Using Elasticities to Derive Optimal Bankruptcy Exemptions

Eduardo Dávila*
NYU Stern

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Abstract

This paper characterizes the optimal bankruptcy exemption for risk averse borrowers who use unsecured contracts but have the possibility of defaulting. It provides a novel general formula — which holds in a wide variety of environments — for the optimal exemption as a function of a few observable sufficient statistics. Knowledge of borrowers' leverage, the sensitivity of the interest rate schedule faced by borrowers with respect to the level of the bankruptcy exemption, the probability of bankruptcy and the change in consumption by bankrupt borrowers is sufficient to determine the optimal bankruptcy exemption. When calibrated to US data, the optimal bankruptcy exemption implied by the model ($100,000) is larger than the average exemption in the US ($70,000), but of the same order of magnitude.

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1 Introduction

Motivation Should we make bankruptcy procedures harsher or more lenient for borrowers who decide not to repay their debts? What is the socially optimal level of bankruptcy exemptions? These are important normative questions — see, for instance, White (2011), who recently documents the large increase in consumer bankruptcy filings in the US in the last decades — that are still subject to debate. This paper shows that a few observable variables are sufficient to answer such questions under a broad array of circumstances.

This paper addresses the problem of optimal bankruptcy design by analytically characterizing the welfare maximizing bankruptcy exemption for risk averse borrowers who use unsecured contracts but can choose to default. The welfare maximizing bankruptcy exemption optimally trades off reduced access to credit ex-ante with improved insurance ex-post.

Main results I derive the main results of the paper in a baseline two period model with risk averse borrowers who only have access to a contract that promises a flat repayment; I subsequently extend the results in many dimensions. Throughout the paper, the bankruptcy exemption m is defined as the dollar amount that a borrower who declares bankruptcy is allowed to keep.

The main contribution of this paper is to characterize a) the welfare change induced by a marginal change in m and b) the optimal bankruptcy exemption m^*, as a function of a few observable sufficient statistics. This characterization provides a clear interpretation of the forces that determine the optimal exemption for a broad range of primitives and directly links the theoretical tradeoffs to a small set of observable variables. Most of the intuition behind the results can be seen in the case with logarithmic utility borrowers who always claim the full bankruptcy exemption. In that case, the optimal exemption m^* (measured in dollars) must satisfy the following equation:

$$m^* = \frac{\beta \pi}{\Lambda \varepsilon_{\tilde{r},m}}, \tag{1}$$

where β is the borrowers' rate of time preference, π is the probability of default in equilibrium, Λ is a measure of borrowers' leverage and $\varepsilon_{\tilde{r},m}$ is the sensitivity of the interest rate schedule faced by borrowers with respect to the bankruptcy exemption measured in dollars. Importantly, all four variables in the formula for m^* have direct empirical counterparts.

Equation (1) captures the key tradeoff regarding the optimal determination of the bankruptcy exemption. On the one hand, if borrowing rates rise quickly with the level of the bankruptcy exemption (high $\varepsilon_{\tilde{r},m}$), it is optimal to set a low exemption level, especially when borrowers' leverage is high (high Λ): a low exemption facilitates the access to credit ex-ante in that case. On the other hand, if default is very frequent in equilibrium (high π), especially when borrowers value consumption relatively more in the terminal period (high β), it is optimal to set a high exemption level, allowing borrowers to consume more when bankrupt. Equation (1) trades off these forces optimally.[1]

The decisions of how much to borrow and when to default do not affect the assessment of marginal interventions nor the formula for m^* directly: the fact that borrowers borrow and default optimally guarantees that the effect on these variables induced by a change in the bankruptcy exemption vanishes from the optimal exemption formula. This fact greatly simplifies the characterization of the optimal exemption.

This paper imposes no restrictions on the shape of the contracts used, but takes the set and shape of the contracts available as given: this is the key friction that the optimal exemption addresses. When the set of available contracts

[1] All variables in equation (1) are endogenous. The logic behind this characterization is identical to the one behind the CAPM, in which the beta of an asset (an endogenous object) becomes a sufficient statistic to determine expected returns, or consumption based asset pricing models, in which the consumption process, independently of how it is generated, is a sufficient statistic for pricing assets.

does not allow agents to reach the first-best allocation, which implies perfect insurance for the risk averse borrowers, an optimally determined bankruptcy exemption improves welfare. Hence, the optimal exemption problem solved in this paper is the second-best problem solved by borrowers who use a given set of contracts but can choose the level of bankruptcy exemptions ex-ante.

Calibration I show the applicability of the theoretical results by calibrating the formula for the optimal exemption to US bankruptcy data and assessing the magnitude of the welfare gains generated by adjusting the bankruptcy exemption. The preferred calibration to US data implies that the optimal bankruptcy exemption should be in the range of $100,000$ dollars, an amount slightly larger than the average exemption in the US (of approximately $70,000$ dollars). For the same calibration, implementing the optimal exemption achieves welfare gains on the order of approximately 0.03% of ex-ante consumption. This calibration exercise is not meant to settle the debate on the level of exemptions. Only further work on the measurement of the key required variables will help us refine the optimal prescription for the level of exemptions.

Extensions I explore several extensions to understand which departures from the baseline model affect the optimal exemption formula. First, allowing for additional margins of adjustment, like labor supply or effort choices, does not modify the optimal exemption formula as long as these choices enter separably into borrowers' utility function. Second, allowing for more general utility specifications for consumption — like Epstein-Zin preferences or state dependent utility — only affects the optimal exemption formula through changes in the borrowers' stochastic discount factor. Third, allowing for multiple traded contracts with arbitrary payoffs is straightforward: a leverage-weighted average of interest rate sensitivities captures then the marginal cost of increased leniency. Fourth, when borrowers are ex-ante heterogeneous but exemptions cannot be individual specific, the optimal exemption becomes a function of a weighted average across borrowers of the marginal effects present in the baseline case. Fifth, whether borrowers are price takers or not determines whether the equilibrium interest rate sensitivity or the interest rate schedule sensitivity is the right measure of the marginal cost of changing exemptions. Sixth, exemptions contingent on aggregate shocks can improve welfare by targeting the same set of sufficient statics identified in the baseline model for each aggregate state. Seventh, the optimal exemption formula in economies with non-zero output gaps accounts for the equilibrium effect of exemptions on aggregate demand. Finally, I show that, in a dynamic context, the optimal exemption is given by an average across periods/states of benefits and costs of the bankruptcy exemption, all weighted according to the borrowers' own stochastic discount factor.

These extensions yield two robust insights. First, the precise determination of the region(s) in which borrowers decide to default does not modify the optimal exemption formula, because borrowers default optimally. Hence, evaluating the welfare implications of changing exemptions does not require explicit modeling of the multiple factors that may influence default decisions.[2] Second, only measures of consumption of bankrupt borrowers are required to assess the welfare benefits of varying exemptions. Preference parameters, like risk aversion, determine how to translate measures of consumption into welfare, but the responses of labor supply and other endogenous choice variables are irrelevant once borrowers' consumption is known, as long as utility of consumption is separable and other economic choices are not subject to frictions.

[2]For instance, default is jointly determined with other endogenous choice variables. Similarly, forward looking borrowers internalize the option value of waiting for uncertainty to be realized before defaulting. Both these considerations, which prevent a simple characterization of default regions, only affect the optimal exemption through the sufficient statistics.

Related Literature

This paper relates to several literatures in economics.[3] First, this paper is most directly related to the literature on **general equilibrium with incomplete markets**, which has studied the possibility of default in very general environments in which markets are exogenously or endogenously incomplete. Zame (1993) and Dubey, Geanakoplos and Shubik (2005) are the first to theoretically analyze the core tradeoff present in this paper. They show that default may be welfare improving in a model with incomplete markets, since it enhances insurance opportunities by introducing new contingencies into contracts. These papers take default penalties as exogenous and do not characterize optimal penalties, which is the approach I adopt in this paper.[4] It is well known that default is only beneficial when markets are initially incomplete. Allowing for default when agents can write fully state contingent contracts, as in Kehoe and Levine (1993), Alvarez and Jermann (2000), or Chien and Lustig (2010), only restricts the contracting space, reducing welfare unequivocally.

Second, this paper is also related to the **quantitative literature** on default and bankruptcy. On the one hand, the papers by S. Chatterjee, D. Corbae, M. Nakajima and J.V. Ríos-Rull (2007) or Livshits, MacGee and Tertilt (2007), among several others, provide a careful quantitative structural analysis of unsecured credit and default from a macroeconomic perspective. Due to their dynamic nature and their rich general equilibrium features, these papers must rely on numerical methods to evaluate the welfare implications of bankruptcy policies. This paper, by contrast, focuses on providing analytical insights into this question. Livshits (2014) provides a recent survey of this growing literature, which has not reached a consensus on the optimal level of exemptions. The work by Gropp, Scholz and White (1997), Gross and Souleles (2002), Fay, Hurst and White (2002), Mahoney (2012), Iverson (2013), Dobbie and Song (2013), and Severino, Brown and Coates (2013), among others, uses microeconometric methodology to understand the implications of actual bankruptcy policies. See White (2007, 2011) for recent surveys of this body of work.

Third, this paper presents a novel application of the **sufficient statistic** approach to the problem of bankruptcy and security design. As stated by Chetty (2009), the central concept of the sufficient statistic approach is to derive formulas for the welfare consequences of policies that are functions of high-level elasticities rather than deep primitives. The optimal policies characterized under this approach are robust to a broad range of environments. Some recent successful applications of this approach are Diamond (1998) and Saez (2001) — hence the title analogy — on income taxation, Shimer and Werning (2007) and Chetty (2008) on unemployment insurance, Arkolakis, Costinot and Rodríguez-Clare (2012) on the welfare gains from trade liberalizations and Basu et al. (2013) on productivity and capital accumulation.

Fourth, this paper is related to the literature on **optimal contracting** and corporate finance, which studies borrower-lender relationships, bankruptcy, and default.[5] Costly state verification, limited commitment or enforcement, moral hazard, adverse selection, and secret cash-flow manipulation are features studied in static and dynamic environments within this extensive literature. Instead of fully optimizing on the shape of the contracts given primitives, in this paper I take the set of contracts available as given, while optimizing on the level of the bankruptcy exemption.

Fifth, the results of this paper also relate to the work in **security design** motivated by risk sharing, compactly

[3] I do not refer to the vast legal literature on bankruptcy; see Hermalin, Katz and Craswell (2007) for a survey.

[4] It is surprising that, despite acknowledging — see section 7 of their paper — that the optimal penalty associated with default is neither zero nor infinite, Dubey, Geanakoplos and Shubik (2005) do not characterize it formally.

[5] Townsend (1979), Diamond (1984), Gale and Hellwig (1985), Bolton and Scharfstein (1990), Bizer and DeMarzo (1992), Hart and Moore (1994, 1998, 1995), Krasa and Villamil (2000), Albuquerque and Hopenhayn (2004), Jappelli, Pagano and Bianco (2005), Rampini (2005), Bisin and Rampini (2006), Hopenhayn and Werning (2008), Grochulski (2010), Gale and Gottardi (2011) are examples of this vast literature. Bolton and Dewatripont (2005), Freixas and Rochet (2008) and Tirole (2005) are textbook references.

summarized in Allen and Gale (1994) and Duffie and Rahi (1995). This literature starts by allowing for some form of market incompleteness and then asks the question of which securities should be introduced in the market to improve risk sharing and welfare. The optimal bankruptcy exemption in this paper solves a restricted optimal security design problem, under a set of particular behavioral (e.g., risk averse borrowers and risk neutral lenders) and environmental (e.g., restricted set of contracts traded) constraints.

Finally, a dynamic version of the environment used in this paper — following Eaton and Gersovitz (1981) —, has become the workhorse model to understand **sovereign default**. If the ex-post penalties imposed on sovereigns that default were enforceable, the results of this paper would also apply to the international context with minor modifications. Formally closer to this paper, Bolton and Jeanne (2007, 2009) use a two-period model of defaultable credit to study the optimal number of creditors and the ability to refinance. See Aguiar and Amador (2013) for a recent survey of the sovereign default literature.

Outline Section 2 lays out the baseline model and analyzes its positive implications and section 3 executes the normative analysis, solving for the optimal bankruptcy exemption. Section 4 calibrates the theoretical formulas to US data and section 5 analyzes multiple extensions. Section 6 concludes. All proofs and derivations are in the appendix.

2 Baseline model

This section presents the main results in a stylized model. Section 5 extends the results in many dimensions. First, I characterize the equilibrium behavior of borrowers and lenders for a given level of the bankruptcy exemption. Subsequently, I characterize the welfare maximizing bankruptcy exemption from an ex-ante perspective.

2.1 Environment

Time is discrete, there are two dates $t = 0, 1$ and there is a unit measure of borrowers and a unit measure of lenders. Because borrowers are risk averse, the results of this paper are more applicable to household borrowing, rather than to corporate borrowing.[6]

Borrowers Borrowers are risk averse and maximize expected utility of consumption with a rate of time preference $\beta > 0$. Their flow utility $U(C)$ satisfies standard regularity conditions: $U'(\cdot) > 0$, $U''(\cdot) < 0$ and $\lim_{C\to 0} U'(C) = \infty$. Unlike many papers in the optimal contracting literature, which assume risk neutrality to simplify the contract design process, the results of this paper exploit interior optimality conditions. Hence, borrowers maximize:

$$\max_{C_0, \{C_1\}_{y_1}, B_0, \{\xi\}_{y_1}} U(C_0) + \beta \mathbb{E}[V(C_1)],$$

where $V(C_1) = \max_{\xi \in \{0,1\}} \left\{ \xi U\left(C_1^{\mathcal{D}}\right) + (1-\xi) U\left(C_1^{\mathcal{N}}\right) \right\}$ and ξ is an indicator for default for every realization y_1. If a borrower decides to repay, $\xi = 0$, while if he decides to default, $\xi = 1$. $C_1^{\mathcal{D}}$ denotes consumption for a borrower who defaults and $C_1^{\mathcal{N}}$ denotes consumption for a borrower who repays.

There is a single consumption good (dollar) in this economy, which serves as numeraire.[7] Every borrower is endowed with y_0 units of the consumption good at $t = 0$. At $t = 1$, every borrower receives a stochastic endowment of y_1 units of the consumption good, whose distribution follows a cdf $F(\cdot)$ with support in $\left[\underline{y_1}, \overline{y_1}\right]$, where $\underline{y_1} \geq 0$ and

[6]There is scope to adapt the results of this paper to the corporate context, in which firms — often assumed risk neutral — engage in risk management with imperfect instruments and become endogenously risk averse, along the lines of Froot, Scharfstein and Stein (1993) and Rampini and Viswanathan (2010).

[7]Allowing for positive inflation is straightforward

$\overline{y_1}$ could be infinite. The realizations of y_1 are iid among borrowers. Therefore, under a law of large numbers, there is no aggregate risk in this economy. All variables are observable by the parties.

Borrowers can trade a single noncontingent contract, i.e., a debt contract. That is, borrowers issue debt with face value B_0, due at $t=1$, and receive from lenders $q_0(B_0,m)B_0$ units of the consumption good at $t=0$. Hence, the gross interest rate faced by borrowers can be defined as $1+r\equiv\frac{1}{q_0}$. When needed, I denote by $\tilde{r}$ the logarithmic interest rate, i.e., $\tilde{r}\equiv\log(1+r)$. In the baseline model, borrowers take into account that the interest rate at which they are able to borrow depends on the amount of debt they take. Hence, the budget constraint at $t=0$ for borrowers is:

$$C_0 = y_0 + q_0(B_0,m)B_0,$$

where the unit price of debt taken $q_0(B_0,m)$ is a function of B_0 and m, as described below.

At $t=1$, once y_1 is realized, borrowers can decide to repay the amount owed B_0 or to default.[8] If they default, they consume $C_1^{\mathcal{D}} = \min\{y_1,m\}$, that is, they keep the bankruptcy exemption of m units of the consumption good, unless m is larger than y_1, in which case they only keep y_1 units. Any positive remainder y_1-m is seized from borrowers and transferred to lenders, although lenders only receive a fraction $\delta\in[0,1)$ of the transferred resources. This loss captures the resource costs associated to the bankruptcy procedure. I assume throughout that there is no ex-post renegotiation of the terms of the contract.

The exemption m takes a value in the interval $[\underline{m},\overline{m}]$. Because the bankruptcy procedure cannot rely on external funds, $\overline{m}\leq\overline{y_1}$. To simplify the exposition, I further restrict m to be greater than the lowest realization of y_1, that is, $\underline{m}>\underline{y_1}$.[9] Therefore, for a given realization of y_1, the budget constraints at $t=1$ when a borrower chooses to repay and when it chooses to default are, respectively:

$$\begin{aligned} C_1^{\mathcal{N}} &= y_1 - B_0 \\ C_1^{\mathcal{D}} &= \min\{y_1,m\} \end{aligned}$$

I assume that borrowers' rate of time preference β, initial endowment y_0 and distribution of future endowments $F(\cdot)$ are such that borrowers borrow in equilibrium, that is, $B_0>0$. The appendix provides an exact sufficient condition.

Lenders Given the exemption level and borrowers default behavior, I assume that lenders supply credit to borrowers according to an interest rate schedule $q_0(B_0,m)$, which depends on the face value of the debt B_0 and the exemption level m. As long as lenders make zero profit and $q_0(B_0,m)$ is well behaved, the main results of this paper are valid without further specifying the lenders' behavior.

That said, to be concrete, I use as a running example the case in which lenders are risk neutral, perfectly competitive and require a given rate of return $1+r^*$, which can differ from the borrowers' rate of time preference β. In that case, $q_0(B_0,m)$ takes the form:

$$q_0(B_0,m) = \frac{\delta\int_{\mathcal{D}}\frac{\max\{y_1-m,0\}}{B_0}dF(y_1) + \int_{\mathcal{N}}dF(y_1)}{1+r^*},$$

where $\mathcal{D}$ represents the default region and $\mathcal{N}$ the no default region, which are determined in equilibrium. See the appendix for how to handle risk averse competitive lenders.

[8] I use the words bankruptcy and default as synonyms in this paper. See White (2011) and Herkenhoff (2012) for how borrowers may default on their obligations without entering in bankruptcy. Allowing borrowers not to repay without declaring bankruptcy does not change the optimal exemption formula, as long as they behave optimally. See the discussion in section 5.

[9] The results extend naturally to the case in which $m<\underline{y_1}$, but the analysis becomes more tedious, since multiple cases have to be analyzed then. Those results are available under request.

Equilibrium definition/regularity conditions An equilibrium, for a given level of exemption m, is defined as a set of consumption allocations C_0, $\{C_1\}_{y_1}$, default decisions $\{\xi\}_{y_1}$, amount of debt issued B_0 and price q_0 such that borrowers default and borrow optimally internalizing that their choices affect the price of the debt and lenders offer an interest rate schedule $q_0(B_0, m)$ while making zero profit.

