(l_0^{sp}, \bar{\omega}^{sp}, \theta^{sp}, 0, r)$. From this condition it follows that

$$\frac{\partial \mathbb{U}}{\partial \bar{b}_0} + \frac{\partial \mathbb{U}}{\partial \theta} \frac{\partial \theta}{\partial \bar{b}_0} < 0 \quad \Rightarrow \quad \eta > 0 \quad \text{and} \quad \bar{b}_0 = 0$$

To sign $\partial \mathbb{U} / \partial \bar{b}_0 + (\partial \mathbb{U} / \partial \theta) (\partial \theta / \partial \bar{b}_0)$ we proceed to compute these derivatives and evaluate at $(l_0^{sp}, \bar{\omega}^{sp}, \theta^{sp}, 0, r)$. One, note that

$$\mathbb{U}(l_0, \bar{\omega}, \theta, \bar{b}_0, \bar{r}) = [e_0 - n_0(l_0 - 1)]U_s + \bar{b}_0 U_{\bar{s}} + [n_0(l_0 - 1) - \bar{b}_0]U_b$$

Then,

$$\frac{\partial \mathbb{U}(l_0^{sp}, \bar{\omega}^{sp}, \theta^{sp}, 0, r)}{\partial \bar{b}_0} = U_{\bar{s}}(\theta^{sp}, r) - U_b(l_0^{sp}, \bar{\omega}^{sp}, \theta^{sp}) = U_{\bar{s}}(\theta^{sp}) - U_b(l_0^{sp}, \bar{\omega}^{sp}, \theta^{sp})$$

where we used that if interest on reserves are equal to the return on the storage technology then $U_{\bar{s}}(\theta^{sp}, r) = U_{\bar{s}}(\theta^{sp})$, from equation (34). On the other hand, from the conditions that describe the planner's allocations we have that

$$\begin{aligned} s_0^{sp} U_s(\theta^{sp}) + b_0^{sp} U_b(l_0^{sp}, \bar{\omega}^{sp}, \theta^{sp}) &= s_0^{ce} U_s(\theta^{ce}) + b_0^{ce} U_b(l_0^{ce}, \bar{\omega}^{ce}, \theta^{ce}) = e_0 U_s(\theta^{ce}) \\ \Rightarrow U_b(l_0^{sp}, \bar{\omega}^{sp}, \theta^{sp}) - U_s(\theta^{sp}) &= \frac{e_0 [U_s(\theta^{ce}) - U_s(\theta^{sp})]}{b_0^{sp}} = -\tau^s U_s(\theta^{sp}) \end{aligned} \quad (\text{A.30})$$

where we used the definition of the optimal tax on storage (27) in the last equality.

Then, from the characterization of the optimal tax on leverage in section 4.1 we have that if $r > (\psi - \alpha) / [\alpha(1 - \psi)]$, or equivalently $\psi(1 + \alpha r) < \alpha(1 + r)$, then

$$\tau^l < 0 \quad \Leftrightarrow \quad n_0 \frac{\partial U_b}{\partial \bar{\omega}} [U_s(\theta^{sp}) - U_b(l_0^{sp}, \bar{\omega}^{sp}, \theta^{sp})] + \left[\frac{\partial U_b}{\partial l_0} \frac{\partial \theta}{\partial \bar{\omega}} - \frac{\partial U_b}{\partial \bar{\omega}} \frac{\partial \theta}{\partial l_0} \right] \frac{\partial \mathbb{U}}{\partial \theta} < 0 \quad (\text{A.31})$$

where we have substituted (A.31) into the expression for the optimal tax on leverage (28).

Two, from Proposition 8 we had that $\partial \theta / \partial \bar{b}_0 > 0$ and evaluating equation (A.30) at

$(l_0^{sp}, \bar{\omega}^{sp}, \theta^{sp}, 0, r)$ we get

$$\frac{\partial \theta(l_0^{sp}, \bar{\omega}^{sp}, \theta^{sp}, 0, r)}{\partial \bar{b}_0} = \frac{\theta^{sp} e_0}{[e_0 - n_0(l_0^{sp} - 1)] n_0(l_0^{sp} - 1)} \quad (\text{A.32})$$

