The Race Between Technology and Human Capital

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Abstract

This paper develops a model in which heterogenous firms invest in R&D to improve technology, and heterogeneous workers invest in human capital to increase their earnings. Both investment technologies have stochastic components, and the balanced growth path has stationary, nondegenerate distributions of technology and human capital.

Technology and human capital are complements in production, so the labor market produces assortative matching between firms and workers: firms with higher productivity employ higher quality workers and pay higher wages. Thus, wage differentials across firms have two sources: differences in firm productivity and differences in labor quality.

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1. OVERVIEW

This paper develops a model in which heterogenous firms invest in R&D to improve technology, and heterogeneous workers invest in human capital to increase their earnings. Both investment technologies have stochastic components, and the balanced growth path has stationary, nondegenerate distributions of technology and human capital.

Technology and human capital are complements in production, so the labor market produces assortative matching between firms and workers: firms with higher productivity employ higher quality workers and pay higher wages. Thus, wage differentials across firms have two sources: differences in firm productivity and differences in labor quality.

The heterogeneous firms produce intermediate goods, which are combined with a Dixit-Stiglitz aggregator to produce the single final good. Final goods are used for consumption and three kinds of investment. Incumbent firms invest to improve their productivity, and they die stochastically. Entering firms pay a fixed cost to obtain an initial technology. Workers invest to increase their human capital.

One goal of the model is to examine the interplay between human accumulation and technological change as contributors to long-run growth. From an empirical point of view, the chicken-and-egg issue makes it difficult to distinguish a single "engine" of growth. A theoretical framework that builds in the symbiotic nature of improvements in the two factors may provide insights for assessing, in particular contexts, the role of each.

A second goal is assess the sources of wage inequality. Empirically, it is not easy to distinguish the importance of technology and human capital differences in generating wage differentials across firms. A theoretical framework that incorporates complementarity between the two may be useful in assessing the importance of each.

The setup here builds on the model of technology growth across firms in Luttmer (*QJE*, 2007), incorporating an active investment decision as Atkeson and Burstein (*JPE*, 2010). On the human capital side, it develops a similar investment model. From a substantive point of view, the model provides a link between the literature on human capital-based growth, as in Alvarez, Buera and Lucas (2008, 2013), Lucas (1988, 2009), Perla and Tonetti (2014), Perla, Tonetti and Waugh (2014), and others, and the literature on technology-driven growth, as in Atkeson and Burstein (2010), Klette and Kortum (2004), Luttmer (2007), and others. The model also has implications for wage inequality across age cohorts of workers, as documented in Deaton and Paxson (1994), and for productivity dynamics in firms, as documented in Bailey et. al. (1992), Bartelsman and Doms (2000), Dunne Roberts and Samuelson (1989), and Hsieh and Klenow (2014). It is also related to the model of technology and wage inequality in Jovanovic (1998).

To start, we will assume that both the average productivity of entering firms and the average initial human capital of new workers grow at a common rate g. We will look for a BGP where the cross-sectional distribution of productivities across firms and of human capital across workers are both lognormal, with constant variances. The entering productivities for new firms and workers can then be made endogenous. Specifically, at each date they will be draws from a distribution that depends on the current cross-sectional distribution.

A. Variables

Exogenous:

population L is exogenous and constant, with birth and death at rate δ_L;

exit rate for firms, $\delta_F > 0$;

productivity of entering firms at date t is lognormally distributed, with a fixed variance and a mean that grows at the constant rate g,

$$\ln X_{it0} \sim N\left(\mu_{EX} + gt, \sigma_{EX}^2\right);$$

human capital of entering workers at date t is lognormally distributed, with fixed variance and a mean that grows at the constant rate g,

$$\ln H_{jt0} \sim N\left(\mu_{EH} + gt, \sigma_{EH}^2\right).$$

Individual decisions:

productivity X_{ita} at age a of incumbent firm i that entered at date t is a geometric Brownian motion, with fixed variance σ_X^2.

The firm's investment decision is the choice of the drift g_X.

human capital H_{jta} at age a of worker j who entered at date t is a geometric Brownian motion, with fixed variance σ_H^2.

The worker's investment decision is the choice of the drift g_H.

Endogenous:

N, the (constant) number of incumbent firms, is determined by free entry.

Q_t is an aggregate variable, "average productivity." It grows at rate g.

Its level depends the cross-sectional distribution of productivities X_{it}.

$x_{ita} = X_{ita}/Q_{t+a}$ is firm i's relative productivity at age a. On a BGP, x_i has a stationary cross-sectional distribution that depends on $\mu_{EX}, g_X, \sigma_{EX}^2, \sigma_X^2$.

Firm i's labor demand and profits at date $t+a$ depend on (x_{ita}, Q_{t+a}).

$h_{jta} = H_{jta}/Q_{t+a}$ is worker j's relative human capital. On a BGP, h_{jta} has a stationary cross-sectional distribution that depends on $\mu_{EH}, g_H, \sigma_{EH}^2$ and σ_H^2.

$W(h, Q)$ is the wage of a worker with relative human capital h in an economy with average productivity Q.

2. THE STATIC MODEL

In this section we solve for the static equilibrium, given L, N, and the distribution functions $F(X)$, $M(H)$.

A. Final good technology

The final good is produced by competitive firms using intermediate goods as inputs. All intermediates enter symmetrically into final good production, but demands for them differ if their prices differ. Specifically, intermediate producers are indexed by their productivity $X > 0$, which determines the price $p(X)$ for their good. Let $f(X)$ denote the density for X across intermediate producers, and let N be the number (mass) of firms. Each final good producer has the CRS technology

$$Y_F = \left[N \int Y(X)^{(\rho-1)/\rho} f(X) dX \right]^{\rho/(\rho-1)}, \tag{1}$$

where $\rho > 1$ is the substitution elasticity. The final goods sector takes the prices $p(X)$ as given. As usual, the price of the final good is

$$p_F = \left[N \int p(X)^{1-\rho} f(X) dX \right]^{1/(1-\rho)}, \tag{2}$$

and input demands are

$$Y^d(X) = \left(\frac{p(X)}{p_F} \right)^{-\rho} Y_F, \qquad \text{all } X. \tag{3}$$

B. Intermediate producers: choice of labor quality

Intermediate producers use heterogeneous labor, differentiated by its human capital level H, as the only input. The output of a firm depends on the size and quality of its workforce, as well as its technology. In particular, if a firm with technology X employs ℓ workers with human capital H, then its output is

$$Y = \ell \phi(H, X),$$

where $\phi(H,X)$ is the CES function

$$\phi(H,X) \equiv \left[\omega H^{(\eta-1)/\eta} + (1-\omega)\, X^{(\eta-1)/\eta}\right]^{\eta/(\eta-1)}, \qquad \eta,\omega \in (0,1)\,. \tag{4}$$

The elasticity of substitution η between technology and human capital is assumed to be less than unity, and ω is the relative weight on human capital. Firms could employ workers with different human capital levels, and in this case their outputs would simply be summed. In equilibrium firms never choose to do so, however, and for simplicity the notation is not introduced.

Let $W(H)$ denote the wage function. For a firm with technology X, the cost of producing one unit of output with labor of quality H is $W(H)/\phi(H,X)$. Optimal labor quality $H^*(X)$ minimizes this expression.