As usual in problems with continuous distributions of shocks, convexity is in general not guaranteed. To ease the exposition throughout, I work under the assumption that the borrowers' problem is convex, so first-order conditions are necessary and sufficient to characterize the optimum. I further assume that borrowers indirect utility $W(m)$ — defined in equation (7) — is also convex in m. The appendix provides sufficient conditions for convexity and the online appendix shows numerically that the model is in practice well behaved.

After understanding how the level of the bankruptcy exemption m affects ex-ante welfare, I solve for the optimal ex-ante exemption. The optimal ex-ante exemption must be interpreted as an optimal commitment device chosen by borrowers — given the set of contracts available — to provide insurance and enforce ex-post repayments, which allows them to borrow ex-ante.

The following remarks highlight the two key features of the economic environment.

Remark. (Given set of contracts/assets) The key friction in this paper is that the set of contracts/assets available to borrowers is given: borrowers do not choose the shape of the traded contracts optimally. This paper does not take a stand on why borrowers do not use a contract or a set of contracts that deliver the first-best outcome of perfect insurance. Transaction costs, hysteresis in contracting, informational frictions or bounded rationality are plausible explanations. For instance, the rich literature on financial contracting referenced above contains elaborated justifications for why debt contracts are prevalent. Taking as given the set of traded contracts/assets, as in the classic research in general equilibrium theory, this paper determines the optimal degree of leniency in bankruptcy. The possibility of bankruptcy introduces new contingencies which improve risk sharing, a mechanism originally pointed out by Zame (1993) and Dubey, Geanakoplos and Shubik (2005).

Remark. (Constant exemption level) In the baseline model, an exemption level that does not depend on the realization of y_1 is optimal even when a nonlinear bankruptcy scheme that depends on y_1 is feasible, because the first-best outcome in this economy involves risk neutral lenders providing a flat consumption profile (full insurance) to risk averse borrowers at $t = 1$. A constant exemption level is not constrained optimal in some of the extensions: I'll point that out whenever that is the case. In those situations, I am solving a second best problem with imperfect instruments. Then, I justify the choice of a constant exemption with the fact that it is the one used in practice.

2.2 Equilibrium characterization

First, I characterize the optimal ex-post default decision by borrowers and then solve for the optimal ex-ante choice of B_0. Finally, I characterize the equilibrium pricing schedules offered by competitive lenders.

Borrowers' default decision At $t = 1$, given his ex-ante choice of B_0, a borrower solves the problem:

$$\max_{\xi \in \{0,1\}} \left\{ \xi U\left(C_1^{\mathcal{D}}\right) + (1-\xi) U\left(C_1^{\mathcal{N}}\right) \right\}$$

Because flow utility is strictly monotonic, this problem is equivalent to $\max_{\{\xi\}} \left\{C_1^{\mathcal{D}}, C_1^{\mathcal{N}}\right\}$. The optimal default decision is given by a threshold on the realization of y_1. When y_1 is high, it is optimal not to default, but when y_1 is sufficiently low, it is preferable to default than to repay the loan. Figure 1 shows graphically the default problem at $t = 1$. The upper envelope of the default and repayment options determines the optimal consumption choice given B_0. The 45 degree line is shown for reference.

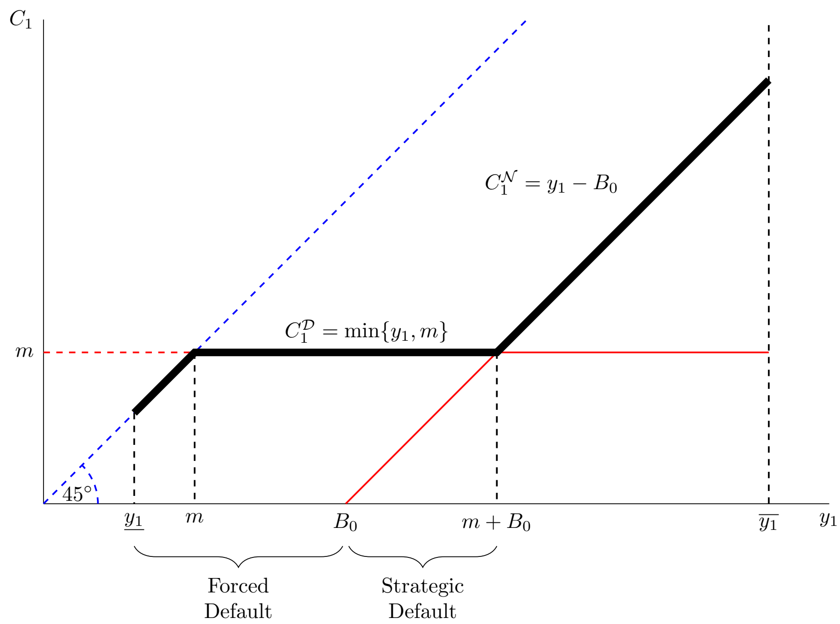

Figure 1: Optimal default decision given B_0

Formally, the optimal default decision is:

$$\begin{cases} \xi = 1, & \text{if } y_1 < m + B_0 \qquad \text{Default} \\ \xi = 0, & \text{if } y_1 \geq m + B_0 \qquad \text{No Default} \end{cases}$$

The default threshold is determined by the indifference condition between the amount to be repaid B_0 and the amount transferred to lenders $y_1 - m$. I assume that an indifferent borrower decides not to default. Given the default decision, the fraction of borrowers that defaults in equilibrium is deterministic and given by $F(m + B_0)$.

This model incorporates forced default, which occurs when a borrower does not have enough resources to fully pay back its debt (it occurs when $B_0 > y_1$), and strategic default, which happens when borrowers have enough resources to fully repay but they decide not to do it (it occurs when $m + B_0 > y_1 > B_0$). This distinction, which often plays a prominent role in discussions about bankruptcy exemptions, is not relevant for the results of this paper.

Borrowers' optimal choice of B_0 Given their optimal ex-post default decision and taking into account the interest rate schedule offered by lenders, borrowers optimally choose how much to borrow. The problem solved by borrowers at $t = 0$, for a given exemption m, is:

$$\max_{B_0} U(y_0 + q_0(B_0, m) B_0) + \beta \left[\int_{\underline{y_1}}^{m} U(y_1) dF(y_1) + \int_{m}^{m+B_0} U(m) dF(y_1) + \int_{m+B_0}^{\overline{y_1}} U(y_1 - B_0) dF(y_1) \right] \tag{2}$$

Under the assumed regularity conditions, the problem solved by borrowers is convex in B_0, so the following first-order condition fully characterizes the solution to (2):

$$U'(C_0) \left[q_0(B_0, m) + \frac{\partial q_0(B_0, m)}{\partial B_0} B_0 \right] = \beta \int_{m+B_0}^{\overline{y_1}} U'(y_1 - B_0) dF(y_1) \tag{3}$$

Intuitively, the left hand side of (3) represents the marginal benefit at $t = 0$ of increasing the face value of the debt by a dollar. This marginal benefit is given by q_0, the amount raised at $t = 0$ per dollar promised at $t = 1$, corrected by how the induced interest rate increase affects the total amount borrowed B_0. Borrowers value this change at their marginal

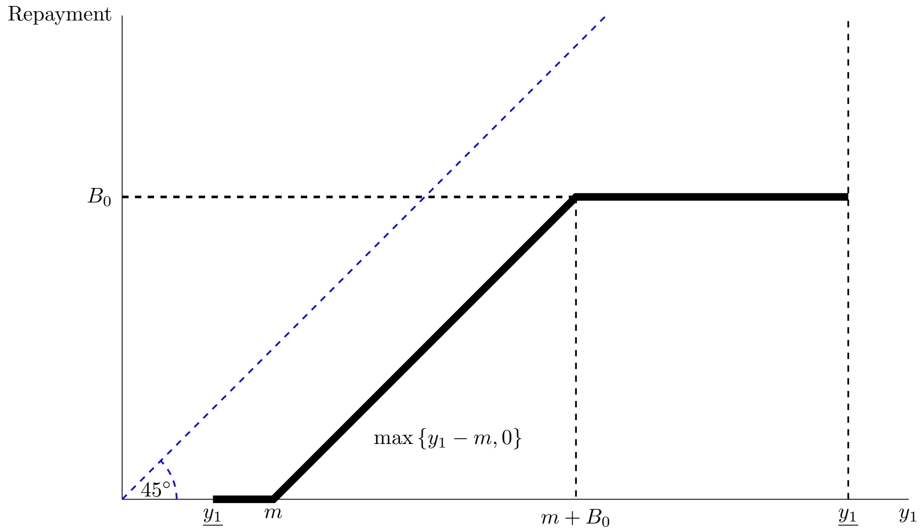

Figure 2: Repayment to lenders given B_0

utility $U'(C_0)$. The right hand side of (3) represents the marginal cost of repaying the debt, given by the marginal utility when the payment is due. This cost is only paid in those states in which borrowers do not default; debt imposes no effective costs on borrowers in those states in which it does not have to be repaid.

Equation (3) allows to characterize analytically how the total amount of credit changes in equilibrium with the level of the bankruptcy exemption m. Unsurprisingly, the sign of $\frac{dB_0}{dm}$ is ambiguous. Formally, $\frac{dB_0}{dm}$ has the following sign:

$$\text{sign}\left(\frac{dB_0}{dm}\right) = \text{sign}\left(U''(C_0)\frac{\partial q_0}{\partial m}B_0\left[q_0 + \frac{\partial q_0}{\partial B_0}B_0\right] + U'(C_0)\left[\frac{\partial q_0}{\partial m} + \frac{\partial^2 q_0}{\partial B_0 \partial m}B_0\right] + \beta U'(m) f(m+B_0)\right)$$

There are three distinct effects that determine the sign of $\frac{dB_0}{dm}$. First, all else equal, an increase in m reduces q_0, which reduces household consumption C_0 and increases the value of $U'(C_0)$; this effect induces borrowers to borrow more. Second, all else equal, an increase in m varies the unit amount that can be raised at $t = 0$, given by the direct price effect $\frac{\partial q_0(B_0,m)}{\partial m}$ and the change in the derivative of the interest rate schedule $\frac{\partial^2 q_0(B_0,m)}{\partial B_0 \partial m}B_0$; this effect is in general ambiguous. Third, all else equal, an increase in m reduces the marginal cost of borrowing because the default region widens, which reduces the likelihood of having to pay back the debt. This effect induces borrowers to borrow more. This term captures the idea that borrowers' decide to borrow more ex-ante anticipating not having to pay back their debts, an effect that is often described as moral hazard.

The empirical literature on this issue finds that high exemptions are often associated with high levels of borrowing. Numerical solutions of the model with standard parametrizations find that B_0 can increase or decrease with m. That said, it is not necessary to take a stance on whether borrowers borrow more or less when exemptions change to understand the effects on welfare of varying bankruptcy exemptions, as long as they choose how much to borrow optimally.

Lenders' interest rate schedule When lenders are risk neutral and perfectly competitive, given borrowers' default decision, they offer the following interest rate schedule for given levels of B_0 and m:

$$q_0(B_0, m) = \frac{\delta \int_m^{m+B_0} \frac{y_1 - m}{B_0} dF(y_1) + \int_{m+B_0}^{\overline{y_1}} dF(y_1)}{1 + r^*} \tag{4}$$

Figure 2 graphically shows the repayment to lenders. The upper envelope between $\max\{y_1 - m, 0\}$ and B_0 represents the effective repayment to lenders. The credit spread is positive, that is $r - r^* > 0$, to account for the possibility of default, and approximately equal to the expected unit loss for lenders, because of risk neutrality.

Two properties of the interest rate schedule are important for the analysis. First, the interest rate schedule decreases with the amount of debt B_0. Second, the interest rate schedule decreases with the level of the bankruptcy exemption. Formally (the exact expressions are in the appendix):

$$\frac{\partial q_0(B_0, m)}{\partial B_0} < 0 \qquad \text{and} \qquad \frac{\partial q_0(B_0, m)}{\partial m} < 0 \tag{5}$$

Intuitively, for a given level of m, the required spread increases with the amount of credit issued. This occurs for two reasons. First, the per unit fraction of liabilities recovered by lenders in default states decreases with the total amount of credit. Second, because the default region widens, more resources are lost as bankruptcy costs. Also, for a given level of B_0, the required spread increases with the level of the bankruptcy exemption. There are again two reasons for this. First, the recovery rate for lenders in default states decreases with the level of the exemption, since borrowers get to keep a higher exemption. Second, because the default region widens, more resources are lost as bankruptcy costs.

We can formally express the equilibrium changes in interest rates induced by changing the bankruptcy exemption in the following way:

$$\frac{dq_0(B_0, m)}{dm} = \frac{\partial q_0(B_0, m)}{\partial B_0} \frac{dB_0}{dm} + \frac{\partial q_0(B_0, m)}{\partial m} \tag{6}$$

The last term in equation (6) is negative, but depending on the sign of $\frac{dB_0}{dm}$, $\frac{dq_0(B_0,m)}{dm}$ can take any sign. As long as borrowing increases with m, that is, $\frac{dB_0}{dm} > 0$, observed interest rates increase in equilibrium, that is, $\frac{dq_0(B_0,m)}{dm} < 0$.

3 Optimal bankruptcy exemption m^*

After characterizing the equilibrium for a given exemption m, I now study the problem of determining the welfare maximizing exemption m^*. Because lenders make zero profit in equilibrium, maximizing borrowers' indirect utility is equivalent to maximizing social welfare in this economy. First, I study how varying the bankruptcy exemption affects ex-ante welfare. Then, I solve for the optimal exemption.

I denote the indirect utility of borrowers, as a function of m, by $W(m)$:

$$W(m) = \begin{array}{c} U(y_0 + q_0(B_0(m), m) B_0(m)) + \\ +\beta \left[\int_{\underline{y_1}}^{m} U(y_1) dF(y_1) + \int_{m}^{m+B_0(m)} U(m) dF(y_1) + \int_{m+B_0(m)}^{\overline{y_1}} U(y_1 - B_0(m)) dF(y_1) \right], \end{array} \tag{7}$$

where $B_0(m)$ is given by the solution to equation (3) and $q_0(B_0(m), m)$ is given by:

$$q_0(B_0(m), m) = \frac{\delta \int_{m}^{m+B_0(m)} \frac{y_1 - m}{B_0(m)} dF(y_1) + \int_{m+B_0(m)}^{\overline{y_1}} dF(y_1)}{1 + r^*}$$

Propositions 1 and 2 present the main results of this paper.

Proposition 1. (Marginal effect of varying m on welfare)

a) The change in welfare induced by a marginal change in the bankruptcy exemption m is given by:

$$\frac{dW}{dm} = U'(C_0) \frac{\partial q_0(B_0(m), m)}{\partial m} B_0 + \int_{m}^{m+B_0} \beta U'\left(C_1^{\mathcal{D}}\right) dF(y_1) \tag{8}$$

b) The change in welfare induced by a marginal change in the bankruptcy exemption m, expressed as a fraction of $t = 0$ consumption, is given by:

$$\frac{\frac{dW}{dm}}{U'(C_0)\,C_0} = -\Lambda \varepsilon_{\tilde{r},m} + \frac{1}{m} \frac{\Pi_m \left\{C_1^{\mathcal{D}}\right\}}{C_0}, \tag{9}$$

where $\Lambda \equiv \frac{q_0 B_0}{y_0 + q_0 B_0}$ is a measure of borrowers' leverage, $\varepsilon_{\tilde{r},m} \equiv \frac{\partial \log(1+r)}{\partial m} = -\frac{\frac{\partial q_0(B_0(m),m)}{q_0}}{\partial m}$ denotes the semi-elasticity of the interest rate schedule offered by lenders with respect to the level of the exemption and $\frac{\Pi_m\{C_1^{\mathcal{D}}\}}{C_0} \equiv \int_m^{m+B_0} \frac{C_1^{\mathcal{D}}}{C_0} \frac{\beta U'(C_1^{\mathcal{D}})}{U'(C_0)} dF(y_1)$ is the price-consumption ratio from the borrowers' perspective of a claim that pays the marginal value of increasing the bankruptcy exemption by one unit.

Proposition 1 characterizes the effect on social welfare of a marginal change in the bankruptcy exemption. The derivation of equation (8) crucially exploits the fact that borrowers borrow and decide when to default optimally. On the one hand, a marginal increase in the exemption m makes borrowing more expensive through a reduction in the price on the debt issued $\frac{\partial q_0}{\partial m}$, which affects the total amount amount of debt outstanding B_0. This change is valued by borrowers according to their $t = 0$ marginal utility $U'(C_0)$. This increase in borrowing costs is the marginal cost of a more lenient bankruptcy procedure. On the other hand, a marginal increase in the exemption m increases the resources that borrowers can keep when they default while claiming the full exemption. Averaging over the pertinent realizations of y_1 and weighting this gain by the marginal utility $\beta U'\left(C_1^{\mathcal{D}}\right)$ in those states, the marginal welfare gain of a more lenient bankruptcy procedure becomes $\beta \int_m^{m+B_0} U'\left(C_1^{\mathcal{D}}\right) dF(y_1)$.

Equation (9) expresses the change in welfare as a money-metric — dividing by $U'(C_0)$ — before normalizing by initial consumption C_0. The term Λ measures borrowers' leverage.[10] The term $\varepsilon_{\tilde{r},m}$ denotes the partial derivative of the interest rate schedule with respect to the bankruptcy exemption. Equation (5) guarantees that $\varepsilon_{\tilde{r},m}$ is strictly positive. $\Pi_m\left\{C_1^{\mathcal{D}}\right\}$ is the price from the borrower's perspective at $t = 0$ of a claim that pays borrowers' consumption only in default states in which borrowers claim the full exemption — those in which $y_1 > m$. $\frac{\Pi_m\{C_1^{\mathcal{D}}\}}{C_0}$ express this price in relative terms to current consumption C_0: this is a measure of the marginal benefit for borrowers of increased leniency. The ratio $\frac{\Pi_m\{C_1^{\mathcal{D}}\}}{C_0}$, which I refer to as the "price-consumption" ratio, is determined by the product of two terms. First, it can be high when the ratio of marginal utilities $\frac{U'(C_1^{\mathcal{D}})}{U'(C_0)}$ is high. Second, it can be high when consumption growth in those states $\frac{C_1^{\mathcal{D}}}{C_0}$ is also high. Both terms are in general related, and tightly linked when utility is CRRA or Epstein-Zin cases, as discussed below.

The optimal bankruptcy exemption m^* can be found as:

$$m^* = \arg\max_m W(m)$$

Under the assumed regularity conditions, the optimal bankruptcy exemption must be a solution to the equation $\frac{dW}{dm} = 0$.