Three, note that if $r > (\psi - \alpha)/[\alpha(1 - \psi)]$ we have that from equation (A.32) that

$$n_0 \frac{\partial U_b}{\partial \bar{\omega}} \frac{\partial \mathbb{U}}{\partial \bar{b}_0} + n_0 \frac{\partial U_b}{\partial \bar{\omega}} \frac{\partial \theta}{\partial \bar{b}_0} \frac{\partial \mathbb{U}}{\partial \theta} < \left\{ -\frac{\partial U_b}{\partial l_0} \frac{\partial \theta}{\partial \bar{\omega}} + \frac{\partial U_b}{\partial \bar{\omega}} \frac{\partial \theta}{\partial l_0} + n_0 \frac{\partial U_b}{\partial \bar{\omega}} \frac{\partial \theta}{\partial \bar{b}_0} \right\} \frac{\partial \mathbb{U}}{\partial \theta}$$

But the term in curly brackets evaluated at $(l_0^{sp}, \bar{\omega}^{sp}, \theta^{sp}, 0, r)$ is zero. In fact, using equations (A.9), (A.21), and (A.33) we have

$$\begin{aligned} & -\frac{\partial U_b}{\partial l_0} \frac{\partial \theta}{\partial \bar{\omega}} + \frac{\partial U_b}{\partial \bar{\omega}} \frac{\partial \theta}{\partial l_0} + n_0 \frac{\partial U_b}{\partial \bar{\omega}} \frac{\partial \theta}{\partial \bar{b}_0} = \\ & \frac{U_b^{sp} \theta^{sp} [\Gamma'(\bar{\omega}^{sp}) - \mu G'(\bar{\omega}^{sp})]}{\Gamma(\bar{\omega}^{sp}) - \mu G(\bar{\omega}^{sp})} \left[\frac{-1}{l_0^{sp} (l_0^{sp} - 1)} - \frac{e_0 + n_0}{l_0^{sp} [e_0 - n_0 (l_0^{sp} - 1)]} + \frac{e_0}{[e_0 - n_0 (l_0^{sp} - 1)] (l_0^{sp} - 1)} \right] = 0 \end{aligned}$$

So we conclude that

$$n_0 \frac{\partial U_b}{\partial \bar{\omega}} \frac{\partial \mathbb{U}}{\partial \bar{b}_0} + n_0 \frac{\partial U_b}{\partial \bar{\omega}} \frac{\partial \theta}{\partial \bar{b}_0} \frac{\partial \mathbb{U}}{\partial \theta} < 0$$

And since $\partial U_b / \partial \bar{\omega} > 0$, then $\partial \mathbb{U} / \partial \bar{b}_0 + (\partial \mathbb{U} / \partial \theta) (\partial \theta / \partial \bar{b}_0) < 0$. Thus, it must be that if $r > (\psi - \alpha)/[\alpha(1 - \psi)]$ then $\eta > 0$ and the optimal QE designs calls for not buying bonds, i.e., $\bar{b}_0 = 0$.

Alternatively, when $r < (\psi - \alpha)/[\alpha(1 - \psi)]$ we can follow the previous line of argument to show that the tax on leverage is positive so

$$n_0 \frac{\partial U_b}{\partial \bar{\omega}} \frac{\partial \mathbb{U}}{\partial \bar{b}_0} + n_0 \frac{\partial U_b}{\partial \bar{\omega}} \frac{\partial \theta}{\partial \bar{b}_0} \frac{\partial \mathbb{U}}{\partial \theta} > \left\{ -\frac{\partial U_b}{\partial l_0} \frac{\partial \theta}{\partial \bar{\omega}} + \frac{\partial U_b}{\partial \bar{\omega}} \frac{\partial \theta}{\partial l_0} + n_0 \frac{\partial U_b}{\partial \bar{\omega}} \frac{\partial \theta}{\partial \bar{b}_0} \right\} \frac{\partial \mathbb{U}}{\partial \theta} = 0$$

where the derivatives are evaluated at $(l_0^{sp}, \bar{\omega}^{sp}, \theta^{sp}, 0, r)$. Thus,

$$\frac{\partial \mathbb{U}}{\partial \bar{b}_0} + \frac{\partial \mathbb{U}}{\partial \theta} \frac{\partial \theta}{\partial \bar{b}_0} > 0$$

since $\partial U_b / \partial \bar{\omega} > 0$. That is, the central bank wants to buy bonds, so $\eta = 0$. Finally, fully differentiating the Lagrangean \mathcal{L} of the central bank's problem and evaluating at the constrained efficient allocation with out QE $(l_0^{sp}, \bar{\omega}^{sp}, \theta^{sp}, 0, r)$, we have that

$$d\mathcal{L} = \lambda \left[\frac{\partial \mathbb{U}}{\partial \bar{b}_0} + \frac{\partial \mathbb{U}}{\partial \theta} \frac{\partial \theta}{\partial \bar{b}_0} \right] d\bar{b}_0 > 0.$$

