Recursive equilibria in dynamic economies with stochastic production*

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Abstract

In this paper we prove the existence of recursive equilibria in stochastic production economies with infinitely lived agents and incomplete financial markets. We consider a general dynamic model with several commodities, which encompasses heterogeneous agent versions of both the Lucas asset pricing model and the stochastic neo-classical growth model as special cases. Our main assumption is that there are atomless shocks to fundamentals that have a purely transitory component and a component that does not depend on last period's shocks directly.

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1 Introduction

The use of so-called recursive equilibria to analyze dynamic stochastic general equilibrium models has become increasingly important in financial economics, macroeconomics, and in public finance. These equilibria are characterized by a pair of functions: a transition function mapping this period's "state" into probability distributions over next period's state, and a "policy function" mapping the current state into current prices and choices (see, e.g., Ljungquist and Sargent (2004) for an introduction.) In applications that consider dynamic stochastic economies with heterogeneous agents and production, it is typically the current exogenous shock together with the capital stock and the beginning-of-period distribution of assets across individuals that defines this recursive state. Following the terminology of stochastic games, we will often refer to recursive equilibria with this minimal "natural" state space as stationary Markov equilibria. Unfortunately, for models with infinitely lived agents and incomplete financial markets no sufficient conditions for the existence of these stationary Markov equilibria can be found in the existing literature. In this paper we consider a dynamic economy with stochastic production, give two examples that illustrate why recursive equilibria might fail to exist, and prove the existence of recursive equilibria for economies with atomless shocks to fundamentals. We assume that these shocks have a purely transitory component affecting endowments and a component that does not depend on last period's shocks directly, which might affect endowments, the production function, and preferences.

There are a variety of reasons for focusing on stationary Markov equilibria. Most importantly, recursive methods can be used to approximate stationary Markov equilibria numerically. Heaton and Lucas (1996), Krusell and Smith (1998), and Kubler and Schmedders (2003) are early examples of papers that approximate stationary Markov equilibria in models with infinitely lived heterogeneous agents. Although an existence theorem for stationary Markov equilibria has not been available, applied research, even if explicitly aware of the problem, needs to focus on such equilibria, as there are no efficient algorithms for the computation of equilibria that are not recursive.[1] For the case of dynamic games, Maskin and Tirole (2001) list several conceptual arguments in favor of stationary Markov equilibria. Duffie et al. (1994) give similar arguments that also apply to dynamic general equilibrium: As prices vary across date events in a dynamic stochastic market economy, it is important that the price process is simple — for instance Markovian on some minimal state space — to justify the assumption that agents have rational expectations.

Unfortunately, due to the non-uniqueness of continuation equilibria, recursive equilibria do not always exist. This intuition was first explained and illustrated in Hellwig (1983) and since then has

[1] While Feng et al. (2013) provide an algorithm for this case, their method can only be used for very small-scale models.

been demonstrated in different contexts. Kubler and Schmedders (2002) give an example showing the non-existence of stationary Markov equilibria in models with incomplete asset markets and infinitely lived individuals. Santos (2002) gives examples of non-existence for economies with externalities. Kubler and Polemarchakis (2004) give examples in models with overlapping generations, which we modify to fit our framework with infinitely lived agents and production. We use these examples to illustrate why one of our main assumptions is needed to obtain existence.

The existence of competitive equilibria for general Markovian exchange economies was shown in Duffie et al. (1994). The authors also prove that the equilibrium process is a stationary Markov process. However, we follow the well established terminology in dynamic games and do not refer to these equilibria as stationary Markov equilibria, because the state also contains consumptions and prices from the previous period.

Citanna and Siconolfi (2010 and 2012) give sufficient conditions for the generic existence of stationary Markov equilibria in models with overlapping generations. Their arguments cannot be extended to models with infinitely lived agents or to models with occasionally binding constraints on agents' choices, and for their argument to work they need to assume a very large number of heterogeneous agents within each generation.

Duggan (2012) and He and Sun (2013) give sufficient conditions for the existence of a Markov equilibrium in stochastic games with uncountable state spaces. Building on work by Nowak and Raghavan (1992), He and Sun (2013) use a result from Dynkin and Evstigneev (1977) to provide sufficient conditions for the convexity of the conditional expectation operator. They show that the assumption of a public coordination device ("sunspot") in Nowak and Raghavan can be replaced by natural assumptions on the exogenous shock to fundamentals.

To show the existence of a recursive equilibrium, we characterize it by a function that maps the recursive state into marginal utilities of all agents. We show that such a function describes a recursive equilibrium if it is a fixed point of an operator that captures the period-to-period equilibrium conditions. Using this characterization, we proceed in two steps to prove the existence of recursive equilibrium. First, we make direct assumptions on the transition probability for the recursive state. Assuming that the probability distribution of next period's state varies continuously with current actions (a "norm-continuous" transition), the operator defined by the equilibrium conditions is a non-empty correspondence on the space of marginal utility functions. Unfortunately, the Fan–Glicksberg fixed point theorem only guarantees the existence of a fixed point in the convex hull of this correspondence. However, following He and Sun (2013) we give conditions that ensure that this is also a fixed point of the original correspondence. For this, we assume that the density of the transition probability is measurable with respect to a sigma algebra that is sufficiently coarse

relative to the sigma algebra representing the total information available to agents. This establishes Theorem 1, which provides a first set of sufficient conditions for the existence of recursive equilibria. In a second step, we provide assumptions on economic fundamentals that guarantee that the endogenous transition probability indeed satisfies these conditions. In particular, we assume that there are atomless shocks to fundamentals that have a purely transitory component affecting endowments and a component that does not depend on last period's shocks directly. The latter might affect endowments, the production function, and preferences. Theorem 2 states that under these assumptions recursive equilibrium exists.
We present our main result for a model without short-lived financial assets (as in Duffie et al. (1994)) — this makes the argument simpler and highlights the economic assumptions necessary for our existence result. We then introduce financial securities together with collateral constraints. In order to define a compact endogenous state space we need to make relatively strong assumptions on endowments and preferences, and to impose constraints on trades. It is subject to further investigations whether these assumptions can be relaxed. While it is well understood that without occasionally binding constraints on trade the existence of recursive equilibrium cannot be established (see, e.g., Krebs (2004)) the assumptions made in this paper are certainly stronger than needed.

In a stationary Markov equilibrium the relevant state-space consists of endogenous as well as exogenous variables that are payoff-relevant, pre-determined, and sufficient for the optimization of individuals at every date event (Maskin and Tirole (2001) give a formal definition of payoff-relevant states for Markov perfect equilibria.) There are several computational approaches that use individuals' "Negishi weights" as an endogenous state instead of the distribution of assets (see, e.g., Dumas and Lyasoff (2012) or Brumm and Kubler (2014)). In models with incomplete financial markets this alternative state does not simplify proving the existence of a recursive equilibrium. Brumm and Kubler (2014) prove existence in a model with overlapping generations, complete financial markets, and borrowing constraints, but the approach does not extend to models with incomplete markets. In this paper we focus on equilibria that are recursive on the "natural" state space — that is to say, the space consisting of the shock, the distribution of assets, and the capital stock.

The rest of the paper is organized as follows: In Section 2 we present the basic model and explain some important special cases. Section 3 contains simple examples that illustrate the difficulties in establishing the existence of recursive equilibria. Section 4 contains an existence proof for recursive equilibria. Section 5 shows how the model can be extended to include one-period financial securities and how in concrete models some of the assumptions made in Section 4 can be relaxed.

2 A general dynamic Markovian economy

We describe the economic model and define recursive equilibrium.

2.1 The model

Time is indexed by $t \in \mathbb{N}_0$. Exogenous shocks $z_t \in \mathbf{Z}$ realize in a complete, separable metric space $\mathbf{Z}$, and follow a first-order Markov process with transition probability $\mathbb{P}(.|z)$ defined on the Borel σ-algebra $\mathcal{Z}$ on $\mathbf{Z}$, that is, $\mathbb{P} : \mathbf{Z} \times \mathcal{Z} \to [0, 1]$. By a standard argument one can construct a filtration $(\mathcal{F}_t)$ so that (z_t) is an $\mathcal{F}_t$-adapted stochastic process. A history of shocks up to some date t is denoted by $z^t = (z_0, z_1, \ldots, z_t)$ and is also called a date event. Whenever convenient, we simply use t instead of z^t. An $\mathcal{F}_t$-adapted stochastic process will be denoted by (x_t).

We consider a production economy with infinitely lived agents. There are H types of agents, $h \in \mathbf{H} = \{1, \ldots, H\}$. At each date event there are L perishable commodities, $l \in \mathbf{L} = \{1, ..., L\}$, available for consumption and production. The individual endowments are denoted by $\omega_h(z^t) \in \mathbb{R}^L_+$ and we assume that they are time-invariant and measurable functions of the current shock alone. We take the consumption space to be the space of adapted and essentially bounded processes. Each agent has a time-separable expected utility function

$$U_h(x) = \mathbb{E}_0 \left[\sum_{t=0}^{\infty} \delta^t u_h \left(z_t, x_t \right) \right],$$

where $\delta \in \mathbb{R}$ is the discount factor, $x_t \in \mathbb{R}^L_+$ denotes the agent's (stochastic) consumption at date t, and x denotes his entire consumption process.

It is useful to distinguish between intertemporal and intraperiod production. Intraperiod production is characterized by a measurable correspondence $\mathbf{Y} : \mathbf{Z} \rightrightarrows \mathbb{R}^L$, where a production plan $y \in \mathbb{R}^L$ is feasible at shock z if $y \in \mathbf{Y}(z)$. For simplicity (and without loss of generality) we assume throughout that each $\mathbf{Y}(z)$ exhibits constant returns to scale so that ownership does not need to be specified.

Intertemporally each type $h = 1, ..., H$ has access to J linear storage technologies, $j \in \mathbf{J} = \{1, ..., J\}$. At a node z each technology (h, j) is described by a column vector of inputs $a^0_{hj}(z) \in \mathbb{R}^L_+$ and a vector-valued random variable of outputs in the subsequent period, $a^1_{hj}(z') \in \mathbb{R}^L_+$. We write $A^0_h(z) = (a^0_{h1}(z), \ldots, a^0_{hJ}(z))$ for the $L \times J$ matrix of inputs and $A^1_h(z') = (a^1_{h1}(z'), \ldots, a^1_{hJ}(z'))$ for the $L \times J$ matrix of outputs. We denote by $\alpha_h(z^t) = (\alpha_{h1}(z^t), ..., \alpha_{hJ}(z^t))^\top \in \mathbb{R}^J_+$ the levels at which the linear technologies are operated at node z^t by agent h.

Each period there are complete spot markets for the L commodities; we denote prices by $p(z^t) = (p_1(z^t), ..., p_L(z^t))$, a row vector. For what follows it will be useful to define the set of stored commodities (or "capital goods") to be

$$\mathbf{L}^K = \{l \in \mathbf{L} : \sum_{h \in \mathbf{H}} \sum_{j \in \mathbf{J}} a^1_{hjl}(z) > 0 \text{ for some } z \in \mathbf{Z}\},$$

and to define $\mathbf{K}^U = \{x \in \mathbb{R}_+^{HL} : x_{hl} = 0 \text{ whenever } l \notin \mathbf{L}^K, h \in \mathbf{H}\}$. We decompose individual endowments into capital goods, f_h, and consumption goods, e_h, and define

$$f_{hl}(z) = \begin{cases} \omega_{hl}(z) & \text{if } l \in \mathbf{L}^K \\ 0 & \text{otherwise,} \end{cases}$$

and $e_h(z) = \omega_h(z) - f_h(z)$.