Proposition 2. (Optimal bankruptcy exemption) *The optimal exemption m^* — expressed in units of the consumption good, i.e., dollars — is characterized by:*

$$m^* = \frac{\frac{\Pi_m\{C_1^{\mathcal{D}}\}}{C_0}}{\Lambda \varepsilon_{\tilde{r},m}}, \tag{10}$$

where $\Lambda \equiv \frac{q_0 B_0}{y_0 + q_0 B_0}$ is a measure of borrowers' leverage, $\varepsilon_{\tilde{r},m} \equiv \frac{\partial \log(1+r)}{\partial m} = \frac{\frac{\partial q_0(B_0,m)}{q_0}}{\partial m}$ denotes the semi-elasticity of the interest rate schedule offered by lenders with respect to the level of the exemption and $\frac{\Pi_m\{C_1^{\mathcal{D}}\}}{C_0} \equiv$

[10] Debt-to-equity ratios, that is, $L \equiv \frac{q_0 B_0}{y_0}$ are frequently used as measures of leverage. The variable Λ is a monotonic transformation of L: $\Lambda = \frac{1}{\frac{1}{L}+1}$.

$\int_m^{m+B_0} \frac{C_1^{\mathcal{D}}}{C_0} \frac{\beta U'\left(C_1^{\mathcal{D}}\right)}{U'(C_0)} dF\left(y_1\right)$ *is the price-consumption ratio from the borrowers' perspective of a claim that pays the marginal value of increasing the exemption by a unit.*

The expression for m^* optimally trades off the marginal benefit of increasing consumption in default states (numerator) against the marginal cost of restricting access to credit (denominator). When m^* is high, borrowers face higher interest rates, which makes borrowing less profitable, at the cost of improved insurance when declaring bankruptcy. A low m^* makes ex-ante borrowing less costly while making bankruptcy more painful.

A high value for m^* is optimal when Λ and $\varepsilon_{\tilde{r},m}$ are large. Intuitively, if interest rates schedules are very sensitive to increasing the bankruptcy exemption, making default more attractive by increasing m^* is very costly in terms of curtailed access to credit; this effect is modulated by the amount borrowed Λ. A low value for m^* is optimal when $\frac{\Pi_m\left\{C_1^{\mathcal{D}}\right\}}{C_0}$, the normalized welfare gain of a marginally higher exemption is large. Although equation (10) must hold at the optimum, it does not provide a characterization of m^* as a function of primitives, because all right hand side variables are endogenous.

CRRA utility To build further intuition, let's assume that borrowers have constant relative risk aversion utility (CRRA) preferences, that is, $U\left(C\right) = \frac{C^{1-\gamma}}{1-\gamma}$, where $\gamma \equiv -C\frac{U''(C)}{U'(C)}$. Assuming a particular utility specification only affects directly the price-consumption ratio $\frac{\Pi_m\left\{C_1^{\mathcal{D}}\right\}}{C_0}$. The marginal cost of increased leniency $\Lambda\varepsilon_{\tilde{r},m}$ does not depend directly on the utility function, only through the effects on B_0 and q_0.

We can thus write:

$$\frac{\Pi_m\left\{C_1^{\mathcal{D}}\right\}}{C_0} = \beta \int_m^{m+B_0} \left(\frac{C_1^{\mathcal{D}}}{C_0}\right)^{1-\gamma} dF\left(y_1\right)$$

Therefore, an increase in leniency m increases $C_1^{\mathcal{D}}$ and has both discount rate and cash flow effects. When the CRRA coefficient γ is greater than one, the discount rate effect $\left(\frac{C_1^{\mathcal{D}}}{C_0}\right)^{-\gamma}$ dominates — this is the standard parametrization, see Campbell (2003) — but, if γ is less than one the consumption growth term $\frac{C_1^{\mathcal{D}}}{C_0}$ dominates. With logarithmic utility ($\gamma = 1$) both effects exactly cancel out. We expect $t = 0$ consumption to be higher than the exemption level, that is $\frac{C_1^{\mathcal{D}}}{C_0} < 1$, so the marginal loss generated by a higher exemption is increasing in the risk aversion parameter γ.[11]

The forces that determine the price-consumption ratio are the same that determine the price-dividend (consumption-wealth) ratio in standard consumption based asset pricing models — see Campbell (2003) for a review. In the classic Lucas (1978) model, price-dividend ratios are constant and equal to the rate of time preference β for investors with logarithmic utility. That same result applies here with one modification: instead of β, the relevant price-consumption ratio becomes $\beta\pi_m$, because we are interested in the price-dividend ratio of a security that only pays in default states with positive recovery by lenders.

Logarithmic utility is an often used benchmark specification for preferences, the price-consumption ratio can be written as:

$$\frac{\Pi_m\left\{C_1^{\mathcal{D}}\right\}}{C_0} = \beta\pi_m,$$

where $\pi_m \equiv \int_m^{m+B_0} dF\left(y_1\right)$ is the unconditional probability that a borrower consumes the bankruptcy exemption. It can alternatively be written as: $\pi_m = \pi_D \cdot \pi_{m|D}$, the unconditional probability of default π_D times the conditional probability that a borrower claims the full exemption $\pi_{m|D}$. Hence, when borrowers have log utility and lenders always claim the full exemption, only the probability of default is needed to assess the marginal benefit of increasing the bankruptcy exemption. When borrowers' risk aversion is larger than unity and $m < C_0$, the logarithmic utility formula provides a lower bound for the optimal exemption that does not require to measure consumption.

[11] See the appendix for a formal argument.

Hence, when borrowers have logarithmic utility, the optimal bankruptcy exemption m^* can be written as:

$$m^* = \frac{\beta \pi_m}{\Lambda \varepsilon_{\tilde{r},m}}$$

In this baseline model, assuming that lenders are risk neutral, we can further rewrite the equation that characterizes the optimal exemption as:

$$m^* = \left(\frac{\beta (1+r^*)}{\Upsilon + \delta (1-\Upsilon)} \right)^{\frac{1}{\gamma}} C_0, \tag{11}$$

where $\Upsilon = \frac{f(m^*+B_0)B_0}{F(m^*+B_0)-F(m^*)}$ is a measure of curvature of the distribution $F(\cdot)$ that can take values in $[0,\infty]$. When $F(\cdot)$ is a uniform, $\Upsilon = 1$. Intuitively, the optimal exemption seeks to equalize consumption at $t=1$ in default states with consumption at $t=0$, although with a correction that depends on the term $\frac{\beta(1+r^*)}{\delta+(1-\delta)\Upsilon}$. Intuitively, all else constant, when $\beta(1+r^*) > 1$, it is optimal to consume more at period 1, which calls for lower exemptions. Similarly, when $\delta + (1-\delta)\Upsilon$, which measures $\frac{\partial \log(1+r)}{\partial m}$, is low, high exemptions are optimal. All else constant, there is no clear comparative static on the bankruptcy cost parameter δ. Two effects compete. On the one hand, a high δ (low bankruptcy costs) amplifies the sensitivity of interest rate schedules to exemption changes because more resources flow from borrowers to lenders in bankruptcy. On the other hand, a high δ dampens the sensitivity of interest rate schedules to exemption changes, because there is no loss associated to defaulting in more states. All these effects are modulated by borrowers' risk aversion: high risk aversion γ pushes towards $\frac{m^*}{C_0} \to 1$. Although equation (11) is helpful to provide intuition behind the sufficient statistics, I focus throughout this paper on equations like (9) and (10), since those hold more generally.

I conclude this section with two remarks.

Remark. (Sufficient statistics) Three observable variables: leverage, the sensitivity of interest rate schedules with respect to the exemption level and the price-consumption ratio, suffice to determine the optimal exemption, independently of the rest of the structure of the model. For log utility borrowers, the price-consumption ratio simplifies to the probability of default when borrowers claim always the full exemption. For instance, the distribution of income shocks or the level of interest rates only affect m^* through these sufficient statistics.

The logic behind these sufficient statistics is similar to the one behind the CAPM, in which the beta of an asset becomes sufficient to determine expected returns. It is also similar to the logic behind consumption based asset pricing models, in which the consumption process, independently of how it is generated, is sufficient to determine asset prices and expected returns.

Remark. (Security design interpretation) The key tradeoff in this paper can be interpreted as a security design problem. Assume that borrowers can choose between two securities, priced by the same set of lenders. They can issue either a noncontingent bond, or a bundle of that noncontingent bond with a put option (whose strike is determined by the optimal default decision). Given their income process, borrowers will prefer one of these two securities; the choice of m adjusts parametrically between both contracts and m^* selects the optimal traded security.

4 Calibration

Although the main contribution of this paper is theoretical, the upshot of the approach followed is that the key theoretical tradeoffs can be quantified by measuring a few observables. To show the applicability of my results in practice, I first calibrate the optimal exemption to US data and then study the magnitude of the welfare gains generated by marginal changes in the bankruptcy exemption.

Optimal exemption

Because it entails minimal informational requirements, I calibrate the optimal exemption for the baseline model assuming that borrowers have CRRA utility.[12] I have just showed that the optimal exemption for CRRA utility borrowers is given by:

$$m^* = \frac{\beta \pi_m \left(\frac{C_1^{\mathcal{D}}}{C_0}\right)^{1-\gamma}}{\Lambda \varepsilon_{\tilde{r},m}} \tag{12}$$

Therefore, four observable variables and two preference parameters, the rate of time preference β and the coefficient of relative risk aversion γ, need to be calibrated. I use a yearly calibration.

Calibrating Λ, β, and γ is straightforward. To calibrate Λ, I target the same average ratio of unsecured debt to personal disposable income as Livshits, MacGee and Tertilt (2007), which is $\frac{q_0 B_0}{y_0} = 8.4\%$. This value implies that $\Lambda = 0.0775$.[13] I use $\beta = 0.96$ as the annual rate of time preference: this is a standard choice. I adopt $\gamma = 10$ for the baseline calibration for risk aversion, but report the results for other levels of risk aversion.

Finding appropriate values for π_m , $C_1^{\mathcal{D}}/C_0$, and $\varepsilon_{\tilde{r},m}$ requires further work. First, π_m can be written as the product of the unconditional probability of default π_D with the probability of claiming the full bankruptcy exemption conditional on defaulting $\pi_{m|D}$. To determine π_D, I use the average probability of filling for chapter 7 bankruptcy, also from Livshits, MacGee and Tertilt (2007), which is 0.8%. This choice should raise no concerns. I determine $\pi_{m|D}$ by using the fact that roughly 90% of bankruptcies are filed as "no-asset" bankruptcies — see Lupica (2012). This implies that 10% of bankrupt borrowers fully claim their exemption.

Second, I use $C_1^{\mathcal{D}}/C_0 = 0.9$ for the change in consumption by bankrupt borrowers. A 10% reduction in consumption may be considered as a large change, but it is within the range of variation documented in Filer and Fisher (2005) using PSID data on changes in food consumption for bankrupt individuals. I purposefully pick a relatively large number for $C_1^{\mathcal{D}}/C_0$ to make m^* sensitive to varying risk aversion; otherwise, the results when $C_1^{\mathcal{D}}/C_0 \approx 1$ are nearly identical to those in the log utility case as long as γ is within a reasonable range.

Finally, calibrating the sensitivity of the interest rate schedule with respect to the exemption $\varepsilon_{\tilde{r},m}$ is not easy. Ideally, we would observe how interest rate schedules vary with changes in exemptions for the same borrower. No paper has been able to follow that approach, precisely because the required pricing data is sensitive for lenders. Instead, I use the average estimate from Gropp, Scholz and White (1997), who estimate the (equilibrium) effect on interest rates of changes in bankruptcy exemptions over time and across states. A value of $\varepsilon_{\tilde{r},m} = 2.5 \cdot 10^{-7}$ is within the reasonable range for the average response of interest rates in the population. This choice implies that increasing the bankruptcy exemption by a hundred thousand dollars increases the equilibrium interest rate by 250 basis points. Conceptually, this value corresponds to $-\frac{\frac{dq_0}{q_0}}{dm}$, as defined in equation (6), and not to $-\frac{\frac{\partial q_0}{q_0}}{\partial m}$, since it does not control for the change in interest rate caused by changes in the level of borrowing $\frac{dB_0}{dm}$, which is found to be positive in the data. Therefore, the chosen value for $\varepsilon_{\tilde{r},m}$ may be upward biased, which means that the optimal exemptions in table 2 could be seen as lower bounds for m^*. Alternatively, we can think that we are calibrating the model in which borrowers are

[12] As shown in section 5, allowing for Epstein-Zin preferences and assuming that the certainty equivalent of $t = 1$ consumption equals C_0 yields identical results.

[13] In actual economies, a substantial fraction of borrowing is collateralized. Under the assumption that collateralized borrowing is fully secured, which implies that the interest rate charged is not sensitive to m, equation (18) derived below implies that the denominator of the optimal exemption only has to account for the fraction of unsecured credit. If collateralized lending is not fully secured, the optimal exemption found in this section should be adjusted downwards. Also, the bulk of exemptions in the US are homestead exemptions. Hence, I'm implicitly assuming that borrowers' endowment at $t = 1$ is in the form of a house.

price takers. As shown in section 5, the equilibrium semielasticity is the correct one in that case.[14]

Table 1 summarizes the choices of parameters and variables used in the baseline calibration.

Parameter/Variable		Value	Parameter/Variable		Value
β	Rate of time preference	0.96	π_D	Probability of default	0.008
$C_1^{\mathcal{D}}/C_0$	Consumption change	0.9	$\pi_{m\|D}$	Probability of no-asset bankruptcy	0.1
Λ	Leverage	0.0775	$\varepsilon_{\tilde{r},m}$	Interest rate sensitivity	$2.5 \cdot 10^{-7}$

Table 1: Calibrated variables

Using equation (12), table 2 presents the optimal exemption m^* for different values of the risk aversion coefficient γ.

$\gamma = 1$ (log)	$\gamma = 5$	$\gamma = 10$	$\gamma = 20$	$\gamma = 50$
$39,638$	$60,416$	$\mathbf{102,314}$	$293,435$	$6,922,079$

Table 2: Optimal exemption m^* (measured in dollars)

This analysis concludes that the actual average bankruptcy exemption across US states, of roughly $70,000$ dollars, is very close to the optimal value implied by the model. The preferred calibration suggests that the optimal exemption should approximately be $100,000$ dollars, although the level of risk aversion has a dramatic effect on m^*. Given that $\varepsilon_{\tilde{r},m}$ could be even lower, the quantitative results of this paper suggest that there may be scope to increase welfare by raising exemptions. In general, this is because the existent empirical research has found low values for interest rate sensitivities.

On the size of welfare gains

The results regarding m^* show that the optimal exemption should be slightly larger than the average exemption across US states. Given that result, a natural question is how large are the welfare gains from changing m at the current level of exemptions. To answer this question, I focus again in the CRRA case and define $\sigma(m) \equiv \frac{\frac{dW}{dm}}{U'(C_0)C_0}$ as the welfare gain, expressed as a fraction of initial consumption, of increasing the bankruptcy exemption starting from a level m. The value of $\sigma(m)$, as shown in section 3, is given by:

$$\sigma(m) = -\Lambda \varepsilon_{\tilde{r},m} + \frac{1}{m} \beta \pi_m \left(\frac{C_1^{\mathcal{D}}}{C_0} \right)^{1-\gamma}$$

For the purposes of this exercise, I freely use $\sigma(m)$ to assess changes of any magnitude, given the difficulty of calibrating $\frac{dW}{dm}$ for all values of m and then integrating over those to actually compute welfare changes. Ideally, if we could obtain empirical counterparts of the different sufficient statistics for all values of m, we could calculate the exact welfare gain or loss induced by a nonmarginal change in m by integrating $\frac{dW}{dm}$ over the relevant range of exemptions. Similarly, we could find the fixed point that yields m^*. Hence, although equation (8) is exact for all values of the bankruptcy exemption m, the calibration becomes an approximation because I impose *constant* values for the sufficient statistics. Hence, the results are more precise for small interventions and should be interpreted otherwise

[14] It is hard to make a case for whether borrowers take interest rates as given or not for unsecured borrowing. A previous version of this paper used price taking as the baseline case.

with caution. The value of $\sigma(m)$ still determines whether a small change in exemptions is welfare improving or not. If $\sigma(m)$ is positive, this paper guarantees that a small increase in exemptions is welfare improving and viceversa.

Using the same set of parameters described in table 2, and starting from the average exemption of $m = 70,000$ dollars, I now show in table 3 the welfare gains associated to a 10,000 dollars increase in m.

$\gamma = 1$ (log)	$\gamma = 5$	$\gamma = 10$	$\gamma = 20$	$\gamma = 50$
-0.0084%	-0.0027%	**0.0089%**	0.062%	1.897%

Table 3: Welfare gain/loss (10,000 dollars change) measured as a fraction of initial consumption

Therefore, table 3 provides the welfare gains, measured as a percent of initial consumption C_0, from increasing the bankruptcy exemption m by 10,000 dollars when $m = 70,000$, for the same parametrization used to derive m^*. As expected, the magnitude of the welfare gain grows with the value of m^*, when $m^* > 70,000$ and viceversa. For instance, for the benchmark calibration, a 10,000 dollars increase in m increases welfare by 0.0089%. Increasing m by 30,000 dollars, which would move the economy from the average 70,000 until the optimum of 100,000, yields a welfare gain of roughly 0.03%; this measures the total welfare loss incurred by not having the optimal exemption.

5 Extensions

The goal of this section is to understand how departures from the baseline model affect the expression for the optimal exemption. In order to do that, I extend the baseline model in multiple dimensions. I address several natural extensions formally and discuss some others before concluding. To ease the exposition, I analyze every extension separately and omit equilibrium definitions and regularity conditions.

5.1 Endogenous income: elastic labor supply and effort choice in frictionless markets

In the baseline model, borrowers' income is exogenously determined. However, it is often argued that creditor friendly bankruptcy procedures reduce borrowers' welfare by distorting labor supply decisions. It is true that seizing a fraction of borrowers' labor income has a negative effect on labor supply. Similarly, large exemptions may distort ex-ante effort choices by borrowers. However, the optimal exemption formula does not change when effort and labor supply are affected by changes in exemptions, as long as those decisions are made optimally.

To capture these concerns, I modify the baseline model in two dimensions. First, I allow borrowers to choose labor supply in both periods. Second, I assume that the distribution of income $F(Y_1; a)$ is a differentiable function of a noncontractible effort choice a, made by borrowers at $t = 0$. This effort choice creates another form of moral hazard. Borrowers can work at given wages w_0 and w_1 — in the background, perfectly competitive/constant returns to scale firms provide labor demand curves. Borrowers' flow utility is now given by a well behaved function $U(C, N; a)$, where N denotes hours worked and a is the quantity of effort exerted. For now, I make no assumptions on the separability between consumption, leisure and effort. For simplicity, I abstract from limits on wage garnishments and assume that all labor income is transferred from borrowers to lenders in bankruptcy.