So we conclude that when $r < (\psi - \alpha)/[\alpha(1 - \psi)]$ a central bank will set positive bond buying program, improving upon the constrained efficient allocation. When \bar{b}_0 it follows from the FOC wrt \bar{r} that the central bank will pay a higher interest on reserves relative to the return on the storage technology. In fact, γ will be strictly positive and the central bank's funding constraint will be binding. \blacksquare

Tables and Figures

Table 1: Planning outcomes and Implementation

ψ	1.0	0.8	0.6	0.4	0.2	0.0
% change in l_0	-8.62%	-5.03%	-1.63%	1.72%	5.13%	8.63%
% change in $\bar{\omega}$	-5.27%	-3.06%	-0.99%	1.04%	3.08%	5.17%
% change in θ	62.01%	27.75%	7.44%	-6.70%	-17.42%	-26.03%
% change in \mathbb{U}^F	0.23%	0.07%	0.01%	0.01%	0.06%	0.16%
% change in \mathbb{U}	0.00%	0.00%	0.00%	0.00%	0.00%	0.00%
τ^l	0.27%	0.15%	0.05%	-0.05%	-0.13%	-0.21%
τ^s	0.00%	-0.05%	-0.03%	0.04%	0.14%	0.27%

Table 2: Outcomes with Quantitative Easing

	Constrained Efficient Allocations	Quantitative Easing with $\tau^s = \tau^l = 0$	Quantitative Easing with τ^s, τ^l Chosen Optimally
% change in l_0	-6.78%	1.68%	-3.05%
% change in $\bar{\omega}$	-4.13%	0.72%	-2.35%
% change in θ	42.19%	43.37%	167.72%
% change in \mathbb{U}^F	0.14%	0.42%	0.98%
% change in \mathbb{U}	0.00%	0.00%	0.00%
\bar{r}		1.16%	1.10%
\bar{s}_0		8.44%	18.02%
τ^l	0.21%		0.17%
τ^s	-0.04%		-0.05%