At $t = 0$ agents have some initial endowment in the capital goods that might be larger than $f_h(z_0)$ and to simplify notation we write the difference as $A_h^1(z_0)\alpha_h(z^{-1})$ for each agent h. We refer to these endowments across agents as the "initial condition".

Given initial conditions $(A_h^1(z_0)\alpha_h(z^{-1}))_{h\in\mathbf{H}} \in \mathbf{K}^U$, we define a sequential competitive equilibrium to be a process of $\mathcal{F}_t$-adapted prices and choices,

$$\left(p_t, (x_{h,t}, \alpha_{h,t})_{h\in\mathbf{H}}, y_t\right)_{t=0}^{\infty}$$

such that markets clear and agents optimize, i.e., (A), (B), and (C) hold.

(A) Market clearing equations:

$$\sum_{h\in\mathbf{H}} (x_h(z^t) + A_h^0(z_t)\alpha_h(z^t) - \omega_h(z_t) - A_h^1(z_t)\alpha_h(z^{t-1})) = y(z^t), \quad \text{for all } z^t$$

(B) Profit maximization:

$$y(z^t) \in \arg\max_{y\in\mathbf{Y}(z_t)} p(z^t)\cdot y$$

(C) Each agent $h = 1, \ldots, H$ maximizes utility:

$$(x_h, \alpha_h) \in \arg\max_{(x,\alpha)\geq 0} U_h(x)$$

$$\text{s.t. } p(z^t)\left(x(z^t) + A_h^0(z_t)\alpha(z^t) - \omega_h(z_t) - A_h^1(z_t)\alpha(z^{t-1})\right) \leq 0, \quad \text{for all } z^t.$$

2.2 Special cases

There are two special cases of the model that are worthwhile discussing in some detail.

In the heterogenous agent version of the Lucas (1978) asset pricing model that is examined in Duffie et al. (1994) there are D Lucas trees available for trade. These are long-lived assets in unit net supply that pay exogenous positive dividends in terms of the single consumption good, which depend on the shock alone. Agents can trade in these trees but are not allowed to hold short positions and there are no other financial securities available for trade. In our model this would amount to assuming that there are $D+1$ commodities (the first D representing the trees), no intraperiod production, and intertemporal production where each agent can store each commodity $l = 1, ..., D$, which then yields one unit of commodity l and a state-contingent amount of commodity $D+1$ (the tree's dividends) per unit stored. Agents only derive utility from consumption of commodity $D+1$

and have positive individual endowments only in this commodity. At $t = 0$, for all $l = 1, ..., D$, agents have initial endowments in commodity l that add up to 1. It is easy to see that a sequential competitive equilibrium for this version of our model will yield the same consumption allocation as a sequential equilibrium in the heterogenous agent Lucas model.

In the Brock–Mirman neo-classical stochastic growth model with heterogenous agents, considered in Krusell and Smith (1998), there is a single capital good that can be used in intraperiod production, together with labor, to produce the single consumption good. This good can be consumed or stored in a linear technology yielding one minus depreciation units of the capital good at all nodes in the subsequent period. Agents derive utility from the consumption good (and possibly from leisure). This is obviously a simple special case of our model. However, unlike Krusell and Smith (1998) we assume that there are finitely many agents.

Of course, in our general framework it is easy to combine the models, include land as a factor of production, or to consider models with irreversible investments.

2.3 Recursive equilibrium

We take as an endogenous state variable the beginning-of-period holdings in capital goods, either obtained from storage or from endowments. We fix an endogenous state space $\mathbf{K} \subset \mathbf{K}^U$ and take $\mathbf{S} = \mathbf{Z} \times \mathbf{K}$. A recursive equilibrium consists of "policy" and "pricing" functions

$$F_\alpha : \mathbf{S} \to \mathbb{R}_+^{HJ}, \quad F_x : \mathbf{S} \to \mathbb{R}_+^{HL}, \quad F_p : \mathbf{S} \to \Delta^{L-1}$$

such that for all initial shocks $z_0 \in \mathbf{Z}$, and all initial conditions $\left(A_h^1(z_0)\alpha_h(z^{-1}) + f_h(z_0)\right)_{h\in\mathbf{H}} \in \mathbf{K}$, there exists a competitive equilibrium,

$$\left(p_t, (x_{h,t}, \alpha_{h,t})_{h\in\mathbf{H}}, y_t\right)_{t=0}^{\infty}$$

such that for all z^t

$$s(z^t) = \left(z_t, \left(A_h^1(z_t)\alpha_h(z^{t-1}) + f_h(z_t)\right)_{h\in\mathbf{H}}\right) \in \mathbf{Z} \times \mathbf{K}$$

and $p(z^t) = F_p(s(z^t))$, $x(z^t) = F_x(s(z^t))$, $\alpha(z^t) = F_\alpha(s(z^t))$.

For computational convenience one typically wants $\mathbf{K}$ to be convex — this will be guaranteed in our existence proof below but for now we do not include the requirement in the definition of recursive equilibrium.

Note that we chose the endogenous state space $\mathbf{K}$ to be a subset of $\mathbf{K}^U$, where $\mathbf{K}^U$ represents the holding of broadly defined capital goods $\mathbf{L}^K$. At the cost of notational inconvenience one could define capital goods and the space of capital holdings agent-wise by

$$\mathbf{L}_h^K = \{l \in \mathbf{L} : \sum_{j\in\mathbf{J}} a_{hjl}^1(z) > 0\}, \ \mathbf{K}_h^U = \{x \in \mathbb{R}_+^{HL} : x_l = 0 \text{ whenever } l \notin \mathbf{L}_h^K\}.$$

The endogenous state space would then satisfy $\mathbf{K} \subset \bigotimes_{h \in H} \mathbf{K}_h^U$, which could be considerably smaller than in the above definition, depending on the application. Similarly, one could make the space of capital holdings depend on the shock $z \in \mathbf{Z}$.

3 Possible non-existence

The following simple examples illustrate why recursive equilibria might fail to exist and help us to motivate our assumptions on exogenous shocks made in Section 4 below. A reader who is primarily interested in the existence proof may wish to skip this section.

The examples are a variation of the examples in Kubler and Polemarchakis (2004) adapted to a model with production and infinitely lived agents. The first example has the advantage that it can be analyzed analytically and all computations are extremely simple. It has the disadvantage that it is non-generic in the sense that non-existence in this example stems from the fact that there is a continuum of continuation equilibria. Preferences and endowments in the second example are more "standard" but we need some tools from computational algebraic geometry to analyze it.

The basic structure of uncertainty and production is the same in both examples. We assume that there are only three possible shock realizations, $z' \in \{1, 2, 3\}$, which are independent of the current shock and equiprobable, thus $\pi(z'|z) = 1/3$ for all $z, z' \in \{1, 2, 3\}$. There are two commodities and two types of agent. As in Section 2, we assume that each agent maximizes time-separable expected utility and to make computations as simple as possible we assume $\delta = 1/2$. Each agent has access to a storage technology. To simplify notation we assume that each agent has his own technology but given our assumptions on endowments below it would be equivalent to assume that each agent has access to both technologies. Agent 1's technology transforms one unit of commodity 1 at given shocks $z = 1$ and $z = 2$ to one unit of commodity 1 in the subsequent period whenever shock 3 occurs. Agent 2's technology transforms one unit of commodity 2 at given shocks $z = 1$ and $z = 2$ to one unit of commodity 2 in the subsequent period whenever shock 3 occurs. At shocks $z = 3$ no storage technology is available,[2] i.e., we have

$$a_1^0(1) = a_1^0(2) = (1, 0), a_1^0(3) = \infty, \qquad a_2^0(1) = a_2^0(2) = (0, 1), a_2^0(3) = \infty$$
$$a_1^1(1) = a_1^1(2) = 0, a_1^1(3) = (1, 0), \qquad a_2^1(1) = a_2^1(2) = 0, a_2^1(3) = (0, 1).$$

3.1 Example 1

In our first example, we assume that the Bernoulli utility functions of agents 1 and 2 are as follows

$$u_1(z = 1, (x_1, x_2)) = u_1(z = 2, x) = -\frac{1}{6x_1}, \quad u_1(z = 3, x) = -\frac{1}{x_1} + x_2,$$

[2] The assumption is made for convenience — all one needs is low enough productivity that guarantees that the activity is not used. In a slight abuse of notation we write $a_h^0(3) = \infty$.

$$u_2(z=1,(x_1,x_2)) = u_2(z=2,x) = -\frac{1}{6x_2}, \quad u_2(z=3,x) = x_1 - \frac{1}{x_2}.$$

Endowments of an agent of type 1 are

$$e_1(z=1) = (e_{11}(1), e_{12}(1)) = (2,0), \quad e_1(z=2) = (0.1,0), \quad e_1(z=3) = (0,2),$$

and endowments of agents of type 2 are

$$e_2(z=1) = (0,0.1), \quad e_2(z=2) = (0,2), \quad e_2(z=3) = (2,0).$$

For simplicity we set up the example completely symmetrically. In shocks 1 and 2 agent 1 only derives utility from consumption of good 1 and is only endowed with good 1, agent 2 only derives utility from good 2 and is only endowed with this good.

It is easy to see that at shocks 1 and 2 there will never be any trade. By assumption, if shock 3 occurs there cannot be any storage. Therefore, the economy decomposes into one-period and two-period "sub-economies". The only non-trivial case is when shock 3 is preceded by either shock 1 or 2. In these two-period economies, agents make a savings decision in the first period and interact in spot markets in the second period.

To analyze the equilibria in these two-period economies, it is useful to compute the individual demands in the second period in shock 3 as functions of the price ratio $\tilde{p} = \frac{p_2(z'=3)}{p_1(z'=3)}$ given amounts of commodity 1 obtained by agent 1's storage, κ_1, and amounts of commodity 2 obtained by agent 2's storage, κ_2. We obtain for agent 1,

$$x_1(\tilde{p}|\kappa) = \begin{cases} (\tilde{p}e_{12}(3) + \kappa_1, 0) & \text{for } \tilde{p}e_{12}(3) - \sqrt{\tilde{p}} + \kappa_1 \leq 0 \\ (\sqrt{\tilde{p}}, e_{12}(3) - \frac{1}{\sqrt{\tilde{p}}} + \frac{\kappa_1}{\tilde{p}}) & \text{otherwise.} \end{cases}$$

and, symmetrically for agent 2,

$$x_2(\tilde{p}|\kappa) = \begin{cases} (0, \frac{e_{21}(3)}{\tilde{p}} + \kappa_2) & \text{for } e_{21}(3) - \sqrt{\tilde{p}} + \tilde{p}\kappa_2 \leq 0 \\ (e_{21}(3) - \sqrt{\tilde{p}} + \tilde{p}\kappa_2, \frac{1}{\sqrt{\tilde{p}}}) & \text{otherwise.} \end{cases}$$

We note first that, in equilibrium, agent 2 never stores in shock 1 and agent 1 never stores in shock 2. To see this, observe that agent 2 stores in shock 1 only if his consumption in good 2 in the subsequent shock 3 is below 0.1. However, $x_2(\tilde{p}|\kappa) \leq 0.1$ and $\kappa \geq 0$ implies $e_{21}(3)/\tilde{p} \leq 0.1$, thus the (relative) price of good 2, $\tilde{p}$, must be at least 20. But then agent 1's consumption of good 1 must be at least $\sqrt{20}$, which violates feasibility. Therefore there cannot be an equilibrium where agent 2 stores in shock 1. The situation for shock 2 is completely symmetric — agent 1 will never store in this shock.

