Large Exchange Economies: A Personal Retrospective on Fifty-Five Years of Research

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Abstract: The requirement that there be "substantially more agents than commodities" for the viability of Walrasian equilibrium theory in a setting not curtailed by convexity assumptions is both implicit and explicit in the last fifty-five years of work in the subject. In this paper I provide a trajectory of this work by a focus on its technical underpinning afforded by a 1941 theorem of Lyapunov. I identify theorems of Steinitz, Hurwicz-Uzawa, Blackwell-Richter, Shapley-Folkman, Loeb, Armstrong-Prikry, Cassels, Rustichini-Yannelis, Sun, Podczeck and those of Kali Rath and Nobusumi Sagara in collaboration with the author. (91 words)

Keywords: Walrasian equilibrium theory, Lyapunov's theorem, number of agents, number of commodities

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An Exact Fatou Lemma for Gelfand Integrals: Equivalence of the Saturation and Fatou Properties*

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Abstract

We establish an exact version of Fatou's lemma for Gelfand integrals of functions and multifunctions in dual Banach spaces without any order structure, and under the saturation property on the underlying measure space. The necessity and sufficiency of saturation for the Fatou property is demonstrated. Our result has a direct application to the equilibrium existence result for saturated economies without convexity assumptions.

Key words: Gelfand integral, Fatou's lemma, Saturated measure space, Weak* upper-closure property.

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Contents

1 Introduction

Fatou's lemma for Lebesgue integration in several (finite) dimensions established by Schmeidler [30] has been extended to Gelfand integration in infinite-dimensional dual Banach spaces by several authors. As demonstrated in [4, 7, 9, 25, 26, 36], under the nonatomicity hypothesis on measure spaces, an approximate or convex version of Fatou's lemma for a sequence of Gelfand integrable functions is known. An exact version of Fatou's lemma, however, fails to hold (see Proposition 3.1) and its failure stems from the well-known counterexample to the Lyapunov convexity theorem in infinite dimensions (see [10, 20, 28]). This observation forces one to remain with the convexity of the "closure" of the Gelfand integral of a multifunction (see [7, 16]) for the derivation of an approximate Fatou lemma under the nonatomicity hypothesis.

Recent work has, however, conclusively established a Lyapunov convexity theorem for separable Banach spaces and their dual spaces (see [19, 21]). The convexity of the Gelfand integral of a multifunction had already been established earlier (see [27, 33]). These exact results have been based in a fundamental way on saturated measure spaces, a concept originating in Maharam [24], and comprehensively explored and characterized in [11, 14, 15]. This then opens the door to the possibility of an exact Fatou lemma for Gelfand integrals. An exact version of Fatou's lemma for Gelfand integrals obtained in [22, 32] presupposes the nonatomic Loeb space setting and a convexity result for the Gelfand integral of a multifunction established in [32] for such spaces. Nonatomic Loeb spaces are the special class of saturated measure spaces, and the parent class of saturated measure spaces is of considerable significance in economic and game theory — as a model for the set of agents, and/or the set of their characteristics and/or the information that is available to them.

Gelfand integrals arise naturally in general equilibrium models with a commodity space modeled as an infinite-dimensional dual Banach space and the space of agents as a finite measure space (see [5, 6, 25, 26, 29, 36]). They also arise in economies modeled by a distribution of the characteristic of agents (see [34, 35]). The first application of Gelfand integrals to game theory was given by [17] in the context of noncooperative games with a nonatomic measure space of players. In particular, Fatou's lemma for a sequence of Gelfand integrable functions has played an important role in demonstrating the existence of competitive equilibria in [25, 26, 29, 36] and that of pure Nash equilibria in [17]. The failure of an exact version of Fatou's lemma, however, leads to an inevitable imposition of the convexity assumption on preferences of agents (see [6, 17, 25, 36]), a rather stringent condition that is clearly unnecessary for economies with a finite-dimensional commodity space (see [2, 3, 13]) and for large games with a finite action set (see [31]).

In this paper, we establish an exact version of Fatou's lemma for Gelfand integrals for functions and multifunctions without order structures in dual Banach spaces under the saturation property on measure spaces. In [23], the authors refer to standard procedures that allow a particular result on a nonatomic Loeb space to be lifted up to a saturated measure space, and in this connection, one can legitimately ask for some formal meta-theorem, or explication as to whether specific properties of a non-atomic Loeb space would be of relevance to such a theorem, or whether more generally, a "particular result" on a particular saturated measure space can be "lifted up" to all saturated measure spaces?[1] What is to be emphasized here is that these

[1]An answer to these questions, inspired by the referee, remains for future work.

procedures are not at issue in the argumentation of the sufficiency result that we offer. Unlike [22], the proof we provide avoids the use of Loeb spaces, or indeed of nonstandard analysis, altogether, and is based on [9], and [16, 18]. And to be sure, it is the necessity of a saturated measure space for the question at hand that is the surprising result if it can be obtained. We also present such a result here.[2] To frame it more generally, we contribute to the literature a result on the necessity and sufficiency of saturation for the Fatou property, a result developed in [20] for the Bochner integral setting. This result, certainly of interest in its own right, also has a direct application to the equilibrium existence result for saturated economies without convexity assumptions (see [29]).

2 Preliminaries

2.1 Gelfand Integration

Let $(\Omega, \mathcal{F}, \mu)$ be a finite measure space, which is assumed to be complete throughout this paper. Denote by $L^1(\mu)$ the space of integrable functions on Ω and by $L^\infty(\mu)$ the space of essentially bounded functions on Ω. Let X be a Banach space with dual X^*. A function $f : \Omega \to X^*$ is *weakly* scalarly measurable* if for every $x \in X$ the scalar function $xf : \Omega \to \mathbb{R}$ defined by $xf(\omega) = \langle x, f(\omega) \rangle$ is measurable. We say that weakly* scalarly measurable functions f and g are *weakly* scalarly equivalent* if $xf(\omega) = xg(\omega)$ for every $x \in X$ a.e. $\omega \in \Omega$ (the exceptional μ-null set depending on x). Let $\mathrm{Borel}(X^*, w^*)$ be the Borel σ-algebra of X^* generated by the weak* topology. If X is a separable Banach space, then X^* is a locally convex Suslin space under the weak* topology (see [37, p. 67]). Hence, under the separability of X, a function $f : \Omega \to X^*$ is weakly* scalarly measurable if and only if it is Borel measurable with respect to $\mathrm{Borel}(X^*, w^*)$ (see [37, Theorem 1])[3].

A weakly* scalarly measurable function f is *Gelfand integrable* over $A \in \mathcal{F}$ if there exists $x_A^* \in X^*$ such that $\langle x, x_A^* \rangle = \int_A xf d\mu$ for every $x \in X$. The element x_A^* is unique (see [10, p. 53]), is called the *Gelfand integral* (or *weak* integral*) of f over A, and is denoted by $\int_A f d\mu$. Denote by $G^1(\mu, X^*)$ (abbreviated to $G^1_{X^*}$) the equivalence classes of Gelfand integrable functions

[2]Some experts, and this includes the referee, may feel that this result is to be expected in light of results on the compactness and convexity of the Gelfand integral of a multifunction. While taking this point, our observation simply referred to the average reader, unlike the expert, not automatically associating the saturation property with the exact Fatou property for Gelfand integration.

[3]This does not necessarily mean that f is *strongly measurable*, that is, f is the dual norm limit of a sequence of simple functions, unless X^* is separable regarding the dual norm topology.

with respect to weak* scalar equivalence, normed by

$$\|f\|_{G^1} = \sup_{x \in U_X} \int |xf| d\mu,$$

where U_X is the closed unit ball in X. This norm is called the *Gelfand norm* and the normed space $(G^1(\mu, X^*), \|\cdot\|_{G^1})$, in general, is not complete.

Denote by $L^\infty(\mu) \otimes X$ the tensor product of $L^\infty(\mu)$ and X. A typical tensor f^* in $L^\infty(\mu) \otimes X$ has a (not necessarily unique) representation $f^* = \sum_{i=1}^n \varphi_i \otimes x_i$ with $\varphi_i \in L^\infty(\mu)$, $x_i \in X$, $i = 1, \dots, n$. A bilinear form on $G^1(\mu, X^*) \times (L^\infty(\mu) \otimes X)$ is given by

$$\langle f, f^* \rangle = \sum_{i=1}^n \int \varphi_i(\omega) \langle x_i, f(\omega) \rangle d\mu = \sum_{i=1}^n \left\langle x_i \int \varphi_i f d\mu \right\rangle$$

for $f \in G^1(\mu, X^*)$ and $f^* = \sum_{i=1}^n \varphi_i \otimes x_i \in L^\infty(\mu) \otimes X$, where $\int \varphi_i f d\mu$ denotes the Gelfand integral of $\varphi_i f \in G^1(\mu, X^*)$. The pair of these spaces $\langle G^1(\mu, X^*), L^\infty(\mu) \otimes X \rangle$ equipped with this bilinear form is a dual system. Thus, it is possible to define the coarsest topology on $G^1(\mu, X^*)$ such that the linear functional $f \mapsto \langle f, f^* \rangle$ is continuous for every $f^* \in L^\infty(\mu) \otimes X$, denoted by $\sigma(G^1_{X^*}, L^\infty \otimes X)$, which is the topology of pointwise convergence on $L^\infty(\mu) \otimes X$. It is evident that $\sigma(G^1_{X^*}, L^\infty \otimes X)$-topology is coarser than the weak topology $\sigma(G^1_{X^*}, (G^1_{X^*})^*)$. A net $\{f_\alpha\}$ in $G^1(\mu, X^*)$ converges to $f \in G^1(\mu, X^*)$ for the $\sigma(G^1_{X^*}, L^\infty \otimes X)$-topology if and only if for every $x \in X$ the net $\{\langle x, f_\alpha(\cdot) \rangle\}$ in $L^1(\mu)$ converges weakly to $\langle x, f(\cdot) \rangle \in L^1(\mu)$. A subset $\mathcal{K}$ of $G^1(\mu, X^*)$ is *integrably bounded* if there exists $\varphi \in L^1(\mu)$ such that $f(\omega) \in \varphi(\omega) U_{X^*}$ for every $f \in \mathcal{K}$ and $\omega \in \Omega$.

The proof of the following result is found in [9, Theorem 3].