Figure 3

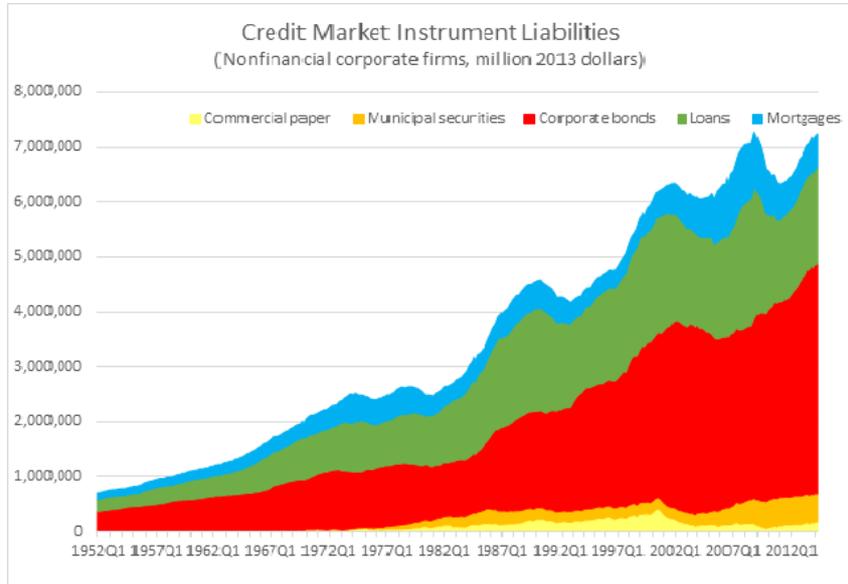
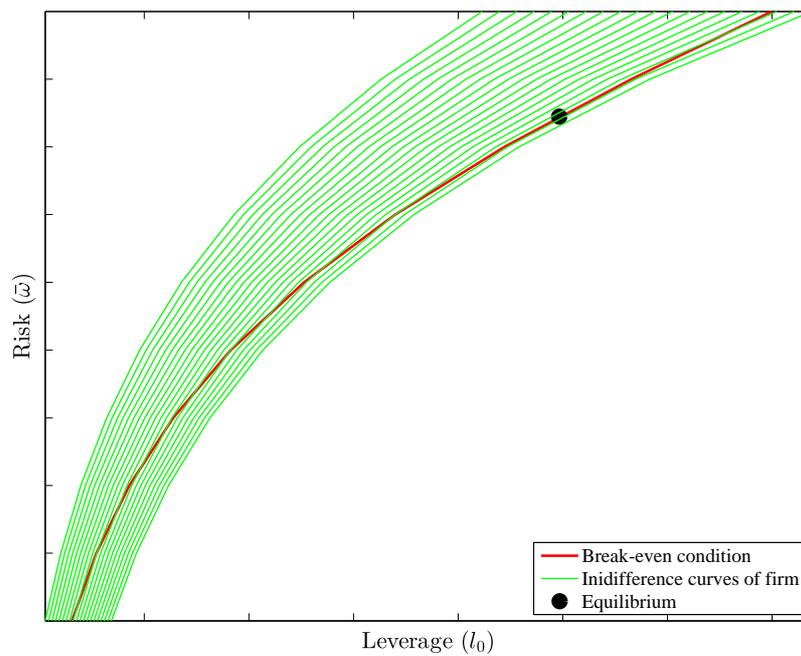
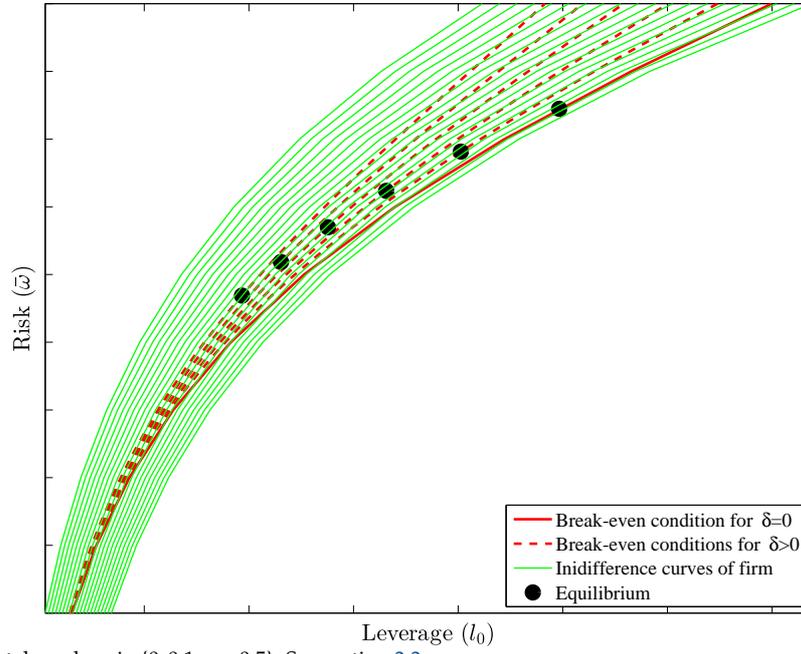


Figure 4: Equilibrium in the Frictionless Benchmark



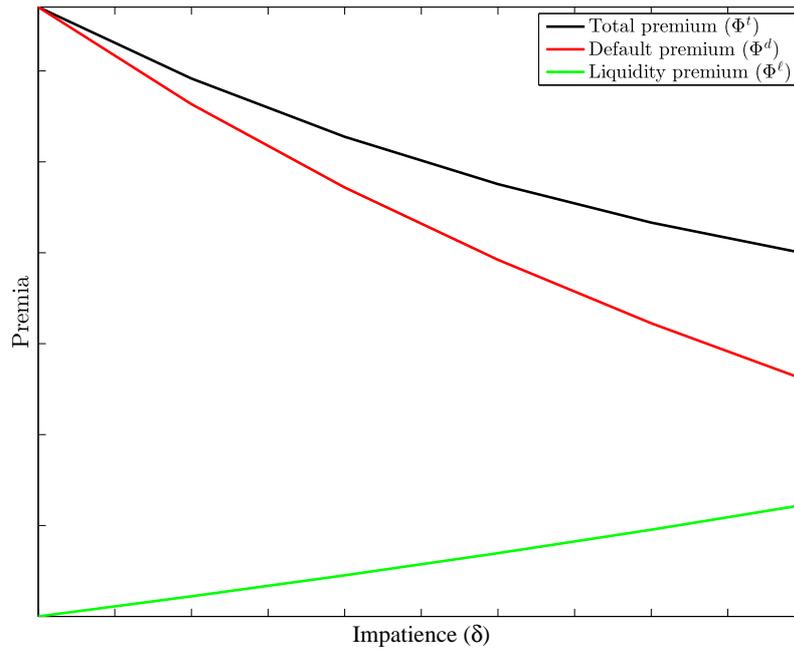
Note: For details see section 3.3.