Conjecture that the wage function has the constant elasticity form

$$W(H) = W_0 H^{1-\varepsilon}, \qquad \varepsilon \in (0,1)\,. \tag{5}$$

Then optimal labor quality is proportional to X, with a constant of proportionality that depends on ε,

$$H^*(X) = a_H X, \tag{6}$$

where

$$a_H \equiv \left(\frac{1-\varepsilon}{\varepsilon}\frac{1-\omega}{\omega}\right)^{\eta/(\eta-1)}. \tag{7}$$

Unit cost is then

$$\frac{W(a_H X)}{\phi(a_H X, X)} = \frac{a_H^{1-\varepsilon}}{\phi_0} W_0 X^{-\varepsilon}, \tag{8}$$

where

$$\phi_0 \equiv \phi(a_H, 1). \tag{9}$$

For $W(H)$ as in (5), the cost minimization problem is concave if (and only if) $\eta \in (0,1)$, as assumed here. The quantity of labor hired is proportional to the target level of output,

$$\ell^*(X;Y_F) = \frac{Y_F}{\phi_0} X^{-1}. \tag{10}$$

Figure 1 displays isoquants for output and expenditure (total wage bill) in quality-quantity space, for the model parameters

$$\eta = 0.5, \qquad \omega = 0.5, \qquad W_0 = 1, \qquad \varepsilon = 0.5,$$

and the technology, output, and expenditure levels

$$\begin{aligned} X_1 &= 0.5, \qquad X_2 = 1, \qquad Y_1 = 0.5, \qquad Y_2 = 1, \\ E_{11} &= 0.7071, \qquad E_{12} = 1.4142, \qquad E_{21} = 0.5, \qquad E_{22} = 1. \end{aligned}$$

The dashed curves are output isoquants for the technology level X_1, and the broken curves are isoquants for $X_2 > X_1$. The four solid curves are expenditure isoquants, and the small circles indicate the four cost-minimizing input mixes. With X fixed, a higher output level Y increases only the quantity ℓ of labor input. With Y fixed, a higher technology level X increases labor quality H and reduces quantity ℓ. Note that cost minimization by firms implies positively assortative matching: firms with better technologies hire workers with more human capital.

C. Intermediate goods: pricing problem

Suppose the wage function has the conjecture form in (5), so unit cost is as in (8). Given the price p_F for the final good, an intermediate firm with productivity X chooses its price $\hat{p}(X)$ to maximize profits. As usual, the optimal price is a markup $\rho/(\rho-1)$ over unit cost. Let $w_0 \equiv W_0/p_F$ denote the scale for the real wage. Then (relative) price, quantity, labor input, and (real) profits for the intermediate firm involve various powers of X,

$$\frac{\hat{p}(X)}{p_F} = \frac{\rho}{\rho-1}\frac{a_H^{1-\varepsilon}}{\phi_0 p_F} W_0 X^{-\varepsilon} \equiv p_0 w_0 X^{-\varepsilon}, \tag{11}$$

$$\hat{Y}(X) = \left(\frac{\hat{p}(X)}{p_F}\right)^{-\rho} Y_F \equiv (p_0 w_0)^{-\rho} Y_F X^{\rho\varepsilon}, \tag{12}$$

$$\hat{\ell}(X) = \frac{\hat{Y}(X)}{\phi_0} X^{-1} \equiv \frac{1}{\phi_0} (p_0 w_0)^{-\rho} Y_F X^{\rho\varepsilon - 1}, \quad (13)$$

$$\frac{\hat{\pi}(X)}{p_F} = \frac{1}{\rho}\frac{1}{p_F}\hat{p}(X)\hat{Y}(X) \equiv \frac{1}{\rho} (p_0 w_0)^{1-\rho} Y_F X^{(\rho-1)\varepsilon}, \quad (14)$$

where Y_F is output of the final good, and the constant

$$p_0 \equiv \frac{\rho}{\rho - 1} \frac{a_H^{1-\varepsilon}}{\phi_0},$$

depends on ε. The price $\hat{p}(X)/p_F$ depends only on w_0 and X, while the quantities $\hat{Y}(X)$, $\hat{\ell}(X)$, and $\hat{\pi}(X)/p_F$ also depend on Y_F. Note that firms with higher X have lower prices, higher sales, and higher profits. The effect of productivity on labor input, in (13), depends on ε, the elasticity of the wage function.

D. State variables

To analyze investment by firms and households, it is convenient to exploit the fact that on a BGP the means of the X's and H's grow at the common, constant rate g, and to look at normalized variables. To this end, define "average productivity"

$$Q \equiv \left[\mathrm{E}\left(X^\beta\right)\right]^{1/\beta}, \qquad \beta \equiv (\rho - 1)/\rho, \quad (15)$$

and the relative values $x \equiv X/Q$, $h \equiv H/Q$. On a BGP Q grows at the constant rate g, and x, h, have stationary distributions. Thus, aggregates depend on Q and individual choices on (x, Q) or (h, Q). It is immediate from the definitions of Q and x that

$$\left[\mathrm{E}\left(x^\beta\right)\right] = 1. \quad (16)$$

Use (11) in the price index (2) to find that the scale for the real wage is

$$p_0 w_0 = N^{1/(\rho-1)} \left[\int X^{\varepsilon(\rho-1)} f(X) dX\right]^{1/(\rho-1)},$$

Conjecture that $\varepsilon = 1/\rho$, which implies

$$p_0 w_0(Q) = N^{1/(\rho-1)} Q^{1/\rho}. \quad (17)$$

From (13), it also implies that the quantity of labor demanded is the same for all technologies. Hence aggregate clearing in the labor market—ignoring heterogeneity across workers—then requires

$$\frac{L}{N} = \frac{1}{\phi_0} \left(p_0 w_0(Q)\right)^{-\rho} Y_F,$$

so output of the final good is

$$Y_F(Q) = \phi_0 L N^{1/(\rho-1)} Q. \tag{18}$$

For firms, use (17) and (18) in (11)-(14) to write relative price, output, labor input, and (real) profits as

$$\begin{aligned} p(x)/p_F &= N^{1/(\rho-1)} x^{-1/\rho}, \\ Y(x,Q) &= \phi_0 \frac{L}{N} Q x, \\ \ell(x) &= \frac{L}{N}, \\ \pi(x,Q)/p_F &= \frac{\phi_0}{\rho} L N^{-1+1/(\rho-1)} Q x^{\beta}. \end{aligned} \tag{19}$$

From (19), revenue and profits are proportional to Qx^{β}. Since more productive firms employ higher quality workers, the wage paid,

$$\begin{aligned} w(x,Q) &= w_0(Q) H^*(xQ)^{1-1/\rho} \\ &= p_0^{-1} N^{1/(\rho-1)} Q^{1/\rho} \left(a_H x Q\right)^{1-1/\rho} \\ &= p_0^{-1} N^{1/(\rho-1)} Q \left(a_H x\right)^{\beta}, \end{aligned}$$

is also proportional to x^{β}. Over time, average real wages grow at the rate g, with part of the growth coming from productivity growth, in $w_0(Q)$, and part from growth in average human capital, in $H = hQ$.