Lemma 2.1. *Let $(\Omega, \mathcal{F}, \mu)$ be a finite measure space and X be a separable Banach space. Then an integrably bounded subset of $G^1(\mu, X^*)$ is relatively sequentially compact*[4] *in the $\sigma(G^1_{X^*}, L^\infty \otimes X)$-topology.*

2.2 Multifunctions

A set-valued mapping from Ω to the family of nonempty subsets of X^* is called a *multifunction*. A multifunction $\Gamma : \Omega \twoheadrightarrow X^*$ is *measurable* if the set $\{\omega \in \Omega \mid \Gamma(\omega) \cap V \neq \emptyset\}$ is in $\mathcal{F}$ for every weakly* open subset V of X^*. A function $f : \Omega \to X^*$ is a *selection* of Γ if $f(\omega) \in \Gamma(\omega)$ a.e. $\omega \in \Omega$. If X is separable, then X^* is a Suslin space, and hence, a multifunction $\Gamma : \Omega \twoheadrightarrow X^*$ with the measurable graph in $\mathcal{F} \times \mathrm{Borel}(X^*, w^*)$ admits a $\mathrm{Borel}(X^*, w^*)$-measurable (or equivalently, weakly* measurable) selection (see [8, Theorem III.22]).

[4]A subset S of a topological space is said to be *relatively sequentially compact* if every sequence in S contains a convergent subsequence.

A multifunction $\Gamma : \Omega \twoheadrightarrow X^*$ is *integrably bounded* if there exists $\varphi \in L^1(\mu)$ such that $\Gamma(\omega) \subset \varphi(\omega)U_{X^*}$ for every $\omega \in \Omega$. If Γ has the measurable graph and is integrably bounded, then it admits a Gelfand integrable selection whenever X is separable. Denote by $\mathcal{S}^1_\Gamma$ the set of Gelfand integrable selections of Γ. The Gelfand integral of Γ is conventionally defined as $\int \Gamma d\mu := \{\int f d\mu \mid f \in \mathcal{S}^1_\Gamma\}$. Denote by $\overline{\mathrm{co}}^*\Gamma$ the multifunction defined by the weak* closure of the convex hull of $\Gamma(\omega)$. Let T be a nonempty set. A (parameterized) multifunction $\Gamma : \Omega \times T \twoheadrightarrow X^*$ is *integrably bounded uniformly in $t \in T$* if there exists $\varphi \in L^1(\mu)$ such that $\Gamma(\omega, t) \in \varphi(\omega)U_{X^*}$ for every $\omega \in \Omega$ and $t \in T$.

The following result is attributed to [16, Theorem 2] (see also [39, Theorem 7.1]).

Lemma 2.2. *Let $(\Omega, \mathcal{F}, \mu)$ be a finite measure space and X be a separable Banach space. If $\Gamma : \Omega \twoheadrightarrow X^*$ is an integrably bounded, weakly* compact-valued multifunction with a measurable graph, then*

$$w^*\text{-cl} \int \Gamma d\mu = \int \overline{\mathrm{co}}^*\Gamma d\mu.$$

2.3 Limit Superior of a Sequence of Sets

The *strong limit superior* (or *strong upper limit*) of a sequence $\{S_n\}$ of nonempty subsets in X^* is defined by

$$\mathrm{Ls}\, S_n = \left\{ x^* \in X^* \mid \exists \{x^*_{n_i}\} : x^* = \lim_{i\to\infty} x^*_{n_i},\ x^*_{n_i} \in S_{n_i}\ \forall i \in \mathbb{N} \right\},$$

where $\{x^*_{n_i}\}$ denotes a subsequence of $\{x^*_n\} \subset X^*$ and the convergence is with respect to the dual norm in X^*. We denote by $w^*\text{-}\lim_n x_n$ the weak* limit point of a sequence $\{x^*_n\}$ in X^*. The *weak* limit superior* (or *weak* upper limit*) of $\{S_n\}$ is defined by

$$w^*\text{-Ls}\, S_n = \left\{ x^* \in X^* \mid \exists \{x^*_{n_i}\} : x^* = w^*\text{-}\lim_{i\to\infty} x^*_{n_i},\ x^*_{n_i} \in S_{n_i}\ \forall i \in \mathbb{N} \right\}.$$

Let $\{\Gamma_n\}$ be a sequence of multifunctions from Ω to X^*. Define the set valued-mapping $w^*\text{-Ls}\,\Gamma_n : \Omega \twoheadrightarrow X^*$ by $\omega \mapsto w^*\text{-Ls}\,\Gamma_n(\omega)$ (which possibly has empty values). A sequence of multifunctions $\{\Gamma_n\}$ is said to be *integrably bounded* if there is $\varphi \in L^1(\mu)$ such that $\Gamma_n(\omega) \subset \varphi(\omega)U_{X^*}$ a.e. $\omega \in \Omega$ for each n; it is said to be *well-dominated* if there is an integrably bounded, weakly* compact-valued multifunction $K : \Omega \twoheadrightarrow X^*$ such that $\Gamma_n(\omega) \subset K(\omega)$ a.e. $\omega \in \Omega$ for each n. Here, K is referred to as a *dominating multifunction* for $\{\Gamma_n\}$. By letting $K = \varphi U_{X^*}$, it is easy to see that $\{\Gamma_n\}$ is well dominated if and only if it is integrably bounded.

The following result is a version of [18, Theorem 1] for Gelfand integrals.

Lemma 2.3. *Let $(\Omega, \mathcal{F}, \mu)$ be a finite measure space and X be a separable Banach space. If $\{f_n\}$ is an integrably bounded sequence in $G^1(\mu, X^*)$ converging to $f \in G^1(\mu, X^*)$ in the $\sigma(G^1_{X^*}, L^\infty \otimes X)$-topology, then*

$$f(\omega) \in \overline{\text{co}}^* w^*\text{-Ls}\, f_n(\omega) \quad a.e.\ \omega \in \Omega.$$

Remark 2.1. Lemma 2.3 is found in [9, Proposition 1] and was shown to follow from [25, Lemma 7]. It is also a special case of [26, Lemma 6.6] in which X^* is replaced with a quasi-complete Lusin space and the Gelfand integrals are replaced with Pettis integrals. Since the dual of a separable Banach space endowed with the weak* topology is a quasi-complete Lusin space (see [26, Fact 6]), the result in [26] is a further extension for [9, 18, 25]. Moreover, as demonstrated in [7, Theorem 5.3], the integral boundedness in Lemma 2.3 can be weakened to the uniform integrability condition, or, more generally, to the Mazur-type tightness condition.

2.4 Saturated Measure Spaces

A finite measure space $(\Omega, \mathcal{F}, \mu)$ is said to be *essentially countably generated* if its σ-algebra can be generated by a countable number of subsets together with the null sets; $(\Omega, \mathcal{F}, \mu)$ is said to be *essentially uncountably generated* whenever it is not essentially countably generated. Let $\mathcal{F}_E = \{A \cap E \mid A \in \mathcal{F}\}$ be the σ-algebra restricted to $E \in \mathcal{F}$. Denote by $L^1_E(\mu)$ the vector subspace of $L^1(\mu)$ whose element is a restriction of each function in $L^1(\mu)$ to E.

Definition 2.1. A finite measure space $(\Omega, \mathcal{F}, \mu)$ is *saturated* if $L^1_E(\mu)$ is nonseparable for every $E \in \mathcal{F}$ with $\mu(E) > 0$. We say that a finite measure space has the *saturation property* if it is saturated.

Saturation is called *super-atomlessness* in [27] because it implies non-atomicity and several equivalent definitions for saturation are known; see [12, 15, 27]. One of the simple characterizations of the saturation property is as follows. A finite measure space $(\Omega, \mathcal{F}, \mu)$ is saturated if and only if $(E, \mathcal{F}_E, \mu)$ is essentially uncountably generated for every $E \in \mathcal{F}$ with $\mu(E) > 0$. If $\mathfrak{m}$ is an uncountable cardinal, then the probability space of $\{0,1\}^{\mathfrak{m}}$ with the uniform measure and that of the product space $[0,1]^{\mathfrak{m}}$ with the Lebesgue measure are typical examples of saturated probability spaces.

Under the saturation property on measure spaces, Lemma 2.2 can be sharpened as follows (see [27, Theorem 4] and [33, Proposition 1]).

Theorem 2.1. *Let X be an infinite-dimensional separable Banach space. Then a finite measure space $(\Omega, \mathcal{F}, \mu)$ is saturated if and only if for every integrably bounded, weakly* compact-valued multifunction $\Gamma : \Omega \twoheadrightarrow X^*$ with the measurable graph, the Gelfand integral $\int \Gamma d\mu$ is weakly* compact and convex in X^*.*

Remark 2.2. Another result on the convexity of the Gelfand integral of a multifunction is available in [26] for the locally convex Hausdorff space setting in countably generated, nonatomic measure spaces that admit a nonatomic disintegration. Note that although the measure spaces dealt in [26] rule out the Lebesgue unit interval as in saturated spaces, they are not saturated.

3 Sequences of Functions

3.1 The Saturation Property and an Exact Fatou Lemma

Under the nonatomicity hypothesis, the closure operation (or the ε-approximation) cannot be removed from the approximate version of Fatou's lemma for a sequence of Gelfand integrable functions in [4, Theorem 1.2], [7, Theorem 5.3], [9, Theorem 1], [25, Lemma 7], and [36, Theorem 1]. Specifically, the following negative result holds.

Proposition 3.1. *For every nonatomic finite measure space $(\Omega, \mathcal{F}, \mu)$ that is not saturated and for every infinite-dimensional Banach space X, there exists an integrably bounded sequence $\{f_n\}$ in $G^1(\mu, X^*)$ with the following properties:*

(i) *There is no $f \in G^1(\mu, X^*)$ such that*

(a) $f(\omega) \in w^*\text{-}\mathrm{Ls}\, f_n(\omega)$ *a.e.* $\omega \in \Omega$.

(b) $\displaystyle\int f d\mu \in w^*\text{-}\mathrm{Ls} \int f_n d\mu$.

(ii) $\displaystyle\mathrm{Ls} \int f_n d\mu \not\subset \int w^*\text{-}\mathrm{Ls}\, f_n d\mu$.

Proof. Since $(\Omega, \mathcal{F}, \mu)$ is not saturated, there exists a Bochner (and hence Gelfand) integrable function $g \in G^1(\mu, X^*)$ such that the range of the indefinite Bochner (and hence, Gelfand) integral $R = \{\int_A g d\mu \in X^* \mid A \in \mathcal{F}\}$ is not convex (see [27, Lemma 4] or [33, Remark 1(2)]), but $\mathrm{cl}\, R$ is convex, see [38, p. 162]. Take a point x^* in $\mathrm{cl}\, R \setminus R$. Then, there exists a sequence $\{A_n\}$ in $\mathcal{F}$ such that $\| \int_{A_n} g d\mu - x^* \|_{X^*} \to 0$. Define $f_n \in G^1(\mu, X^*)$ by $f_n = g\chi_{A_n}$. Since $f_n(\omega) \in \{\mathbf{0}, g(\omega)\}$ for each n, the sequence $\{f_n\}$ is integrably bounded and $w^*\text{-}\mathrm{Ls}\, f_n(\omega) \subset \{\mathbf{0}, g(\omega), \{\mathbf{0}, g(\omega)\}, \emptyset\}$. Moreover, the Gelfand integral $\int f_n d\mu$ converges strongly to x^* in X^*. Note that any Gelfand integrable selection from $w^*\text{-}\mathrm{Ls}\, f_n$ is of the form $g\chi_A$ with $A \in \mathcal{F}$ because weak* measurability coincides with Borel(X^*, w^*)-measurability in view of the separability of X. Suppose that for some $f \in G^1(\mu, X^*)$ conditions (a) $f(\omega) \in w^*\text{-}\mathrm{Ls}\, f_n(\omega)$ a.e. $\omega \in \Omega$; and (b) $\int f d\mu \in w^*\text{-}\mathrm{Ls} \int f_n d\mu$ hold true. Integrating both sides of (a) together with (b) in view of $\mathrm{Ls} \int f_n d\mu = x^*$ yields $x^* \in \int w^*\text{-}\mathrm{Ls}\, f_n d\mu \subset R$, a contradiction.