We now consider a two-period economy with the initial shock equal to 1 where agent 2 does not store, i.e., $\kappa_2 = 0$. If also $\kappa_1 = 0$, then the equilibrium conditions for the second period spot market have a continuum of solutions: any $\tilde{p}$ satisfying $e_{12}(3)^{-2} = 1/4 \leq \tilde{p} \leq e_{21}(3)^2 = 4$ is a possible spot market equilibrium. However, we now show that in the two-period economy only $\tilde{p} = 4$ is consistent with agent 1's intertemporal optimization. For $\tilde{p} = 4$, agent 1's consumption at shock 3 is given by $x_1(z' = 3) = (2, 1.5)$. If agent 1's consumption in good 1 drops below 2 he will always store positive amounts, and by feasibility it cannot be above 2 without storage.

To see that this equilibrium is unique, first observe that there cannot be another equilibrium with identical consumption for agent 1 in good 1. To see that there cannot be an equilibrium with $\kappa_1 > 0$, observe that for $\kappa_1 > 0$ the only possible spot equilibrium would have $x_{11} = 2 + \kappa_1$; however, the Euler equation implies that $\kappa_1 > 0$ is then inconsistent with intertemporal optimality.

When the economy starts in shock 2, the situation is completely symmetric, with only one possible equilibrium with $\kappa_1 = \kappa_2 = 0$, $\tilde{p} = \frac{1}{4}$ and agent 1's consumption given by $x_1(z' = 3) = (0.5, 0)$.

Thus, in every competitive equilibrium we have $\kappa_1 = \kappa_2 = 0$ and consumption and prices in shock 3 differ depending on whether the realization of the previous shock was 1 or 2. Therefore, there is no recursive equilibrium.

3.2 Example 2

In the second example we assume that agents of type 1 have Bernoulli utility

$$u_1(z = 1, (x_1, x_2)) = u_1(z = 2, x) = -\frac{1}{12}x_1^{-2}, \quad u_1(z = 3, x) = -\frac{1}{2}x_1^{-2} - 32x_2^{-2},$$

and agents of type 2 have Bernoulli functions

$$u_2(z = 1, (x_1, x_2)) = u_2(z = 2, x) = -\frac{1}{12}x_2^{-2}, \quad u_2(z = 3, x) = -32x_1^{-2} - \frac{1}{2}x_2^{-2}.$$

Endowments of an agent of type 1 are

$$e_1(z = 1) = (\frac{1}{15}(5 + 2\sqrt{5}), 0), \quad e_1(z = 2) = (0.01, 0), \quad e_1(z = 3) = (0, 1),$$

and endowments of agents of type 2 are

$$e_2(z = 1) = (0, 0.01), \quad e_2(z = 2) = (0, \frac{1}{15}(5 + 2\sqrt{5})), \quad e_2(z = 3) = (1, 0).$$

Contrary to Example 1, in this example the assumption that endowments lie on the boundary is made to simplify the algebra and is not crucial for the non-existence result.

The first part of the argument is exactly as in the first example: at shocks 1 and 2 there will never be any trade, and when shock 3 occurs there cannot be any storage. Therefore, we only have to consider two-period economies where agents make a savings decision in the first period and interact on spot markets in the second period. There are two such economies depending on whether the shock in the first period is 1 or 2.

To analyze the equilibria in the two-period economies, we note again that, in equilibrium, agent 2 will never store in shock 1 and agent 1 will never store in shock 2. Agent 2 stores in shock 1 only if his consumption in good 2 in the subsequent period is below 0.01. From $x_{22} < 0.01$, $\kappa \geq 0$, and the budget constraint of agent 2, we get: $x_{21} > 1 - p_2/p_1 \cdot 0.01$. Also, the following first order condition must hold in shock 3:

$$\frac{4x_{22}}{x_{21}} = \left(\frac{p_1}{p_2}\right)^{\frac{1}{3}}.$$

Combined, these conditions imply that the (relative) price of good 2 must certainly be larger than 64 — assuming $p_2/p_1 < 64$ results in a contradiction:

$$\frac{1}{9} = \frac{0.04}{0.36} > \frac{4 \cdot 0.01}{1 - p_2/p_1 \cdot 0.01} > \frac{4x_{22}}{x_{21}} = \left(\frac{p_1}{p_2}\right)^{\frac{1}{3}} > \left(\frac{1}{64}\right)^{\frac{1}{3}} = \frac{1}{4}$$

However, if $p_2/p_1 \geq 64$, then agent 1's consumption of good 1 must certainly be above $1+\frac{1}{15}(5+2\sqrt{5})$, which violates feasibility.

To determine the full equilibrium it is useful to first focus on possible equilibrium allocations and prices in shock $z' = 3$, given an amount stored from the previous period. We focus on the case in which agent 1 operated the storage technology (in shock 1); the case where agent 2 operated the technology (in shock 2) is analogous by symmetry.

We normalize the price of good 2, $p_2 = 1$, and define $\tilde{p} := \sqrt[3]{p_1(z' = 3)}$. Then, given goods from storage, κ_1, and substituting the market clearing conditions, equilibrium can be described by the following polynomial system of equations — the first order conditions of agent 1 and 2, and the budget constraint of agent 1:

$$\begin{aligned} -4x_{11}\tilde{p} + x_{12} &= 0 \\ -\frac{1}{4}(1 + \kappa_1 - x_{11})\tilde{p} + (1 - x_{12}) &= 0 \\ \tilde{p}(x_{11} - \kappa_1) + (x_{12} - 1) &= 0 \end{aligned}$$

The lexicographic Gröbner basis (see Kubler and Schmedders (2010a) for an introduction to this method and Kubler and Schmedders (2010b) for its application to general equilibrium models) reveals that agent 1's consumption in good 1 is determined by the following equations:

$$225x_{11}^3 + 195(-\kappa_1 - 1)x_{11}^2 + (-29\kappa_1^2 - 58\kappa_1 + 35)x_{11} + (-\kappa_1^3 - 3\kappa_1^2 - 67\kappa_1 - 1) = 0 \qquad (1)$$

At $\kappa_1 = 0$ the three solutions are given by $x_{11}^1 = \frac{1}{5}, x_{11}^2 = \frac{1}{15}(5 - 2\sqrt{5}), x_{11}^3 = \frac{1}{15}(5 + 2\sqrt{5})$. Figure 1 depicts the relation between the equilibrium consumption, x_{11}, and κ_1. The figure clearly shows that for $\kappa_1 = 0$ there are three different values of x_{11} that are consistent with equilibrium.

Moving to the two-period model, one obviously obtains that if the initial shock is 1, with $e_{11}(z = 1) = \frac{1}{15}(5+2\sqrt{5})$ one equilibrium is $\kappa_1 = \kappa_2 = 0$, $x_{11}(z = 1) = e_{11}(z = 1)$, $x_{11}(z' = 3) = e_{11}(z = 1)$. Despite the fact that at $\kappa_1 = 0$ we found 3 solutions to Equation (1) above, it turns out that this

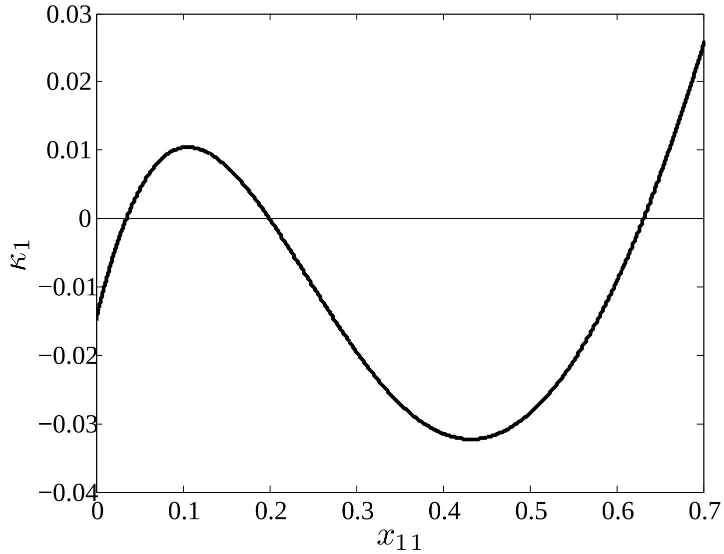

Figure 1: Equilibrium consumption and storage

is the only equilibrium for the two-period economy. To prove this formally, one can use Gröbner bases to compute the solutions to the system

$$e_{11}(z=1) - \kappa_1 - x_{11} = 0 \text{ together with Equation (1)}$$

and observe that there is only one real solution (again see Kubler and Schmedders (2010a) for details.) The economic reason for the uniqueness is simple — if agent 1 expected an equilibrium with low consumption he would start saving; in fact he would save so much that there would be a unique spot equilibrium. At this point, however, consumption would be too high, so the only possible equilibrium is that agent 1 does not save precisely because he expects that the equilibrium that must occur will have high consumption for him. It is useful to write out the entire second period allocation for this equilibrium. We have

$$x_{11} = \frac{1}{15}(5+2\sqrt{5}), x_{12} = \frac{2}{15}(5+\sqrt{5}), \quad x_{21} = \frac{2}{15}(5-\sqrt{5}), x_{22} = \frac{1}{15}(5-2\sqrt{5}) \tag{2}$$

A symmetric argument holds true for agent 2: if the initial shock is 2, the only equilibrium is that he stores zero resulting in the equilibrium consumption allocation

$$x_{11} = \frac{1}{15}(5-2\sqrt{5}), x_{12} = \frac{2}{15}(5-\sqrt{5}), \quad x_{21} = \frac{2}{15}(5+\sqrt{5}), x_{22} = \frac{1}{15}(5+2\sqrt{5}). \tag{3}$$

To summarize, we have shown that each of the two sub-economies has a unique equilibrium. If the initial shock is 1, agent 1's consumption in the second period is high, if the initial shock is 2, his consumption is low.

There is no recursive equilibrium since prices and consumption at shock $z' = 3$ depend on the shock in the previous period. The "capital stock" among the agents alive at the beginning of the period plays no role and is in fact zero in equilibrium, prices rather being determined by the previous shock. If the economy is in shock 1, agent 1 is rich and he decides not to save for next period's shock 3 only if he expects equilibrium prices of good 2 to be high, i.e., he expects his consumption to be high. If he were to expect the equilibrium that is bad for him (i.e., with low consumption) he would start saving so much that the "bad" equilibrium disappears. Thus the only possible outcome in shock 3 is the good equilibrium. If, on the other hand, the economy is in shock 2, agent 2 is rich and the same argument applies. In shock 3 each agent's consumption depends on his endowments in the previous period, not because of savings but because of expectations.

4 Existence

In this section we prove the existence of a recursive equilibrium. Section 4.1 shows how to characterize recursive equilibrium via marginal utility functions. Section 4.2 proves existence making direct assumptions on the transition probability for the recursive state. Assumptions on economic fundamentals that guarantee these conditions are provided in Section 4.3.

4.1 Characterizing recursive equilibria

We now characterize recursive equilibrium via a function that maps the recursive state into marginal utilities of all agents. We show that such a function describes a recursive equilibrium if it is a fixed point of an operator that captures the period-to-period equilibrium conditions.

We make the following assumption on preferences and endowments:

Assumption 1

1. *Endowments are bounded above and below: There are* $\underline{\omega}$, $\overline{\omega} \in \mathbb{R}_+$ *with* $\underline{\omega} > 0$ *such that for all shocks* z, *all goods* l, *and all agents* h

$$\underline{\omega} < \omega_{hl}(z) < \overline{\omega}.$$

2. *The agents' discount factors satisfy* $\delta \in (0,1)$.