3. INTERMEDIATE PRODUCERS: INVESTMENT

Let X_{ita} denote the stochastic process for the productivity of incumbent firm i that enters at date t, as function of its age a. It is a geometric Brownian motion, with parameters (g_X, σ_X^2) , where the firm chooses the drift g_X, which is its investment decision, and the variance σ_X^2 is fixed. Since Q_t grows at the rate g, so the firm's relative productivity $x_{ita} \equiv X_{ita}/Q_t$ is also a geometric Brownian motion, with parameters $(g_X - g, \sigma_X^2)$.

A. Investment by incumbents

Consider a firm's choice of g_X. Recall from (19) that the (real) profit flow for a firm with state (x, Q) is proportional to Qx^β. The cost of investment is paid in goods and, we will assume, as in Atkeson and Burstein, that the cost is scaled by current profitability. Thus, the cost of investment for that firm, if it chooses drift g_X, is $\psi_X(g_X)Qx^\beta$, where the function ψ_X is strictly increasing and strictly convex.

Since there is no fixed cost of operating, there is no voluntary exit, and the firm operates until the exogenous exit shock hits. Let $V^F(x, Q)$ denote the value of the firm as a function of the state. The HJB equation for the firm is

$$\begin{aligned}(r + \delta_F)\, V^F(x, Q) \;&=\; \max_{g_X} \left\{ \left[\frac{\phi_0}{\rho} L N^{-1+1/(\rho-1)} - \psi_X(g_X) \right] Q x^\beta \right. \\ &\quad \left. + (g_X - g)\, x V_x^F + \frac{1}{2}\sigma_X^2 x^2 V_{xx}^F + gQV_Q^F \right\}.\end{aligned}$$

It is straightforward to show that V^F has the homogeneous form $V^F(x, Q) = v_F Q x^\beta$, so the HJB equation can be written as

$$\begin{aligned}(r + \delta_F)\, v_F \;&=\; \max_{g_X} \left[\frac{\phi_0}{\rho} L N^{-1+1/(\rho-1)} - \psi_X(g_X) \right. \\ &\quad \left. + (g_X - g)\, \beta v_F + \frac{1}{2}\sigma_X^2 \beta\, (\beta - 1)\, v_F + g v_F \right]. \end{aligned} \tag{20}$$

The first-order condition for investment is

$$\psi_X'(g_X) = \beta v_F, \tag{21}$$

so g_X is independent of (x, Q) , and using g_X in (20) we find that

$$v_F \;=\; \frac{\phi_0 L N^{-1+1/(\rho-1)}/\rho - \psi_X(g_X)}{(r + \delta_F - g) - \beta\,(g_X - g) - \beta\,(\beta - 1)\,\sigma_X^2/2}$$

$$= \quad \frac{1}{\beta}\frac{\phi_0 L N^{-1+1/(\rho-1)}/\rho - \psi_X(g_X)}{\overline{g}_X - g_X} \tag{22}$$

where

$$\overline{g}_X \equiv g + \frac{1}{\beta}\left(r + \delta_F - g\right) - \frac{1}{2}\left(\beta - 1\right)\sigma_X^2, \tag{23}$$

and we require $g_X < \overline{g}_X$ so the firm has finite value. Since $\overline{g}_X > g$, average productivity at incumbent firms could be growing faster than productivity at entrant firms. Nevertheless, incumbents are exiting at a sufficiently rapid rate so that aggregate growth is coming entirely from growth in entrant productivity.

Use (22) in (21) to write the FOC for g_X as

$$\frac{\phi_0}{\rho} L N^{-1+1/(\rho-1)} = \psi_X(g_X) + \left(\overline{g}_X - g_X\right)\psi_X'(g_X). \tag{24}$$

Assume that the technology depreciates at a fixed rate $\delta_X \geq 0$ if there is no investment. The following assumption—Inada conditions in ψ_X—insures that (24) has a unique soluion.

Assumption X: The cost function $\psi_X(g_X)$ is continuously differentiable, strictly increasing, and strictly convex. In addition, $\psi_X(-\delta_X) = 0$ and $\psi_X'(-\delta_X) = 0$, where $\delta_X \geq 0$, and $\lim_{g_X \to \nu_X} \psi_X'(g_X) = +\infty$, where $-\delta_X < 0 < \nu_X \leq \overline{g}_X$.

Under Assumption X the RHS of (24) is a strictly increasing function of g_X, taking the value zero at $g_X = -\delta_X$, and diverging as $g_X \to \nu_X$.

An increase in $\phi_0 L N^{-1+1/(\rho-1)}/\rho$, the slope of the profit function, increases the solution g_X, as does anything that decreases $\overline{g}_X$. Hence an increase in g or a decrease in N increases g_X.

B. The distribution of relative productivity

Since the investment choice g_X is independent of (x, Q), it follows that X_{ita} is a geometric Brownian motion with parameters (g_X, σ_X^2). Assume that the initial values for each cohort of entrants are lognormally distributed, with a fixed variance σ_{EX}^2 and a mean that grows at the constant rate g over time. Thus, for the cohort of age a at date $t+a$,

$$\ln X_{ita} \sim N(\mu_{EX} + gt + \left(g_X - \sigma_X^2/2\right) a, \sigma_{EX}^2 + \sigma_X^2 a), \qquad \text{all } t, a,$$

where μ_{EX} is the mean of log productivity for entrants at $t = 0$.

Since Q_t grows at the constant rate g, relative productivity $x_{ita} = X_{ita}/Q_{t+a}$ for the cohort also has a lognormal distribution. Define $z_{ita} = \ln x_{ita}$ and

$$\mu_{z0} \equiv \mu_{EX} - \ln Q_0, \qquad \text{and} \qquad \gamma_z \equiv g_X - \frac{1}{2}\sigma_X^2 - g,$$

where Q_0 must be determined. Then z_{ita} has a normal distribution that does not depend on t.

$$z_{ita} \sim N\left(\mu_z(a), \Sigma_z^2(a)\right), \qquad \text{all } t, a,$$

where

$$\begin{aligned} \mu_z(a) &= \mu_{z0} + \gamma_z a, \\ \Sigma_z^2(a) &= \sigma_{EX}^2 + \sigma_X^2 a. \end{aligned} \tag{25}$$

The distribution of z across firms of all ages is a mixture of normals. In particular, since the exit rate $\delta_F > 0$ is fixed, the cohort of age a gets weight $\delta_F e^{-\delta_F a}$, all $a \geq 0$, and

$$F(z) = \int_0^\infty \delta_F e^{-\delta_F a} \Phi\left(z; \mu_{z0} + \gamma_z a, \sigma_{EX}^2 + \sigma_X^2 a\right) da, \tag{26}$$

where $\Phi(z; m, s^2)$ is a normal cdf with parameters (m, s^2). Hence the mean of the mixed distribution is

$$\begin{aligned} \overline{\mu}_z &= \int_{-\infty}^\infty \delta_F e^{-\delta_F a} \mu_z(a) da \\ &= \mu_{z0} + \gamma_z \frac{1}{\delta_F}. \end{aligned} \tag{27}$$

The variance for the cohort of age a grows like a^2. Since the exit rate is exponential in a, the variance of the mixed distribution is finite.