Moreover, it is easy to see that the inclusion $\mathrm{Ls}\int f_n d\mu \subset \int w^*\text{-}\mathrm{Ls}\, f_n d\mu$ does not hold. $\square$

Since the inclusion $\mathrm{Ls}\int f_n d\mu \subset w^*\text{-}\mathrm{Ls}\int f_n d\mu$ is always true, condition (ii) of Proposition 3.1 is stronger than the negation: $w^*\text{-}\mathrm{Ls}\int f_n d\mu \not\subset \int w^*\text{-}\mathrm{Ls}\, f_n d\mu$. The failure of an exact Fatou lemma leads, as an inevitable consequence, to the introduction of the saturation property on measure spaces.

Theorem 3.1. *Let $(\Omega, \mathcal{F}, \mu)$ be a saturated finite measure space and X be a separable Banach space. If $\{f_n\}$ is an integrably bounded sequence in $G^1(\mu, X^*)$, then:*

(i) *There exists $f \in G^1(\mu, X^*)$ such that*

   (a) $f(\omega) \in w^*\text{-}\mathrm{Ls}\, f_n(\omega)$ *a.e.* $\omega \in \Omega$.

   (b) $\displaystyle\int f d\mu \in w^*\text{-}\mathrm{Ls}\int f_n d\mu$.

(ii) $\displaystyle w^*\text{-}\mathrm{Ls}\int f_n d\mu \subset \int w^*\text{-}\mathrm{Ls}\, f_n d\mu$.

Proof. (i): By Lemma 2.1, the sequence $\{f_n\}$ has a subsequence that converges to some $f_0 \in G^1(\mu, X^*)$ in the $\sigma(G^1_{X^*}, L^\infty \otimes X)$-topology. Thus,

$$\left\langle x, \int f_0 d\mu \right\rangle = \lim_{i\to\infty} \int \langle x, f_{n_i}(\omega)\rangle d\mu = \left\langle x, \lim_{i\to\infty} \int f_{n_i} d\mu \right\rangle$$

for every $x \in X$. Hence, $\int f_0 d\mu \in w^*\text{-}\mathrm{Ls}\int f_n d\mu$. By Lemma 2.3, $f_0(\omega) \in \overline{\mathrm{co}}^*\, w^*\text{-}\mathrm{Ls}\, f_n(\omega)$ a.e. $\omega \in \Omega$. By Lemma 2.2, we have

$$\int f_0 d\mu \in \int \overline{\mathrm{co}}^*\, w^*\text{-}\mathrm{Ls}\, f_n d\mu = w^*\text{-}\mathrm{cl}\int w^*\text{-}\mathrm{Ls}\, f_n d\mu.$$

On the other hand, the saturation of $(\Omega, \mathcal{F}, \mu)$ guarantees that the Gelfand integral $\int w^*\text{-}\mathrm{Ls}\, f_n d\mu$ is weakly* compact and convex by Theorem 2.1. Hence, the above inclusion yields $\int f_0 d\mu \in \int w^*\text{-}\mathrm{Ls}\, f_n d\mu$. Therefore, there exists a Gelfand integrable selection f of $w^*\text{-}\mathrm{Ls}\, f_n$ such that $\int f_0 d\mu = \int f d\mu \in w^*\text{-}\mathrm{Ls}\int f_n d\mu$.

(ii): Take any $x^* \in w^*\text{-}\mathrm{Ls}\int f_n d\mu$. Then there exists a subsequence $\{f_{n_i}\}$ such that $w^*\text{-}\lim_i \int f_{n_i} d\mu = x^*$. It follows from condition (i) that there exists $f \in G^1(\mu, X^*)$ such that (a) $f(\omega) \in w^*\text{-}\mathrm{Ls}\, f_{n_i}(\omega)$ a.e. $\omega \in \Omega$; and (b) $\int f d\mu = w^*\text{-}\lim_j \int f_{n_j} d\mu$, where $\{f_{n_j}\}$ is a further subsequence of $\{f_{n_i}\}$. Integrating both sides of condition (a) together with condition (b) yields $x^* = \int f d\mu \in \int w^*\text{-}\mathrm{Ls}\, f_n d\mu$. $\square$

Remark 3.1. For the nonatomic Loeb space setting, as demonstrated in [22, Theorem 2.4], the integral boundedness condition in Theorem 3.1 can be weakened to the uniform integrability condition and the exact Fatou lemma is fortified with the case for a sequence of scalar measures instead of a single scalar measure μ.

3.2 A Characterization of the Saturation Property

Definition 3.1. Let $\{f_n\}$ be a sequence in $G^1(\mu, X^*)$ such that $w^*\text{-Ls}\int f_n d\mu$ is nonempty.

(i) $\{f_n\}$ satisfies the *weak* upper closure property* if there exists $f \in G^1(\mu, X^*)$ such that

  (a) $f(\omega) \in w^*\text{-Ls}\, f_n(\omega)$ a.e. $\omega \in \Omega$.

  (b) $\displaystyle\int f d\mu \in w^*\text{-Ls}\int f_n d\mu$.

(ii) $\{f_n\}$ satisfies the *Fatou property* if

$$w^*\text{-Ls}\int f_n d\mu \subset \int w^*\text{-Ls}\, f_n d\mu.$$

Theorem 3.2. *Let $(\Omega, \mathcal{F}, \mu)$ be a nonatomic finite measure space and X be an infinite-dimensional separable Banach space. Then the following conditions are equivalent:*

(i) *$(\Omega, \mathcal{F}, \mu)$ has the saturation property.*

(ii) *Every integrably bounded sequence in $G^1(\mu, X^*)$ has the weak* upper-closure property.*

(iii) *Every integrably bounded sequence in $G^1(\mu, X^*)$ has the Fatou property.*

Proof. (i) ⇒ (ii): See Theorem 3.1(i).

(ii) ⇒ (iii): See the proof of Theorem 3.1(ii).

(iii) ⇒ (i): Suppose that a nonatomic finite measure space $(\Omega, \mathcal{F}, \mu)$ is not saturated. It follows from Proposition 3.1 that there exists an integrably bounded sequence $\{f_n\}$ in $G^1(\mu, X^*)$ such that $\text{Ls}\int f_n d\mu \not\subset \int w^*\text{-Ls}\, f_n d\mu$. Hence, $\{f_n\}$ fails to satisfy the Fatou property. □

4 Sequences of Multifunctions

4.1 The Saturation Property and an Exact Fatou Lemma

The following result removes the closure operation from [36, Theorem 1] under the saturation property.

Theorem 4.1. *Let $(\Omega, \mathcal{F}, \mu)$ be a saturated finite measure space and X be a separable Banach space. If $\Gamma_n : \Omega \twoheadrightarrow X^*$ is an integrably bounded sequence of multifunctions, then*

$$w^*\text{-Ls} \int \Gamma_n d\mu \subset \int w^*\text{-Ls}\, \Gamma_n d\mu.$$

Proof. If $w^*\text{-Ls} \int \Gamma_n d\mu = \emptyset$, then the result is trivially true. Thus, without loss of generality, we may assume that $w^*\text{-Ls} \int \Gamma_n d\mu \neq \emptyset$. Take any $x^* \in w^*\text{-Ls} \int \Gamma_n d\mu$. Then there is a sequence $\{x^*_{n_i}\}$ of X^* such that $x^*_{n_i} \in \int \Gamma_{n_i} d\mu$ for each i with $w^*\text{-}\lim_i x^*_{n_i} = x^*$. Hence, there is an integrably bounded sequence $\{f_{n_i}\}$ in $G^1(\mu, X^*)$ such that $x^*_{n_i} = \int f_{n_i} d\mu$ and $f_{n_i} \in \mathcal{S}^1_{\Gamma_{n_i}}$ for each i. It follows from Theorem 3.1 that

$$x^* = w^*\text{-}\lim_{i \to \infty} x^*_{n_i} \in w^*\text{-Ls} \int f_n d\mu \subset \int w^*\text{-Ls}\, f_n d\mu \subset \int w^*\text{-Ls}\, \Gamma_n d\mu.$$

Therefore, the desired inclusion holds. □

The following result extends the fact that integration preserves upper semicontinuity in [32, Theorem 10] for nonatomic Loeb spaces to the case for saturated measure spaces.

Corollary 4.1. *Let $(\Omega, \mathcal{F}, \mu)$ be a saturated finite measure space, T be a metric space, and X be a separable Banach space. If $\Gamma : \Omega \times T \twoheadrightarrow X^*$ is a weakly* closed-valued multifunction satisfying*

(i) $\Gamma(\omega, \cdot\,) : T \twoheadrightarrow X^*$ *is weakly* upper semicontinuous for every $\omega \in \Omega$,*

(ii) $\Gamma(\,\cdot\,, t) : \Omega \twoheadrightarrow X^*$ *has a measurable graph for every $t \in T$ and is integrably bounded uniformly in $t \in T$,*

then $\int \Gamma(\omega, \cdot\,) d\mu : T \twoheadrightarrow X^*$ is weakly upper semicontinuous.*

Proof. Let $\{t_n\}$ be a sequence in T converging to t. Since $\int \Gamma(\omega, t) d\mu$ is weakly* compact by Theorem 2.1, it suffices to show that the integral has the closed graph, that is, $w^*\text{-Ls} \int \Gamma(\omega, t_n) d\mu \subset \int \Gamma(\omega, t) d\mu$. Let $\Gamma_n(\omega) = \Gamma(\omega, t_n)$ and apply Theorem 4.1 to $\{\Gamma_n\}$ in view of $w^*\text{-Ls}\, \Gamma(\omega, t_n) \subset \Gamma(\omega, t)$ to yield the result. □