3. *The Bernoulli functions* $u_h : \mathbf{Z} \times \mathbb{R}^L_{++} \to \mathbb{R}$ *are assumed to be measurable in* z *and strictly increasing, strictly concave, and* C^2 *in* x. *They satisfy a strong Inada condition: for each* $z \in \mathbf{Z}$ *and all* $x \in \mathbb{R}^L_+ \setminus \mathbb{R}^L_{++}$ *along any sequence* $x^n \to x$, $u_h(z, x^n) \to -\infty$. *Moreover, utility is bounded above in the sense that there exists a* $\bar{u}$ *such that for all* $h \in \mathbf{H}$, $u_h(z,x) \leq \bar{u}$ *for all* $z \in \mathbf{Z}$, $x \in \mathbb{R}^L_{++}$.

The assumption that all commodities enter an agent's utility function is made for convenience and can be relaxed. Similarly, the assumption that individual endowments are strictly positive in all commodities is very strong, but can be replaced by an alternative assumption. We discuss these assumptions in Section 5.2.

As in Duffie et al. (1994), Assumption 1 implies that there is a $\underline{c} > 0$ such that, independently of prices, an agent will never choose consumption that is below $\underline{c}$ in any component, i.e. $x_l(s^t) \geq \underline{c}$ for all commodities l whenever $(x(s^t))$ is an optimal consumption process for any agent. The reason is that an agent can always consume his endowments (he cannot sell them on financial markets in advance), and we therefore must have, for any shock z and for any x that is sufficiently small in one component,

$$u_h(z,x) + \frac{\delta \overline{u}}{1-\delta} < u_h(\omega_h(z)) + \mathbb{E}_z \left[\sum_{t=1}^{\infty} \delta^t u_h(\omega_h(z_t)) \right],$$

where $\bar{u}$ is the upper bound on Bernoulli utility.

We therefore define

$$\mathbf{C} = \{x \in \mathbb{R}^L_+ : x_l \geq \underline{c} \text{ for all } l = 1, ..., L\}.$$

The lower bound on consumption implies an upper bound on marginal utility, which we define by

$$\bar{m} = \max_{h \in \mathbf{H}} \sup_{x \in \mathbf{C}, z \in \mathbf{Z}} \|D_x u_h(z,x)\|_\infty.$$

We make the following assumptions on production possibilities:

Assumption 2 *For each shock z the production set $\mathbf{Y}(z) \subset \mathbb{R}^L$ is assumed to be closed, convex-valued, to contain $\mathbb{R}^L_-$, exhibit constant returns to scale, i.e., $y \in \mathbf{Y}(z) \Rightarrow \lambda y \in \mathbf{Y}(z)$ for all $\lambda \geq 0$, and to satisfy $\mathbf{Y}(z) \cap -\mathbf{Y}(z) = \{0\}$. In addition, production is bounded above: There is a $\bar{\kappa} \in \mathbb{R}_+$ so that for all $\kappa \in \mathbf{K}^U$, $h \in \mathbf{H}$, $z \in \mathbf{Z}$, $l \in \mathbf{L}^K$, and for all $\alpha_h \in \mathbb{R}^J_+$*

$$\sum_{h \in \mathbf{H}} \left(A^0_h(z)\alpha_h - \kappa_h - e_h(z) \right) \in \mathbf{Y}(z) \Rightarrow \sup_{z'} \sum_{h \in \mathbf{H}} (f_{hl}(z') + \sum_{j \in \mathbf{J}} a^1_{hjl}(z')\alpha_{hj}) \leq \max[\bar{\kappa}, \sum_{h \in \mathbf{H}} \kappa_{hl}].$$

While the first part of Assumption 2 is standard, the second part is a strong assumption on the interplay of intra- and inter-period production. For each capital good, the economy can never grow above $\bar{\kappa}$ when starting below that limit. The assumption is made for convenience and ensures boundedness of consumption. Stronger specific assumptions on the correspondence $\mathbf{Y}$ might lead to a relaxation of the second part of the assumption.

We define

$$\mathbf{K} = \{\kappa \in \mathbf{K}^U : \sum_{h \in \mathbf{H}} \kappa_{hl} \leq \bar{\kappa}, \kappa_{hl} \geq \underline{\omega} \text{ for all } l \in \mathbf{L}^K \text{ and all } h \in \mathbf{H}\} \tag{4}$$

and take the state space to be $\mathbf{S} = \mathbf{Z} \times \mathbf{K}$ with Borel σ-algebra $\mathcal{S}$.

Define $\mathbf{\Xi}$ to be the set of storage decisions across agents, α, that ensure that next period's endogenous state lies in $\mathbf{K}$, i.e.,

$$\mathbf{\Xi} = \{\alpha \in \mathbb{R}^{HJ}_+ : \left(f_h(z') + A^1_h(z')\alpha_h\right)_{h \in \mathbf{H}} \in \mathbf{K} \text{ for all } z' \in \mathbf{Z}\}.$$

Assumptions 1 and 2 allow us to give the following simple sufficient condition for the existence of a recursive equilibrium.

LEMMA 1 *A recursive equilibrium exists if there are bounded functions* $M : \mathbf{S} \to \mathbb{R}_+^{HL}$ *such that for each* $s = (z, \kappa) \in \mathbf{S}$ *there exist prices* $\bar{p} \in \Delta^{L-1}$*, production plans* $\bar{y} \in \mathbf{Y}(z)$*, and optimal actions* $(\bar{x}_h, \bar{\alpha}_h)$ *for each agent* $h \in \mathbf{H}$ *with* $D_x u_h(z, \bar{x}_h) = M_h(s)$ *and* $\bar{\alpha} \in \Xi$ *such that*

$$\begin{aligned}(\bar{x}_h, \bar{\alpha}_h) \quad \in \quad & \arg \max_{x \in \mathbb{R}_+^L, \alpha \in \mathbb{R}_+^J} u_h(z, x) + \delta \mathbb{E}_s \left[M_h(s') A_h^1(z') \alpha \right] \text{ s.t.} \\ & -\bar{p} \cdot (x - \kappa_h - e_h(z) + A_h^0(z)\alpha) \geq 0\end{aligned}$$

where

$$s' = \left(z', \left(A_h^1(z')\bar{\alpha}_h + f_h(z') \right)_{h \in \mathbf{H}} \right),$$

production plans are optimal, i.e.,

$$\bar{y} \in \arg \max_{y \in \mathbf{Y}(z)} \bar{p} \cdot y,$$

and markets clear, i.e.,

$$\sum_{h \in \mathbf{H}} (\bar{x}_h + A_h^0(z)\bar{\alpha}_h - e_h(z) - \kappa_h) = \bar{y}.$$

This alternative characterization of a recursive equilibrium in terms of M-functions is useful because in order to show the existence it suffices to provide a fixed point argument in the space of these marginal utility functions. This is at the heart of our existence proof below. To prove the lemma note that if the conditions in the lemma are satisfied then there exist $((\bar{x}_{ht}, \bar{\alpha}_{ht})_{h \in \mathbf{H}}, \bar{p}_t)$ such that markets clear, budget equations hold, and such that the following first order conditions hold for each agent $h \in \mathbf{H}$:

$$D_x u_h(z_t, \bar{x}(z^t)) - \lambda_h(z^t)\bar{p}(z^t) \quad = \quad 0 \tag{5}$$

$$\bar{\alpha}(z^t) \perp \left(-\lambda_h(z^t)\bar{p}(z^t)A_h^0(z_t) + \mathbb{E}_{z^t} \left[D_x u_h(\bar{x}_h(z^{t+1}))A_h^1(z_{t+1}) \right] \right) \quad \geq \quad 0. \tag{6}$$

To prove Lemma 1, it suffices to show that these conditions are sufficient for (x_{ht}, α_{ht}) to be a solution to the agents' infinite horizon problem. Although it is a standard argument we give it for completeness. Following Duffie et al. (1994), assume that for any agent h, given prices, a budget feasible policy $(\bar{x}_{ht}, \bar{\alpha}_{ht})$ satisfies (5) and (6). Suppose there is another budget feasible policy (x_{ht}, α_{ht}). Since the value of consumption in 0 only differs by the value of production plans, strict concavity of $u_h(z, .)$ together with the gradient inequality implies that

$$u_h(z_0, \bar{x}_h(z^0)) \geq u_h(z_0, x_h(z^0)) + D_x u_h(z_0, \bar{x}_h(z^0))A_h^0(z_0)(\alpha_h(z^0) - \bar{\alpha}_h(z^0)). \tag{7}$$

We show by induction, that for any T, we have

$$\begin{aligned}\mathbb{E}_0 \left[\sum_{t=0}^{\infty} \delta^t u_h(z_t, \bar{x}_h(z^t)) \right] \geq \quad & \mathbb{E}_0 \left[\textstyle\sum_{t=0}^{T} \delta^t u_h(z_t, x_h(z^t)) \right] + \mathbb{E}_0 \left[\textstyle\sum_{t=T+1}^{\infty} \delta^t u_h(z_t, \bar{x}_h(z^t)) \right] + \\ & \delta^T \mathbb{E}_0 \left[D_x u_h(z_T, \bar{x}_h(z^T))A_h^0(z_T)(\alpha_h(z^T) - \bar{\alpha}_h(z^T)) \right].\end{aligned} \tag{8}$$

We have already shown, in (7), that (8) holds for $T = 0$. To obtain the induction step for $\bar{\alpha}_j > 0$, we use the first order conditions to substitute $\delta\mathbb{E}_{z^{t-1}}\left[D_x u_h(z_t, \bar{x}_h(z^t)) a^1_{hj}(z_t)(\alpha_{hj}(z^{t-1}) - \bar{\alpha}_{hj}(z^{t-1}))\right]$ for $D_x u_h(\bar{x}(z^{t-1})) a^0_{hj}(z_{t-1})(\alpha_{hj}(z^{t-1}) - \bar{\alpha}_{hj}(z^{t-1}))$, and then apply the budget constraint and the law of iterated expectations. When $\bar{\alpha}_j = 0$, it is clear that $\alpha_j \geq \bar{\alpha}_j$ and since

$$\delta\mathbb{E}_{z^{t-1}}\left[D_x u_h(z_t, \bar{x}_h(z^t)) a_{hj}(z_{t-1}, z_t)\right] \geq D_x u(\bar{x}(z^{t-1})) a_{hj}(z_{t-1}, 0),$$

the induction step follows.

The second term on the right hand side of (8) will converge to zero as $T \to \infty$ since u is bounded above and consumption is bounded below (as functions M are bounded above). The third term will converge to zero because consumption is bounded below and production is bounded by Assumption 2.

4.2 A first existence result

Using the characterization of recursive equilibrium given in Lemma 1, we now prove the existence of a recursive equilibrium by making direct assumptions on the transition probability for the recursive state. Assuming that the probability distribution of the next period's state varies continuously with current actions, the operator defined by the equilibrium conditions is a non-empty correspondence on the space of marginal utility functions. By the Fan–Glicksberg fixed-point theorem there exists a fixed point of the convex hull of this correspondence. Making an additional assumption that is inspired by He and Sun (2013), we can show that this implies the existence of a recursive equilibrium.