C. Entry

Entry costs are also paid in goods. At date t, a potential entrant can invest $I_E Q_t$ units of goods and obtain a new product. Hence the entry condition is

$$I_E Q_t \geq \mathrm{E}\left[V_F\left(X_{it0}/Q_t, Q_t\right)\right] = v_F Q_t \mathrm{E}\left[x_{i0}^{\beta}\right],$$

with equality if firms enter. Entry is strictly positive on the BGP, and from (25), the distribution of relative productivity for entrants is constant. Hence the entry condition is

$$I_E = v_F \mathrm{E}\left[x_{i0}^{\beta}\right]. \tag{28}$$

4. HOUSEHOLDS

Individuals, who are finite-lived, are organized into infinitely-lived dynastic household, with each household comprising a representative cross-section of the population. Individual members of a dynasty pool their earnings, and the dynasty allocates family income to consumption and investment in human capital. There is a continuum of identical households of total mass one.

A. Consumption

Individual household members die at a constant rate δ_L, and are replaced by an equal inflow of new members, so the size of each household, L, is constant. Each household member supplies one unit of labor inelastically, so L is also aggregate labor supply.

All household members share equally in consumption, and the household has the usual constant-elasticity preferences

$$U = \int_0^\infty e^{-\hat{r}t} \frac{1}{1-\theta} c(t)^{1-\theta} dt,$$

where $\hat{r} > 0$ is the pure rate of time preference and $1/\theta > 0$ is the elasticity of intertemporal substitution. On the BGP, per capita consumption grows at the rate g, so the real interest rate is

$$r = \hat{r} + \theta g. \tag{29}$$

Household income also grows at the rate g, so its PDV is finite if and only if $r > g$. The following restriction ensures that this is so.

Assumption G: Assume

$$\hat{r} > (1 - \theta)\, g.$$

B. Investment in human capital

New entrants into the workforce at date t have initial human capitals H_{it0} that are lognormally distributed with a mean that grows at the rate g over time time. That is, $\ln H_{it0} \sim N(\mu_{EH} + gt, \sigma_{EH}^2)$. Each individuals then makes investments continuously over his lifetime to maximize the expected (net) discounted value of lifetime earnings. The investment process is like the one for firms. Specifically, the individual chooses the drift g_H for his human capital, and pays the associated cost. The variance σ_H^2 for the process is fixed.

Recall the definition of relative human capital, $h \equiv H/Q$. The pair of state variables (h, Q) is convenient for analyzing the individual's investment problem. Recall from (5) and (17) that the individual's (real) wage rate is proportional to Qh^β. Assume the cost of investment is scaled like the wage, so the cost for an individual with state (Q, h) who chooses drift g_H is $\psi_H(g_H)Qh^\beta$, where the function ψ is strictly increasing and strictly convex.

Let $V^L(h^\beta, Q)$ denote the expected discounted value of earnings over the rest of this individual's life, if he follows an optimal investment plan. Since Q grows at the rate g and h is a geometric Brownian motion with parameters $(g_H - g, \sigma_H^2)$, the HJB equation is

$$
\begin{aligned}
(r+\delta_L)\,V^L(h,Q) &= \max_{g_H}\Big\{\left[p_0^{-1}N^{1/(\rho-1)} - \psi_H(g_H)\right]Qh^\beta \\
&\quad + (g_H - g)\,hV_h^L + \frac{1}{2}\sigma_H^2 h^2 V_{hh}^L + gQV_Q^L\Big\}.
\end{aligned}
$$

Again, it is easy to show V^L has the homogeneous form $V^L(h,Q) = v_L Q h^\beta$, so the HJB equation can be written as

$$
\begin{aligned}
(r+\delta_L)\,v_L &= \max_{g_H}\Big[\frac{1}{p_0}N^{1/(\rho-1)} - \psi(g_H) \\
&\quad + (g_H - g)\,\beta v_L + \frac{1}{2}\sigma_H^2\beta\,(\beta-1)\,v_L + g v_L\Big].
\end{aligned} \tag{30}
$$

The FOC for optimal investment is

$$
\psi_H'(g_H) = \beta v_L, \tag{31}
$$

so g_H is independent of (h,Q), and using g_H in (30) we find that

$$
\begin{aligned}
v_L &= \frac{p_0^{-1}N^{1/(\rho-1)} - \psi(g_H)}{(r+\delta_L - g) - \beta\,(g_H - g) - \beta\,(\beta-1)\,\sigma_H^2/2} \\
&= \frac{1}{\beta}\frac{p_0^{-1}N^{1/(\rho-1)} - \psi(g_H)}{\overline{g}_H - g_H}
\end{aligned} \tag{32}
$$

where

$$
\overline{g}_H \equiv g + \frac{1}{\beta}\,(r+\delta_L-g) - \frac{1}{2}\,(\beta-1)\,\sigma_H^2. \tag{33}
$$

We require $g_H < \overline{g}_H$, so expected net earnings are finite. Since $\overline{g}_H > g$, wage growth for experienced workers could be faster than overall wage growth. Nevertheless, older workers are retiring at a sufficiently rapid rate so this effect is not contributing to aggregate wage growth.

Use (32) in (31) to write the first order condition as

$$
p_0^{-1}N^{1/(\rho-1)} = \psi_H(g_H) + (\overline{g}_H - g_H)\,\psi_H'(g_H). \tag{34}
$$

Assume human capital depreciates at a fixed rate $\delta_H \geq 0$, if there is no investment. As before, it is convenient to put Inada conditions on ψ_H.

Assumption H: The cost function $\psi_H(g_H)$ is continuously differentiable, strictly increasing, and strictly convex. In addition, $\psi_H(-\delta_H) = 0$ and $\psi'_H(-\delta_H) = 0$, where $\delta_H \geq 0$, and $\lim_{g_H \to \nu_H} \psi'_H(g_H) = +\infty$, where $-\delta_H < 0 < \nu_H \leq \overline{g}_H$.

Under Assumption H the RHS of (34) is a strictly increasing function of g_H, taking the value zero at $g_H = -\delta_H$, and diverging as $g_H \to \nu_H$. Hence, given N, there is a unique value g_H satisfying (34).

An increase in $p_0^{-1} N^{1/(\rho-1)}$, the slope of the wage function, increases g_H, as does anything that decreases $\overline{g}_H$. Hence an increase in g or N increases g_H.

C. The distribution of relative human capital

Since the investment choice g_H is independent of (h, Q), the human capital H_{jta} of an individual j born at date t, as function of his age a, is a geometric Brownian motion with parameters (g_H, σ_H^2). The initial values for each cohort of newborns are lognormally distributed, with a fixed variance σ_{EH}^2 and a mean that grows at the constant rate g. Thus,

$$\ln H_{jta} \sim N(\mu_{EH} + gt + \left(g_H - \sigma^2/2\right) a, \sigma_{EH}^2 + \sigma_H^2 a),$$

where μ_{EH} is the mean of log human capital for new entrants to the workforce at $t = 0$.