Figure 5: Comparative Statics on δ .



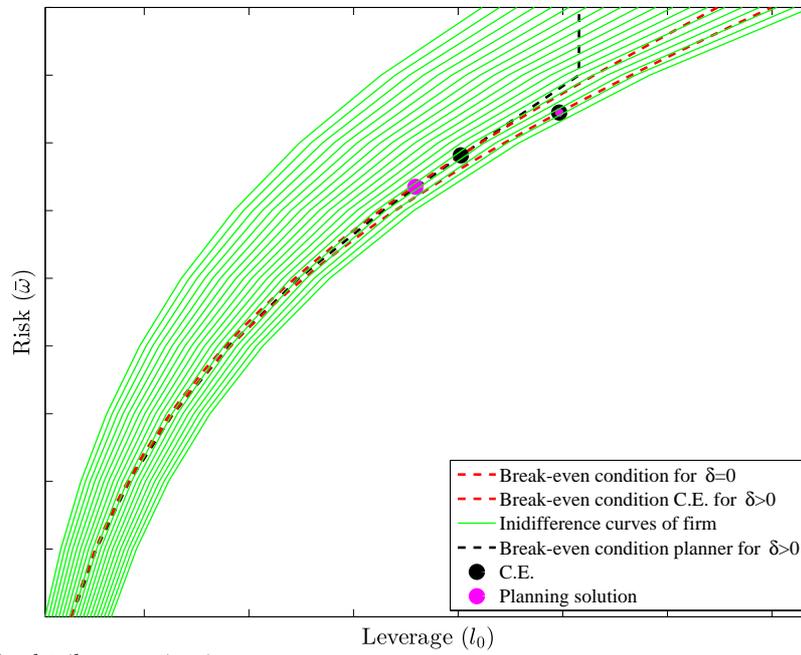
Note: δ take values in $\{0, 0.1, \dots, 0.5\}$. See section 3.3.

Figure 6: Bond Premia Decomposition



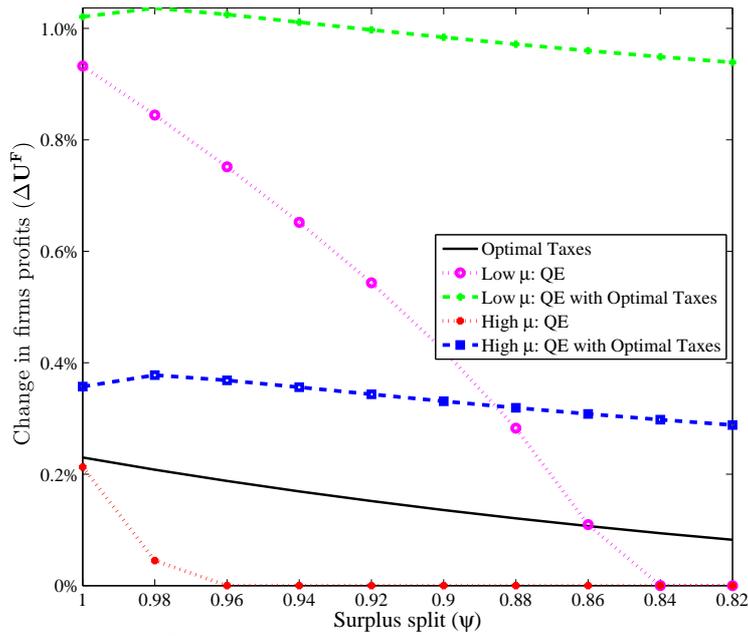
Note: For details see section 3.3.

Figure 7: Constrained Efficient Equilibrium



Note: For details see section 4.

Figure 8: Effect of Quantitative Easing



Note: For details see section 5.