4.2 A Characterization of the Saturation Property

Definition 4.1. A sequence $\Gamma_n : \Omega \twoheadrightarrow X^*$ of multifunctions satisfies the *Fatou property* if

$$w^*\text{-Ls} \int \Gamma_n d\mu \subset \int w^*\text{-Ls}\, \Gamma_n d\mu.$$

Theorem 4.2. *Let X be an infinite-dimensional separable Banach space. A nonatomic finite measure space $(\Omega, \mathcal{F}, \mu)$ has the saturation property if and only if every integrably bounded sequence of multifunctions $\Gamma_n : \Omega \twoheadrightarrow X^*$ has the Fatou property.*

Proof. By Theorem 4.1, the Fatou property holds for integrably bounded sequence of multifunctions whenever the underlying finite measure space is saturated. For the converse implication, suppose that a nonatomic finite measure space $(\Omega, \mathcal{F}, \mu)$ is not saturated. It follows from Proposition 3.1 that there exists an integrably bounded sequence of Gelfand integrable functions such that the Fatou property fails. □

Similar to a sequence of Gelfand integrable functions, the weak* upper closure property for a sequence of multifunctions can be defined. To this end, let $\{\Gamma_n\}$ be a sequence of multifunctions from Ω to X^* with a Gelfand integrable selection such that $w^*\text{-Ls} \int \Gamma_n d\mu$ is nonempty. We say that $\{\Gamma_n\}$ satisfies the *weak* upper closure property* if there exists a multifunction $\Gamma : \Omega \twoheadrightarrow X^*$ with a Gelfand integrable selection such that (i) $\Gamma(\omega) \in w^*\text{-Ls}\,\Gamma_n(\omega)$ a.e. $\omega \in \Omega$; and (ii) $\int \Gamma d\mu \in w^*\text{-Ls} \int \Gamma_n d\mu$. As demonstrated in [7, Theorem 6.3], an approximate version of the weak* upper-closure property holds for a tight sequence of Gelfand integrable multifunctions with weakly* compact, convex values whenever $(\Omega, \mathcal{F}, \mu)$ is nonatomic. The equivalence of saturation and the weak* upper closure property for a tight sequence of multifunctions with weakly* compact, nonconvex values is an intriguing open question.

Note

We kindly received on March 31, 2015 from Professor Greinecker a manuscript authored by Greinecker-Podczeck on the subject of this note. We hope to relate it to our results in subsequent work.

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An Exchange Economy with Differentiated Commodities and a Saturated Measure Space of Consumers[1]

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Abstract

We study core and competitive equilibria of a large exchange economy over the commodity space $ca(K)$, the space of singed measures on a compact metric space K. We prove the existence and core equivalence theorems without assuming convexity of preferences for a model with the saturated measure space of consumers.

JEL classification: D51

Key Words: Large exchange economy, commodities of Borel measures, existence and core equivalence of competitive equilibria, saturated measure space of consumers.

1 Introduction

In this paper, we will seek a satisfactory theory of competitive equilibria of an exchange economy consisting of the existence and its equivalence with core for a general model with differentiated commodities in a similar way as established by Aumann [2, 3] on the finite dimensional commodity spaces. It is well known that Aumann proved his theorems without assuming that the consumers' preferences are convex. Since this is the issue with which our paper is concerned, we will explain below why similar results have been considered to be very difficult to be obtained in the models with differentiated commodities.

The market equilibrium model with differentiated commodities would be probably dated back at least to the duopoly model of price competition of Hotelling [11]. Then the hedonic price theory of Rosen [40] and the commodity characteristics model of Lancaster [24] followed. The unified approach to handle the differentiated commodities was proposed by Mas-Colell [27]. In his model, the differentiated commodities are represented by elements of the commodity space $ca(K)$, the space of signed measures on a compact metric space K which stands for the set of commodity characteristics; see Section 2 for details.

We notice that this is one of the first general equilibrium model on an infinite dimensional commodity space with a continuum of traders for which the existence and the core equivalence of the competitive equilibria were both demonstrated simultaneously. As the pioneering work, the paper of Bewley [4, 5][1] which proved the existence and core equivalence of competitive equilibrium for an exchange economy with a measure space of consumers and the commodity space ℓ^∞ should also be cited.

Mas-Colell's formulation seems to have established its status as the "standard model" for representing the differentiated commodities in the market models. Indeed 4 years later of the Mas-Colell's paper, O. Hart [8] applied this formalism to the theory of monopolistically competitive equilibrium. For other economic applications to the analysis of competitive or monopolistically competitive markets, we refer [14, 26, 34, 35, 37, 46].

Mas-Colell was forced to introduce a continuum of traders, since he included the indivisibility of differentiated commodities in order to circumvent the problem which is now familiar among the mathematical theorists that the norm interior of the positive orthant of $ca(K)$ is empty. The indivisible commodities necessarily make the consumption set to be nonconvex, hence they potentially disturb the upper hemi-continuity of the individual demand correspondence. In order to restore the upper hemi-continuity of the aggregate (mean) demand correspondence which is necessary for the fixed point argument, he invoked the "regularizing effect" of the total demand by aggre-

[1]Bewley's '91 paper was first written in 1970 for his Ph.D. thesis.

gation over the consumers. This technique works only for the model with the continuum of traders, not for any model with finite number of consumers, where the aggregation is just simply summation. This is the reason why Mas-Colell needed the continuum of consumers.

Jones [13] introduced an alternative condition of the "bounded marginal rate of substitution" between the differentiated commodities[2] and replaced the condition of indivisible commodities of Mas-Colell by it. This condition is free from the problem of the discontinuous demands, and allowed him to prove the existence of competitive equilibrium for finite consumers economy on the commodity space $ca(K)$. The relationship between the indivisibility and the bounded marginal rate conditions is further elaborated by [21]. Jones [12] also proved the existence theorem for the model on $ca(K)$ with a continuum of consumers. However, he did not prove the core equivalence in this paper.

Up to this point, we have encountered typical examples of the equilibrium model on an infinite dimensional commodity space with a continuum of traders. We have many more examples of this type of market models; Khan-Yannelis [23] and Noguchi [30] proved the existence of a competitive equilibrium for the economies with a measure space of agents in which the commodity space is a separable Banach space whose positive orthant has norm interior points. Bewley [5] (which was already mentioned), Noguchi [31] and Suzuki [49] proved the equilibrium existence theorems for the economies with a measure space of consumers on the commodity space ℓ^{∞}, the space of bounded sequences. Among the papers discussing economies on the commodity space $ca(K)$ which were already cited, we have to pick Martin-da-Rocha [26], Ostroy and Zame [34] and Podczeck [37]. Since Ostroy–Zame [34] is probably most closely related with our paper, we will came back later to their paper.

These authors anticipated in their works that there are significant technical difficulties for extensions of the Aumann's theorems to infinite dimensional commodity spaces. Since the Negishi method for the existence proof (e.g., Mas-Colell [29]) can not be applied to the models with the continuum of traders, they invoked some procedures of finite approximation. For instance, Bewley [5] and Martin-da-Rocha [26] approximated the large, infinite-dimensional economy by finite-consumers, infinite-dimensional subeconomies. Khan–Yannelis [23] and Noguchi [30] approximated their economies by subeconomies with norm compact consumption sets. Noguchi [31], Ostroy–Zame [34], Podczeck [37] and Suzuki [49] proved their theorems by approximating the large, infinite-dimensional economy by large, finite-dimensional subeconomies. These finite approximation methods required the common mathematical techniques; the Fatou's lemma and/or the Lyapunov's convexity theorem.

[2]Mas-Colell [29] generalized this condition and he called it *proper*.

In order for the Fatou's Lemma to be applied on the finite dimensional spaces, for instance, the only condition which is required for the demand correspondences is that they are integrably bounded. The infinite dimensional version of Fatou's lemma, however, requires that the demand correspondences are contained in a convex valued correspondence (see for example, Yannelis ([52] Theorem 5.2). Since the Lyapunov convexity theorem fails on the infinite dimensional spaces, this means that the convex valuedness of the demand correspondences itself is strongly wanted. Indeed, Bewley, Khan–Yannelis, Ostroy–Zame, Podczeck and Suzuki assumed that the preferences are convex. Noguchi and Martin-da-Rocha assumed that a commodity vector does not belong to the convex hull of its preferred set. These assumptions obviously enfeeble the impact of the Aumann's classical result which revealed the "convexifying effect" of large numbers of the economic agents. This is the problem which the present paper tackles.

A strategy of Mas-Colell [27] and Jones [12] to avoid (not solve) this problem is that they defined their economies as a probability measure μ on the set of agents' characteristics $\mathcal{P} \times \Omega$, where $\mathcal{P}$ is the space of preferences and Ω is the set of initial endowments. Then the competitive equilibrium of this economy is also defined as a probability measure ν (and a price vector π which is an element of the dual of the commodity space) on $X \times \mathcal{P} \times \Omega$, where the set X is a consumption set which is assumed to be identical among all consumers. A technical advantage of the "distributional form" is that it can dispense with the Fatou's lemma nor the Lyapunov convexity theorem in the course of proofs, hence they could prove the existence of equilibria without convexity assumptions on preferences[3]. On the other hand, the distribution approach is not entirely appropriate when one discusses the core. Indeed, in order to define the core, Mas-Colell had to use a representation of the measure (economy); A (measurable) mapping $\mathcal{E}$ from the unit interval $[0,1]$ to $\mathcal{P} \times \Omega$ is a representation of the economy μ if it satisfies $\mu = \bar{\ell} \circ \mathcal{E}$, where $\bar{\ell}$ is the Lebesgue measure on $[0,1]$.

We will present the model as that of the "individual form" in this paper, hence the economy will be given as a measurable mapping $\mathcal{E}$ from an atomless measure space $(A, \mathcal{A}, \lambda)$ to $\mathcal{P} \times \Omega$. Then the core and competitive equilibria will be defined in the standard manner. In order to obtain the existence theorem without the convexity of preferences for the economy of the individual form, we will need an alternative strategy.

As already stated, Ostroy–Zame [34] also discussed an exchange economy of the individual form with the commodity space $ca(K)$. They examined thoroughly the competitive nature of the core of non-atomic markets, and concluded that if the markets are economically or physically "thick", there

[3] Bewley [5] and Suzuki [47, 48, 50] also applied this method to the model on the commodity space ℓ^∞ (infinite time horizon model). However, Bewley assumed the convexity of preferences, since he approximated the economy by finitely many *consumers* economies.

exists a competitive equilibrium and the core equivalence theorem holds. A market is economically thick if the preferences are weak*-continuous and satisfy the condition of the bounded marginal rate of substitution. It is called physically thick if each individual endowment vector (measure) is absolutely continuous with respect to the total endowment vector ([34], p.603). We will prove the existence and the core equivalence theorems for a model on $ca(K)$ which is economically thick. The main difference between their paper and the present one is that they worked within a class of economies with convex preferences.