Since average productivity Q_t grows at the constant rate g, relative human capital $h_{ita} = H_{ita}/Q_{t+a}$ also has a lognormal distribution. Define $\zeta_{ita} = \ln h_{ita}$, and

$$\mu_{\zeta 0} \equiv \mu_{EH} - \ln Q_0, \qquad \text{and} \qquad \gamma_\zeta \equiv g_H - \frac{1}{2}\sigma_H^2 - g.$$

Then

$$\zeta \sim N\left(\mu_\zeta(a), \Sigma_\zeta^2(a)\right), \qquad \text{all } t, a,$$

where

$$\begin{aligned} \mu_\zeta(a) &= \mu_{\zeta 0} + \gamma_\zeta a, \qquad (35) \\ \Sigma^2_\zeta(a) &= \sigma^2_{EH} + \sigma^2_H a. \end{aligned}$$

As with technology, the distribution of h in the whole population is a mixture of normals. Since the exit rate is $\delta_L > 0$,

$$M(\zeta) = \int_0^\infty \delta_L e^{-\delta_L a} \Phi\left(\zeta; \mu_{\zeta 0} + \gamma_\zeta a, \sigma^2_{EH} + \sigma^2_H a\right) da, \qquad (36)$$

where Φ is a normal cdf. The mixed distribution has mean

$$\overline{\mu}_\zeta = \mu_{\zeta 0} + \gamma_\zeta \frac{1}{\delta_L}, \qquad (37)$$

and the variance is finite.

5. THE BALANCED GROWTH PATH

To complete the description of a BGP, we must impose market clearing for every level of human capital and determine the values for various endogenous constants.

a. Market clearing for labor

Proposition 1 shows that if the cdf's $F(z)$ and $M(\zeta)$ are the mixtures of normals in (26) and (36), then a constant elasticity wage function clears the market for labor at every human capital level if $\varepsilon = 1/\rho$ and the parameters of the stationary distributions conform in a certain sense. The resulting labor demand in (13) is constant across firms.

PROPOSITION 1: Suppose the distributions of relative technology and relative human capital (in logs), the cdf's $F(z)$ and $M(\zeta)$, are the mixtures of normals in (26) and (36), and let N, L, be the mass of firms and the size (mass) of the workforce.

Then the wage function in (5) clears the market for every kind of labor if

$$\varepsilon = 1/\rho, \tag{38}$$

and

$$\begin{aligned} \mu_{\zeta 0} &= \mu_{z0} + \ln a_H, & \frac{\gamma_\zeta}{\delta_L} &= \frac{\gamma_z}{\delta_F}, \\ \sigma^2_{EH} &= \sigma^2_{EX}, & \frac{\sigma^2_H}{\delta_L} &= \frac{\sigma^2_X}{\delta_F}, \end{aligned} \tag{39}$$

where a_H is defined in (7).

PROOF: For the wage function in (5), a firm with relative productivity z chooses labor with relative human capital

$$\zeta^*(z) = z + \ln a_H.$$

As shown in section 2D, for $\varepsilon = 1/\rho$, every firm demands the same quantity of labor, which in equilibrium must be L/N. Hence market clearing for all levels for human capital requires

$$LM(z + \ln a_H) = N \int_{-\infty}^{z} \frac{L}{N} f(z) dz, \qquad \text{all } z,$$

or

$$M(z + \ln a_H) = F(z), \qquad \text{all } z. \tag{40}$$

Use (39) and the change of variable $\delta_L a = \delta_F b$ to write M as

$$M(z + \ln a_H) = \int_0^\infty \delta_F e^{-\delta_F b} \Phi\left(z; \mu_{z0} + \gamma_z b, \sigma^2_{EX} + \sigma^2_X b\right) db, \qquad \text{all } z,$$

so the required condition holds. ■

b. Levels

With $\varepsilon = 1/\rho$, the constants a_H and ϕ_0 in (7) and (9) are determined. The interest rate r in (29) depends on the exogenous parameters $g, \hat{r}$, and θ. It remains to determine N, Q_0, g_X, v_F, g_H and v_H.

Under Assumption X, the FOC (24) has a unique solution $g_X(N)$, with $\lim_{N\to 0} g_X(N) = \nu_X$, $\lim_{N\to\infty} g_X(N) = -\delta_X$, and

$$g_X'(N) = -\frac{\rho-2}{\rho-1}\frac{\phi_0 L}{\rho}\frac{N^{-2+1/(\rho-1)}}{\overline{g}_X - g_X(N)}\frac{1}{\gamma''(g_X(N))} < 0. \tag{41}$$

The normalized value of the firm in (22) then depends on N directly and also indirectly, through g_X. Call this value $v_F(N)$,

$$v_F(N) \equiv \frac{1}{\beta}\frac{\phi_0 L}{\rho}\frac{N^{-(\rho-2)/(\rho-1)} - \gamma\left[g_X(N)\right]}{\overline{g}_X - g_X(N)}. \tag{42}$$

Use the envelop theorem to find that

$$v_F'(N) \equiv -\frac{\rho-2}{\rho-1}\frac{1}{\beta}\frac{\phi_0 L}{\rho}\frac{N^{-2+1/(\rho-1)}}{\overline{g}_X - g_X(N)}.$$

Recall from (16) that $1 = \mathrm{E}(x^\beta) = \mathrm{E}_F\left[e^{\beta z}\right]$, and use (26) to find that $\mathrm{E}_F\left[e^{\beta z}\right]$ is increasing in both μ_{z0} and $g_X(N)$. For any N, let $\mu_{z0}(N)$ denote the value for which $\mathrm{E}_F\left[e^{\beta z}\right] = 1$. (This can be thought of as determining Q_0.) Since $g_X(N)$ is a decreasing function, $\mu_{z0}(N)$ is increasing. Moreover, since $g(N)$ takes values in a bounded set, so does $\mu_{z0}(N)$.

Use $v_F(N)$, $\mu_{z0}(N)$, and the expressions for $\mu_z(0)$ and $\Sigma_z^2(0)$ in (25) to write the free entry condition (28) as

$$I_E = v_F(N)\exp\left[\beta\mu_{z0}(N) + \frac{1}{2}\beta^2\sigma_{EX}^2\right]. \tag{43}$$

For $\rho > 2$, the term $N^{-(\rho-2)/(\rho-1)}$ in v_F diverges to $+\infty$ as $N \to 0$ and converges to 0 as $N \to +\infty$, while the other terms in (43) have finite ranges. Hence there is at least one solution. Moreover, for $\rho > 2$, $v_F(N)$ is a decreasing function. Therefore, unless $\mu_{z0}(N)$ is strongly increasing over some range, the solution is unique.

Given N, the values for g_X, v_F, μ_{z0}, and γ_z are determined. Then use (39) to determine γ_ζ and μ_ζ. The definitions of γ_ζ and γ_z imply that

$$g_H = g + \gamma_\zeta + \frac{1}{2}\sigma_H^2$$

or

$$\frac{1}{\delta_L}(g_H - g) = \frac{1}{\delta_F}(g_X - g) \tag{44}$$

Thus, the requirement $g_H \in (-\delta_H, \overline{g}_H)$ puts bounds on the allowable range for γ_z. Assuming the required condition holds, the cost function ψ_H must be reverse engineered so that the FOC (34) holds. The normalized value of the wage stream is then given by (32).

Clearing in the final goods market determines the level of consumption, C, as a residual: output minus investment by incumbent firms, entrants, and workers,

$$\begin{aligned} C(t) &= Y_F(t) - I_X(t) - I_N(t) - I_H(t) \\ &= \left[\phi_0 L N^{1/(\rho-1)} - N\psi_X(g_X)\mathrm{E}\left(x^\beta\right) - \delta_F N I_E - L\psi_H(g_H)\mathrm{E}\left(h^\beta\right)\right] Q(t) \\ &\equiv c_0 Q(t), \qquad\qquad \text{all } t. \end{aligned} \tag{45}$$

Hence consumption grows like Q, at rate g.