Borrowers now solve:

$$\max_{C_0, \{C_1\}_{y_1}, B_0, \{\xi\}_{y_1}, N_0, \{N_1\}_{y_1}, a} U(C_0, N_0; a) + \beta \mathbb{E}_a [V(C_1, N_1)]$$

$$\text{s.t.} \qquad C_0 = y_0 + w_0 N_0 + q_0 B_0; \quad C_1^{\mathcal{N}} = y_1 + w_1 N_1^{\mathcal{N}} - B_0; \quad C_1^{\mathcal{D}} = \min\{y_1, m\},$$

where $V(C_1, N_1) = \max_{\xi \in \{0,1\}} \left\{ \xi \max_{C_1^{\mathcal{D}}, N_1^{\mathcal{D}}} U\left(C_1^{\mathcal{D}}, N_1^{\mathcal{D}}\right) + (1-\xi) \max_{C_1^{\mathcal{N}}, N_1^{\mathcal{N}}} U\left(C_1^{\mathcal{N}}, N_1^{\mathcal{N}}\right) \right\}$. As in the baseline model, there are three regions depending on the realization of y_1. First, the no default region, denoted by $\mathcal{N}$. Second, the default region in which borrowers keep the full exemption, denoted by $\mathcal{D}_m$. Third, the default region in which borrowers do not exhaust the full exemption, denoted by $\mathcal{D}_y$. Borrowers decide not to work at all when bankrupt, because all their labor income can be garnished — a cap to wage garnishments would only reduce their labor supply partially, but the conclusions would remain unchanged.

The logic used to characterize the default region is identical to the baseline model. First, I define the $t = 1$ indirect utility of borrowers after choosing optimally consumption and labor supply as $\tilde{V}(y_1; B_0, w_1)$, that is:

$$\tilde{V}(y_1; B_0, w_1) \equiv \max U\left(C_1^{\mathcal{N}}, N_1^{\mathcal{N}}\right) \text{ s.t. } \quad C_1^{\mathcal{N}} = y_1 + w_1 N_1^{\mathcal{N}} - B_0$$

Given y_1, a static consumption-leisure choice characterizes borrowers' labor supply, that is:

$$w_1 \frac{\partial U}{\partial C}\left(C_1^{\mathcal{N}}, N_1^{\mathcal{N}}\right) = -\frac{\partial U}{\partial N}\left(C_1^{\mathcal{N}}, N_1^{\mathcal{N}}\right)$$

Second, the default region is characterized by a threshold $\tilde{y}_1$ such that:

$$U(m) = \tilde{V}(\tilde{y}_1; B_0, w_1)$$

When $y_1 \geq \tilde{y}_1$, it is optimal for a borrower to repay, but when $y_1 < \tilde{y}_1$, it is optimal to default — see figure A.1 in the appendix for a graphical representation. In this case, lenders pricing schedules also depend directly on borrowers effect choice a, that is, we have $q_0(B_0, m, a)$. Given the optimal default decision, borrowers' behavior is characterized by three additional optimality conditions. First, an Euler equation for borrowing:

$$\frac{\partial U}{\partial C}(C_0, N_0; a)\left[q_0 + \frac{\partial q_0(B_0, m, a)}{\partial B_0} B_0\right] = \beta \int_{\mathcal{N}} \frac{\partial U}{\partial C}\left(C_1^{\mathcal{N}}, N_1^{\mathcal{N}}\right) dF(y_1; a)$$

Second, an optimal effort choice:

$$\frac{\partial U}{\partial a}(C_0, N_0; a) + \frac{\partial U}{\partial C}(C_0, N_0; a) \frac{\partial q_0(B_0, m, a)}{\partial a} B_0 + \beta \int V(C_1, N_1) \frac{\partial f(y_1; a)}{\partial a} dy_1 = 0$$

Third, an optimal consumption-leisure choice at $t = 0$:

$$w_0 \frac{\partial U}{\partial C}(C_0, N_0; a) = -\frac{\partial U}{\partial N}(C_0, N_0; a)$$

From the optimality conditions, it is easy to show that changes in the exemption m modify both borrower's consumption, labor supply and effort choices. In particular, when m increases, borrowers' labor supply is lower because the default region, in which no labor is supplied, grows.

Proposition 3. (Endogenous income: frictionless markets)

a) *The marginal welfare change from varying the optimal exemption m when labor supply is endogenous and borrowers have an effort choice is given by:*

$$\frac{dW}{dm} = \frac{\partial U}{\partial C_0}(C_0, N_0; a) \frac{\partial q_0(B_0, m, a)}{\partial m} B_0 + \beta \int_{\mathcal{D}_m} \frac{\partial U}{\partial C_1}\left(C_1^{\mathcal{D}}, 0\right) dF(y_1; a) \tag{13}$$

b) The optimal exemption m^* when labor supply is endogenous and borrowers have an effort choice is given by:

$$m^* = \frac{\frac{\Pi_m\left\{C_1^{\mathcal{D}}\right\}}{C_0}}{\Lambda \varepsilon_{\tilde{r},m}} \tag{14}$$

where $\Lambda \equiv \frac{q_0 B_0}{y_0 + q_0 B_0}$, $\varepsilon_{\tilde{r},m} \equiv \frac{\partial \log(1+r)}{\partial m} = -\frac{\frac{\partial q_0(B_0,m,a)}{q_0}}{\partial m}$ *and* $\frac{\Pi_m\left\{C_1^{\mathcal{D}}\right\}}{C_0} \equiv \beta \int_{\mathcal{D}_m} \frac{C_1^{\mathcal{D}}}{C_0} \frac{\frac{\partial U}{\partial C_1}\left(C_1^{\mathcal{D}}, 0\right)}{\frac{\partial U}{\partial C_0}(C_0, N_0; a)} dF(y_1; a) \, dF(y_1)$.

The intuition behind proposition (14) is simple and powerful: as long as borrowers optimally choose their labor supply and effort, changes in these variables do not modify the optimal exemption formula — the same argument applies to any other static endogenous variable. If the utility of consumption is separable from the disutility of working and providing effort, equations (13) and (14) are identical to their counterparts in the baseline model. In general, the marginal benefit of the bankruptcy exemption depends on the values of N and a through its effect on the marginal utility of consumption. Although there are is a rich literature studying the separability properties of the utility function, e.g., Attanasio and Weber (1989) or Aguiar and Hurst (2007), separable utility of consumption is often seen as a reasonable benchmark.

This is an important takeaway of this paper: we only need to measure consumption to determine the welfare consequences of bankruptcy policies. Labor supply responses to changes in exemptions are irrelevant once the consumption response has been accounted for.[15]

5.2 Non-pecuniary utility loss

In the baseline model, borrowers ruthlessly default whenever the pecuniary benefits of doing so are greater than the pecuniary cost. Perhaps because of stigma or social pressure, it is often argued that borrowers may feel "bad" about not paying back their liabilities, even when defaulting seems beneficial using exclusively pecuniary considerations. Alternatively, lenders can hound bankrupt borrowers, which may also create a non-pecuniary utility loss. I now capture that possibility by allowing borrowers to have state dependent utility. In particular, I assume that the utility of a borrower who consumes C units of the consumption good in bankruptcy is given by $U(\phi C)$, where $\phi \in [0,1]$. As in the baseline mode, renegotiation remains unfeasible.

Borrowers now solve:

$$\max_{C_0, \{C_1\}_{y_1}, B_0, \{\xi\}_{y_1}} U(C_0) + \beta \mathbb{E}[V(C_1)],$$

where $V(C_1) = \max_{\xi \in \{0,1\}} \left\{\xi U\left(\phi C_1^{\mathcal{D}}\right) + (1-\xi) U\left(C_1^{\mathcal{N}}\right)\right\}$. The logic used to characterize the default region is identical to the baseline model. The optimal default decision is given by:

$$\begin{cases} \xi = 1, & \text{if } y_1 < \phi m + B_0 \qquad \text{Default} \\ \xi = 0, & \text{if } y_1 \geq \phi m + B_0 \qquad \text{No Default} \end{cases}$$

With the exception of the change in the default region, the expression that characterizes B_0 is analogous to the one in the baseline model, that is:

$$U'(C_0)\left[q_0 + \frac{\partial q_0(B_0, m)}{\partial B_0} B_0\right] = \beta \int_{\phi m + B_0}^{\overline{y_1}} U'(y_1 - B_0)\, dF(y_1)$$

Proposition 4. (Non-pecuniary utility loss)

a) *The marginal welfare change from varying the optimal exemption m when bankrupt borrowers experience a non-pecuniary utility loss is given by:*

$$\frac{dW}{dm} = U'(C_0)\frac{\partial q_0(B_0(m), m)}{\partial m} B_0 + \beta \int_m^{\phi m + B_0} \phi U'\left(\phi C_1^{\mathcal{D}}\right) dF(y_1)$$

b) The optimal exemption m^* when bankrupt borrowers experience a non-pecuniary utility loss is given by:

$$m^* = \frac{\frac{\Pi_m\left\{C_1^{\mathcal{D}}\right\}}{C_0}}{\Lambda \varepsilon_{\tilde{r},m}},$$

[15] Similar insights are widely used in the consumption based asset pricing literature. Marginal utility of consumption is the only information needed to price any asset when utility is separable.

where $\Lambda \equiv \frac{q_0 B_0}{y_0+q_0 B_0}$, $\varepsilon_{\tilde{r},m} \equiv \frac{\partial \log(1+r)}{\partial m} = -\frac{\frac{\partial q_0(B_0,m)}{q_0}}{\partial m}$ *and* $\frac{\Pi_m\left\{C_1^{\mathcal{D}}\right\}}{C_0} \equiv \beta \int_m^{\phi m+B_0} \frac{\phi C_1^{\mathcal{D}}}{C_0} \frac{U'\left(\phi C_1^{\mathcal{D}}\right)}{U'(C_0)} dF(y_1)$.

As expected, for the standard case of $\gamma > 1$, low values of ϕ push towards higher optimal exemptions because, all else constant, borrowers' marginal utility is higher when bankrupt. The presence of non-pecuniary costs of default modifies the default region and the expression for the price-consumption ratio, but the formula for the optimal bankruptcy exemption remains unchanged. This conclusion extends to any form of state dependent utility. State dependent utility only modifies m^* through the sufficient statistics identified in this paper.

5.3 Epstein-Zin utility

The results of the baseline model remain valid when borrowers have nonexpected utility. I analyze here the Epstein-Zin case to disentangle the effects of risk aversion versus intertemporal substitution. While risk aversion plays an important role in pricing the marginal benefit of a bankruptcy exemption increase, intertemporal substitution plays an important role shaping the sensitivity of credit demand to interest rates. All result can be easily extended to more general Kreps-Porteus preferences or other types of well behaved nonexpected utility preferences.

Borrowers' utility is now given by:

$$V_0 = \left[\left(1-\hat{\beta}\right) C_0^{1-\frac{1}{\psi}} + \hat{\beta}\left(\mathbb{E}\left[C_1^{1-\gamma}\right]\right)^{\frac{1-\frac{1}{\psi}}{1-\gamma}}\right]^{\frac{1}{1-\frac{1}{\psi}}},$$

where the parameter γ is the coefficient of relative risk aversion and ψ represents the elasticity of intertemporal substitution for a given nonstochastic consumption path. Given B_0, borrowers default decision is identical to the baseline model. Taking that into account, borrowers now solve:

$$\max_{B_0} \left[\begin{array}{c} \left(1-\hat{\beta}\right)(y_0+q_0 B_0)^{1-\frac{1}{\psi}} + \\ \hat{\beta}\left(\int_{\underline{y_1}}^{m} (y_1)^{1-\gamma} dF(y_1) + \int_m^{m+B_0} (m)^{1-\gamma} dF(y_1) + \int_{m+B_0}^{\overline{y_1}} (y_1-B_0)^{1-\gamma} dF(y_1)\right)^{\frac{1-\frac{1}{\psi}}{1-\gamma}} \end{array}\right]^{\frac{1}{1-\frac{1}{\psi}}}$$

Borrowers choose how much to borrow as in the baseline model. Their choice of B_0 is given by:

$$(C_0)^{-\frac{1}{\psi}}\left[q_0 + \frac{\partial q_0(B_0,m)}{\partial B_0} B_0\right] = \beta Q^{\gamma-\frac{1}{\psi}} \int_{m+B_0}^{\overline{y_1}} (y_1-B_0)^{-\gamma} dF(y_1), \tag{15}$$

where Q denotes the certainty equivalent of consumption at $t=1$ (the appendix contains the exact expression) and $\beta \equiv \frac{\hat{\beta}}{1-\hat{\beta}}$. It is not hard to show that the sensitivity of credit demand to interest rates is controlled by ψ (see the derivation in the appendix).

Proposition 5. (Epstein-Zin utility)

a) *The marginal welfare change from varying the optimal exemption m when borrowers have Epstein-Zin preferences is given by:*

$$\frac{dW}{dm} = V_0^{\frac{1}{\psi}}\left[\left(1-\hat{\beta}\right)(C_0)^{-\frac{1}{\psi}} \frac{\partial q_0(B_0(m),m)}{\partial m} B_0 + \hat{\beta} Q^{\frac{\gamma-\frac{1}{\psi}}{1-\gamma}} \int_m^{m+B_0} \left(C_1^{\mathcal{D}}\right)^{-\gamma} dF(y_1)\right]$$

b) The optimal exemption m^* *when borrowers have Epstein-Zin preferences is given by:*

$$m^* = \frac{\frac{\Pi_m\left\{C_1^{\mathcal{D}}\right\}}{C_0}}{\Lambda \varepsilon_{\tilde{r},m}} \tag{16}$$

where $\Lambda \equiv \frac{q_0 B_0}{y_0+q_0 B_0}$, $\varepsilon_{\tilde{r},m} \equiv \frac{\partial \log(1+r)}{\partial m} = -\frac{\frac{\partial q_0(B_0,m)}{q_0}}{\partial m}$ *and* $\frac{\Pi_m\left\{C_1^{\mathcal{D}}\right\}}{C_0} \equiv \left(\frac{Q}{C_0}\right)^{\gamma-\frac{1}{\psi}} \beta \int_m^{m+B_0} \left(\frac{C_1^{\mathcal{D}}}{C_0}\right)^{1-\gamma} dF(y_1)$.

Using a more general form of preferences only modifies the expression for the price-consumption ratio $\frac{\Pi_m\{C_1^{\mathcal{D}}\}}{C_0}$, which is now given by the product of two terms. The first term $\beta \int_m^{m+B_0} \left(\frac{C_1^{\mathcal{D}}}{C_0}\right)^{1-\gamma} dF(y_1)$ is the same as in the CRRA case studied above. The second term $\left(\frac{Q}{C_0}\right)^{\gamma-\frac{1}{\psi}}$ can be interpreted as a correction to the rate of time preference that captures a preference for early versus late resolution of uncertainty. Although Q is a complex object, equation (16) provides intuition on the optimal exemption. For instance, when $Q < C_0$, a situation in which borrowers face large consumption risks, high values of $\gamma - \frac{1}{\psi}$, which represent a preference for early resolution of uncertainty, make consumption at $t = 0$ more valuable, which calls for lower exemptions all else constant. Intuitively, borrowers enjoy more early consumption, so they value more cheaper access to credit ex-ante. As expected, when $\gamma = \frac{1}{\psi}$, the second term of the price-consumption ratio cancels out, recovering the CRRA formulas.

Intuitively, it is fair to say that the elasticity of intertemporal substitution is important to determine the demand for credit — because ψ controls directly the sensitivity of the demand for credit to interest rates — but risk aversion, through the classic CRRA term, and the preference for early versus later resolution of uncertainty, play an important role when assessing the welfare effects of varying bankruptcy exemptions.[16] More generally, changes in preferences only modify m^* through the sufficient statistics identified in this paper.

5.4 Multiple contracts

In the baseline model, borrowers can exclusively trade a single noncontingent contract. Now borrowers can use multiple contracts with general payoffs. Every contract $j = 1, \ldots, J$ has an arbitrary payoff scheme $z_j(y_1)$, which can take positive or negative values depending on the realization of y_1. I assume, without loss of generality, that B_{0j} and q_{0j} are positive for all contracts. Given these assumptions, the distinction between borrowers and lenders is now blurred, although I continue to use the same nomenclature for both groups of agents.

I make two assumptions about the behavior of lenders. First, if needed, lenders can fully commit to pay borrowers at $t = 1$. Second, lenders are able to observe borrowers' portfolio choices at $t = 0$, allowing the interest rate schedule offered for contract j to depend on the whole portfolio of a borrower, that is:

$$q_{0j}(B_{01}, \ldots, B_{0J}, m)$$

Borrowers' budget constraints now read as:

$$C_0 = y_0 + \sum_{j=1}^{J} q_{0j}(B_{01,\ldots}B_{0J}, m) B_{0j}; \qquad C_1^{\mathcal{N}} = y_1 - \sum_{j=1}^{J} z_j(y_1) B_{0j}; \qquad C_1^{\mathcal{D}} = \min\{y_1, m\}$$

The baseline model is a special case of this one when $J = 1$ and $z_1(y_1) = 1$. If the set of securities spans all realizations of y_1, this formulation also nests the complete markets benchmark.[17] Allowing for fully or partially secured contracts in this environment is straightforward.

The logic used to characterize the default region is identical to the baseline model. Borrowers default for those realizations of y_1 in which $C_1^{\mathcal{D}} > C_1^{\mathcal{N}}$. A set of equalities given by $\min\{y_1, m\} = y_1 - \sum_{j=1}^{J} z_j(y_1) B_{0j}$ defines the thresholds for three regions that depend on the realization of y_1. First, the no default region, denoted by $\mathcal{N}$. Second,

[16] These results relate to the findings of Tallarini (2000), who shows that the EIS determines quantity dynamics but risk aversion is the relevant parameter for asset valuations.

[17] The complete markets extension is more natural when $F(\cdot)$ has finite support. Starting from the complete markets benchmark, when all Arrow-Debreu contracts are available, it is easy to show that allowing for bankruptcy is welfare reducing. Intuitively, the contingent contracts for states in which borrowers default cease to be traded, because lenders would require an arbitrarily large interest rate, so $\frac{\partial \log(1+r_j)}{\partial m} \approx \infty$.