In this paper, we will assume the space $(A, \mathcal{A}, \lambda)$ of consumers to be a *saturated* or *super-atomless* measure space (Definition 2.1 in Section 2). The saturated (or super-atomless) measure spaces have very nice properties from which many crucial mathematical results including Fatou's lemma and Lyapunov theorem follow (Keisler–Sun [15], Khan–Sagara [17, 18, 19], Khan–Sagara–Suzuki [20], Podczeck [39] and Sun–Yannelis [45]). As noted above, both of them failed for maps which take their values in infinite dimensional vector spaces when the measure space is simply atomless.

With help of these mathematical results, the existence of competitive equilibria can be proved for the economies with the saturated measure space of consumers without assuming the convexity of preferences[4]. This has been indeed achieved by Sagara–Suzuki [44] by applying the Fatou's lemma due to Khan–Sagara–Suzuki [20]. They proved the existence theorems both for the spaces $ca(K)$ and ℓ^∞ systematically using the standard finite dimensional approximations.

However in the present paper, the existence of equilibria will be proved by another method (Theorem 2.1). It will be explained briefly as follows. Let $\mathcal{E} : (A, \mathcal{A}, \lambda) \to \mathcal{P} \times \Omega$ be an economy. By the Jones's theorem there exists a distributional equilibrium (π, ν) for the distributional economy $\mu = \mathcal{E}_*\lambda \equiv \lambda \circ \mathcal{E}^{-1}$, the direct image measure. What we want to obtain is a representation $\phi : A \to X$ with $(\phi, \mathcal{E})_*\lambda = \nu$. The representation can be indeed obtained by a crucial mathematical property of the saturated space (Fact 2 in the proof of Theorem 2.1). The pair (π, ϕ) is then the desired equilibrium.

This method has been well known in the game theory literatures (e.g., Carmona–Podczeck [6], Keisler–Sun [15], Khan–Rath–Yu–Zhang [16] and Noguchi [32]), and and called "back and forward" argument between the

[4]Rustichini and Yannelis [43] and Podczeck [36] also proved the existence of competitive equilibria for the economies with the continuum of consumers and infinite dimensional commodity spaces. Rustichini–Yannelis obtained their results by assuming that the cardinality of the consumers space is larger than the commodity space. Podczeck proved his theorem by imposing a somewhat complicated mathematical structure on the consumers space. The advantage of our formulation is simplicity. The saturation is an intrinsic property of a measure space, not what is imposed "by hands" from outside. However, we want to emphasize that the Podczeck formalism is an economic origin of the saturated space (super-atomless in Podczeck's terminology); see Sagara–Suzuki [44] for details.

individual and the distributional forms of equilibria. It is simpler than the direct proof using the Fatou's Lemma when the distributional equilibria have been already obtained and it also provides a clue to explore the relationship between the individual and the distributional equilibria; see [16, 44, 50].

The second result of our paper is the core equivalence theorem (Theorem 2.2). Rustichini and Yannelis [42] discussed core equivalence theorems for an exchange economy of the individual form with infinite dimensional commodity spaces. For a space whose positive orthant has the empty norm interior, they proved the core equivalence theorems under the assumption of the "existence of extremely desirable commodities" which is a generalization of the Jones' bounded marginal rate of substitutions. Although they have the spaces L_p, $(1 \leq p < +\infty)$ in their minds as concrete examples, they indicated how their result could cover $ca(K)$ (Remark 7.4, p.324). Since they imposed an additional technical assumption (Assumption A.11, p.320), the direct comparison between their results and that of our paper seems to be difficult. They concentrated on the core equivalence only, and did not discuss the existence. As examples of other important papers on the core equivalence, we refer [38, 51]. These papers are concerned with the implications of the topological separability of the commodity space to the core equivalence, which is not our issue of this paper.

For the proof of the core equivalence theorem, we follow Ostroy–Zame [34]. We will keep the saturation there. It allows us to use the Lyapunov theorem proved by Khan–Sagara [17]. Consequently we can avoid the physical thickness assumption in the proof of Ostroy–Zame. However, our reason for the saturation is more fundamental. Ostroy–Zame proved their core equivalence theorem *without* using the convexity of preferences. Indeed, they stated that for their core equivalence theorem,

> convexity of individual preferences is indeed superfluous. However, approximate versions of the Lyapunov convexity theorem or of Fatou's lemma are not strong enough to guarantee the *existence* of Walrasian equilibrium ([34], p.597, italic by Ostroy–Zame),

hence they worked with the core equivalence in the framework of the convex preferences. Our reason to keep the saturation is the same as that of Ostroy–Zame. We want to discuss the core equivalence in the framework where the core is guaranteed to be nonempty. To this end, they keep the convexity; we keep the saturation.

In the next section we will present the model and the statements of our theorems. Section 3 is devoted to the proofs. Section 4 concludes by discussing the related works.

2 Model and Results

2.1 Equilibria of the Economy

Let (K, d) be a compact metric space. Following Mas-Colell [27] and Jones [12], the economic interpretation of K is that it is a space of the commodity characteristics. Hence each $t \in K$ represents the complete list of characteristics which describes the commodity. A (differentiated) commodity bundle ξ is defined as a signed measure on K, hence an element of $ca(K)$. In particular, the Dirac measure δ_t is the (one unit of) commodity bundle which contains a characteristics $t \in K$. For a Borel measurable subset J of K, $\xi|_J$ means the restriction of the measure ξ to J.

For $\xi \in ca(K)$, $\xi \geq \mathbf{0}$ means that $\xi(B) \geq 0$ for every $B \in \mathcal{B}(K)$, where $\mathcal{B}(K)$ is the set of all Borel measurable subsets of K. $\xi > \mathbf{0}$ means that $\xi \geq \mathbf{0}$ and $\xi \neq \mathbf{0}$. The non-negative orthant $ca_+(K)$ of $ca(K)$ is the space of (Borel) measures on K, $\mathcal{M}(K) = ca_+(K) \equiv \{\xi \in ca(K) | \ \xi \geq \mathbf{0}\}$. The differentiated consumption vectors are elements of the set $\mathcal{M}(K)$. In this paper, all consumers are assumed to have an identical consumption set

$$X = \mathcal{M}(K).$$

As usual, a preference relation $\succsim \subset X \times X$ is a complete, transitive and reflexive binary relation on X, and we denote $(\xi, \zeta) \in \succsim$ by $\xi \succsim \zeta$. $\xi \prec \zeta$ means that $(\xi, \zeta) \notin \succsim$. Let $\mathcal{P}$ be a collection of allowed preference relations. We assume that $\mathcal{P}$ satisfies (i) through (iii).

Assumption (PR).

(i) $\succsim \in \mathcal{P}$ is closed in $X \times X$ in the weak* topology,

(ii) For all $\succsim \in \mathcal{P}$, $\xi, \zeta \in X$, if $\xi < \zeta$ then $\xi \prec \zeta$,

(iii) For all $\succsim \in \mathcal{P}$ and for every $\rho > 1$, there exists $\epsilon > 0$ such that if $diameter(J) < \epsilon$ and $\xi, \zeta \in X$ satisfy $\zeta(J) \geq \xi(J)$, then $\xi|_J + \xi|_{K \setminus J} \prec \rho\zeta|_J + \xi|_{K \setminus J}$.

Assumption (i) is the continuity with respect to the weak* topology and (ii) is the monotonicity. The assumption (iii) says that each consumer is willing to accept any trade in which 'terms' (namely ρ) are strictly greater than one over a sufficiently close characteristics of the traded commodities. This condition is called by Ostroy-Zame [34] by uniform substitutability. The economy is called *economically thick* if it satisfies the conditions (i) and (iii) of (PR).

Jones [12] proposed a condition of slightly different form of (iii), which reads (*iv*) For all $\succsim \in \mathcal{P}$ and for every $\rho > 1$, there exists $\epsilon > 0$ such that for all $r > 0$, for all $\xi \in X$ and for all $s, t \in K$ with $d(s,t) < \epsilon$, $\xi + r\delta_s \prec \xi + \rho r \delta_t$.

It is easily seen that the condition (*iii*) implies the condition (*iv*). Jones [12, 13] showed that this assumption will hold if the utility functions representing the preferences are smooth and the derivatives satisfy some bounded conditions. See Jones [12, 13] for more explanations for these conditions. If K is a finite set, then the conditions (*iii*) and (*iv*) is automatically satisfied. Note that the parameter $\epsilon > 0$ in (*iii*) and (*iv*) can be taken uniformly over the preferences by the assumption (PC) below.

Since $\mathcal{P} \subset \mathcal{F}(X \times X)$ by (*i*), we can endow $\mathcal{P}$ with the topology of closed convergence on $\mathcal{F}(X \times X)$. We also assume

Assumption (PC). $\mathcal{P}$ is compact with respect to the topology of closed convergence in $\mathcal{F}(X \times X)$.

It follows from Assumptions (PR)(*i*) and (PC), nearby commodities are considered to be uniformly good substitutes. Note that X is not a metric space, hence it is not true that $\mathcal{F}(X \times X)$ is a compact metric space. Jones [12], however, showed that a compact subset of $\mathcal{F}(X \times X)$ is indeed metrizable.

An initial endowment is assumed to be a nonnegative vector ω of $X = \mathcal{M}(K)$. We assume that the set Ω of all allowed endowments was a weak* compact subset of $\mathcal{M}(K)$, say, $\Omega = \{\omega \in \mathcal{M}(K) | \ \omega(K) \leq \hat{\omega}\}$ for some fixed number $\hat{\omega} > 0$.

A finite measure space $(A, \mathcal{A}, \lambda)$ is called essentially generated by a family $\mathcal{G} \subset \mathcal{A}$ if the smallest σ-algebra containing $\mathcal{G}$ together with the λ-null sets is $\mathcal{A}$ itself. It is essentially countably (uncountably) generated if $\mathcal{G}$ is a countable (uncountable) set. For a measure space $(A, \mathcal{A}, \lambda)$, $(E, \mathcal{A}_E, \lambda_E)$ is the restriction to a subset $E \in \mathcal{A}$.

Definition 2.1. A (finite) measure space $(A, \mathcal{A}, \lambda)$ is *saturated* (or *super-atomless*) if and only if $(E, \mathcal{A}_E, \lambda_E)$ is essentially uncountably generated for every $E \in \mathcal{A}$ with $\lambda(E) > 0$.

Typical examples of the (homogeneous) saturated measure spaces are the atomless Loeb spaces (Loeb [25]), the product spaces of the form $[0,1]^{\mathbf{m}}$ and $\{0,1\}^{\mathbf{m}}$, where $\mathbf{m}$ is an uncountable cardinal, $[0,1]$ equipped with the Lebesgue measure, and $\{0,1\}$ the "half-half" measure. The cardinalities of $[0,1]^{\mathbf{m}}$ and $\{0,1\}^{\mathbf{m}}$ are more than the continuum, however, there exists an atomless Loeb space with the continuum cardinal. Moreover, Podczeck [39] constructed a saturated measurable structure on $[0,1]$ by "enriching" the Lebesgue σ-algebra. Therefore we can assume that the measure space of consumers is saturated. Sagara and Suzuki [44] discussed systematically the models with the saturated measure space of consumers. See also Suzuki [50].