Uniqueness

Suppose $\delta_X = \delta_H = 0$ and both cost functions are quadratic: $\psi_X(g_X) = \psi_{X0} g_X^2/2$, and $\psi_H(g_H) = \psi_{H0} g_H^2/2$. Then the FOC for g_X in (24) requires

$$\frac{\phi_0}{\rho} L N^{-1+1/(\rho-1)} = \left(\overline{g}_X + \frac{1}{2} g_X\right) g_X \psi_{X0},$$

or

$$\frac{1}{2} g_X^2 + \overline{g}_X g_X - \frac{\phi_0}{\psi_{X0}\rho} L N^{-1+1/(\rho-1)} = 0.$$

The roots of this quadratic are real and of opposite sign,

$$g_X(N) = -\overline{g}_X \pm \sqrt{\overline{g}_X^2 + 2\phi_0 L N^{-1+1/(\rho-1)}/\psi_{X0}\rho}.$$

The negative root violates the lower bound on g_X, so only the positive root is of interest. That root is less than $\overline{g}_X$ if

$$2\overline{g}_X > \sqrt{\overline{g}_X^2 + 2\phi_0 L N^{-1+1/(\rho-1)}/\psi_{X0}\rho},$$

or

$$\overline{g}_X^2 > \frac{2}{3}\frac{\phi_0}{\psi_{X0}\rho} L N^{-1+1/(\rho-1)},$$

which holds if ψ_{X0} is large enough.

[What can be said about $\mu_{z0}(N)$?]

Use $g_X(N)$ and the quadratic cost function in (43) to determine N, which in turn gives $g_X(N)$. Use (44) to determined g_H, and then determine the coefficient ψ_{H0} for the second quadratic from the FOC (34),

$$\frac{1}{2} g_H^2 \psi_{H0} + (\overline{g}_H - g_H)\, g_H \psi_{H0} = p_0^{-1} N^{1/(\rho-1)},$$

or

$$\psi_{H0} = \frac{p_0^{-1} N^{1/(\rho-1)}}{g_H\left(\overline{g}_H - g_H/2\right)},$$

and

$$\begin{aligned} \beta v_L &= \frac{p_0^{-1} N^{1/(\rho-1)}}{\overline{g}_H - g_H}\left(1 - \frac{g_H}{2\overline{g}_H - g_H}\right) \\ &= \frac{p_0^{-1} N^{1/(\rho-1)}}{\overline{g}_H - g_H/2} \end{aligned}$$

[To be completed.]

6. EFFICIENCY

A. Allocation of labor

The allocation of labor at each date is efficient. Since the labor market is perfectly competitive and labor is supplied inelastically, this conclusion is not surprising. Each producer has an incentive to reduce output, to exploit his market power, tending to reduce labor demand and wages. But since labor is inelastically supplied, wages fall enough to ensure full employment.

To see that allocation of labor across producers is efficient, first note that because better technologies and higher human capital are complements in production, efficiency clearly requires assortative matching.

Given the distribution functions $F(x)$ and $G(h)$ for relative productivity and human capital, and the total masses L and N of labor and firms, function allocating $\ell(x)$ units of labor to firm x leads to the mapping $\hat{h}(\cdot)$ if

$$N \int^x \ell(v) f(v) dv = LG(h(x)), \qquad \text{all } x,$$

or

$$N\ell(x)f(x) = Lg(h(x))h'(x), \qquad \text{all } x. \tag{46}$$

An efficient allocation of labor chooses $\ell(x)$, or equivalently $h'(x)$, to maximize total output Y. Thus, it solves the calculus of variations problem

$$\max_{\{\ell(x)\}} \int N \left[\phi(h(x), x)\ell(x)\right]^\beta f(x) dx, \qquad \text{s.t. } (46),$$

or, since N and L are fixed,

$$\max \int \left[\phi(h(x), x) g(h(x)) h'(x)\right]^\beta (f(x))^{1-\beta}\, dx \equiv \max \int \Omega\left[x, h(x), h'(x)\right] dx.$$

The usual Euler equation

$$\frac{\partial \Omega}{\partial h} = \frac{d}{dx}\frac{\partial \Omega}{\partial h'}, \qquad \text{all } x,$$

here implies

$$\beta \left(\phi g h'\right)^\beta f(x)^{1-\beta} \left(\frac{\phi_h}{\phi} + \frac{g'}{g}\right) = \frac{d}{dx}\left[\beta \left(\phi g\right)^\beta f(x)^{1-\beta} \left(h'\right)^{\beta - 1}\right],$$

or

$$h'\left(\frac{\phi_h}{\phi} + \frac{g'}{g}\right) = \beta\left(\frac{\phi_h}{\phi}h' + \frac{\phi_x}{\phi} + \frac{g'}{g}h'\right) + (1-\beta)\left(\frac{f'}{f} - \frac{h''}{h}\right),$$

or

$$\frac{h''}{h} + \frac{\phi_h}{\phi}h' + \frac{g'}{g}h' - \frac{f'}{f} = \frac{\beta}{1-\beta}\frac{\phi_x}{\phi}, \qquad \text{all } x. \tag{47}$$

The resource constraint (46) for the labor market implies

$$\frac{\ell'(x)}{\ell(x)} + \frac{f'}{f} = \frac{g'}{g}h' + \frac{h''}{h'}, \qquad \text{all } x.$$

Use this fact in (47) to get

$$\frac{\ell'(x)}{\ell(x)} = \frac{1}{\phi}\left[\frac{\beta}{1-\beta}\phi_x - \phi_h h'\right], \qquad \text{all } x. \tag{48}$$

Consider linear allocation functions, $h(x) = ax$. Since $\phi_X(a,1)/\phi_H(a,1) = a^{1/\eta}(1-\omega)/\omega$, the term on the right in (48) vanishes for the value $a = a_H$ defined in (7), and (48) holds for any uniform labor allocation, $\ell(x) = \overline{\ell}$. The required level is determined by (46),

$$\ell(x) = \overline{\ell} = \frac{L}{N}, \qquad \text{all } x,$$

which agrees with (19).

Note that normalized output is

$$\begin{aligned} y &= \left[\int N\left[\phi(a_H x, x)\frac{L}{N}\right]^{\beta} f(x)dx\right]^{1/\beta} \\ &= N^{1/\beta}\phi_0\frac{L}{N}\mathrm{E}\left[x^{\beta}\right]^{1/\beta} \\ &= \phi_0 L N^{1/(\rho-1)} \end{aligned} \tag{49}$$

where the last line uses (16).

B. Investment

To ask whether investment is efficient, consider a joint perturbation to investments in technology and human capital that preserves the constant elasticity wage function. Under such a perturbation (40) must hold at every date. To this end, we will fix a small perturbation $\varepsilon_X > 0$ to g_X, for all firms at all dates, and choose the perturbations to human capital so that (40) holds.