First note that the exogenous transition probability $\mathbb{P}$ implies, given $\alpha \in \Xi$, a transition probability $\mathbb{Q}(.|s, \alpha)$ on $\mathcal{S}$: Given α across all agents, and next period's shock z', the next period's endogenous state is given by

$$\left(f_h(z') + A^1_h(z')\alpha_h\right)_{h\in\mathbf{H}}.$$

To prove the existence of a recursive equilibrium we first make additional assumptions directly on $\mathbb{Q}$. To state them we need the following definition from He and Sun (2013):

Given a measure space $(\mathbf{S}, \mathcal{S})$ with an atomless probability measure λ and a sub-σ-algebra $\mathcal{G}$, for any non-negligible set $B \in \mathcal{S}$, let $\mathcal{G}^B$ and $\mathcal{S}^B$ be defined as $\{B \cap B' : B' \in \mathcal{G}\}$ and $\{B \cap B' : B' \in \mathcal{S}\}$. A set $B \in \mathcal{S}$ is said to be a $\mathcal{G}$-atom if $\lambda(B) > 0$ and given any $B_0 \in \mathcal{S}^B$, there exists a $B_1 \in \mathcal{G}^B$ such that $\lambda(B_0 \triangle B_1) = 0$.

The following assumptions are from He and Sun (2013)[3] — in Section 4.3 below we give assumptions on fundamentals that imply Assumption 3 and therefore ensure existence.

[3] Assumptions 3.1. and 3.2. correspond to the assumptions made by He and Sun (2013) on the transition probability representing the law of motion of the states. Assumption 3.3. corresponds to their crucial sufficient condition for existence, called the "coarser transition kernel".

Assumption 3

1. *For any sequence* $\alpha^n \in \mathbf{\Xi}$ *with* $\alpha^n \to \alpha^0 \in \mathbf{\Xi}$

$$\sup_{B \in \mathcal{S}} |\mathbb{Q}(B|s, \alpha^n) - \mathbb{Q}(B|s, \alpha^0)| \to 0.$$

2. *For all* (s, α), $\mathbb{Q}(.|s, \alpha)$ *is absolutely continuous with respect to some fixed probability measure on* $(\mathbf{S}, \mathcal{S})$, λ, *with Radon–Nikodym derivative* $q(.|s, \alpha)$.

3. *There is a sub* σ*-algebra* $\mathcal{G}$ *of* $\mathcal{S}$ *such that* $\mathcal{S}$ *has no* $\mathcal{G}$ *atom and* $q(.|s, \alpha)$ *and* $A^1(.)$ *are* $\mathcal{G}$*-measurable for all* $s = (z, \kappa)$ *and all* $\alpha \in \mathbf{\Xi}$.

The first existence result of this paper is as follows:

Theorem 1 *Under Assumptions 1–3 a recursive equilibrium exists.*

To prove the result let $\mathbf{M}$ be the set of all measurable functions $M : \mathbf{S} \to \mathbb{R}_+^{HL}$ that are λ-essentially bounded above by $\bar{m}$ and below by 0. They lie in the space $L_\infty^m(\mathbf{S}, \mathcal{S}, \lambda)$ of essentially bounded and measurable (equivalence classes of) functions from $\mathbf{S}$ to $\mathbb{R}^m$ with $m = HL$. Following Nowak and Raghavan (1992) and Duggan (2012), we endow L_∞^m with the weak* topology $\sigma(L_\infty^m, L_1^m)$. The set $\mathbf{M}$ is then a non-empty, convex, and compact subset of a locally convex, Hausdorff topological vector space. Moreover, since $\mathbf{S}$ is a separable metric space, L_1^m is separable, and consequently $\mathbf{M}$ is metrizable in the weak* topology. We can therefore work with sequences rather than nets of functions in $\mathbf{M}$.

Given any $\bar{M} = (\bar{M}^1, \ldots, \bar{M}^H) \in \mathbf{M}$, we define

$$E_h^{\bar{M}}(s, x, \alpha, \alpha^*) = u_h(z, x) + \delta \mathbb{E}_s \left[\bar{M}^h(s') \cdot A_h^1(z')\alpha \right] \tag{9}$$

with

$$s' = \left(z', \left(f_h(z') + A_h^1(z')\alpha_h^* \right)_{h \in \mathbf{H}} \right).$$

In the definition of $E_h^{\bar{M}}$, the $\alpha \in \mathbb{R}_+$ stands for the choice of agent h, while the vector $\alpha^* \in \mathbb{R}_+^H$ is taken by individuals as given, in particular its influence on the state-transition. Lemma 2 states properties of the function $E_h^{\bar{M}}$ that we need in Lemma 3. It is the direct analogue of Lemma 1 in Duggan (2012).

Lemma 2 *Given any* $\bar{M} \in \mathbf{M}$, *the functions* $E_h^{\bar{M}}(., x, \alpha, \alpha^*)$ *are measurable in* s. *For given* s *the functions are jointly continuous in* x, α, α^* *and* $\bar{M}$.

The next lemma is the key result in this subsection and it guarantees the existence of a policy that satisfies the equilibrium conditions.[4]

[4]In our setup this result plays the same role as the result that there always exists a mixed strategy Nash equilibrium for the stage game in the stochastic game setup.

LEMMA 3 *For each* $\bar{M} \in \mathbf{M}$ *and* $s = (z, \kappa) \in \mathbf{S}$ *there exists a* $\bar{x} \in \mathbb{R}_+^{HL}$, $\bar{\alpha} \in \mathbf{\Xi}$, $\bar{y} \in \mathbf{Y}(z)$ *and* $\bar{p} \in \Delta^{L-1}$ *such that*

$$\sum_{h \in \mathbf{H}} (\bar{x}_h - e_h(z) - \kappa_h + A_h^0(z)\bar{\alpha}_h) = \bar{y}, \tag{10}$$

for each agent h

$$(\bar{x}_h, \bar{\alpha}_h) \in \arg \max_{x \in \mathbf{C}, \alpha \in \mathbb{R}_+^J} E_h^{\bar{M}}(s, x, \alpha, \bar{\alpha}) \text{ s.t.} \tag{11}$$
$$-\bar{p} \cdot (x - e_h(z) - \kappa_h + A_h^0(z)\alpha) \geq 0,$$

and

$$\bar{y} \in \arg \max_{y \in \mathbf{Y}(z)} \bar{p} \cdot y. \tag{12}$$

Proof.

Given $s \in \mathbf{S}$, define compact sets

$$\mathbf{A} = \{\alpha \in \mathbb{R}_+^J : f_{hl}(z') + \sum_{j \in \mathbf{J}} a_{hlj}^1(z')\alpha_j \leq \bar{\kappa} \text{ for all } h \in \mathbf{H}, l \in \mathbf{L} \text{ and all } z' \in \mathbf{Z}\}, \tag{13}$$

$$\tilde{\mathbf{C}}(s) = \{x \in \mathbf{C} : \frac{1}{2}x - \sum_{h \in \mathbf{H}} (e_h(z) + \kappa_h) \in \mathbf{Y}(z)\},$$

and

$$\widetilde{\mathbf{Y}}(s) = \{y \in \mathbf{Y}(z) : y + \sum_{h \in \mathbf{H}} (e_h(z) + \kappa_h) \geq 0\}.$$

To ensure compactness of $\mathbf{A}$ it is without loss of generality to assume that for each agent h there is a commodity l and a shock z' such that $\sum_{j \in \mathbf{J}} a_{hlj}^1(z') > 0$.

Define for each agent h

$$\Phi^h : \Delta^{L-1} \times \mathbf{\Xi} \rightrightarrows \tilde{\mathbf{C}}(s) \times \mathbf{A}$$

by

$$\Phi^h(p, \alpha^*) = \arg \max_{x \in \tilde{\mathbf{C}}(s), \alpha \in \mathbf{A}} E_h^{\bar{M}}(s, x, \alpha, \alpha^*) \text{ s.t.}$$
$$-p \cdot (x - e_h(z) - \kappa_h + A_h^0(z)\alpha) \geq 0$$

By a standard argument, the correspondence Φ is convex-valued, non-empty valued, and upper-hemicontinuous. Define the producer's best response $\Phi^{H+1} : \Delta^{L-1} \rightrightarrows \widetilde{\mathbf{Y}}(s)$ by

$$\Phi^{H+1}(p) = \arg \max_{y \in \widetilde{\mathbf{Y}}(s)} p \cdot y$$

and define a price player's best response,

$$\Phi^0 : (\tilde{\mathbf{C}}(s) \times \mathbf{A})^H \times \tilde{\mathbf{Y}}(s) \rightrightarrows \Delta^{L-1}$$

by

$$\Phi^0((x_h, \alpha_h)_{h \in \mathbf{H}}, y) = \arg \max_{p \in \Delta^{L-1}} p \cdot \left(y - \sum_{h \in \mathbf{H}} (x_h - e_h(z) - \kappa_h + A_h^0(z)\alpha_h) \right).$$

It is easy to see that this correspondence is also upper-hemicontinuous, non-empty, and convex valued. Finally, define

$$\Phi^{H+2} : \mathbf{A}^H \rightrightarrows \mathbf{\Xi}$$

by

$$\Phi^{H+2}(\alpha^*) = \arg\min_{\alpha \in \mathbf{\Xi}} \|\alpha - \alpha^*\|_2.$$

By Kakutani's fixed point theorem there exists a fixed point to the correspondence $X_{h=0}^{H+2}\Phi^h$, which we denote by $(\bar{x}, \bar{\alpha}, \bar{y}, \bar{\alpha}^*, \bar{p})$. By a standard argument, we must have

$$\sum_{h \in \mathbf{H}} (\bar{x}_h - e_h(z) - \kappa_h + A_h^0(z)\bar{\alpha}_h) = \bar{y}.$$

Therefore consumption solves the agent's problem for all $x \in \mathbf{C}$ — the upper bound imposed by requiring $x \in \tilde{\mathbf{C}}(s)$ will never bind. Production maximizes profits among all $y \in \mathbf{Y}(z)$ since the upper bound can also never bind. In addition, Assumption 2 implies that the upper bound on each α^h cannot be binding — if it were some agents would consume below $\underline{c}$ in some commodities. Therefore each agent maximizes utility subject to $x \in \mathbf{C}$, $\alpha \in \mathbb{R}_+^J$. □

We can define correspondences $s \rightrightarrows \mathbf{N}_{\bar{M}}(s)$ to contain all $(x_h)_{h\in\mathbf{H}}$ such that there exist $(\alpha_h)_{h\in\mathbf{H}} \in \mathbf{\Xi}$, $p \in \Delta^{L-1}$ and $y \in \mathbf{Y}(z)$ that satisfy Equations (10), (11), and (12) and $s \rightrightarrows \mathbf{P}_{\bar{M}}(s)$ by $\mathbf{P}_{\bar{M}}(s) = \{(D_x u_h(z, x_h))_{h\in\mathbf{H}} : x \in \mathbf{N}_{\bar{M}}(s)\}$. We define $co(\mathbf{P}_{\bar{M}})$ by requiring $co(\mathbf{P}_{\bar{M}})(s)$ to be the convex hull of $\mathbf{P}_{\bar{M}}(s)$. Let $\mathbf{R}(\bar{M})$ be the set of (equivalence classes of) measurable selections of $\mathbf{P}_{\bar{M}}$, and $co(\mathbf{R}(\bar{M}))$ the set of measurable selections of $co(\mathbf{P}_{\bar{M}})$. In the next lemma we consider the correspondence $co(\mathbf{R}) : \mathbf{M} \rightrightarrows \mathbf{M}$ defined by $co(\mathbf{R})(M) = co(\mathbf{R}(M))$ and establish that it has a closed graph and non-empty, convex values. The lemma is almost directly from Nowak and Raghavan (1992) — we include a detailed proof for completeness.