Let $(A, \mathcal{A}, \lambda)$ be an atomless measure space of consumers. For the definition of the economy, we follow Aumann [2, 3], and Hildenbrand [10].

Definition 2.2. An economy $\mathcal{E}$ is a (Borel) measurable map $\mathcal{E} : A \to \mathcal{P} \times \Omega,\ a \mapsto (\succsim_a, \omega(a))$. An economy is called saturated if $(A, \mathcal{A}, \lambda)$ is saturated.

Since $\omega(a) \in \Omega$, the Gelfand integral $\int_A \omega(a) d\lambda$ exists. We need a further assumptions on the endowments distributions.

Assumption (AE) (Adequate endowments). *support* $\left(\int_A \omega(a) d\lambda\right) = K$.

The assumption (AE) says that every commodity characteristics are available in the market. Let $C(K)$ be the set of all continuous functions on K, and we denote the constant zero function by $\mathbf{0}$. For $\pi \in C(K)$, $\pi \geq \mathbf{0}$ means that $\pi(t) \geq 0$ for all $t \in K$, $\pi > \mathbf{0}$ means that $\pi \geq \mathbf{0}$ and $\pi \neq \mathbf{0}$. Finally $\pi \ggg \mathbf{0}$ means that $\pi \geq \epsilon$ for some $\epsilon > 0$.

Let $C_+(K) = \{\pi \in C(K) |\ \pi \geq \mathbf{0}\}$. The price vector is a nonzero element of the space $C_+(K)$. Then for $\xi \in X$ and $\pi \in C_+(K)$, we denote $\pi\xi = \int_K \pi(t) d\xi(t)$. A Gelfand integrable map $\xi : A \to X$ is called an allocation. An allocation is said to be feasible if $\int_A \xi(a) d\lambda \leq \int_A \omega(a) d\lambda$. It is called exactly feasible if $\int_A \xi(a) d\lambda = \int_A \omega(a) d\lambda$.

We now state the next definition of the competitive equilibrium.

Definition 2.3. A pair (π, ξ) of a price vector $\pi (\neq \mathbf{0}) \in C_+(K)$ and an allocation $\xi : A \to X$ is called a competitive equilibrium of the economy $\mathcal{E}$ if the following conditions hold,

(E-1) $\pi\xi(a) \leq \pi\omega(a)$ and $\xi(a) \succsim \zeta$ whenever $\pi\zeta \leq \pi\omega(a)$ a.e,

(E-2) $\int_A \xi(a) d\lambda = \int_A \omega(a) d\lambda$.

The first main result of the present paper now reads;

Theorem 2.1. *Let $\mathcal{E}$ be an economy which is saturated and satisfies the assumptions (PR), (PC) and (AE). Then there exists a competitive equilibrium (π, ξ) for $\mathcal{E}$.*

2.2 Core of the Economy

Let $\mathcal{E} : A \to \mathcal{P} \times X,\ a \mapsto (\succsim_a, \omega(a))$ be an economy. A non-null measurable set (coalition) $C \subset A$ is called a coalition.

Definition 2.4. A coalition $C \subset A$ is said to block an allocation ξ if there exists an allocation ζ such that

(C-1) $\int_C \zeta(a) d\lambda \leq \int_C \omega(a) d\lambda$,

(C-2) $\xi(a) \prec_a \zeta(a)$ on C.

The definition of the core is now given by the standard manner.

Definition 2.5. A feasible allocation $\xi : a \to X$ belongs to a core of an economy $\mathcal{E}$ if there exist no measurable sets $C \subset A$ with $\lambda(C) > 0$ which block ξ.

Ostroy and Zame [34] observed that the next assumption (globally bounded rates of substitution) will make the argument extremely simple, although it is a rather strong condition.

Assumption (GB). There exists a constant $M > 0$ such that for all $\xi, \eta, \zeta \in \mathcal{M}(K)$ with $\zeta - \xi + \eta \geq \mathbf{0}$ and $M\|\xi\| < \|\eta\|$, it follows that $\zeta \prec_a \zeta - \xi + \eta$ a.e.

The second main result of this paper is;

Theorem 2.2. *Let $\mathcal{E}$ be an economy which is saturated and satisfies the assumptions (PR), (PC), (GB) and (AE). A feasible allocation $\xi : A \to X$ belongs to the core of the economy $\mathcal{E}$ if and only if there exists a price vector $\pi \in C_+(K)$ such that (π, ξ) is a competitive equilibrium for $\mathcal{E}$.*

It follows from Theorem 2.1 that Theorem 2.2 is not vacuous.

3 Proofs of Theorems

3.1 Proof of Theorem 2.1

According to Mas-Colell [27], Jones [12] and Suzuki [47, 48] a probability measure μ on the measurable space $(\mathcal{P}\times\Omega, \mathcal{B}(\mathcal{P}\times\Omega))$ is called a distributional economy. The marginals of μ will be denoted by subscripts, for instance, $\mu_{\mathcal{P}}$ denotes the marginal on $\mathcal{P}$, and so on. A probability measure ν on $X \times \mathcal{P}\times\Omega$ is called an allocation distribution if $\nu_{\mathcal{P}\times\Omega} = \mu$. An allocation distribution is called (exactly) feasible if $\int_X i(\xi) d\nu_X = \int_\Omega i(\omega) d\mu_\Omega$. Since $i(\xi) = \xi$ for all ξ, hereafter we will denote $\int_X i(\xi) d\nu_X = \int_X \xi d\nu_X$, and so on. The Gelfand integrals $\int_X \xi d\nu_X$ and $\int_\Omega \omega d\mu_\Omega$ exist by Diestel and Uhl [7], pp.53-4. The distributional equilibrium is defined as follows (Mas-Colell (*ibid*) etc).

Definition 3.1. A pair (π, ν) of a price vector $\pi \in C_+(K)$ and an allocation distribution ν on $X\times\mathcal{P}\times\Omega$ is called a competitive equilibrium of the economy μ if the following conditions hold,

(D-1) $\nu(\{(\xi, \succsim, \omega) \in X \times \mathcal{P}\times\Omega | \ \pi\xi \leq \pi\omega$ and $\xi \succsim \zeta$ whenever $\pi\zeta \leq \pi\omega\}) = 1$,

(D-2) $\int_X \xi d\nu_X = \int_\Omega \omega d\mu_\Omega$,

(D-3) $\nu_{\mathcal{P}\times\Omega} = \mu$.

Let $(A, \mathcal{A}, \lambda)$ be an atomless probability measure space. For a measurable map $f : A \to \mathcal{P}\times\Omega$, the direct image measure $\lambda\circ f^{-1}$ is denoted by $f_*\lambda$.

Definition 3.2. For an economy μ, a measurable map $\mathcal{E} : A \to \mathcal{P}\times\Omega$ such that $\mu = \mathcal{E}_*\lambda$ is called a representation of μ. The representation is called saturated if the measure space $(A, \mathcal{A}, \lambda)$ is saturated.

Note that a representation is not unique even if it exists. Since $\mathcal{P} \times \Omega$ is a compact metric space, the representations of μ exists by Keisler-Sun [15], Lemma 2.1. Moreover, since the saturated measure spaces are atomless, the saturated representations also exist. Let (π, ν) be a distributional equilibrium of an economy μ. Suppose for a moment that there exists a norm bounded (hence weak* compact metric) subset Z of X with $\mathit{support}(\nu_X) \subset Z$. Then there also exists a (pair of) measurable map(s) $(\xi, \mathcal{E}) : A \to X \times \mathcal{P}\times\Omega$ which satisfies $\nu = (\xi, \mathcal{E})_*\lambda$. Then the (pair of) map(s) $(\xi, \mathcal{E})$ is called a representation of ν, and $\xi : A \to X$ is nothing but an allocation. The saturated representation exists for ν by the same reason for that of μ. First we show

Lemma 3.1. *Let $(A, \mathcal{A}, \lambda)$ be an atomless measure space, $\mathcal{E} : A \to \mathcal{P} \times \Omega$ be a representation of μ and $\xi : A \to X$ a measurable mapping. Define $\nu = (\xi, \mathcal{E})_*\lambda$. Then ξ is an equilibrium allocation of $\mathcal{E}$ if and only if ν is an equilibrium distribution of μ.*

Proof. Suppose that ξ is an equilibrium allocation of $\mathcal{E}$. Then there exists a price vector $\pi \in C_+(K)$ with $\lambda(E) = 1$ and $\int_A \xi(a)d\lambda = \int_A \omega(a)d\lambda$, where

$$E = \{a \in A|\ \pi\xi(a) = \pi\omega(a) \text{ and } \xi(a) \succsim_a \zeta \text{ whenever } \pi\zeta \le \pi\omega(a)\}.$$

Let $F = \{(\xi, \succsim, \omega) \in X \times \mathcal{P} \times \Omega|\ \pi\xi = \pi\omega \text{ and } \xi \succsim \zeta \text{ whenever } \pi\zeta \le \pi\omega\}$. Then $(\xi, \mathcal{E})(E) = F$, hence $\nu(F) = (\xi, \mathcal{E})_*\lambda(F) = \lambda(E) = 1$, which proves the condition (D-1). Since $\xi_*\lambda = \nu_X$ and $\omega_*\lambda = \nu_\Omega = \mu_\Omega$, we have from the change of variable formula $\int_X \xi d\nu_X = \int_\Omega \omega d\mu_\Omega$. Hence the condition (D-2) is met. Finally, the condition (D-3) follows from $\nu_{\mathcal{P}\times\Omega} = \mathcal{E}_*\lambda = \mu$. The converse is also proved in a similar way. □

We now prove Theorem 2.1. Let $\mathcal{E} : A \to \mathcal{P} \times \Omega$ be a saturated economy which satisfies the assumption (PR), (PC) and (AE). We can induce a distributional economy $\mu = \mathcal{E}_*\lambda$. Let $C_{++}(K) = \{\pi \in C(K)|\ \pi \ggg \mathbf{0},\ \|\pi\| = 1\}$. Jones [12] proved

Fact 1. There exists a distributional equilibrium $(\pi, \nu) \in C_{++}(K)\times\mathcal{M}(X\times \mathcal{P} \times \Omega)$ for μ.