Consider the perturbed distributions

$$\begin{aligned}\widetilde{F}(z;t) &= \int_0^\infty \Phi\left(z\ ;\ \mu_{z0}+\gamma_z b+\chi_z(b;t),\sigma_{EX}^2+\sigma_X^2 b\right)e^{-\delta_F b}\delta_F db,\\ \widetilde{M}(\zeta;t) &= \int_0^\infty \Phi\left(\zeta\ ;\ \mu_{\zeta 0}+\gamma_\zeta a+\chi_\zeta(a;t),\sigma_{EH}^2+\sigma_H^2 a\right)e^{-\delta_L a}\delta_L da, \qquad \text{all } t,\end{aligned}$$

where $\chi_z(b;t)$ and $\chi_\zeta(a;t)$ denote the cumulative perturbations to growth for the cohorts of firms of age b and workers with experience a, both at date t. Use (39) and the change of variable $\delta_F b=\delta_L a$ to write $\widetilde{F}$ as

$$\widetilde{F}(z;t)=\int_0^\infty \Phi\left(z\ ;\ \mu_{\zeta 0}-\ln\alpha_H+\gamma_\zeta a+\chi_z\left(\frac{\delta_L}{\delta_F}a;t\right),\sigma_{EH}^2+\sigma_H^2 a\right)e^{-\delta_L a}\delta_L da.$$

Then $\widetilde{F}(\zeta-\ln a_H;t)=\widetilde{M}(\zeta;t)$, all ζ,t, if

$$\chi_\zeta\left(a;t\right)=\chi_z\left(\frac{\delta_L}{\delta_F}a;t\right), \qquad \text{all } a,t\geq 0. \tag{50}$$

Since the perturbation to technology growth is constant across firms and over time, the cumulative perturbation for a firm of age b at date t is

$$\chi_z(b,t)=\varepsilon_X \min\left\{b,t\right\},$$

and (50) holds if and only if

$$\chi_\zeta\left(a;t\right)=\varepsilon_X \min\left\{a\delta_L/\delta_F,t\right\}, \qquad \text{all } a,t\geq 0. \tag{51}$$

Since χ_ζ is the integral of flow perturbations, clearly (51) holds for $t=0$ or $a=0$. Suppose $\delta_L/\delta_F<1$, which is the empirically relevant case. Then the required (flow) perturbations to human capital growth are

$$\varepsilon_H(a;t)=\begin{cases}\varepsilon_X & \text{if } t\leq a\delta_L/\delta_F,\\ \varepsilon_X\delta_L/\delta_F, & \text{otherwise.}\end{cases} \tag{52}$$

At any date t, sufficiently old workers, those with age $a>t\delta_F/\delta_L$, get the perturbation $\varepsilon_H=\varepsilon_X$, while all others get the perturbation $\varepsilon_H=\varepsilon_X\delta_L/\delta_F$. If $\delta_L/\delta_F\geq 1$, reverse the roles of X and H in the argument.

In the long run the two perturbations are proportional, with a ratio equal to the ratio of the exit rates, δ_L/δ_F. If $\delta_L/\delta_F < 1$, the perturbation to human capital growth must initially be larger for older cohorts, to compensate for the fact that the inflow of new labor is slower than the inflow of new firms. Note that the perturbation does not affect the long run growth rate g.

What is the effect of these changes on output and investment costs? Since N, L are unchanged, and the labor allocation across technologies still satisfies $H = a_H X$, we can use (18) to find that the change in output is

$$\Delta_{YF}(t) = \frac{\Delta_Q(t)}{Q(t)} Y_F(t),$$

where Δ_Q is the change in Q. On the perturbed path, the total cost of investment in technology at t is

$$\widetilde{I}_X(t) = \gamma(g_X + \varepsilon_X) N Q(t),$$

where we have used the fact that (16) continues to hold on the perturbed path. Hence the increase is

$$\Delta_{IX}(t) \approx \left[\frac{\gamma'(g_X)}{\gamma(g_X)} \varepsilon_X + \frac{\Delta_Q}{Q} \right] I_X(t), \qquad \text{all } t.$$

Similarly, the increases in the cost of investment in human capital and entry costs are

$$\begin{aligned} \Delta_{IH}(t) &\approx \left[\frac{\psi'(g_H)}{\psi(g_H)} \left[e^{-\delta_F t} + \left(1 - e^{-\delta_F t}\right) \frac{\delta_L}{\delta_F} \right] \varepsilon_X + \frac{\Delta_Q}{Q} \right] I_H(t), \\ \Delta_{IN}(t) &\approx \frac{\Delta_Q}{Q} I_N(t), \qquad\qquad \text{all } t. \end{aligned}$$

where the first line uses the fact that $e^{-\delta_F t}$ is the fraction of the workforce that gets the larger perturbation at date t.

Hence the net gain from the perturbation is

$$\begin{aligned} J \equiv & \int_0^\infty e^{-rt} \left\{ \left[Y_F(t) - I_X(t) - I_H(t) - I_N(t) \right] \frac{\Delta_Q(t)}{Q(t)} \right. \\ & \left. - \varepsilon_X \left[\frac{\gamma'(g_X) I_X}{\gamma(g_X)} + \frac{\psi'(g_H) I_H}{\psi(g_H)} \left[\frac{\delta_L}{\delta_F} + \left(1 - \frac{\delta_L}{\delta_F}\right) e^{-\delta_F t} \right] \right] \right\} dt. \end{aligned} \tag{53}$$

Recall from (45) that the term in brackets in the first line of (53) is simply consumption, $C(t) = c_0 Q_0 e^{gt}$. Hence the first line in (53) is

$$J_1 = c_0 Q_0 \int_0^\infty e^{(g-r)t} \frac{\Delta_Q}{Q} dt. \tag{54}$$

Recall that on the original BGP

$$[Q(t)]^\beta = \mathrm{E}\left[X(t)^\beta\right].$$

Let $\widetilde{X}(t)$ denote the perturbed technology and let $\widetilde{x}(t) = \widetilde{X}(t)/Q(t)$ denote the perturbed technology relative to $Q(t)$. Then on the perturbed path

$$\left[\widetilde{Q}(t)\right]^\beta = \mathrm{E}\left\{\left[\widetilde{X}(t)\right]^\beta\right\} = \mathrm{E}\left\{[\widetilde{x}(t)]^\beta\right\} Q(t)^\beta, \qquad \text{all } t \geq 0.$$

Hence on the perturbed path

$$\left[\frac{\widetilde{Q}(t)}{Q(t)}\right]^\beta = \int_0^\infty \mathrm{E}\left\{[\widetilde{x}(b,t)]^\beta\right\} e^{-\delta_F b} \delta_F db, \qquad \text{all } t,$$

where the expression on the right uses the linearity of the expectation operator. For each age cohort b, and each date t, $\widetilde{x}(b,t)$ is lognormally distributed, with parameters $[\mu_z(b) + \varepsilon_X \min\{b,t\}, \Sigma_z^2(b)]$. Recall that for a lognormally distributed random variable x with parameters (m, s^2), $\mathrm{E}\left[x^\beta\right] = \exp\{\beta m + \beta^2 s^2/2\}$. Hence

$$\begin{aligned} \mathrm{E}\,[\widetilde{x}(b,t)]^\beta &= \exp\left[\beta\mu_z(b) + \beta\varepsilon_X \min\{b,t\} + \beta^2 \Sigma_z^2(b)/2\right] \\ &= \mathrm{E}\,[x(b)]^\beta e^{\beta\varepsilon_X \min\{b,t\}}. \end{aligned}$$