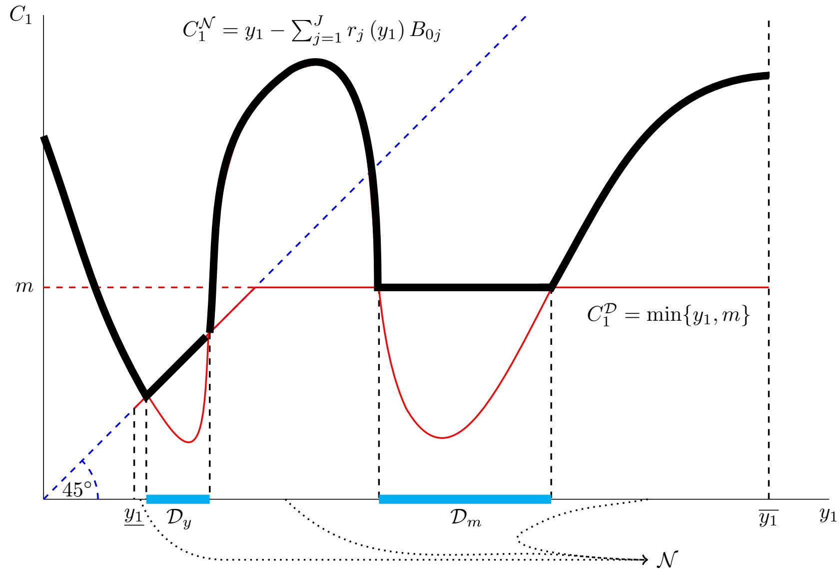

Figure 3: Optimal default decision with multiple contracts with arbitrary payoffs

the default region in which borrowers keep the full exemption, denoted by $\mathcal{D}_m$. Third, the default region in which borrowers do not exhaust the full exemption, denoted by $\mathcal{D}_y$. Figure 3, which is the counterpart of figure 1 in the baseline model, shows a possible scenario. This figure illustrates how the optimal exemption formula is invariant to whether borrowers default for high or low realizations of y_1. Both forced and strategic default also occur in this more general case too.

In the case of risk neutral competitive lenders, assuming that all claimants split borrowers' payments proportionally, interest rate schedules are:

$$q_{0j}(B_{01},\ldots,B_{0J},m) = \frac{\eta_j \int_{\mathcal{D}_m}\left(\frac{y_1-m}{B_{0j}}\right)dF(y_1) + \int_{\mathcal{N}} z(y_1)\,dF(y_1)}{1+r^*}, \quad \forall j,$$

where $\eta_j \equiv \frac{z_j(y_1)B_{0j}}{\sum_{j=1}^{J} z_j(y_1)B_{0j}}$ is the recovery rate in bankruptcy. Different assumptions about η_j do not change the main results of this paper.

Borrowers' behavior can be characterized by a set of J optimality conditions:

$$U'(C_0)\left[q_{0j} + \sum_{j=1}^{J}\frac{\partial q_{0j}(B_{01,\ldots}B_{0J},m)}{\partial B_{0j}}B_{0j}\right] = \beta \int_{\mathcal{N}} z_j(y_1)\,U'\left(C_1^{\mathcal{N}}\right)dF(y_1), \quad \forall j$$

A marginal change in B_{0j} affects the interest rate charged to all other contracts.

Proposition 6. (Multiple contracts with arbitrary payoffs)

a) *The marginal welfare change from varying the optimal exemption m when borrowers can use J contracts with arbitrary payoffs is given by:*

$$\frac{dW}{dm} = U'(C_0)\sum_{j=1}^{J}\frac{\partial q_{0j}(B_{01,\ldots}B_{0J},m)}{\partial m}B_{0j} + \beta\int_{D_m} U'\left(C_1^{\mathcal{D}}\right)dF(y_1) \tag{17}$$

b) The optimal exemption m^* when borrowers can use J contracts with arbitrary payoffs is given by:

$$m^* = \frac{\frac{\Pi_m\left\{C_1^{\mathcal{D}}\right\}}{C_0}}{\sum_{j=1}^{J}\Lambda_j \varepsilon_{\tilde{r}_j,m}} \tag{18}$$

where $\Lambda_j \equiv \frac{q_{0j}B_{0j}}{y_0+\sum_{j=1}^{J} q_{0j}B_{0j}}$, $\varepsilon_{\tilde{r}_j,m} \equiv \frac{\partial \log(1+r_j)}{\partial m} = -\frac{\frac{\partial q_0(B_{01},\ldots B_{0J},m)}{q_0}}{\partial m}$ *and* $\frac{\Pi_m\{C_1^{\mathcal{D}}\}}{C_0} \equiv \beta \int_{\mathcal{D}_m} \frac{C_1^{\mathcal{D}}}{C_0}\frac{U'(C_1^{\mathcal{D}})}{U'(C_0)} dF(y_1)$.

The main difference with respect to the baseline model is that the cost of having more lenient exemptions is now given by a weighted average of the interest rate sensitivities for all contracts traded with respect to the bankruptcy exemption. The weight given to a contract j is determined by the fraction of the amount raised by that contract as a function of total consumption. Intuitively, the welfare costs associated with higher rates are larger for those contracts which account for a larger fraction of borrowers portfolios. Allowing for multiple contracts with arbitrary payoffs only changes the benefits of varying exemptions through changes in the default region.

5.5 Heterogeneous borrowers

In the baseline model, borrowers are ex-ante symmetric. I now assume that borrowers are ex-ante heterogeneous across multiple dimensions: they may have different preferences, endowments, distribution of shocks, etc. I index borrowers by i and assume that they are distributed according to a well behaved distribution with cdf $G(i)$.

I make two assumptions. First, lenders are able to price each (group of) borrower(s) i separately. There are no cross-subsidies among different types of borrowers. Second, I restrict the analysis to a single constant bankruptcy exemption. Allowing for individual exemptions that condition on individual characteristics, it is optimal to replicate the outcome of the baseline model for each type of borrower. I rule out that possibility. In this environment, allowing for nonlinear exemptions is welfare improving.

When borrowers are ex-ante heterogeneous and only a single exemption can be chosen, I adopt a social welfare function that maximizes a weighted sum of individual utilities, Hence, social welfare W is now given by:

$$W = \int \lambda(i) W(i) \, dG(i),$$

where $\lambda(i)$ denote welfare weights.

Proposition 7. (Heterogeneous borrowers)

a) *The marginal welfare change from varying the optimal exemption m when when borrowers are ex-ante heterogeneous is given by:*

$$\frac{dW}{dm} = \int \lambda(i) \left[U_i'(C_{0i}) \frac{\partial q_{0i}(B_{0i},m)}{\partial m} B_{0i} + \beta_i \int_m^{m+B_{0i}} U_i'\left(C_{1i}^{\mathcal{D}}\right) dF_i(y_{1i}) \right] dG(i)$$

b) The optimal exemption m^* when borrowers are ex-ante heterogeneous is given by:

$$m^* = \frac{\int h_i \frac{\Pi_{m,i}\{C_{1i}^{\mathcal{D}}\}}{C_{0i}} dG(i)}{\int h_i \varepsilon_{\tilde{r}_i,m} \Lambda_i dG(i)}$$

where $h_i \equiv \lambda(i) U_i'(C_{0i})$, $\Lambda_i \equiv \frac{q_{0i}B_{0i}}{y_{0i}+q_{0i}B_{0i}}$, $\varepsilon_{\tilde{r}_i,m} \equiv \frac{\partial \log(1+r_i)}{\partial m} = -\frac{\frac{\partial q_{0i}(B_{0i},m)}{q_{0i}}}{\partial m}$ *and* $\frac{\Pi_{m,i}\{C_{1i}^{\mathcal{D}}\}}{C_0} \equiv \int_m^{m+B_{0i}} \frac{C_{1i}^{\mathcal{D}}}{C_{0i}} \frac{\beta_i U_i'(C_{1i}^{\mathcal{D}})}{U_i'(C_{0i})} dF_i(y_{1i})$.

As expected, the optimal exemption now contains a weighted average of marginal costs and benefits in the cross-section of borrowers. The weights h_i are a combination of the social welfare weight and the marginal utility of consumption. When $h_i = 1$ for all borrowers, social welfare is simply a sum of certainty equivalents — this is an often used benchmark in welfare analysis. In that case, the optimal m^* can written as:

$$m^* = \frac{\mathbb{E}_G\left[\frac{\Pi_{m,i}\{C_{1i}^{\mathcal{D}}\}}{C_{0i}}\right]}{\mathbb{E}_G\left[\varepsilon_{\tilde{r}_i,m}\Lambda_i\right]},$$

where $\mathbb{E}_G[\cdot]$ denotes cross-sectional averages. Therefore, the optimal exemption formula is identical to the one in the baseline model, but now using cross sectional averages for marginal benefits and costs of varying exemption levels. Using the product of cross-sectional averages in the denominator $\mathbb{E}_G[\varepsilon_{\tilde{r}_i,m}]\mathbb{E}_G[\Lambda_i]$ may bias m^* upwards if the covariance between interest rate schedule sensitivities and leverage ratios is positive, and negatively if it is negative. With heterogeneity, the variables that determine m^* remain unchanged, although now cross-sectional averages are needed.

5.6 Price taking borrowers

In the baseline model, borrowers internalize that the interest rate they pay depends on the amount they borrow. Alternatively, we can assume that borrowers can borrow at a constant rate, at least within a given range. I now explore that possibility. I adopt an equilibrium notion in which borrowers provide a loan demand, given interest rates, and lenders provide a load supply schedule, which determines a standard competitive equilibrium.[18]

The ex-post default decision is identical to the baseline model. The ex-ante behavior of borrowers is now captured by the following Euler equation:

$$U'(C_0)\,q_0 = \beta \int_{m+B_0}^{\overline{y_1}} U'(y_1 - B_0)\,dF(y_1) \tag{19}$$

This condition provides a demand for credit, which combined the supply schedule $q_0(B_0, m)$ characterizes the equilibrium. In this case, when borrowers' EIS is greater than unity ($\psi > 1$), it can be shown that the demand for credit increases with low interest rates and viceversa. It can also be shown that the equilibrium interest rates always increase with the exemption level, that is, $\frac{dq_0}{dm} < 0$. Unsurprisingly, the sign of $\frac{dB_0}{dm}$ remains indeterminate.

Proposition 8. (Price taking borrowers)

a) *The marginal welfare change from varying the optimal exemption m when when borrowers are price takers is given by:*

$$\frac{dW}{dm} = U'(C_0)\frac{dq_0}{dm}B_0 + \int_m^{m+B_0} \beta U'\left(C_1^{\mathcal{D}}\right)dF(y_1)$$

b) The optimal exemption m^* when borrowers are price takers is given by:

$$\frac{\frac{dW}{dm}}{U'(C_0)\,C_0} = -\Lambda\varepsilon_{\tilde{r},m} + \frac{1}{m}\frac{\Pi_m\left\{C_1^{\mathcal{D}}\right\}}{C_0},$$

where $\Lambda \equiv \frac{q_0 B_0}{y_0 + q_0 B_0}$, $\varepsilon_{\tilde{r},m} \equiv \frac{d\log(1+r)}{dm} = -\frac{\frac{dq_0}{q_0}}{dm}$ *and* $\frac{\Pi_m\left\{C_1^{\mathcal{D}}\right\}}{C_0} \equiv \beta\int_m^{m+B_0}\frac{C_1^{\mathcal{D}}}{C_0}\frac{U'\left(C_1^{\mathcal{D}}\right)}{U'(C_0)}dF(y_1)$.

There is an important difference between the results of the baseline model, in which borrowers internalize the effect of borrowing on interest rates, and the results in which borrowers are price takers: the relevant variable to measure the sensitivity of interest rates to changes in exemptions is now the full equilibrium response of interest rates, which also includes the change in interest rates induced by equilibrium changes in total borrowing. Equation (6) relates both terms in equilibrium. Intuitively, because price taking borrowers fail to internalize that higher borrowing increases interest rates, this effect must be accounted for when measuring welfare. This extension shows that the assumptions on the behavior of agents are crucial to derive the results in this paper.

[18] The problem of price takers borrowers has two local optima. One is characterized by an interior first-order condition. The second one entails borrowing as much as possible. When the feasible set of B_0 is unbounded above, borrowers credit demand is infinite at any given rate, preventing the existence of an equilibrium. I assume instead that borrowers judiciously set an upper bound on the total amount of credit so that the interior optimum is also global. Eaton and Gersovitz (1981) use a similar approach to guarantee an interior solution in a related environment. In practice, we observe that credit card lenders offer fixed rates and credit limits, which corresponds with this behavior. A previous version of this paper used the price taking case as the benchmark: detailed results for that case are available under request.

5.7 Bankruptcy exemptions contingent on aggregate risk

I now allow for aggregate shocks and ask how bankruptcy exemptions should vary depending on the state of economy. As long as agents use contracts that do not condition on aggregate shocks, it is optimal to set bankruptcy exemptions which vary with the realization of the aggregate state. Assume that borrowers income now takes the form $y_1 = A(\omega) + \tilde{y}_1(\omega)$. The realization of the aggregate shock is denoted by ω, which can take a finite number of values $\omega \in \Omega$ with probability $p(\omega)$. $A(\omega)$ and $\tilde{y}_1(\omega)$ denote aggregate and idiosyncratic endowment shocks. The distribution of idiosyncratic shocks $F_\omega(\cdot)$ can vary with ω, freeing the correlation patterns between aggregate and idiosyncratic shocks. I also allow for exemptions $m(\omega)$ contingent on the value of the aggregate shock.

Given ω, the logic used to characterize the default region is identical to the baseline model. Hence, borrowers solve:

$$\max_{B_0} U(y_0 + q_0 B_0) + \beta \sum_{\omega} p(\omega) \left[\begin{array}{c} \int_{\underline{y_1(\omega)}}^{m(\omega)} U(y_1(\omega))\, dF_\omega(y_1(\omega)) \\ + \int_{\underline{m(\omega)}}^{m(\omega)+B_0} U(m(\omega))\, dF_\omega(y_1(\omega)) \\ + \int_{m(\omega)+B_0}^{\overline{y_1(\omega)}} U(y_1 - B_0)\, dF_\omega(y_1(\omega)) \end{array} \right]$$

So they borrow according to:

$$U'(C_0)\left[q_0 + \frac{\partial q_0(B_0, \{m(\omega)\})}{\partial B_0} B_0\right] = \beta \sum_{\omega} p(\omega) \left[\int_{m(\omega)+B_0}^{\overline{y_1(\omega)}} U'(y_1 - B_0)\, dF_\omega(y_1(\omega))\right]$$

Under natural assumptions, for instance, when lenders are risk neutral, the interest rate schedule offered can be written as:

$$q_0(B_0, \{m(\omega)\}) = \sum_{\omega} p(\omega)\, q_0^{\omega}(B_0, m(\omega)),$$

where $q_0^{\omega}(B_0, m(\omega))$ is the interest rate schedule of a security that only pays in state ω. This allows to write $\frac{\partial q_0(B_0,\{m(\omega)\})}{\partial m(\omega)} = p(\omega) \frac{\partial q_0^{\omega}(B_0,m(\omega))}{\partial m(\omega)}$

Proposition 9. (Bankruptcy exemption contingent on aggregate risk)

a) *The marginal welfare change from varying the optimal state contingent exemption $m(\omega)$ is given by:*

$$\frac{\partial W(\{m(\omega)\})}{\partial m(\omega)} = p(\omega)\left[U'(C_0)\frac{\partial q_0^{\omega}(B_0, m(\omega))}{\partial m(\omega)} B_0 + \beta \int_{m(\omega)}^{m(\omega)+B_0} U'\left(C_1^{\mathcal{D}}\right) dF_\omega(y_1(\omega))\right], \qquad \forall \omega \tag{20}$$

b) The optimal state contingent exemptions $m^*(\omega)$ are given by:

$$m^*(\omega) = \frac{\frac{\Pi_{m(\omega)}\left\{C_1^{\mathcal{D}}\right\}}{C_0}}{\Lambda \varepsilon_{\tilde{r},m(\omega)}}, \quad \forall \omega$$

where $\Lambda \equiv \frac{q_0 B_0}{y_0 + q_0 B_0}$, $\varepsilon_{\tilde{r},m(\omega)} \equiv \frac{\partial \log(1+r^{\omega})}{\partial m(\omega)} = -\frac{\frac{\partial q_0^{\omega}(B_0,m(\omega))}{q_0^{\omega}}}{\partial m}$ *and* $\frac{\Pi_{m(\omega)}\left\{C_1^{\mathcal{D}}\right\}}{C_0} \equiv \int_{m(\omega)}^{m(\omega)+B_0} \frac{C_1^{\mathcal{D}}}{C_0} \frac{\beta U'\left(C_1^{\mathcal{D}}\right)}{U'(C_0)} dF_\omega(y_1(\omega))$.

The set of optimal state contingent exemptions is jointly determined by (20). The marginal gain/loss of varying the exemption in state ω is now proportional to the probability $p(\omega)$ of that state occurring. Conditional on the state occurring, the tradeoff is identical to the baseline model. The pro- or countercyclicality of exemptions depends on how price-consumption ratios and interest rate sensitivities vary across states ω. If, all else constant, bankruptcies increase in recessions, as we expect to occur, $\Pi_{m(\omega)}\left\{C_1^{\mathcal{D}}\right\}$ is high in aggregate downturns. This calls for countercyclical penalties under standard utility specifications, as long as interest rate sensitivities $\varepsilon_{\tilde{r},m(\omega)}$ do not vary significantly across aggregate states, the natural assumption for risk neutral lenders.[19]

[19]There is scope to explore how optimal exemptions must be determined in a coinsurance problem between which risk averse borrowers and lenders who face credit constraints (which makes them effectively risk averse).