Since $\pi \in C_{++}(K)$, there exists an $\epsilon > 0$ such that $\pi(t) \ge \epsilon$ for all $t \in K$. Then $\epsilon\xi(K) \le \pi\xi \le \pi\omega \le \omega(K) \le \hat{\omega}$, so that $\xi(K) \le \hat{\omega}/\epsilon$ for all $\xi \in \mathit{support}(\nu_X)$. Then we can assume without loss of generality that

$support(\nu_X) \subset Z$ for an weak* compact metric (hence a complete and separable metric) set $Z \subset X$, hence μ has the representations. Let $(\pi, \xi(a))$ be a representation of μ. Keisler and Sun [15] proved

Fact 2. Suppose that a finite measure space $(A, \mathcal{A}, \lambda)$ is saturated, Z and Y are complete separable metric spaces. Then for every measure $\nu \in \mathcal{M}(Z \times Y)$ and measurable function $\mathcal{E} : A \to Y$ with $\mathcal{E}_*\lambda = \nu_Y$, there exists a measurable function $\xi : A \to Z$ which satisfies $(\xi, \mathcal{E})_*\lambda = \nu$.

Since $\mathcal{E}_*\lambda = \mu = \nu_{\mathcal{P}\times\Omega}$, we have from Fact 2 a measurable map ξ with $\nu = (\xi, \mathcal{E})_*\lambda$. Then ξ is an equilibrium allocation by Lemma 3.1. □

3.2 Proof of Theorem 2.2

The proof that a competitive equilibrium allocation is also a core allocation is standard. We skip it. Suppose that $\xi : A \to X$ be a core allocation.

Let $\bar{\omega} \equiv \int_A \omega(a)d\lambda$, and consider the measure space $(K, \mathcal{B}(K), \bar{\omega})$. Let $L^1(\bar{\omega})$ be the space of Gelfand integrable functions on $(K, \mathcal{B}(K), \bar{\omega})$. Then we can identify $L^1(\bar{\omega})$ as the subspace of $ca(K)$ of Borel measures which are absolutely continuous with respect to $\bar{\omega}$ via the Radon-Nikodym theorem as follows. Let $\phi \in ca(K)$ be a Borel measure which is absolutely continuous with respect to $\bar{\omega}$. Then for each Borel set $B \in \mathcal{B}(K)$, we have $\phi(B) = \int_B \psi(t)d\bar{\omega}$, where $\psi(t)$ is the Radon-Nikodym derivative of the measure $\phi(B)$. We denote this as $\phi = \psi\bar{\omega}$, $\psi \in L^1(\bar{\omega})$ and identify ϕ and ψ. Note that $\|\phi\| = \|\psi\|$.

Let $\mathcal{Z}$ be the space of pairs (C, ζ) such that C is a non-null measurable subset of A and $\zeta : C \to L^1(\bar{\omega})$ is a Bochner integrable function with $\xi(a) \prec_a \zeta(a)\bar{\omega}$ a.e on C. Let $\mathcal{Q}$ be the (net) preferred set:

$$\mathcal{Q} = \left\{ \int_C \zeta(a)\bar{\omega}d\lambda - \int_C \omega(a)d\lambda \,\middle|\, (C, \zeta) \in \mathcal{Z} \right\}.$$

Since $(A, \mathcal{A}, \lambda)$ is saturated, we can show that $\mathcal{Q}$ is a convex subset of $ca(K)$ by the same argument of Hildenbrand ([10], Proposition 5, p.62) using a Lyapunov theorem of Khan-Sagara ([17], Theorem 4.1). Note that $L^1(\bar{\omega})$ is separable, since the σ-algebra $\mathcal{B}(K)$ is countably generated because of the space K being a compact metric space[5].

Let $\mathcal{C}$ be the cone generated by $\mathcal{Q}$. We will find a linear functional $\pi \in L^1(\bar{\omega})^* = L^\infty(\bar{\omega})$ which supports the cone $\mathcal{C} \cap L^1(\bar{\omega})$; an appropriate representative of the equivalence class of π will provide the equilibrium price vector.

[5]Indeed, this means that $(K, \mathcal{B}(K), \bar{\omega})$ is not saturated; a measure space $(A, \mathcal{A}, \lambda)$ is saturated if and only if for every $E \in \mathcal{A}$ with $\lambda(E) > 0$, the space $L^1_E(\lambda)$ of the integrable functions on E is non-separable. See Khan-Sagara ([18], Definition 3.4).

We now define

$$\Psi = \{\psi \in L^1(\bar{\omega})| \ \ M\|\psi^+\| < \|\psi^-\|\},$$

where M is given in the assumption (GB), and ψ^+ and ψ^- are the positive and negative part of ψ, respectively. Obviously Ψ is an open cone in $L^1(\bar{\omega})$. We will show that $\mathcal{C} \cap \Psi = \emptyset$. Suppose not. Then there exists $(C, \zeta) \in \mathcal{Z}$ such that

$$\int_C \zeta(a)\bar{\omega} d\lambda - \int_C \omega(a) d\lambda = \psi^+\bar{\omega} - \psi^-\bar{\omega}, \ \ M\|\psi^+\| < \|\psi^-\|.$$

Since ψ^+ and ψ^- are disjoint, it follows that $\psi^+\bar{\omega} \leq \int_C \zeta(a)\bar{\omega} d\lambda$. Since Ψ is open and $\zeta(a)$ is Bochner integrable, we may assume without loss of generality that the map $\zeta(a)$ is a simple map. Hence there exists $\zeta_1 \dots \zeta_n \in L^1(\bar{\omega})$ and a finite partition $C_1 \dots C_n$ of C such that $\zeta(a) = \zeta_i$ on C_i. Then by the Riesz decomposition theorem ([1], p.319), we can find $\mathbf{0} \leq s(a) = s_i \in L^1(\bar{\omega})$ with $\mathbf{0} \leq s_i \leq \zeta_i$, $i = 1 \dots n$ and $\sum_{i=1}^n \lambda(C_i)s_i = \psi^+$. Let $f(a) = s_i\bar{\omega}$ for $a \in C_i$. Then $\mathbf{0} \leq f(a) \leq \zeta(a)\bar{\omega}$ for all $a \in C$ and $\int_C f(a)d\lambda = \psi^+\bar{\omega}$. Define

$$h(a) = \zeta(a)\bar{\omega} - f(a)\bar{\omega} + \left(\frac{M\|f(a)\|}{\|\psi^-\|}\right)\psi^-\bar{\omega}.$$

Then one obtains that

$$\begin{aligned}
\int_C h(a)d\lambda - \int_C \omega(a)d\lambda &= \int_C \zeta(a)\bar{\omega}d\lambda - \int_C \omega(a)d\lambda \\
&- \int_C f(a)\bar{\omega}d\lambda + \left(\frac{M\int_C \|f(a)\bar{\omega}\|d\lambda}{\|\psi^-\|}\right)\psi^-\bar{\omega} \\
&= \psi^+\bar{\omega} - \psi^-\bar{\omega} - \psi^+\bar{\omega} + \left(\frac{M\|\int_C f(a)\bar{\omega}d\lambda\|}{\|\psi^-\|}\right)\psi^-\bar{\omega} \\
&\leq \left(\frac{M\|\psi^+\|}{\|\psi^-\|}\right)\psi^-\bar{\omega} - \psi^-\bar{\omega} \leq \mathbf{0},
\end{aligned}$$

where in the second equality, we used $\|\int_C f(a)\bar{\omega}d\lambda\| = \int_C f(a)\bar{\omega}d\lambda(K) = \int_C f(a)\bar{\omega}(K)d\lambda = \int_C \|f(a)\bar{\omega}\|d\lambda$. Therefore $h(a)$ is feasible on C. The assumption (GB) implies that $h(a) \succsim_a \zeta(a)$ and hence $\xi(a) \prec_a h(a)$ a.e. on C. This contradicts that ξ is a core allocation. We conclude that $\mathcal{Q} \cap \Psi = \emptyset$. By the Hahn-Banach separation theorem (Aliprantis–Border [1]), there exists $\pi \in L^1(\bar{\omega})^* = L^\infty(\bar{\omega})$ such that $\pi\eta \geq 0$ for every $\eta \in \mathcal{Q}$. The monotonicity of preferences implies that $\pi \geq \mathbf{0}$. We will show that some representative of the (usual "almost everywhere") equivalence class of π is an equilibrium price of the core allocation ξ. To this end, choose any bounded Borel function $\bar{\pi}$ representing π.

Ostroy and Zame ([34], Lemma 2, p.617) proved

Fact 3. Let ξ be a feasible allocation and $\bar{\pi}$ a bounded Borel function on K. Then the following statements are equivalent:

(i) there is a bounded Borel function π^* such that $\pi^* = \bar{\pi}$ a.e and (π^*, ξ) satisfies the conditions (E-1) and (E-2) of Definition 3;

(ii) For almost all $a \in A$, if $\alpha \in L^1(\bar{\omega})$ satisfies that $\xi(a) \prec_a \alpha$, then $\bar{\pi}\omega(a) < \bar{\pi}\alpha$.

Then on account of Fact 3, it will suffice to show that there does not exist a measure $\alpha \in L^1(\bar{\omega})$ such that $\xi(a) \prec_a \alpha$ and $\bar{\pi}\alpha \le \bar{\pi}\omega(a)$ a.e. Suppose on the contrary, there existed such a vector α. The separability of $L^1(\bar{\omega})$ and the continuity of preferences would make possible for us to find a set $C \subset A$ of positive measure and a measure $\beta \in L^1(\bar{\omega})$ such that $\xi(a) \prec_a \beta$ and $\bar{\pi}\beta < \bar{\pi}\omega(a)$ for almost all $a \in C$. Define a map $\bar{\zeta} : C \to L^1(\bar{\omega})$ by $\bar{\zeta}(a) = \beta$ for all $a \in C$. Then the pair $(C, \bar{\zeta}) \in \mathcal{Z}$, hence $\lambda(C)\beta - \int_C \omega(a)d\lambda \in \mathcal{Q}$. Consequently we have $0 \le \bar{\pi}(\lambda(C)\beta - \int_C \omega(a)d\lambda) = \int_C \bar{\pi}(\beta - \omega(a))d\lambda$, which contradicts that $\bar{\pi}\beta < \bar{\pi}\omega(a)$ a.e on C. We conclude that for almost all $a \in A$, there does not exist a measure $\alpha \in L^1(\bar{\omega})$ such that $\xi(a) \prec_a \alpha$ and $\bar{\pi}\alpha \le \bar{\pi}\omega(a)$ a.e. Fact 3 now implies that there exists a bounded Borel set π^* that agree with $\bar{\pi}$ almost everywhere and supports ξ as an equilibrium price.