The integral above is then

$$\begin{aligned} \left[\frac{\widetilde{Q}(t)}{Q(t)}\right]^\beta &= \int_0^t \mathrm{E}\,[x(b)]^\beta e^{\beta\varepsilon_X b} e^{-\delta_F b} \delta_F db \\ &\quad + e^{\beta\varepsilon_X t} \int_t^\infty \mathrm{E}\,[x(b)]^\beta e^{-\delta_F b} \delta_F db, \qquad \text{all } t, \end{aligned}$$

Hence

For the second line in (53), recall that aggregate investments in the two kinds of capital are

$$
\begin{aligned}
I_X(t) &= N\gamma(g_X)Q(t)\mathrm{E}\left(x^\beta\right), \\
I_H(t) &= L\psi(g_H)Q(t)\mathrm{E}\left(h^\beta\right), \qquad \text{all } t,
\end{aligned}
$$

and recall from (16), (21) and (31) that

$$
\mathrm{E}\left(x^\beta\right) = 1, \qquad \mathrm{E}\left(h^\beta\right) = a_H^\beta, \qquad \gamma'(g_X) = \beta v_F, \qquad \psi'(g_H) = \beta v_H.
$$

Since $Q(t)$ grows at the rate g, use these facts and (52) to find that the second term in (53) is

$$
\begin{aligned}
J_2 &\equiv \beta Q_0 \varepsilon_X \int_0^\infty e^{-(r-g)t} \left\{ N v_F + L v_H a_H^\beta \left[\frac{\delta_L}{\delta_F} + \left(1 - \frac{\delta_L}{\delta_F}\right) e^{-\delta_F t} \right] \right\} dt \\
&= \frac{\beta Q_0 \varepsilon_X}{r-g} \left\{ N v_F + L v_H a_H^\beta \left[\frac{\delta_L}{\delta_F} + \frac{r-g}{\delta_F + r - g} \left(1 - \frac{\delta_L}{\delta_F}\right) \right] \right\} \\
&= \frac{Q_0 \varepsilon_X}{r-g} \beta \left[N v_F + L v_H a_H^\beta \frac{\delta_L + r - g}{\delta_F + r - g} \right]. \qquad (55)
\end{aligned}
$$

Hence the perturbation is welfare improving if and only it

$$
c_0 \left(r-g\right) \int_0^\infty e^{(g-r)t} \frac{\Delta_Q}{Q} dt > \varepsilon_X \beta \left[N v_F + L v_H a_H^\beta \frac{\delta_L + r - g}{\delta_F + r - g} \right].
$$

The term on the right is the annualized income flow from technology and human capital, multiplied by $\beta = 1 - 1/\rho$, one minus the markup. If the stated inequality holds, a positive perturbation raises welfare, and if the inequality runs the other way, a negative perturbation raises welfare. Investment is efficient if and only if the two terms are equal, and there doesn't seem to be any reason why they would be. [Add a numerical example.]

7. CONCLUSIONS

[To be completed]

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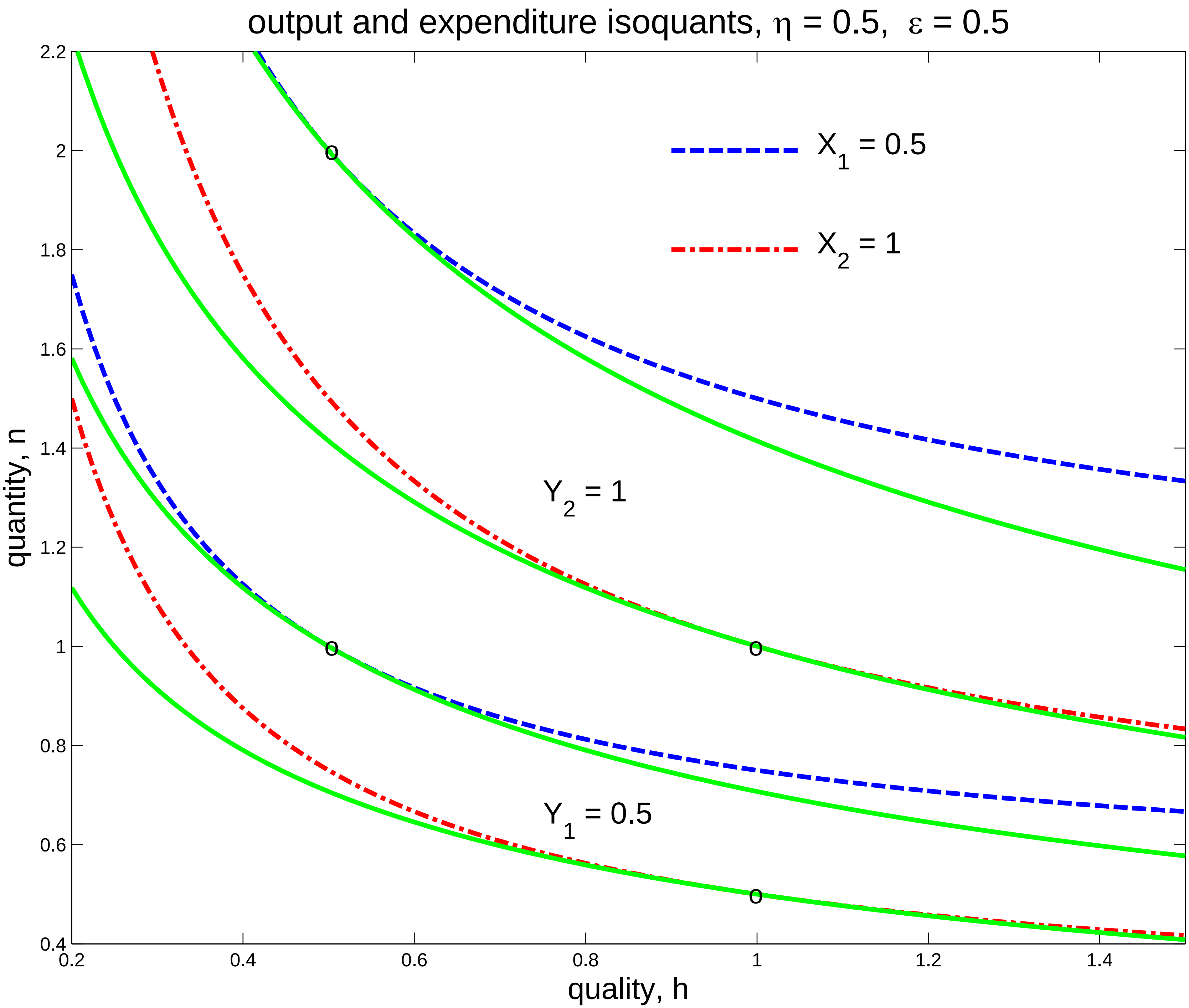
output and expenditure isoquants, η = 0.5, ε = 0.5
X1 = 0.5
X2 = 1
Y2 = 1
Y1 = 0.5
quantity, n
quality, h
2.2
2
1.8
1.6
1.4
1.2
1
0.8
0.6
0.4
0.2
0.4
0.6
0.8
1
1.2
1.4