5.8 Endogenous income: labor wedges and aggregate demand

The derivation of the optimal exemption formula in section 5.1 assumed no frictions on the production side of the economy. I now show how the existence of a welfare relevant labor wedge modifies the optimal exemption formula. I keep the same structure for aggregate shocks as in the previous extension, but return to the constant exemption benchmark. For simplicity, output is only produced at $t = 1$ with a constant returns to scale production function $Y = AN$, where A is a constant productivity parameter. At the first-best, we expect the condition $w_1(\omega) = A$ to hold. I assume instead that $w_1(\omega) = (1+\tau(\omega))A$, where $\tau(\omega)$ is the labor wedge in this economy, which can emerge under nominal rigidities and imperfect monetary policy.[20] There is one-to-one relation between the labor wedge and the output gap. When the labor wedge is positive, the economy is infra utilizing its resources, and viceversa when negative. Firms profits are returned back to households. As before, all labor income is garnished in bankruptcy, so borrowers opt for not working then.[21]

Given ω, the logic used to characterize the default region is identical to the baseline model. Hence, borrowers solve:

$$\max_{B_0} U(C_0) + \beta \sum_{\omega} p(\omega) \mathbb{E}\left[V(C_1(\omega), N_1(\omega)) \mid \omega\right],$$

$$\text{s.t.} \qquad C_0 = y_0 + q_0 B_0; \quad C_1^{\mathcal{N}}(\omega) = y_1 + w_1(\omega) N_1^{\mathcal{N}}(\omega) - B_0 + \pi_1(\omega); \quad C_1^{\mathcal{D}} = \min\{y_1, m\},$$

$V(C_1, N_1) = \max_{\xi \in \{0,1\}} \left\{\xi \max_{C_1^{\mathcal{D}}, N_1^{\mathcal{D}}} U\left(C_1^{\mathcal{D}}, N_1^{\mathcal{D}}\right) + (1-\xi) \max_{C_1^{\mathcal{N}}, N_1^{\mathcal{N}}} U\left(C_1^{\mathcal{N}}, N_1^{\mathcal{N}}\right)\right\}$. As before, conditional on the aggregate shock, three regions for default arise. Borrowers' labor supply and borrowing choices are characterized by:

$$U'(C_0)\left[q_0 + \frac{\partial q_0(B_0, m)}{\partial B_0} B_0\right] = \beta \sum_{\omega} p(\omega) \left[\int_{\mathcal{N}(\omega)} U\left(C_1^{\mathcal{N}}(\omega), N_1^{\mathcal{N}}(\omega)\right) dF_\omega(y_1(\omega))\right]$$

$$w_1(\omega) \frac{\partial U}{\partial C}\left(C_1^{\mathcal{N}}, N_1^{\mathcal{N}}\right) = -\frac{\partial U}{\partial N}\left(C_1^{\mathcal{N}}, N_1^{\mathcal{N}}\right)$$

This last condition, combined with the (effective) labor demand function $w_1(\omega) = (1+\tau(\omega))A$, pins down the equilibrium on the production side economy.

Proposition 10. (Endogenous income: labor wedges and aggregate demand)

a) *The marginal welfare change from varying the optimal exemption m when there is a nonzero labor wedge is given by:*

$$\frac{dW}{dm} = U'(C_0) \frac{\partial q_0}{\partial m} B_0 + \beta \sum_{\omega} p(\omega) \left[\begin{array}{c} \int_{\mathcal{D}_m(\omega)} U'\left(C_1^{\mathcal{D}}(\omega), 0\right) dF_\omega(y_1(\omega)) \\ + \int_{\mathcal{N}(\omega)} \frac{\partial U(C_1(\omega), N_1(\omega))}{\partial C_1(\omega)} dF_\omega(y_1(\omega)) \tau(\omega) \frac{dY(\omega)}{dm} \end{array}\right],$$

b) The optimal exemption m^* when there is a nonzero labor wedge is given by:

$$m^* = \frac{\frac{\Pi_m\left\{C_1^{\mathcal{D}}\right\}}{C_0} + \frac{\Pi_{\mathcal{N}}\{\tau(\omega)\}}{C_0}}{\Lambda \varepsilon_{\tilde{r},m}} \tag{21}$$

[20] A full-fledged New-Keynesian microfoundation is straightforward to develop, so I use the New-Keynesian terminology when describing the results. See, for instance, Gali, Gertler and Lopez-Salido (2007), Chari, Kehoe and McGrattan (2007) or Shimer (2009) to understand how labor wedges may arise. For the analysis of this section, I am assuming that labor wedges are caused by welfare reducing frictions and not by (efficient) preference shocks.

[21] There is scope to further analyze the interaction of bankruptcy exemptions and aggregate demand effects in a quantitative dynamic model. See Dobbie and Goldsmith-Pinkham (2014) for microeconometric work along those lines.

where $\Lambda \equiv \frac{q_0 B_0}{y_0+q_0 B_0}$, $\varepsilon_{\tilde{r},m} \equiv \frac{\partial \log(1+r)}{\partial m} = -\frac{\frac{\partial q_0}{q_0}}{\partial m}$, $\frac{\Pi_m\left\{C_1^{\mathcal{D}}\right\}}{C_0} = \beta \sum_\omega p(\omega) \left[\int_{\mathcal{D}_m(\omega)} \frac{C_1^{\mathcal{D}}(\omega)}{C_0} \frac{\frac{\partial U\left(C_1^{\mathcal{D}}(\omega),0\right)}{\partial C_1(\omega)}}{U'(C_0)} dF_\omega\left(y_1(\omega)\right)\right]$ *and* $\frac{\Pi_{\mathcal{N}}\left\{\tau(\omega)\frac{dY(\omega)}{\frac{dm}{m}}\right\}}{C_0} = \beta \sum_\omega p(\omega) \left[\int_{\mathcal{N}} \frac{\frac{\partial U(C_1(\omega),N_1(\omega))}{\partial C_1(\omega)}}{U'(C_0)} dF_\omega\left(y_1(\omega)\right) \frac{\tau(\omega)}{C_0} \frac{dY(\omega)}{\frac{dm}{m}}\right]$.

A new term $\frac{\Pi_{\mathcal{N}}\left\{\tau(\omega)\frac{dY(\omega)}{\frac{dm}{m}}\right\}}{C_0}$ that was not present in the baseline model emerges in the numerator of the optimal exemption formula. In general, high exemptions are welfare improving when $\mathbb{C}ov\left(\tau(\omega), \frac{dY(\omega)}{dm}\right) > 0$. For instance, if when there is a demand shortfall, that is, $\tau(\omega) > 0$, high bankruptcy exemptions boost aggregate demand, that is, $\frac{dY(\omega)}{dm} > 0$, there is a additional rational to increase bankruptcy exemptions, since they provided an additional macroeconomic hedge. Using the language of Farhi and Werning (2014), the second term in the numerator of equation (21) can be interpreted as a macroprudential correction to the optimal exemption.

5.9 Dynamics

Finally, I extend the results to a dynamic environment. I assume that time is discrete and there is a finite horizon: $t = 0, \ldots, T$.[22] At every point in time, borrowers trade a single noncontingent one period contract.[23] The income process for y_t has a Markov structure. Employment, health or family shocks, which previous literature has shown to be key drivers of bankruptcy, are all captured in the stochastic process for y_t. In case of default, borrowers debt is fully discharged but they can't borrow in the defaulting period and recover access to credit markets with a stochastic probability α. Chatterjee et al. (2007) show that these are reasonable assumptions in the context of the US unsecured credit system. I again restrict my attention to a noncontingent exemption which is not time varying. More sophisticated instruments, as choosing the fraction of debt discharged or the time of borrowers' exclusion from financial markets, can be welfare improving in this environment. Independently of whether those features are chosen optimally or not or whether they are important to explain borrowers' behavior, they only affect optimal exemptions through the characterized sufficient statistics. As in the baseline mode, renegotiation remains unfeasible.

Hence, borrowers maximize:

$$\max \mathbb{E}\left[\sum_{t=0}^{T} \beta^t U(C_t)\right]$$

Borrowers' consumption when they do not default is given by $C_t^{\mathcal{N}} = y_t + q_t B_t - B_{t-1}$. In default states, borrowers consume $C_t^{\mathcal{D}} = \min\{y_t, m\}$ and, when excluded from credit markets, they simply consume their income y_t.

From a $t = 0$ perspective, we can write borrowers' problem recursively as:

$$V_{\mathcal{N},0}(B_{-1}, y_0; m) = \max_{B_0} U(y_0 + q_0 B_0) + \beta \mathbb{E}\left[\max\left\{V_{\mathcal{N},1}(B_0, y_1; m), V_{\mathcal{D},1}(y_1; m)\right\}\right],$$

where $V_{\mathcal{N},1}(B_0, y_1; m)$ and $V_{\mathcal{D},1}(B_0, y_1; m)$, which denote indirect utility at $t = 1$ for borrowers' who repay and default, respectively, are explicitly characterized in the appendix. Indirect utility in default $V_{\mathcal{D},1}$ is independent of B_0 because of the assumption that all debt is fully discharged.

I characterize default at $t = 1$ and borrowing at $t = 0$. The results for other periods follow directly. The logic used to characterize the default region is identical to the baseline model. Borrowers default decision is determined by an indifference condition, given by:

$$V_{\mathcal{N},1}(B_0, \hat{y}_1; m) = V_{\mathcal{D},1}(\hat{y}_1; m), \tag{22}$$

[22] Provided the distribution of wealth remains bounded, the results extend to infinite horizon economies.

[23] There is scope to extend the results of this section to longer maturity contracts.

where $\hat{y}_1$ denote indifference thresholds. Once again, I denote the repayment region by $\mathcal{N}_t$, the default region in which borrowers claim the full exemption by $\mathcal{D}_{m,t}$ and the default region in which borrowers do not claim the full exemption by $\mathcal{D}_{y,t}$. All three regions are formed in general by a union of intervals. The value functions in equation (22) embed the forward looking nature of the default decision.

Borrowers choose B_0 according to:

$$U'(y_0+q_0B_0)\left[q_0+\frac{\partial q_0(B_0,m)}{\partial B_0}B_0\right]=\beta\int_{\mathcal{N},1}U'(y_1+q_1B_1-B_0)\,dF(y_1)$$

The characterization of default and borrowing decisions is identical in other periods.

Proposition 11. (Dynamics)

a) *The marginal welfare change from varying the optimal exemption m in the dynamic model is given by:*

$$\frac{dW}{dm}=\sum_{t=0}^{T-1}\mathbb{E}_{\mathcal{N},t}\left[\beta^tU'\left(C_t^{\mathcal{N}}\right)\frac{\partial q_t}{\partial m}B_t\right]+\sum_{t=1}^{T}\mathbb{E}_{\mathcal{D}_m,t}\left[\beta^tU'\left(C_t^{\mathcal{D}}\right)\right]$$

b) The optimal exemption m^* in the dynamic model is given by:

$$m^*=\frac{\sum_{t=1}^{T}\frac{\Pi_{m,t}\left\{C_t^{\mathcal{D}}\right\}}{C_0}}{\sum_{t=0}^{T-1}\Pi_{\mathcal{N},t}\left\{g_t\Lambda_t\varepsilon_{\tilde{r}_t,m}\right\}},$$

where $\mathbb{E}_{\mathcal{N}_t}[\cdot]$ denotes the $t=0$ expectation of being in a no default state in which borrowers have access to credit, $\mathbb{E}_{\mathcal{D}_m,t}[\cdot]$ denotes the $t=0$ expectation of defaulting in a given state, $\Lambda_t\equiv\frac{q_tB_t}{y_t+q_tB_t-B_{t-1}}$, $g_t\equiv\frac{C_t}{C_0}$, $\varepsilon_{\tilde{r}_t,m}\equiv\frac{\partial\log(1+r_t)}{\partial m}=-\frac{\frac{\partial q_t}{q_t}}{\partial m}$ and $\frac{\Pi_{m,t}\left\{C_t^{\mathcal{D}}\right\}}{C_0}\equiv\mathbb{E}_{\mathcal{D}_m,t}\left[\frac{C_t^{\mathcal{D}}}{C_0}\frac{\beta^tU'\left(C_t^{\mathcal{D}}\right)}{U'(C_0)}\right]$ and $\Pi_{\mathcal{N},t}\{x\}\equiv\mathbb{E}_{\mathcal{D},t}\left[\frac{\beta^tU'\left(C_t^{\mathcal{D}}\right)}{U'(C_0)}x\right]$.

As expected, the optimal exemption becomes a weighted average across periods/states of the marginal benefits/losses. The numerator of m^*, which captures the marginal benefit of increased leniency, is the price of a claim to an asset that pays consumption in the default states in which investors claim the full exemption as a ratio of initial consumption. The denominator, which captures marginal losses, is the $t=0$ price of a weighted average of interest rate schedule sensitivities with respect to the bankruptcy exemption — with weights given by the product of consumption growth g_t and leverage ratios Λ_t. Intuitively, the optimal formula in the dynamic model trades off marginal welfare gains and losses using borrowers' stochastic discount factor. The results of the static model can be interpreted as the steady state of a dynamic model.

I would like to make three final observations. First, the precise condition that determines the decision of declaring bankruptcy does not affect the formula for the optimal exemption. For instance, whether borrowers' go bankrupt after high or low income realizations does not change the formula for optimal exemptions. The option value to wait before defaulting, present in a model like this, complicates the characterization of the default decision, but it only affect the optimal exemption through the characterized sufficient statistics. Second, if borrowers are allowed to save part of their exemption when bankrupt, the price-consumption ratio must be corrected by a term that depends on borrowers propensity to consume (see the appendix for an exact expression). Hence, all results of the paper implicitly assume an (approximate) unit propensity to consume out of the bankruptcy exemption. This should entail little loss of generality, since we expect bankrupt borrowers to have done poorly and have high propensities to consume. Third, from the derivation of proposition 11, it is easy to see that allowing for allowing borrowers to stop repayments (default) without declaring bankruptcy does not affect the formula for the optimal exemption. That possibility will be priced by lenders and borrowers will exercise it optimally, so it will only affect exemptions through the characterized sufficient statistics, in particular the sensitivity of the interest rate schedule.

5.10 Final remarks

I would like to make two final remarks.

Remark. (Externalities) It is straightforward to allow for externalities associated with bankruptcy. Externalities introduce a wedge in the optimal exemption formula, with negative externalities pushing for lower exemptions. Needless to say, only spillovers which are not be priced by lenders modify the optimal exemption formula. For instance, if default creates a social loss of Θ units (expressed in borrowers' ex-ante utility), social welfare is given by $W(m) - \Theta F(m + B_0)$ and the optimal exemption becomes:

$$m^* = \frac{\frac{\Pi_m\left\{C_1^{\mathcal{D}}\right\}}{C_0}}{\Lambda \varepsilon_{q_0} + \frac{\Theta f(m+B_0)}{U'(C_0)C_0}}$$

Similarly, a planner that acknowledges that borrowers make mistakes when borrowing — for instance, because they are too optimistic or pessimistic about their income realizations — would also tilt the optimal exemption. These internalities would also introduce wedges into the optimal exemption formula.

Remark. (Market structure and selection) The results of this paper exploit the competitive nature of the equilibrium. More generally, there is scope to extend the results of this paper to situations in which adverse selection, credit rationing or imperfect competition on the lenders' side matter in equilibrium, as in, for instance, Rothschild and Stiglitz (1976), Stiglitz and Weiss (1981) or Bizer and DeMarzo (1992). Developing tractable models of these issues are relevant questions by itself and remain outside the scope of this paper but, in general, changes in the market structure have the potential to create wedges that may modify the optimal exemption formula.

6 Conclusion

This paper has provided a novel theoretical characterization of optimal bankruptcy exemptions for a large class of models of unsecured credit. Importantly, this characterization uses measurable sufficient statistics. Knowledge of borrowers' leverage, the sensitivity of the interest rate schedule faced by borrowers with respect to the level of exemptions, the probability of default and the change in the consumption of bankrupt borrowers is sufficient to assess the adequacy of bankruptcy reforms and to characterize the optimal exemption in a wide variety of circumstances. Other features of the environment need not be specified once these variables are known. Using the theoretical results directly, for the preferred calibration to US data, this paper concludes that the optimal bankruptcy exemption should be approximately $100,000$ dollars. This value is slightly larger than the average exemption level in the US, but it is of same order of magnitude.

The conclusions of this paper will help to foster further research on both structural modeling and microeconometric work on bankruptcy. Regarding structural macroeconomic modeling, computing the sufficient statistics found in this paper for different models will allow us to understand through which channels different assumptions on primitives affect the level of optimal exemptions. Empirical work that identifies the sensitivity of interest rate schedules with respect to changes in exemptions or that provides precise measures of consumption for bankrupt borrowers will be essential to assess which level of exemptions is appropriate.

Appendix

Proofs: Section 2

When borrowers are risk neutral, differentiating equation (4) yields:

$$\frac{\partial q_0(B_0,m)}{\partial B_0} = \frac{-\delta \int_m^{m+B_0} \frac{y_1-m}{B_0^2} dF(y_1) - (1-\delta) f(m+B_0)}{1+r^*} < 0$$
$$\frac{\partial q_0(B_0,m)}{\partial m} = \frac{-\delta \int_m^{m+B_0} \frac{1}{B_0} dF(y_1) - (1-\delta) f(m+B_0)}{1+r^*} < 0$$

Given their optimal ex-post default decision, borrowers solve $\max_{B_0} J(B_0;m)$, where:

$$J(B_0,m) = U(y_0 + q_0(B_0,m)B_0) + \beta \left[\int_{\underline{y_1}}^{m} U(y_1) dF(y_1) + \int_{m}^{m+B_0} U(m) dF(y_1) + \int_{m+B_0}^{\overline{y_1}} U(y_1 - B_0) dF(y_1) \right] \tag{23}$$

A sufficient condition for borrowers to borrow in equilibrium is the following:[24]

$$\left.\frac{dJ}{dB_0}\right|_{B_0=0} = U'(y_0) q_0(0,m) - \beta \int_{\underline{y_1}}^{\overline{y_1}} U'(y_1) dF(y_1) > 0$$

This condition holds for different combinations of fundamentals. For example, when the initial endowment y_0 is sufficiently low, when borrowers are very impatient, $\beta \to 0$, or when the level of expected future income is sufficiently large and not too stochastic (to prevent the precautionary savings effect from dominating).

The first-order condition to the borrowers' problem is the following:[25]

$$\frac{dJ}{dB_0} = U'(C_0)\left[q_0 + \frac{\partial q_0}{\partial B_0} B_0 \right] - \beta \int_{m+B_0}^{\overline{y_1}} U'(y_1 - B_0) dF(y_1) = 0 \tag{24}$$

When lenders are risk neutral:

$$U'(C_0)\left[\frac{(\delta - 1) f(m+B_0) B_0 + \int_{m+B_0}^{\overline{y_1}} dF(y_1)}{1+r^*} \right] = \beta \int_{m+B_0}^{\overline{y_1}} U'(y_1 - B_0) dF(y_1)$$

Note that when $m + B_0 \to \overline{y_1}$, $\frac{dJ}{dB_0} < 0$, which guarantees that there is an interior solution to the problem. Note also that, when bankruptcy is costless $(\delta \to 1)$, a by choosing m^* equal to the full insurance benchmark, we can replicate the perfect insurance outcome between risk neutral lenders and risk averse borrowers.