Finally, since the market is economically thick in the sense of Ostroy and Zame, all of the conditions for Theorem 3 of Ostroy–Zame ([34], p.604) but the convexity of preferences are satisfied, hence all equilibrium prices belong to a norm compact subset of $C(K)$ which implies $\pi^* \in C(K)$ (notice that the convexity is not required for this result). □

4 Concluding Remarks

4.1 Mas-Colell's Model with Differentiated and Indivisible Commodities

The model of Mas-Colell [27] is described as follows. There exist differentiated commodities and only one homogeneous commodity in the market, hence the consumption set which is assumed to be identical for all consumers is defined by

$$X^M = H \times D, \;\; H = \mathbb{R}_+, \;\; D = \{\xi \in \mathcal{M}(K) |\; \xi(B)(\le \hat{\xi}) \in \mathbb{N} \text{ for all } B \in \mathcal{B}(K)\}.$$

A typical element of X^M is denoted by (x, ξ), $x \in H$ and $\xi \in D$. Notice that the differentiated commodities are assumed to be integer valued, or they are indivisible. The set $H = \mathbb{R}_+$ is unbounded, but D is assumed to be bounded above by $\hat{\xi} > 0$. $\hat{\xi}$ is intended to be a very large number. The set of all allowed preferences is denoted as $\mathcal{P}^M$. The assumptions on $\mathcal{P}^M$ are

Assumption (M-PR).

(i') $\succsim \in \mathcal{P}^M$ is closed in $X^M \times X^M$ in the weak* topology,

(ii') For all $\succsim \in \mathcal{P}^M$, if $x > z$ and $\xi \geq \zeta$, then $(x, \xi) \prec (z, \zeta)$,

(iii') For all $\succsim \in \mathcal{P}^M$ and for all $(x, \xi) \in X^M$, there exists $\rho > 0$ such that $(x, \xi) \prec (\rho, \mathbf{0})$,

(iv') For all $\succsim \in \mathcal{P}^M$ and for all $\xi, \zeta \in D$, if $x > 0$ then $(0, \zeta) \prec (x, \xi)$.

The assumption (ii') is the monotonicity of the homogeneous goods, and (iii') is the overriding desirability for the homogeneous good. The assumption (iv') is needed even if there exist no commodity differentiations. It is related to the indivisibility of the commodities. Further elaborations of this assumption are pursued by Khan-Sagara-Suzuki [21]. See also Suzuki [50].

The set Ω^M of initial endowments are bounded subset of X^M,

$$\Omega^M = \{(e, \omega) \in X^M | \ e, \omega(K) \leq \hat{\omega}\}.$$

Of course the constant $\hat{\omega} > 0$ is intended to be far smaller than the upper bound $\hat{\xi}$ of X^M. The distributional economy $\mu \in \mathcal{M}(\mathcal{P}^M \times \Omega^M)$ and the distributional equilibrium (π, ν), $\nu \in \mathcal{M}(X^M \times \mathcal{P}^M \times \Omega^M)$ were defined in the proof of Theorem 1. Mas-Colell required a similar assumption with ours for the endowment distribution.

Assumption (M-AE). *support* $\left(\int_{\Omega_D^M} \omega d\mu_{\Omega_D^M}\right) = K$.

Then he proved

Theorem M1. Let μ be an economy which is saturated and satisfies the assumptions (M-PR), (PC) and (M-AE). Then there exists a competitive equilibrium (π, ν) for μ.

Since $D \subset ca(K)$ is bounded, hence compact metric space with respect to the weak* topology, the distribution $\nu \in \mathcal{M}(X^M \times \mathcal{P}^M \times \Omega^M)$ has a representation $((x, \xi), \mathcal{E}) : A \to X^M \times \mathcal{P}^M \times \Omega^M$. The core of the distributional economy μ is defined with respect to the representation $((x, \xi), \mathcal{E})$. Mas-Colell also proved

Theorem M2. Let μ be an economy which satisfies the assumptions (M-PR), (PC), and (M-AE). A feasible allocation distribution ν belongs to the core of the economy μ if and only if there exists a price vector $\pi(\neq \mathbf{0}) \in C_+(K)$ such that $(\pi, (x, \xi))$ is a competitive equilibrium for $\mathcal{E}$.

Since $(\pi, (x, \xi))$ is a competitive equilibrium for $\mathcal{E}$ if and only if (π, ν) is a competitive equilibrium for μ, Theorem M2 establishes the core equivalence of the distributional equilibrium and shows that the core do not depend on representations.

Let $(A, \mathcal{A}, \lambda)$ be a saturated measure space of consumers, and consider an economy $\mathcal{E}^M : A \to \mathcal{P}^M \times \Omega^M$. A feasible allocation map $(x, \xi) : A \to X^M$ and the competitive equilibrium $(\pi, (x, \xi))$ are defined in obvious ways. Exactly as in the same way as Theorem 1, we can prove *via* Theorem M1,

Theorem 4.1. *Let $\mathcal{E}^M$ be an economy which is saturated and satisfies the-assumptions (M-PR), (PC) and (AE). Then there exists a competitive equilibrium $(\pi, (x, \xi))$ for $\mathcal{E}^M$.*

Let (x, ξ) be a core allocation of the economy $\mathcal{E}^M$. Applying Theorem M2 to the distributional economy $\mu \equiv \mathcal{E}^M_* \lambda$ and its core distribution $\nu \equiv (x, \xi)_* \lambda$, we obtain

Theorem 4.2. *Let $\mathcal{E}^M$ be an economy which satisfies the assumptions (M-PR), (PC), and (AE). A feasible allocation $(x, \xi) : A \to X^M$ belongs to the core of the economy $\mathcal{E}^M$ if and only if there exists a price vector $\pi(\neq \mathbf{0}) \in C_+(K)$ such that $(\pi, (x, \xi))$ is a competitive equilibrium for $\mathcal{E}^M$.*

4.2 Noguchi-Zame Model with Externalities

Noguchi and Zame (2006) discussed a distributional economy with consumption externalities and finitely many commodities. Their fundamental idea of incorporating the externalities into the model is to apply the normal form of large games formulated by Mas-Colell [28].

There exist ℓ commodities in the market[6] and the common consumption set is assumed to be $X = \mathbb{R}^\ell_+$. A consumption vector is denoted by $\boldsymbol{x} \in X$.

The consumers are specified by their types (traits) denoted by t. Let T be a complete and separable metric space of traits and τ a probability measure on T. $(T, \mathcal{B}(T), \tau)$ is the measure space of *observable population characteristics*. Then the observable consumption of society is represented by a probability measure σ on $X \times T$, hence $\sigma \in \mathcal{M}(X \times T)$. We write the set of consumption distributions with $\sigma_T = \tau$ by $\mathcal{D}(\tau)$.

A (strict) preference relation $\prec$ is an irreflexive, transitive and negatively transitive binary relation on $X \times \mathcal{D}(\tau)$. Let $\mathcal{P}$ be the set of allowed preferences (Noguchi-Zame required the preferences are monotone, or $(\boldsymbol{x}, \sigma) \prec (\boldsymbol{z}, \sigma)$ whenever $\boldsymbol{x} < \boldsymbol{z}$).

The set Ω of initial endowments is a consumption set itself, $\Omega = X$. The space of consumer characteristics is then $\mathcal{C} = T \times \mathcal{P} \times \Omega$. An economy μ

[6] An attractive feature of their model is that it admits the indivisible commodities. In what follows, however, we will neglect the indivisible commodities and the production possibility in their model for expository simplicity.

is naturally defined by a distribution (probability measure) on $\mathcal{C}$ such that $\mu_T = \tau$. The marginal distribution μ_Ω is nothing but the initial endowments distribution.

The (distributional) equilibrium (π, ν) of the economy μ is now should be obvious; it is a pair (π, ν) consisting of a price vector $\pi \in \mathbb{R}^\ell_+$ together with a probability measure ν on $X \times \mathcal{C} = X \times T \times \mathcal{P} \times \Omega$ such that the following conditions hold,

(NZ-1) $\nu(\{(\boldsymbol{x}, t, \prec, \omega)|\ \pi\boldsymbol{x} \leq \pi\omega \text{ and } (\boldsymbol{x}, \nu_{X\times T}) \succsim (\boldsymbol{z}, \nu_{X\times T})$
$\text{whenever } \pi\boldsymbol{z} \leq \pi\omega\}) = 1,$

(NZ-2) $\int_X \boldsymbol{x} d\nu_X = \int_\Omega \omega d\mu_\Omega$,

(NZ-3) $\nu_\mathcal{C} = \mu$.

As always, the condition (NZ-3) ensures the consistency between the economy and the allocation distribution. Moreover in this model, it ensures that $\nu_T = \mu_T = \tau$, hence $\nu_{X\times T} \in \mathcal{D}(\tau)$.

Noguchi-Zame [33] proved that the equilibrium exists under very general assumptions (Theorem 1, p.150). They also showed that given a representation $\mathcal{E} : T \to \mathcal{C}$ of the economy μ, the distributional equilibrium allocation ν admits the representation $\xi : T \to X$ with $\nu = (\xi, \mathcal{E})_* \tau$ when the preferences are strictly convex (Theorem 2, p.151).

It is an interesting open question to extend their results to infinite dimensional commodity spaces. Furthermore, the relation between the distributional and the individual equilibria is a very important problem to be explored. Khan and Sun [22] proposed the concept of symmetric equilibria.

Definition 4.1. An equilibrium distribution ν is called *symmetric* if there exists a measurable map $\sigma : \mathcal{C} \to X$ such that $\nu(Graph(\sigma)) = 1$, where $Graph(\sigma) = \{(\boldsymbol{x}, t, \succsim, \omega)|\ \boldsymbol{x} = \sigma(t, \succsim, \omega)\}$.

If an equilibrium is symmetric, then the consumers with the identical characteristics consume the identical consumption vector. Noguchi and Zame gave a sufficient condition (strict convexity) for the symmetricity of equilibria.

Let a distributional economy μ and its equilibrium distribution ν be given. A probability space $(A, \mathcal{A}, \lambda)$ realizes ν, or $(A, \mathcal{A}, \lambda)$ is a realization of ν, if every individual economy $\mathcal{E} : A \to \mathcal{C}$ which represents μ has a measurable map $\xi : A \to X$ such that $\nu = (\xi, \mathcal{E})_* \lambda$. The next result was first appeared in the large atomless games (Khan-Rath-Yu-Zhang [16]).

Theorem 4.3. *Let an atomless distributional economy μ and its equilibrium distribution ν be given. Then the following conditions are equivalent.*

(a) ν is symmetric,

(b) every atomless probability space is a realization of ν,

(c) every atomless non-saturated probability space is a realization of ν,

(d) the measure space $([0,1], \mathcal{B}([0,1]), \bar{\ell})$ *(here* $\bar{\ell}$ *is the Lebesgue measure on the Borel* σ*-algebra) is a realization of* ν*.*

As a corollary, we obtain that an atomless probability space $(A, \mathcal{A}, \lambda)$ realizes a non-symmetric equilibrium of an atomless distributional economy μ if and only if it is saturated. For the proof of Theorem 5, see Khan-Rath-Yu-Zhang [16]. They proved their result for the Nash equilibria of the large distributional game of Mas-Colell [28] type. For the corresponding result for the market equilibria, see Sagara-Suzuki [44].

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