LEMMA 4 *For each $\bar{M} \in \mathbf{M}$ the correspondence $\mathbf{P}_{\bar{M}}(s)$ is measurable and compact valued and the correspondence $co(\mathbf{R}) : \mathbf{M} \rightrightarrows \mathbf{M}$ is non-empty, convex, weak* compact valued, and upper-hemicontinuous.*

Proof.

For given $s \in \mathbf{S}$ the set of allocations $x \in \mathbb{R}_+^{HL}$, $\alpha \in \mathbf{\Xi}$, $y \in Y(z)$, and prices $p \in \Delta^{L-1}$ satisfying (10), (11), and (12) can be described as solutions to a system of equations and inequalities. It is easy to notice that the correspondence mapping s to equilibrium allocations and prices, $s \rightrightarrows \widetilde{\mathbf{N}}_{\bar{M}}(s)$, is non-empty and compact-valued. Simplifying, the correspondence can be written in the following way:

$$\begin{aligned} \widetilde{\mathbf{N}}_{\bar{M}}(s) =& \{x \in X : f(s,x) = 0,\ g(s,x) \geq 0\} \cap \tilde{X}(s) \\ =& \{x \in X : g(s,x) = 0\} \cap \{x \in X : g(s,x) \geq 0\} \cap \tilde{X}(s) \end{aligned}$$

where the functions f and g are Caratheodory, and $\tilde{X}(s)$ is lower measurable and compact valued. Moreover, since production is bounded above and endowments are bounded (see Assumptions 1 and 2), we can choose X to be compact. By Corollary 18.8 in Aliprantis and Border (2006) the correspondence $s \rightrightarrows \{x \in X : f(s,x) = 0\}$ is measurable. The map $s \rightrightarrows \{x \in X : g(s,x) \geq 0\}$ is closed-valued and also measurable by Lemma 18.7, Corollary 18.8, and Lemma 18.4 (1) therein: $\{x \in X : g(s,x) \geq 0\} = \{x \in X : g(s,x) = 0\} \cup \{x \in X : g(s,x) < 0\}$. By Lemma 18.4 (3) we get the measurability of the correspondence $s \rightrightarrows \widetilde{\mathbf{N}}_{\bar{M}}(s)$, and hence also of $s \rightrightarrows \mathbf{P}_{\bar{M}}(s)$ by continuity of the functions $D_x u_h$.

By the selection theorem of Kuratowski and Ryll-Nardzewski, $P_{\bar{M}}$ has a measurable selector (see Theorem 18.13 in Aliprantis and Border(2006)). Consequently, the map $M \rightrightarrows co(\mathbf{R})(M)$ is non-empty valued, and obviously it is also convex-valued. Take $M^n \to M$ as $n \to \infty$, $M^n, M \in \mathbf{M}$ and $v^n \to v$ such that $v^n \in co(\mathbf{R})(M^n)$ for each n. We assume that both sequences converge in the weak* topology $\sigma(L^m_\infty, L^m_1)$ on the space $\mathbf{M}$. We need to show $v \in co(\mathbf{R})(M)$. Notice that for given s, $M \rightrightarrows \mathbf{P}_M(s)$ has a closed graph and that Theorem 17.35 (2) in Aliprantis and Border (2006) implies the correspondence $M \rightrightarrows co(\mathbf{P}_M(s))$ has a closed graph as well. Moreover, there exists a sequence $\hat{v}^n$ of finite convex combinations of $\{v^p : p = 1, 2, ...\}$ such that $\hat{v}^n$ converges to v almost surely, i.e., $\hat{v}^n(s) \to v(s)$ for every $s \in S \setminus S_1$ where the set S_1 is of measure zero. Given the closed graph of $M \rightrightarrows co(\mathbf{P}_M(s))$ it is now easy to show that $v(s) \in co(\mathbf{P}^\epsilon_M(s))$ for any ϵ-neighborhood of $co(\mathbf{P}_M(s))$, $\epsilon > 0$, and any $s \in S - S_1$. This proves that $M \rightrightarrows co(\mathbf{R})(M)$ is upper-hemicontinuous. Similarly it can be shown that $M \rightrightarrows co(\mathbf{R})(M)$ is closed-valued and thus by compactness of $\mathbf{M}$, $co(\mathbf{R})$ is weak* compact valued. □

To complete the proof of the existence of a recursive equilibrium, i.e., the proof of Theorem 1, we apply the argument from He and Sun (2013).

Given a measure space $(\mathbf{S}, \mathcal{S})$ with an atomless probability measure λ and a sub-σ-algebra $\mathcal{G}$, and an integrably bounded and closed valued correspondence,

$$F : S \rightrightarrows \mathbb{R}^m$$

define $co(F)$ pointwise by requiring $co(F)(s)$ to be the convex hull of $F(s)$ and

$$\mathcal{I}_F^{\mathcal{S},\mathcal{G}} = \{\mathbb{E}\left[f|\mathcal{G}\right] : f \text{ is an } \mathcal{S}\text{-measurable selection of } F\}.$$

Dynkin and Evstigenev (1976) prove the following result:

Lemma 5 *If $\mathcal{S}$ has no $\mathcal{G}$-atom, then $\mathcal{I}_F^{\mathcal{S},\mathcal{G}} = \mathcal{I}_{co(F)}^{\mathcal{S},\mathcal{G}}$.*

Lemma 4 and the Fan–Glicksberg fixed point theorem imply that there exists a $\hat{M} \in \mathbf{M}$ such that $\hat{M} \in co(\mathbf{R})(\hat{M})$. Assumption 3 and Lemma 5 imply that there is a sub-σ-algebra $\mathcal{G}$ of $\mathcal{S}$ such that there is a $M^* \in \mathbf{R}(\hat{M})$ such that $\mathbb{E}\left[M^*|\mathcal{G}\right] = \mathbb{E}\left[\hat{M}|\mathcal{G}\right]$. Therefore we must have for each agent

h and all (s,α) that

$$\begin{aligned}\int_{\mathbf{S}} M_h^*(s')A_h^1(z')d\mathbb{Q}(s'|s,\alpha) &= \int_{\mathbf{S}} M_h^*(s')A_h^1(z')q(s'|s,\alpha)d\lambda(s') = \int_{\mathbf{S}} \mathbb{E}\left[M_h^* A_h^1(z')q_{s,\alpha}|\mathcal{G}\right]d\lambda \\ &= \int_{\mathbf{S}} \mathbb{E}\left[M_h^*|\mathcal{G}\right]A_h^1(z')q_{s,\alpha}d\lambda = \int_{\mathbf{S}} \mathbb{E}\left[\hat{M}_h|\mathcal{G}\right]A_h^1(z')q_{s,\alpha}d\lambda \\ &= \int_{\mathbf{S}} \hat{M}_h(s')A_h^1(z')d\mathbb{Q}(s'|s,\alpha).\end{aligned}$$

It is then clear that for all s, $\mathbf{P}_{\hat{M}}(s) = \mathbf{P}_{M^*}(s)$ and M^* must be a $\mathcal{S}$-measurable selection of $\mathbf{P}_{M^*}(s)$. We can now construct a "constrained" recursive equilibrium from M^*, which is a competitive equilibrium where agents are constrained to choose consumption in $\mathbf{C}$. As we argued above, it will never be optimal to choose consumption below $\underline{c}$ — by convexity we obtain the same equilibrium if agents are choosing consumption in $\mathbb{R}_+^L$ and hence have proven the existence of a recursive equilibrium.

4.3 A second existence theorem

So far, we have shown the existence of a recursive equilibrium under Assumptions 1–3. However, Assumption 3 is not on fundamentals, but on the transition probability $\mathbb{Q}$ for exogenous and endogenous states. We now provide assumptions on fundamentals that guarantee Assumption 3 and thus the existence of a recursive equilibrium.

In particular, we now assume that the space of exogenous shocks can be decomposed into three complete, separable metric spaces, $\mathbf{Z} = \mathbf{Z}_0 \times \mathbf{Z}_1 \times \mathbf{Z}_2$ with Borel σ-algebra $\mathcal{Z} = \mathcal{Z}_0 \otimes \mathcal{Z}_1 \otimes \mathcal{Z}_2$, and the shock is given by $z = (z_0, z_1, z_2)$. Moreover, for each $i = 1, 2, 3$ there is a measure μ_{z_i} on $\mathbf{Z}_i$ and there are conditional densities $r_{z_0}(z_1|z, z_1')$, $r_{z_1}(z_1'|z)$, and $r_{z_2}(z_2'|z, z_0', z_1')$ such that for any $B \in \mathcal{Z}$ we have

$$\mathbb{P}(B|z) = \int_{\mathbf{Z}_1}\int_{\mathbf{Z}_0}\int_{\mathbf{Z}_2} \mathbb{1}_B(z')r_{z_2}(z_2'|z, z_0', z_1')r_{z_0}(z_0'|z, z_1')r_{z_1}(z_1'|z)d\mu_{z_2}(z_2')d\mu_{z_0}(z_0')d\mu_{z_1}(z_1').$$

To ensure continuity of the state-transition in Assumption 3.1, we assume that the shock z_0 is purely transitory, has a continuous density, and only affects agents' f-endowments. Moreover, given z_2 and z_3, there is a diffeomorphism from $\mathbf{Z}_0$ to a subset of $\mathbf{K}$. More precisely, we make the following assumptions:

Assumption 4

1. *z_0 is purely transitory, i.e., for all $z_0, \hat{z}_0 \in \mathbf{Z}_0$ and all $(z_1, z_2) \in \mathbf{Z}_1 \times \mathbf{Z}_2$,*

$$\mathbb{P}(.|z_0, z_1, z_2) = \mathbb{P}(.|\hat{z}_0, z_1, z_2).$$

2. *$\mathbf{Z}_0$ is a Euclidean space, μ_{z_0} is Lebesgue, and the density $r_{z_0}(.|z, z_1')$ is continuous for (almost) all (z, z_1') and vanishing at the boundary of its support in $\mathbf{Z}_0$.*

3. *For each agent h, and all $(z_1, z_2) \in Z_1 \times Z_2$, $f_h(., z_1, z_2)$ is a diffeomorphism from $\mathbf{Z}_0$ to a subset of $\mathbf{K}$ with a non-empty interior. All other fundamentals are independent of z_0, i.e., for all h, we can write $e_h(z) = e_h(z_1, z_2)$, $A_h^1(z) = A_h^1(z_1, z_2)$, $u_h(z, .) = u_h((z_1, z_2), .)$, $\mathbf{Y}(z) = \mathbf{Y}(z_1, z_2)$.*

Assumption 4.3. can be slightly relaxed in that we can allow $A_h^1(z)$ to depend on z_0 if we assume that for all $\alpha \geq 0$, $f_h(., z_1', z_2') + A_h^1(., z_1', z_2')\alpha$ is a diffeomorphism from $\mathbf{Z}_0$ to a subset of $\mathbb{R}^{HL}$. For simplicity we take $A_h^1(.)$ to be independent of z_0.

To ensure that the z_2 shock ensures convexity in the conditional expectation operator we make the following assumption:

Assumption 5 *Conditionally on next period's z_1' the shock z_2' is independent of both z_0' and the current shock z, i.e., we can write $r_{z_2}(z_2'|z, z_0', z_1') = r_{z_2}(z_2'|z_1')$. The measure μ_{z_2} is an atomless probability measure on $\mathbf{Z}_2$ and, for each agent h, $A_h^1(z)$ and $f_h(z)$ do not depend on z_2.*

This construction was first used in Duggan (2012). It is clear that this is a strict generalization of a "simple sunspot". The shock z_2 can affect fundamentals $(e_h)_{h\in\mathbf{H}}$ and $\mathbf{Y}$ in arbitrary ways.