The second order condition, which establishes convexity, is the following:

$$\frac{dJ}{dB_0^2} = \underbrace{U''(C_0)\left[q_0 + \frac{\partial q_0}{\partial B_0} B_0 \right]^2}_{<0} + \underbrace{U'(C_0) \frac{\partial \left(q_0 + \frac{\partial q_0}{\partial B_0} B_0 \right)}{\partial B_0}}_{\gtreqless 0} + \underbrace{\beta U'(m) f(m+B_0)}_{>0} + \underbrace{\beta \int_{m+B_0}^{\overline{y_1}} U''(y_1 - B_0) dF(y_1)}_{<0} \leq 0 \tag{25}$$

Note that $\frac{\partial \left(q_0 + \frac{\partial q_0}{\partial B_0} B_0 \right)}{\partial B_0} = 2\frac{\partial q_0}{\partial B_0} + \frac{\partial q_0^2}{\partial B_0^2} B_0$, which in the case of risk neutral lenders equals $\frac{f(m+B_0)\left[-(2-\delta) - (1-\delta) B_0 \frac{f'(m+B_0)}{f(m+B_0)} \right]}{1+r^*}$. Hence, a sufficient condition for the second term to be negative is that $B_0 \frac{f'(m+B_0)}{f(m+B_0)} \geq -\frac{2-\delta}{1-\delta}$. When numerically solving the model, for usual distributions, the borrowers' problem is convex, despite the positive third term in equation (25).

[24] It is possible that $\lim_{B_0 \to 0^+} q_0(B_0,m) \neq \frac{1}{1+r^*}$, which might make the borrowers' problem nonconvex — this happens for instance when lenders are risk neutral. I assume throughout that the relevant global optimum is the local one in the borrowing region. Formally, $J(0,m) < J\left(B_0^*, m\right)$, where B_0^* is defined by equation (3).

[25] I use the notation $\frac{\partial q_0}{\partial B_0}$ throughout instead of $\frac{\partial q_0(B_0,m)}{\partial B_0}$ to simplify the notation. It should cause no confusion.

The curvature of the utility function induced by borrowers' risk aversion prevents to find simple sufficient conditions for convexity, as the one used by Bernanke, Gertler and Gilchrist (1999) in a related environment.

By differentiating equation (24), we can find $\frac{dB_0}{dm}$:

$$\frac{dB_0}{dm} = \frac{\overbrace{U''(C_0)\frac{\partial q_0}{\partial m}B_0\left[q_0+\frac{\partial q_0}{\partial B_0}B_0\right]}^{>0} + \overbrace{U'(C_0)\left[\frac{\partial q_0}{\partial m}+\frac{\partial^2 q_0}{\partial B_0 \partial m}B_0\right]}^{\gtreqless 0} + \overbrace{\beta U'(m)f(m+B_0)}^{>0}}{-\frac{dJ}{dB_0^2}}$$

When numerically solving the model, for usual distributions, the level of borrowing B_0 increases with m. The amount raised by borrowers at $t=0$ can also increase or decrease with m. Formally, $\frac{d(q_0 B_0)}{dm} = \left[\frac{\partial q_0}{\partial m}+\frac{\partial q_0}{\partial B_0}\frac{dB_0}{dm}\right]B_0 + q_0\frac{dB_0}{dm}$, which can take any sign.

To determine whether $\frac{C_1^{\mathcal{D}}}{C_0} > 1$, we can rewrite the optimality condition for debt as $U'(C_0) = \beta(1+\hat{r})\pi_{\mathcal{N}}\mathbb{E}\left[U'\left(C_1^{\mathcal{N}}\right)|\mathcal{N}\right]$, where $1+\hat{r} = \left(q_0+\frac{\partial q_0(B_0,m)}{\partial B_0}B_0\right)^{-1}$. We can also express $\mathbb{E}\left[U'\left(C_1^{\mathcal{N}}\right)|\mathcal{N}\right] \approx U'\left(\mathbb{E}\left[C_1^{\mathcal{N}}|\mathcal{N}\right]\right)(1+\kappa)$, where $\kappa > 0$ corrects for Jensen's inequality. Combining both expressions, we can conclude that $U'(C_0) = \alpha U'\left(\mathbb{E}\left[C_1^{\mathcal{N}}|\mathcal{N}\right]\right) < \alpha U'\left(C_1^{\mathcal{D}}\right)$, where $\alpha \equiv \beta(1+\hat{r})\pi_{\mathcal{N}}(1+\kappa)$. So a sufficient (not necessary) condition for $\frac{C_1^{\mathcal{D}}}{C_0} < 1$ is that $\alpha \le 1$. Borrowers impatience (low β) and small precautionary savings motives (low κ), which are the conditions needed for borrowers to borrow, guarantee that $\alpha \le 1$.

Allowing for competitive risk averse lenders is straightforward. We can think of many competitive lenders owned by a set of well diversified investors, who pin down the stochastic discount factor effectively used by lenders. The interest rate schedule takes the shape:

$$q_0(B_0,m) = \frac{\delta\int_{\mathcal{D}}\frac{\max\{y_1-m,0\}}{B_0}dH(y_1)+\int_{\mathcal{N}}dH(y_1)}{1+r^*},$$

where $H(\cdot)$ is a well behaved cdf (an absolutely continuous change of measure with respect to $F(\cdot)$). A similar approach can be followed in the case with aggregate shocks. Allowing for risk averse competitive lenders affects the equilibrium through changes in interest rates schedules.

Proofs: Section 3

Proposition 1. (Marginal effect on welfare)

a) Starting from equation (7), we can write, applying repeatedly Leibniz rule and using the optimality condition for B_0 and the default decision:

$$\begin{aligned}\frac{dW}{dm} &= \left[U'(C_0)\frac{\partial q_0}{\partial m}B_0+\beta\int_m^{m+B_0}U'\left(C_1^{\mathcal{D}}\right)dF(y_1)\right]\\ &+\underbrace{\left[U'(C_0)\left[q_0+\frac{\partial q_0}{\partial B_0}B_0\right]-\beta\int_{m+B_0}^{\overline{y_1}}U'(y_1-B_0)dF(y_1)\right]}_{=0}\frac{dB_0}{dm}\\ &+\underbrace{\beta\left[U(m)f(m+B_0)\left(1+\frac{dB_0}{dm}\right)-U(m)f(m+B_0)\left(1+\frac{dB_0}{dm}\right)\right]}_{=0},\end{aligned} \tag{26}$$

where we need to use equation (5) in the paper. As in most normative exercises, it is hard to guarantee the convexity of the problem in general, although numerical solutions are well behaved. Formally, we can write:

$$\frac{d^2J}{dm^2} = d\frac{\left(U'(C_0)\frac{\partial q_0}{\partial m}B_0\right)}{dm}+\beta\int_m^{m+B_0}U''\left(C_1^{\mathcal{D}}\right)dF(y_1)+\beta U'\left(C_1^{\mathcal{D}}\right)\left(f(m+B_0)\left(1+\frac{dB_0}{dm}\right)-f(m)\right),$$

where:

$$d\frac{\left(U'\left(C_0\right)\frac{\partial q_0}{\partial m}B_0\right)}{dm}=U''\left(C_0\right)\frac{d\left(q_0B_0\right)}{dm}\frac{\partial q_0}{\partial m}B_0+U'\left(C_0\right)\left[\frac{\partial q_0}{\partial m}\frac{dB_0}{dm}+\left[\frac{\partial^2 q_0}{\partial m^2}+\frac{\partial^2 q_0}{\partial m\partial B_0}\frac{dB_0}{dm}\right]B_0\right]$$

b) By rewriting equation (8), we can find:

$$\frac{\frac{dW}{dm}}{U'\left(C_0\right)C_0}=\underbrace{\frac{\frac{\partial q_0}{q_0}}{\partial m}}_{\equiv-\varepsilon_{\tilde{r},m}}\underbrace{\frac{q_0B_0}{C_0}}_{\equiv\Lambda}+\frac{1}{m}\underbrace{\beta\int_m^{m+B_0}\frac{C_1^{\mathcal{D}}}{C_0}\frac{U'\left(C_1^{\mathcal{D}}\right)}{U'\left(C_0\right)}dF\left(y_1\right)}_{\equiv\frac{\Pi_m\left\{C_1^{\mathcal{D}}\right\}}{C_0}},$$

where $\varepsilon_{\tilde{r},m}\equiv\frac{\partial\log(1+r)}{\partial m}=-\frac{\frac{\partial q_0\left(B_0(m),m\right)}{q_0}}{\partial m}$.

Proposition 2. (Optimal bankruptcy exemption)

Setting $\frac{dW}{dm}=0$ characterizes the optimal exemption. If the problem is well behaved, m^* should yield a unique solution. Formally, solving for m^* in equation (9) immediately yields (10).

In the logarithmic utility case:

$$\frac{\Pi_m\left\{C_1^{\mathcal{D}}\right\}}{C_0}=\beta\int_m^{m+B_0}\frac{C_1^{\mathcal{D}}}{C_0}\frac{U'\left(C_1^{\mathcal{D}}\right)}{U'\left(C_0\right)}dF\left(y_1\right)=\beta\underbrace{\int_m^{m+B_0}dF\left(y_1\right)}_{\equiv\pi_m},$$

since $U'\left(C\right)=\frac{1}{C}$.

Note that we can write: $-\frac{\partial q_0(\cdot,\cdot)}{\partial m}B_0=\frac{\pi_m}{1+r^*}\left[\varepsilon+\left(1-\varepsilon\right)\Upsilon\right]$, where $\Upsilon=\frac{f(m+B_0)B_0}{\pi_m}$. Substituting into equation (8), we can find:

$$\frac{\left(1+r^*\right)\frac{dW}{dm}}{U'\left(C_0\right)\pi_m}=-\left[\varepsilon+\left(1-\varepsilon\right)\Upsilon\right]+\beta\left(1+r^*\right)\left(\frac{m}{C_0}\right)^{-\gamma}=0$$

Solving for m^* yields equation (11) in the paper. When the distribution $F\left(\cdot\right)$ is a uniform, $F\left(m+B_0\right)-F\left(m\right)=f\left(m+B_0\right)B_0$, so $\Upsilon=1$.

Proofs: Section 5

Proposition 3. (Endogenous income:frictionless markets)

a) Applying repeatedly Leibniz' rule and using the optimality condition B_0, the default decision and first-order condition on labor supply $N_1^{\mathcal{N}}$ and N_0 and the optimal effort choice a, we can write:

$$\frac{dW}{dm}=\frac{\partial U}{\partial C_0}\left(C_0,N_0;a\right)\frac{\partial q_0\left(B_0,m,a\right)}{\partial m}B_0+\beta\int_{D_m}\frac{\partial U}{\partial C_1}\left(C_1^{\mathcal{D}},0\right)dF\left(y_1;a\right),$$

where $\frac{\partial U}{\partial C_0}\left(C_0,N_0;a\right)$ denotes the partial derivative or flow utility with respect to consumption — equivalently with $\frac{\partial U}{\partial C_1}\left(C_1^{\mathcal{D}},0\right)$. Note that the derivation of the optimality conditions in the text use the fact that $\frac{d\tilde{V}}{dB_0}=-\frac{\partial U}{\partial C}\left(C_1^{\mathcal{N}},N_1^{\mathcal{N}}\right)$.

Figure A.1 illustrates graphically how to characterize the default region when borrowers' labor supply decision is nontrivial. Although it is impossible to solve explicitly for the default region, its characterization is conceptually identical to the baseline model.

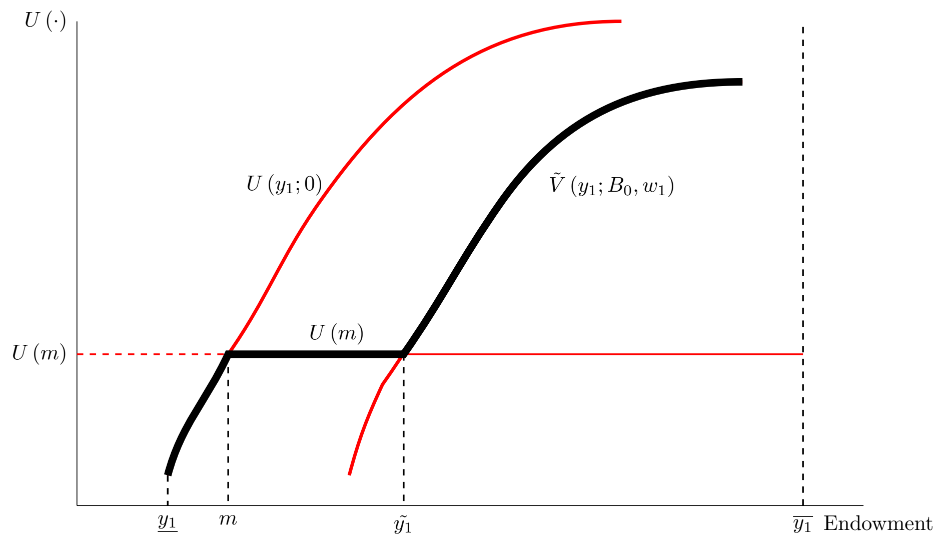

Figure A.1: Optimal default decision given B_0

b) Setting $\frac{dW}{dm}=0$ and solving for m yields directly m^*.

Proposition 4. (Non-pecuniary penalties/state contingent utility)

a) The bankruptcy region is characterized a new indifference condition $\phi m = y_1 - B_0$. The derivation of $\frac{dW}{dm}$ follows the same steps as the baseline model.

b) Setting $\frac{dW}{dm}=0$ and solving for m yields directly m^*.

Proposition 5. (Epstein-Zin utility)

a) The derivation of $\frac{dW}{dm}$ follows the same steps as the baseline model. I define Q, which denotes the certainty equivalent of consumption at $t=1$, as:

$$Q \equiv \left(\int_{\underline{y_1}}^{m} (y_1)^{1-\gamma} dF(y_1) + \int_{m}^{m+B_0} (m)^{1-\gamma} dF(y_1) + \int_{m+B_0}^{\overline{y_1}} (y_1 - B_0)^{1-\gamma} dF(y_1) \right)^{\frac{1}{1-\gamma}}$$

Taking derivatives in equation (15), assuming that $\frac{\partial q_0}{\partial B_0} \approx 0$, yields:

$$\left.\frac{dB_0}{dq_0}\right|_{\frac{\partial q_0}{\partial B_0}\approx 0} = \frac{(C_0)^{-\frac{1}{\psi}}\left[1-\frac{\Lambda}{\psi}\right]}{-SOC},$$

where $SOC = \beta\left[(q_0)^2 \frac{1}{\psi}(y_0+q_0B_0)^{-\frac{1}{\psi}-1} + \int_{m+B_0}^{\overline{y_1}} \gamma(y_1-B_0)^{-\gamma-1} dF(y_1) + (y_1-B_0)^{-\gamma} f(m+B_0)\right] < 0.$ This shows that a change in interest rates (keeping interest rate schedule elasticities fixed) has an income and a substitution effect. The substitution effect, which determines the sensitivity of borrowing to interest rate changes, dominates when $\psi > \Lambda$. Hence, $\psi > 1$, the standard parametrization in asset pricing models, implies that low interest rates increase borrowing and viceversa.

b) Setting $\frac{dW}{dm}=0$ and solving for m yields directly m^*.

Proposition 6. (Multiple contracts with arbitrary payoffs)

a) As stated in the paper, there are three regions depending on the realization of y_1. The indifference condition between the regions $\mathcal{D}_m$ and $\mathcal{N}$ in the range $y_1 > m$ is given by $m = y_1 - \sum_{j=1}^{J} z_j(y_1) B_{0j}$. The indifference condition between the regions $\mathcal{D}_y$ and $\mathcal{N}$ in the range $y_1 \leq m$ is given by $0 = \sum_{j=1}^{J} z_j(y_1) B_{0j}$. When $\sum_{j=1}^{J} z_j(m) B_{0j} > 0$, at $y_1 = m$ there is a new boundary separating the $\mathcal{D}_m$ and $\mathcal{D}_y$ regions — this doesn't occur for instance in figure 3 in the text. This characterization decomposes the possible set of realizations in $\left[\underline{y_1}, \overline{y_1}\right]$ into multiple nonoverlapping intervals.

Indirect utility as a function of the exemption is written as:

$$W(m) \equiv U\left(y_0 + \sum_j q_{0j}(B_{01}, \ldots, B_{0J}, m) B_{0j}\right)$$
$$+\beta\left[\int_{\mathcal{D}_y} U(y_1)\, dF(y_1) + \int_{\mathcal{D}_m} U(m)\, dF(y_1) + \int_{\mathcal{N}} U\left(y_1 - \sum_j z_j(y_1) B_j(y_1)\right) dF(y_1)\right]$$

The derivation of equation (17) follows the same steps as the baseline model. It requires the use of J optimality conditions for the J different contracts traded.

b) Setting $\frac{dW}{dm} = 0$ and solving for m yields directly m^*.

Proposition 7. (Heterogeneous borrowers)

a) The derivation of $\frac{dW}{dm}$ follows the same steps as the baseline model. It requires the use of as many optimality conditions regarding the choice of B_0 and default as the number of borrowers.

b) Setting $\frac{dW}{dm} = 0$ and solving for m yields directly m^*. Note that $\mathbb{E}_G\left[\varepsilon_{\tilde{r}_i,m}\Lambda_i\right] = \mathbb{E}_G\left[\varepsilon_{\tilde{r}_i,m}\right]\mathbb{E}_G\left[\Lambda_i\right] + \mathbb{C}ov_G\left[\varepsilon_{\tilde{r}_i,m}, \Lambda_i\right]$.

Proposition 8. (Price taking borrowers)

a) The derivation of $\frac{dW}{dm}$ follows the same steps as the baseline model. Note that now the term corresponding to $\frac{\partial q_0}{\partial B_0} B_0 \frac{dB_0}{dm} U'(C_0)$ does not cancel out in equation (26). By differentiating equation (19), and denoting by $B_0^b(q_0)$ the solution to that loan demand equation, we can write:

$$\frac{\partial q_0}{\partial B_0^b} = \frac{-\beta\left[(q_0)^2 U''(C_0) + \int_{m+B_0}^{\overline{y_1}} U''\left(y_1 - B_0^b\right) dF(y_1) + U'(m) f\left(m + B_0^b\right)\right]}{U'\left(y_0 + q_0^b B_0^b\right)\left[1 - \frac{\Lambda}{\psi}\right]},$$

where the numerator is the second order condition to the borrowers' problem. And:

$$\frac{\partial q_0}{\partial m} = \frac{-\beta U'(m) f\left(m + B_0^b\right)}{U'\left(y_0 + q_0^b B_0^b\right)\left[1 - \frac{\Lambda}{\psi}\right]} < 0$$

Combining these two equations that characterize loan demand with equation (5), which characterizes loan supply, it is easy to show that $\frac{dq_0}{dm} < 0$ and $\frac{dB_0}{dm}$ is indeterminate.

b) Setting $\frac{dW}{dm} = 0$ and solving for m yields directly m^*.