The following is the main result of the paper.

Theorem 2 *Under Assumptions 1, 2, 4, and 5 there exists a recursive equilibrium.*

To prove the theorem we show that Assumptions 4 and 5 imply Assumption 3 if state-transitions and the state space are reformulated appropriately. It is easy to notice that since the shock z_0 is purely transitory and does not affect any fundamentals except (f_h), the realization of this shock is reflected in the value of the endogenous state κ and except for the value of κ it is irrelevant for current endogenous variables and the future evolution of the system. Therefore, departing slightly from our previous notation, we take $\mathbf{S} = \mathbf{Z}_1 \times \mathbf{Z}_2 \times \mathbf{K}$ with Borel σ-algebra $\mathcal{S}$. Furthermore we write $\underline{\mathbf{S}} = \mathbf{Z}_1 \times \mathbf{K}$ for the space that includes only the z_1- shock component and the holdings in capital goods; we denote the Borel σ-algebra of $\underline{\mathbf{S}}$ by $\underline{\mathcal{S}}$. For each $B \in \mathcal{S}$ take

$$\mathbb{Q}(B|s, \alpha) = \mathbb{P}\left(\{z' \in \mathbf{Z} : \left((z_1', z_2'), (f_h(z') + A_h^1(z')\alpha_h)_{h\in\mathbf{H}}\right) \in B\}|z\right).$$

We have the following lemma:

Lemma 6 *Under Assumpion 4* , $\mathbb{Q}(.|s, \alpha)$ *satisfies Assumption 3.1.*

Proof. By Assumption 5, it suffices to show norm-continuity for the marginal transition function on $\underline{\mathcal{S}}$, i.e., that for any sequence $\alpha^n \in \boldsymbol{\Xi}$ with $\alpha^n \to \alpha^0 \in \boldsymbol{\Xi}$

$$\sup_{B\in\underline{\mathcal{S}}} |\mathbb{Q}_{\underline{s}}(B|s, \alpha^n) - \mathbb{Q}_{\underline{s}}(B|s, \alpha^0)| \to 0.$$

To show this, we first define for given $(z, z_1') \in \mathbf{Z} \times \mathbf{Z}_1$ and $\alpha \in \boldsymbol{\Xi}$ a diffeomorphism $g_{(z,z_1',\alpha)}$ that maps $\mathbf{Z}_0$ into its range $\bar{\mathbf{K}}_{(z,z_1',\alpha)} = g_{(z,z_1',\alpha)}(\mathbf{Z}_0) \subseteq \mathbf{K}$ with $g_{(z,z_1',\alpha)}(z_0) = (f_h(z_0', z_1') + A_h^1(z_1')\alpha_h)_{h\in\mathbf{H}}$

and

$$r_\kappa(\kappa'|z,z_1',\alpha) := \begin{cases} r_{z_0}(g^{-1}_{(z,z_1',\alpha)}(\kappa')|z,z_1') \cdot \left|J(g^{-1}_{(z,z_1',\alpha)}(\kappa'))\right| & \text{if } \exists\ z_0 : g_{(z,z_1',\alpha)}(z_0) = \kappa' \\ 0 \quad \text{otherwise,} \end{cases}$$

where $|J(\cdot)|$ denotes the determinant of the Jacobian.

Denoting by μ_κ the Lebesgue measure on $\mathbf{K}$, for $B \in \underline{\mathcal{S}}$ and $\alpha^n \in \Xi$ with $\alpha^n \to \alpha^0 \in \Xi$, we have

$$\begin{aligned} \mathbb{Q}_{\underline{s}}(B|s,\alpha^n) &= \int_{\mathbf{Z}_1}\int_{\mathbf{Z}_0} \mathbb{1}_B\left[z_1', (f_h(z') + A_h^1(z')\alpha_h^n)_{h\in\mathbf{H}}\right] r_{z_0}(z_0'|z,z_1') r_{z_1}(z_1'|z) d\mu_{z_0}(z_0') d\mu_{z_1}(z_1') \\ &= \int_{\mathbf{Z}_1}\int_{\mathbf{Z}_0} \mathbb{1}_B\left[z_1', g_{(z,z_1',\alpha^n)}(z_0')\right] r_{z_0}(z_0'|z,z_1') r_{z_1}(z_1'|z) d\mu_{z_0}(z_0') d\mu_{z_1}(z_1') \\ &= \int_{\mathbf{Z}_1}\int_{\bar{\mathbf{K}}_{(z,z_1',\alpha^n)}} \mathbb{1}_B\left[z_1', \kappa'\right] r_\kappa(\kappa'|z,z_1',\alpha^n) r_{z_1}(z_1'|z) d\mu_\kappa(\kappa') d\mu_{z_1}(z_1') \\ &= \int_{\mathbf{Z}_1}\int_{\mathbf{K}} \mathbb{1}_B\left[z_1', \kappa',\right] r_\kappa(\kappa'|z,z_1',\alpha^n) r_{z_1}(z_1'|z) d\mu_\kappa(\kappa') d\mu_{z_1}(z_1') \\ &= \int_B r_\kappa(\kappa'|z,z_1',\alpha^n) r_{z_1}(z_1'|z) d\mu_\kappa(\kappa') d\mu_{z_1}(z_1'), \end{aligned}$$

where we used Fubini's theorem for the first equality and the change of variables theorem for the third equality. By Scheffe's lemma and the convergence $r_\kappa(\kappa'|z,z_1',\alpha^n) \to r_\kappa(\kappa'|z,z_1',\alpha^0)$, we get norm-continuity of $\alpha \to \mathbb{Q}_{\underline{s}}(\cdot|s,\alpha)$. □

In the above proof, we have shown that for all (s,α) the marginal distribution of $\mathbb{Q}(.|s,\alpha)$ on $\underline{\mathbf{S}}$ is absolutely continuous with respect to the product measure $\eta = \mu_\kappa \times \mu_{z_1}$ and has a Radon–Nikodym derivative $q_{\underline{s}}(z_1',\kappa'|s,\alpha) = r_\kappa(\kappa'|z,z_1',\alpha) r_{z_1}(z_1'|z)$.

The following result is directly from He and Sun (2013), Proposition 4.

Lemma 7 *Defining the measure $\lambda(.)$ by*

$$\lambda(B) = \int_{\underline{\mathbf{S}}}\int_{\mathbf{Z}_2} \mathbb{1}_B[\underline{s}, z_2] r_{z_2}(z_2|z_1) d\mu_{z_2}(z_2) d\eta(\underline{s})$$

and taking $\mathcal{G} = \underline{\mathcal{S}} \otimes \{\emptyset, \mathbf{Z}_2\}$, Assumption 5 implies that $\mathbb{Q}(.|s,\alpha)$ satisfies Assumptions 3.2 and 3.3.

The proof of Theorem 2 now follows directly from the argument above, i.e., the result follows directly from Theorem 1.

4.4 The examples revisited

Assumption 5 and especially Assumption 4 are strong assumptions regarding the underlying economy. Assumption 4 guarantees that agents' current choices lead to a non-degenerate distribution over the endogenous state next period, conditional on the exogenous shock. This is in contrast with standard models where current choices often pin down next period's endogenous state deterministically. In stochastic games it is well known that the assumption of a so-called deterministic transition creates serious problems for the existence of Markov equilibria (see Duggan (2012) for an explanation and in particular Levy (2013) for an example of non-existence in a model with a deterministic transition). We want to argue that our examples from Section 3 above show that the same is true in general equilibrium models and that it is not possible to expect a general existence result without some form of continuity in the state-transition.

The situation is slightly complicated by the fact that our stylized examples of non-existence violate Assumption 1, Assumption 4, and Assumption 5 above. However, it is easy to see that Assumption 1 alone cannot restore existence in the examples. In particular, Example 2 goes through if endowments are in the interior of the consumption set; the specific endowments were simply chosen to make the examples as simple as possible. It is much more difficult to determine whether both Assumptions 4 and 5 are necessary for obtaining general existence.

In particular Assumption 5 alone guarantees existence in Example 2 and in any simple perturbation of Example 2. To see this, we modify the example slightly and assume that the shock has a continuous component, $Z = \{1, 2, 3\} \times [0, 1]$. Endowments, technology, and Bernoulli utility in the discrete shocks 1 and 2 do not depend on $\zeta \in [0, 1]$, the continuous component of the shock. The same is true for Bernoulli utilities in the discrete shock 3, however endowments are now as follows

$$e_1(z = (3, \zeta)) = (0, 1 + \zeta\bar{\zeta}), e_2(z = (3, \zeta)) = (1 + \zeta\bar{\zeta}, 0) \text{ for some } \bar{\zeta} \geq 0.$$

To simplify the analysis we also assume that ζ affects inter-period production — this is not consistent with Assumption 5, but this plays no role in our example. We assume that $a_1^1(3, \zeta) = (1 + \zeta\bar{\zeta}, 0)$ and $a_1^2(3, \zeta) = (0, 1 + \zeta\bar{\zeta})$.

Note that for $\bar{\zeta} = 0$ the additional exogenous shock is a pure sunspot not affecting economic fundamentals but possibly playing the role of a coordination device since in equilibrium agents can base their decisions on the realization of this shock. We show that both for $\bar{\zeta} = 0$ and for $\bar{\zeta} > 0$ the additional shock restores existence.

Clearly there is still a sequential competitive equilibrium where spot-prices depend on the realization of the shock in the previous period and agents never save. By homotheticity of utilities each shock $(z = 3, \zeta)$ just scales the economy: given storage, equilibrium prices are all identical and consumption just shifts up. However, if we allow spot prices to change with the realization of the shock, as is the case in the definition of recursive equilibrium, it is easy to obtain the existence of a recursive equilibrium. The construction is as follows:

Without loss of generality we focus on the case where shock 1 occured in the previous period and agent 2 does not save, so $\kappa_1 \geq 0, \kappa_2 = 0$. Denote by $\underline{x_{11}} = \frac{1}{15}(5 - 2\sqrt{5})$ the smallest real solution of Equation (1) at $\kappa_1 = 0$ and by $\overline{x}_{11} \sim 0.6609$ the largest real solution of that equation at $\kappa_1 = 0.01$. For each $\kappa_1 \in [0, 0.01]$ define $\eta(\kappa_1)$ as a unique solution in $[0, 1]$ of the following equation

$$\eta(\kappa_1)\frac{1}{x_L^3} + (1 - \eta(\kappa_1))\frac{1}{x_H^3} = \frac{1}{\underline{x_{11}}^3} + \frac{\kappa_1}{0.01}\left(\frac{1}{\overline{x_{11}}^3} - \frac{1}{\underline{x_{11}}^3}\right),$$

where x_L is the smallest and x_H is the largest solution to Equation (1) at κ_1. The policy functions in a recursive equilibrium are then as follows: given κ_1 if $0 \leq \zeta \leq \eta(\kappa_1)$ agent 1 consumes $(1+\zeta\bar{\zeta})x_L$ and if $\eta(\kappa_1) < \zeta \leq 1$ agent 1 consumes $(1+\zeta\bar{\zeta})x_H$.

It can easily be verified that expected marginal utility is a continuous and decreasing function in κ_1 and that there always exists a solution for the optimal savings in shock 1. Since shock 2 is completely symmetric there exists a recursive equilibrium.

Since in many applied models Assumption 5 is likely to be much weaker than Assumption 4, the fact that this assumption alone guarantees existence in our Example 2 is an important observation. However, note that the assumption does not suffice for existence in Example 1 — even a pure sunspot cannot help agents to coordinate if $\kappa_1 = \kappa_2 = 0$ and recursive equilibrium does not exit. The reason is simple — in Example 2 the equilibrium correspondence in shock 3 (equilibrium prices and allocations as a map from (κ_1, κ_2)) is not convex valued, thus introducing a continuous shock allows us to convexify the correspondence and find a continuous selection. In Example 1 this correspondence is convex valued to begin with. But there is no continuous selection and hence a continuous shock cannot restore existence.

Assumption 5 cannot restore existence in cases where the convex hull of the equilibrium correspondence does not admit a continuous selection. In the class of examples considered in Section 3 this is only possible for non-generic economies (since it requires indeterminacy of equilibria in a static economy) but there is no reason to believe that in a full-fledged dynamic economy this is also the case.

Assumption 4, on the other hand, does guarantee existence in both Examples 1 and 2. It guarantees that Lemma 2 holds and this is sufficient to establish existence in simple examples where the infinite horizon economy decomposes into two-period economies. Existence in the two examples fails precisely because $E^{\bar{M}}(s, x, \alpha, \alpha^*)$ is never continuous in α^* if $\bar{M}$ is not continuous itself. But it is easy to see that in both Example 1 and Example 2 the map from (κ_1, κ_2) to equilibrium marginal utilities in shock 3 does not have a continuous selection. Given any measurable selection, a version of Assumption 4 guarantees continuity of $E^{\bar{M}}$ and the existence of a recursive equilibrium. Concretely, it suffices to assume in both examples that conditional on shock 3, the joint distribution of agent 1's endowments in commodity 1 and agent 2's endowments in commodity 2 is uniform on $[0, \epsilon]^2$. While this does not exactly satisfy Assumption 4 (e.g., the density of the shock to agents' f-endowments does not vanish at the boundary), one can verify that it guarantees continuity of $E_h^{\bar{M}}(s, x, \alpha, \alpha^*)$ in α^* for any (measurable) selection $\bar{M}$ of the equilibrium correspondence in shock 3 (i.e., the map for κ to consumptions and marginal utilities).

To summarize, the failure of existence in the examples in Section 3 stems from the fact that these examples violate Assumption 4. While in simple examples existence might be restored without assuming continuity of the transition, the examples demonstrate that in general one cannot expect existence without a form of Assumption 4. The examples do not show why Assumption 5 is necessary for existence but it is clear that our proof critically relies on convexity, which cannot be obtained without Assumption 5. Whether or not existence of Markov equilibria can be shown with a different approach and without Assumption 5 is subject to further research.

5 Extensions

So far, we have considered the case without trade in one-period financial assets. In this section, we describe a simple way of incorporating financial markets into our framework. We also discuss an alternative formulation of Assumption 1 that might be more suitable for some applications.

5.1 Financial markets

We now assume that agents can trade in financial markets, in addition to undertaking intertemporal storage. There are D one-period securities, $d = 1, ..., D$, in zero net supply, each being characterized by its payoff $b_d : \mathbf{Z} \to \mathbb{R}^L_+$ — a bounded and measurable function of the shock. At each z^t securities are traded at prices $\rho(z^t)$; we denote an agent's portfolio by $\theta_h(z^t) \in \mathbb{R}^D$.

In order to establish the existence of a recursive equilibrium we need to restrict agents' portfolio choices. Let $\mathbf{K}$ be defined as in (4) above and $\mathbf{A}$ as in (13). Each agent h faces a constraint on trades in asset markets and storage decisions (α, θ), given by a convex and closed set $\mathbf{\Theta}_h \subset \mathbb{R}^J_+ \times \mathbb{R}^D$, which satisfies that whenever $\alpha \in \mathbf{A}$ and $(\alpha, \theta) \in \mathbf{\Theta}_h$ then

$$A^1_h(z')\alpha + \sum_{d=1}^{D} \theta_d b_d(z') \geq 0 \text{ for all } z' \in \mathbf{Z}.$$

Without loss of generality we assume that trade is possible in all financial securities, i.e., for each d there is an agent h and an $\alpha \in \mathbf{A}$ so that for some $\theta_d < 0$, $(\alpha, \theta) \in \mathbf{\Theta}_h$.

Note that collateral constraints of the form

$$A^1_h(z')\alpha + \sum_{d=1}^{D} \min(\theta_d, 0) b_d(z') \geq 0 \text{ for all } z' \in \mathbf{Z},$$

are one example of constraints that satisfy our assumption, but there could be many others. However, this is a somewhat nonstandard formulation of a collateral constraint since agents cannot borrow against the value of their future production — they need to borrow against future production directly.

As before the endogenous state-space is given by $\mathbf{K}$. A recursive equilibrium is given by maps from the state $s \in \mathbf{S} = \mathbf{Z} \times \mathbf{K}$ to prices of commodities and financial securities as well as consumption, investment, and portfolio choices across all agents. The analogous result to Lemma 1 above is now as follows:

A recursive equilibrium exists if there are functions $M : \mathbf{S} \to \mathbb{R}^{HL}_+$ such that for each $s \in \mathbf{S}$ there exist prices $(\bar{p}, \bar{\rho}) \in \Delta^{D+L-1}$, a production plan $\bar{y} \in \mathbf{Y}(z)$ for each agent h, optimal actions $(\bar{x}_h, \bar{\alpha}_h, \bar{\theta}_h)$ with $D_x u_h(\bar{x}_h) = M_h(s)$, and

$$\begin{aligned}(\bar{x}_h, \bar{\alpha}_h, \bar{\theta}_h) \;\in\; & \arg \max_{x \in \mathbb{R}^L_+, (\alpha,\theta) \in \mathbf{\Theta}_h} \\ & u_h(z, x) + \delta \mathbb{E}_s \left[M_h(s') \cdot \left(\sum_j a^1_{hj}(z')\alpha_j + \sum_d b_d(z')\theta_d \right) \right] \text{ s.t.} \\ & -\bar{q} \cdot \theta - \bar{p} \cdot (x - \kappa_h - e_h(z) + A^0_h(z)\alpha) \geq 0,\end{aligned}$$

where

$$s' = \left(z', \left(A^1_h(z')\bar{\alpha}_h + \sum_d \bar{\theta}_{hd} b_d(z') + f_h(z') \right)_{h \in \mathbf{H}} \right),$$

production plans are optimal, i.e.,

$$\bar{y} \in \arg \max_{y \in \mathbf{Y}(z)} \bar{p} \cdot y,$$

and markets clear, i.e.,

$$\sum_{h \in \mathbf{H}} (\bar{x}_h + A^0_h(z)\bar{\alpha}_h - e_h(z) - \kappa_h) = \bar{y}$$

and

$$\sum_{h \in \mathbf{H}} \bar{\theta}_h = 0.$$

The proof is similar to the proof of Lemma 1.

Assumptions 4.3 and Assumption 5 now need to be extended in that we assume in addition that for each asset d, $b_d(z)$ does not depend on z_0 and does not depend on z_2, i.e., it is only a function of z_1. The definition of the transition probability $\mathbb{Q}$ now reads as

$$\mathbb{Q}(B|s, \alpha, \theta) = \mathbb{P}\left(\{z' \in \mathbf{Z} : [z'_1, z'_2, (f_h(z') + A^1_h(z')\alpha_h + \sum_d \theta_{hd} b_d(z'))_{h \in \mathbf{H}}] \in B\} | z \right).$$

With the additional assumptions, it is easy to see that Lemma 6 holds as stated. The proofs of Lemma 2 and 4 are almost identical as those in the case without financial securities. To prove the analogue of Lemma 3, one can bound the set of admissible portfolios, and proceed as in the proof in Section 4.

5.2 Assumptions on endowments and preferences

Standard formulations of both the Lucas asset pricing model and the neoclassical stochastic growth model violate our Assumption 1. Agents are not endowed with capital or Lucas trees and they do not derive utility from consuming them.

The first part of the assumption, $\omega_h(z) \in \mathbb{R}^L_{++}$, cannot be relaxed easily since we need to require positive f-endowments in order to guarantee Assumption 3.1. However, while they have to be positive, f-endowments can be arbitrarily small. Therefore, this assumption can be interpreted as a small perturbation to the original model that ensures existence. An alternative is to require that agents operate all technologies at a positive level, i.e., to require that for some $\epsilon > 0$ agents face the additional constraint that $\alpha_j \geq \epsilon$ for all j. With this alternative assumption one then needs to make similar assumptions on $A_h^1(z')$ as we did on $f_h(z)$. In concrete examples (e.g., the neoclassical growth model with one consumption good) this additional constraint might never be binding (e.g., if agents can have zero labor endowments with positive probability, then the Inada condition on utility will always force them to have positive savings). It is beyond the scope of this paper to work out precise conditions for existence in these cases where we allow agents' endowments to lie on the boundary of $\mathbb{R}_+^L$.

The second part of the assumption, that $u_h(z,.)$ is strictly increasing in all L commodities, seems much more problematic. However, this assumption can be substantially relaxed once one makes concrete assumptions on the technology. Assume that utility only depends on a subset of commodities (for example, the "consumption-goods" that cannot be obtained through storage) $\mathbf{L}^C = \{1,...,L^C\}$, $L^C \leq L$, and that it is strictly increasing, strictly concave, C^2, satisfies the strong Inada condition, and is bounded as a function of these commodities. For this case, Lemma 3 needs to be slightly reformulated to ensure that the M-functions are defined even for commodities that do not enter the utility directly: a recursive equilibrium exists if there are bounded functions $M : \mathbf{S} \to \mathbb{R}_+^{HL}$ such that for each $s \in \mathbf{S}$ there exist prices $\bar{p} \in \Delta^{L-1}$ and optimal actions $(\bar{x}_h, \bar{\alpha}_h)$ for each agent $h \in \mathbf{H}$ with $\bar{\alpha} \in \mathbf{\Xi}$ such that

$$D_x u_h(z, \bar{x}_h) = (M_{h1}(s), \ldots, M_{hL^C}(s)) \text{ and } (M_{h,L^C+1}(s), ..., M_{h,L}(s)) = \xi_h(p_{L^C+1}, \ldots, p_L),$$

with $\xi_h = \frac{\partial u_h(z,\bar{x}_h)/\partial x_1}{p_1}$ and such that

$$\begin{aligned}(\bar{x}_h, \bar{\alpha}_h) \quad \in \quad & \arg \max_{x \in \mathbb{R}_+^L, \alpha \in \mathbb{R}_+^J} u_h(z,x) + \delta \mathbb{E}_s \left[M_h(s') A_h^1(z')\alpha \right] \text{ s.t.} \\ & -\bar{p} \cdot (x - \kappa_h - e_h(z) + A_h^0(z)\alpha) \geq 0\end{aligned}$$

where

$$s' = \left(z', \left(A_h^1(z')\bar{\alpha}_h + f_h(z') \right)_{h \in \mathbf{H}} \right),$$

production plans are optimal and markets clear.

The existence proof of the previous section can then be applied if assumptions on production ensure that, in the "two-period problem", the prices of the goods that do not enter the utility functions are bounded below by zero and uniformly above by some constant. More precisely, in Lemma 3 we need to require that there exist $p \in \Delta^{L-1}$ such that for each agent h and each l, it holds that $0 \leq \xi_h p_l \leq \bar{m}$, with ξ_h defined as above. For standard models, e.g., the neoclassical growth model or the Lucas model, this can be seen to hold fairly easily.

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