Proposition 9. (Bankruptcy exemption contingent on aggregate risk)

a) The problem that characterizes the optimal set of exemptions is given by:$\max_{\{m_\omega\}} W(\{m_\omega\})$. The derivation of $\frac{\partial W}{\partial m_\omega}, \forall \omega$ follows the same steps as the baseline model. It uses borrowers' optimality of default decisions given the realization of ω and optimality of the choice of B_0.

b) Setting $\frac{dW}{dm_\omega} = 0$ and solving for m_ω yields directly m_ω^* for every ω.

Proposition 10. (Endogenous income: labor wedge)

a) We can write ex-ante welfare as:

$$W(m) = \max U(y_0 + q_0 B_0) + \beta \sum_\omega p(\omega)\left[\begin{array}{c} \int_{\mathcal{D}_y} U(y_1, 0)\, dF_\omega(y_1(\omega)) + \int_{\mathcal{D}_m} U(m, 0)\, dF_\omega(y_1(\omega)) \\ + \int_{\mathcal{N}} U\left(y_1 + f(N(\omega)) - B_0, N_1^{\mathcal{N}}(\omega)\right) dF_\omega(y_1(\omega)) \end{array}\right]$$

Using optimality conditions we can write:

$$\frac{dW}{dm}=U'(C_0)\frac{\partial q_0}{\partial m}B_0+\beta\sum_{\omega}p(\omega)\left[\begin{array}{c}\int_{\mathcal{D}_m}U'\left(C_1^{\mathcal{D}}(\omega),0\right)dF_{\omega}(y_1(\omega))\\ +\int_{\mathcal{N}}\left[\frac{\partial U\left(C_1(\omega),N_1^{\mathcal{N}}(\omega)\right)}{\partial C_1(\omega)}A+\frac{\partial U\left(C_1(\omega),N_1^{\mathcal{N}}(\omega)\right)}{\partial N_1(\omega)}\right]\frac{d\overline{N}(\omega)}{dm}dF_{\omega}(y_1(\omega))\end{array}\right]$$

$$\frac{dW}{dm}=U'(C_0)\frac{\partial q_0}{\partial m}B_0+\beta\sum_{\omega}p(\omega)\left[\begin{array}{c}\int_{\mathcal{D}_m}U'\left(C_1^{\mathcal{D}}(\omega),0\right)dF_{\omega}(y_1(\omega))\\ +\int_{\mathcal{N}}\frac{\partial U\left(C_1(\omega),N_1^{\mathcal{N}}(\omega)\right)}{\partial C_1(\omega)}dF_{\omega}(y_1(\omega))\tau(\omega)\frac{dY(\omega)}{dm}\end{array}\right]$$

Where we can use the following facts: $\frac{dY}{dm}=A\frac{dN}{dm}$, and because of constant returns to scale $\frac{dY}{dN_1^{\mathcal{N}}}=\frac{dY}{d\overline{N}}=A$, where $\overline{N}$ is aggregate labor supply.

b)We can thus write:

$$\frac{\frac{dW}{dm}}{C_0U'(C_0)}=\frac{\frac{\partial q_0}{q_0}}{\partial m}\frac{q_0B_0}{C_0}+\frac{1}{m}\beta\sum_{\omega}p(\omega)\left[\begin{array}{c}\int_{\mathcal{D}_m}\frac{C_1^{\mathcal{D}}(\omega)}{C_0}\frac{\frac{\partial U\left(C_1^{\mathcal{D}}(\omega),0\right)}{\partial C_1(\omega)}}{U'(C_0)}dF_{\omega}(y_1(\omega))\\ +\int_{\mathcal{N}}\frac{\frac{\partial U\left(C_1(\omega),N_1^{\mathcal{N}}(\omega)\right)}{\partial C_1(\omega)}}{U'(C_0)}dF_{\omega}(y_1(\omega))\frac{\tau(\omega)}{C_0}\frac{dY(\omega)}{\frac{dm}{m}}\end{array}\right]$$

And solving for m yields directly m^* in equation (21).

Proposition 11. (Dynamics)

a) The problem solved by borrowers at $t=0$ can be written as:

$$V_{\mathcal{N},0}(0,y_0;m)=\max_{B_0}U(y_0+q_0B_0)+\beta\mathbb{E}\left[\max\left\{V_{\mathcal{N},1}(B_0,y_1;m),V_{\mathcal{D},1}(y_1;m)\right\}\right],$$

where $V_{\mathcal{N},1}(B_0,y_1;m)$ denotes indirect utility at $t=1$ when a borrower repays and $V_{D,1}(y_1;m)$ denotes the indirect utility at $t=1$ when a borrower just defaulted, given by:

$$V_{\mathcal{N},1}(B_0,y_1;m)=\max_{B_1}U(y_1+q_1B_1-B_0)+\beta\mathbb{E}\left[\max\left\{V_{\mathcal{N},2}(B_1,y_2;m),V_{\mathcal{D},2}(B_1,y_2;m)\right\}\right]$$
$$V_{\mathcal{D},1}(y_1;m)=U(\min\{y_1,m\})+\beta\left[\alpha\mathbb{E}\left[V_{\mathcal{D}_\alpha,2}(y_2;m)|y_1\right]+(1-\alpha)\mathbb{E}\left[V_{\mathcal{N},2}(0,y_2;m)|y_1\right]\right],$$

where $V_{\mathcal{D}_\alpha,2}(y_2;m)$ denotes the indirect utility at $t=2$ for a borrower who has no access to credit markets.

$$V_{\mathcal{D}_\alpha,2}(y_2;m)=U(y_2)+\beta\left[\alpha\mathbb{E}\left[V_{\mathcal{D}_\alpha,3}(y_3;m)|y_2\right]+(1-\alpha)\mathbb{E}\left[V_{\mathcal{N},3}(0,y_3;m)|y_2\right]\right]$$

The definition of these value functions for future periods is straightforward.

Hence, the derivative $\frac{dW}{dm}$ can be written, after using extensively optimality conditions, as:

$$\frac{dW}{dm}=U'(C_0)\frac{dq_0}{dm}B_0+\beta\left[\int_{\mathcal{N},1}\frac{dV_{\mathcal{N},1}(B_0,y_1;m)}{dm}dF(y_1)+\int_{\mathcal{D},1}\frac{dV_{\mathcal{D},1}(y_1;m)}{dm}dF(y_1)\right],\tag{27}$$

where

$$\frac{dV_{\mathcal{N},1}(B_0,y_1;m)}{dm}=U'\left(C_1^{\mathcal{D}}\right)\frac{dq_0}{dm}B_1+\beta\left[\begin{array}{c}\int_{\mathcal{N},2}\frac{dV_{\mathcal{N},2}(B_1,y_2;m)}{dm}dF(y_2|y_1)\\ +\int_{\mathcal{D},2}\frac{dV_{\mathcal{D},2}(y_2;m)}{dm}dF(y_2|y_1)\end{array}\right]$$

$$\frac{dV_{\mathcal{D},1}(y_1;m)}{dm}=U'(m)\mathbb{I}(y_1>m)+\beta\left[\begin{array}{c}\alpha\mathbb{E}\left[\left.\frac{dV_{\mathcal{D}_\alpha,2}(y_2;m)}{dm}\right|y_1\right]\\ +(1-\alpha)\mathbb{E}\left[\left.\frac{dV_{\mathcal{N},2}(0,y_2;m)}{dm}\right|y_1\right]\end{array}\right]$$

$$\frac{dV_{\mathcal{D}_\alpha,2}(y_2;m)}{dm}=\beta\left[\alpha\mathbb{E}\left[\left.\frac{dV_{\mathcal{D}_\alpha,3}(y_3;m)}{dm}\right|y_2\right]+(1-\alpha)\mathbb{E}\left[\left.\frac{dV_{\mathcal{N},3}(0,y_3;m)}{dm}\right|y_2\right]\right]$$

Substituting the last three expressions into equation (27) and iterating forward, we can write:

$$\frac{dW}{dm} = \sum_{t=0}^{T-1} \mathbb{E}_{\mathcal{N},t} \left[\beta^t U' \left(C_t^{\mathcal{N}} \right) \frac{dq_t}{dm} B_t \right] + \sum_{t=1}^{T} \mathbb{E}_{D_m,t} \left[\beta^t U' \left(C_t^{\mathcal{D}} \right) \right],$$

where $\mathbb{E}_{\mathcal{N},t}[\cdot]$ denotes the $t=0$ expectation of being in a no default situation at a given state/period and $\mathbb{E}_{D_m,t}[\cdot]$ denotes the $t=0$ expectation of defaulting while consuming the bankruptcy exemption in a given state.

The previous equation can be rewritten as:

$$\frac{\frac{dW}{dm}}{U'(C_0)\, C_0} = -\sum_{t=0}^{T-1} \Pi_{\mathcal{N},t} \left\{ g_t \Lambda_t \varepsilon_{\tilde{r}_t,m} \right\} + \frac{1}{m} \sum_{t=1}^{T} \frac{\Pi_{m,t} \left\{ C_t^{\mathcal{D}} \right\}}{C_0}, \tag{28}$$

where all new variables are defined in proposition 11. If part of the exemption can be saved, that is, $C_t^{\mathcal{D}} + S_t = \min\{y_t, m\}$, the second term in equation (28) becomes $\frac{1}{m} \sum_{t=1}^{T} \frac{\Pi_{m,t} \left\{ \mu_t C_t^{\mathcal{D}} \right\}}{C_0}$, where $\mu_t = \frac{1}{1 - \frac{S_t}{m}}$. When household propensity to consume is close to one, $\mu_t \approx 1$.

b) Setting $\frac{dW}{dm} = 0$ and solving for m yields directly m^*.

Externalities

Assume that social welfare now includes an additional cost in terms of utility that is only paid when there is default. Hence the optimal exemption solves:

$$\max_m W^S \equiv W(m) - \Theta F(B_0 + m)$$

Where $W(m)$ is the same as in equation (7), $\Theta > 0$ is a scalar capturing the internalized social cost and $F(B_0 + m)$ is the probability of default. The change in welfare induced by a change in m is given by:

$$\frac{dW}{dm} = U'(C_0) \frac{\partial q_0}{\partial m} B_0 + \beta \int_{\underline{y_1}}^{m+B_0} y_1 U' \left(C_1^{\mathcal{D}} \right) dF(y_1) - \Theta f(m + B_0)$$

The optimal exemption is given by:

$$m^* = \frac{\frac{\Pi_m \left\{ C_1^{\mathcal{D}} \right\}}{C_0}}{\Lambda \varepsilon_{q_0} + \frac{\Theta f(m+B_0)}{U'(C_0) C_0}}$$

Because, $\Theta f(m + B_0) > 0$, when there are externalities associated with bankruptcy, the optimal m^* is lower, to reduce bankruptcy in equilibrium.

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Online appendix (not for publication)

Numerical example

I illustrate the equilibrium of the Epstein-Zin version of the model with a numerical example, using the parameters given in table 4. The distribution $F(\cdot)$ is a log-normal with parameters μ and σ. This is not a calibration exercise and merely an illustration of the analytical results derived in the paper.

Preferences	$\gamma = 10$	$\psi = 1.5$	$\beta = 0.96$	$r^* = 4\%$
Endowments	$y_0 = 55$	$\mu = 4.9$	$\sigma = 0.095$	
Bankruptcy	$\delta = 0.1$			

Table 4: Parameters numerical example

Figure A.2 shows how the interest rate schedule $q_0(B_0, m)$ changes with B_0 and m respectively, keeping one of the two variables constant. For the first figure, I assume that $m = 80$ and, for the second one, I assume that $B_0 = 23$. It corresponds with equation (4) in the text. As shown analytically is equation (5), both functions are downward sloping.

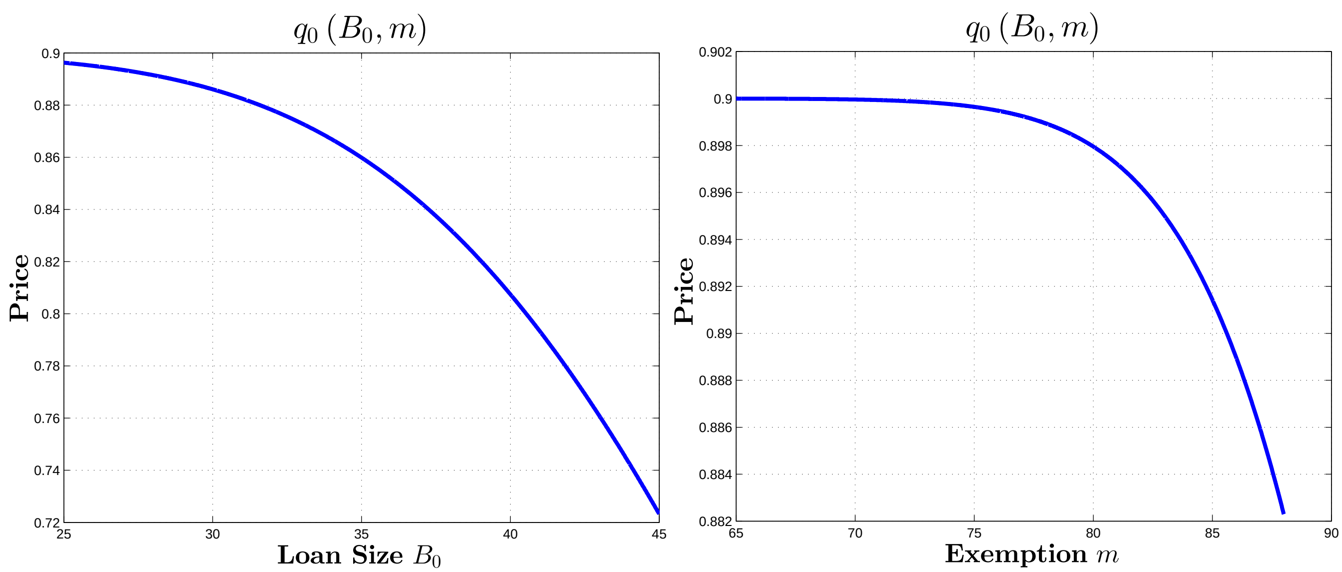

Figure A.2: Interest rate schedule $q_0(B_0, m)$

Figure A.3 shows the function maximized by borrowers at $t = 0$ when they have to choose B_0 for a given level of m. I assume $m = 80$. It corresponds with equation (2) in the text and (23) in the appendix.

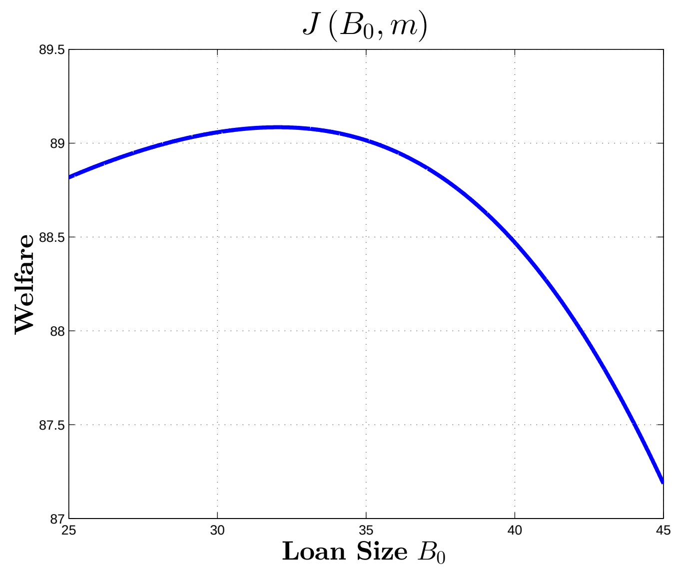

Figure A.3: Borrowers objective function $J(B_0, m = 80)$

The left plot in figure A.4 shows the optimal choice of B_0 for different levels of m. It corresponds with equation (3) in the text. As described in the text, $\frac{dB_0}{dm}$ can take any sign depending on the strength of the effects mentioned. I chose this particular parametrization to illustrate how $B_0(m)$ can increase or decrease on m — we observe a form of "Laffer curve". The right plot in figure A.4 shows borrowers welfare for a different levels of m. It corresponds with equation (7) in the text. The optimal exemption m^* is the value that maximizes that expression.

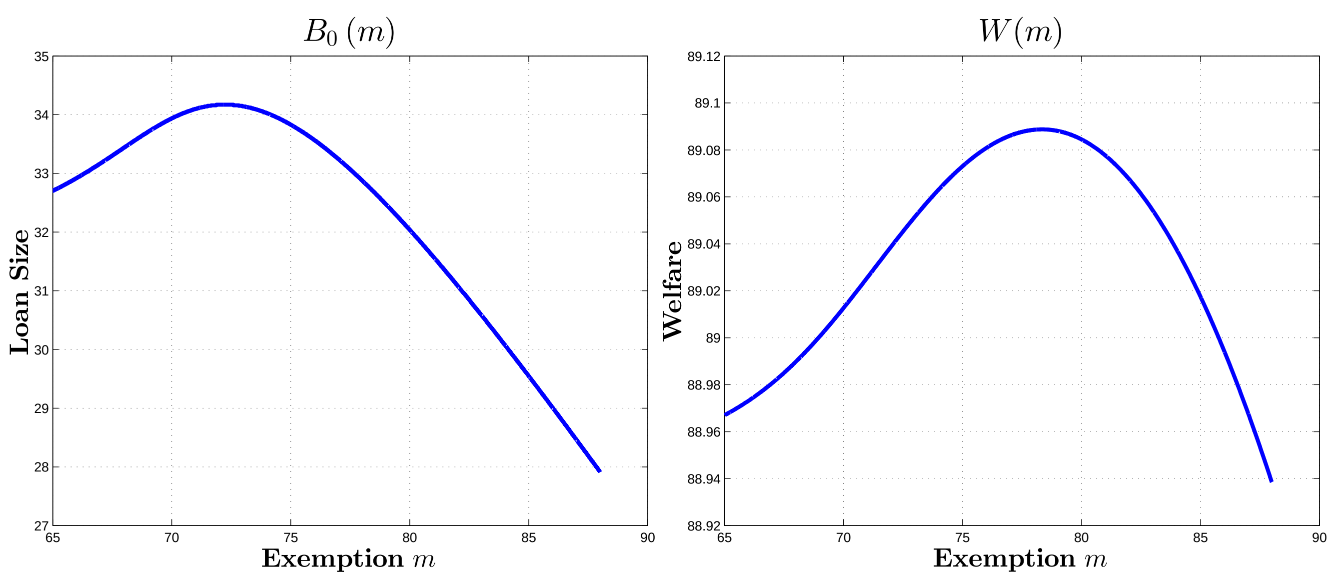

Figure A.4: Optimal borrowing choice $B_0(m)$ and indirect utility $W(m)$

In this example, $m^* = 78.35$, $B_0(m^*) = 32.74$, $\Lambda = 0.34$ and $\pi_D = 2.29\%$.