Back to Fundamentals: Abstract Competitive Equilibrium

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Abstract
We propose a new abstract definition of competitive equilibrium: a profile of alternatives and a public ordering (prestige, price or a social norm) such that each agent prefers his assigned alternative to all lower-ranked ones. The equilibrium operates in an abstract setting built upon a concept of convexity borrowed from Convex Geometry. The "magic" of linear equilibrium prices is put into perspective by establishing an analogy between linear functions in the standard convexity and "primitive orderings" in the abstract convexity. We apply the concept to a variety of convex economies and relate it to Pareto efficiency.

Keywords: Competitive Equilibrium, Social Norms, Convex Geometry.

We acknowledge financial support in the form of ERC grant 269143. We are grateful to Jacopo Perego, Debraj Ray and Rani Spiegler for helpful comments.

1. Introduction

In this paper, we return to the fundamental concept of competitive equilibrium and extend the notion to a more abstract setting. The extension is based on the idea that competitive equilibrium is a method of creating harmony in an interactive situation with a feasibility restriction and self-interested agents, using a public ordering of the alternatives. This ordering either limits the choices available to the agents or systematically influences their preferences. In the standard economic setting, it is specified by prices that apply equally to all agents. These prices determine consumers' choice sets and producers' preferences. We propose an analogous solution concept adjusted to fit more abstract situations in which valuation using prices is replaced by valuation according to a public ordering.

The road to the construction of the equilibrium notion starts with a discussion of the notion of convexity. In the Euclidean setting, the algebraic notion of convexity is central to the standard analysis of competitive equilibrium. As we primarily consider settings that lack an algebraic structure, we will employ a more abstract form of convexity. Definitions of convexity involve the primitive phrase "b is between the elements $a^1, \ldots, a^L$". In an Euclidean space, this means that b is an algebraic convex combination of a^1,...,a^L. However, the phrase has a more common meaning and is even used in daily conversation. For example, one can say that Game Theory is between Mathematics and Economics and that Switzerland is culturally between Germany, Italy and France. Therefore, the first step will be to borrow a formal concept of convexity from the existing literature of Convex Geometry (see Edelman and Jamison (1985)). This concept specifies, for each set A, a set $K(A)$ of elements that are "between elements in A" and defines a set A as *convex* if $K(A) = A$. We then present a new characterization result for convex geometries based upon a collection of "primitive orderings": a set $K(A)$ contains any element which, for every primitive ordering, is ranked above some element in A (and which element that is may depend on the ordering). We say that such a set of primitive orderings *generates* the convexity. The primitive orderings play an analogous role to that of linear functions in the case of standard convexity.

The second step involves defining the economic object to be studied. A *convex economy* is a model that consists of: (*i*) a set of agents; (*ii*) a set of elements from which each agent chooses; (*iii*) the agents' preference relations over the set of elements; (*iv*) a feasibility constraint on choice profiles; and (*v*) a set of primitive orderings that generates a notion of convexity.

After establishing this framework we proceed to analyze several definitions of competitive equilibrium. The first definition is of *unrestricted competitive equilibrium* (UCE). This definition is in the spirit of Shapley and Scarf (1974)'s concept of equilibrium for the housing economy model. A UCE is defined as a profile of choices together with an arbitrary ordering on the set of elements that satisfies two conditions: (i) each choice assigned to an agent is preferred by him to all lower-ordered alternatives; and (ii) the profile is feasible. We refer to the equilibrium ordering as a *public ordering* whose interpretation will be discussed later. In our setting, it plays an analogous role to that of prices in the standard economic setting. It is shown that every Pareto-efficient outcome is supported by some UCE but UCE may be Pareto-inefficient.

We then proceed to discuss restrictions on equilibrium public orderings by imposing connections between them and the underlying convexity notion. In particular, we introduce the concept of a *primitive competitive equilibrium* (PCE), which is an UCE with the additional requirement that the public ordering must be one of the primitive orderings (which generate the convexity). This is analogous to the requirement that prices in the standard setting be linear. We apply these solution concepts to convex economies in which agents have convex preferences. In particular, we study the relationship between PCE outcomes and Pareto efficiency (the first and second fundamental welfare theorems).

Finally, we consider six examples of convex economies, each with an underlying economic story. These examples are intended to demonstrate the richness of the approach.

2. **Convex Geometry**

The basic concepts of convex geometry are given in Edelman and Jamison (1985). Let X be a set (finite unless stated otherwise) whose members we call *elements*. Convexity is defined through an operator $K : 2^X \to 2^X$ with the interpretation that $K(A)$ is the set of elements that are "between elements in A" (including the elements of A themselves). In the standard analysis, K is the convex hull operator. A set A is *convex* if $K(A) = A$.

Note that this concept of convexity allows us to say that "c is between a and b" (by stating that $c \in K(\{a,b\})$ but not a phrase like "c is a quarter of the way from a to b" (expressed as $c = 0.75a + 0.25b$ in the standard algebraic convexity).

A *convex geometry* is an operator K that satisfies the following properties:

(A1) $A \subseteq K(A)$ and $K(\emptyset) = \emptyset$ ("extensitivity").

(A2) $A \subseteq B$ implies $K(A) \subseteq K(B)$ ("monotonicity").

(A3) $K(K(A)) = K(A)$ ("idempotence").

(A4) If A is convex, $a, b \notin A$ and $a \in K(A \cup b)$, then $b \notin K(A \cup a)$ ("anti-exchange").

A1 captures the degenerate sense in which each element in a set is between the set's elements. A2 means that an element which is between elements of a set is also between elements of any larger set. One direction of A3, $K(K(A)) \supseteq K(A)$, follows from A1. The other direction, $K(K(A)) \subseteq K(A)$, means that any element which is between elements that are themselves between elements of A is also between elements of A. A4 states that if (i) A is convex; (ii) a and b are not in A; and (iii) a is between b and elements of A, then it is impossible for b to be between a and elements of A. Of course, all four properties hold for the standard Euclidean case where K is the convex hull operator. A non-example is the operator $K(A) \equiv X$ which satisfies A1, A2 and A3 but not A4 (since Ø is convex and $K(\{a\}) = K(\{b\}) = X$).

A key to our discussion of competitive equilibrium is a new representation theorem of convex geometries. The theorem generalizes a property that holds regarding the standard convex geometry in Euclidean spaces: *A point is in the convex hull of a set if and only if, for every linear ordering, there is a weakly lower element in the set.* In other words, a point is outside the convex hull of a set if there is a linear ordering that places it below all members of the set.

To illustrate, in Figure 1, w is not in the convex hull of $\{x, y, z\}$ since there is a linear ordering (depicted by the dashed line) that ranks w below x, y and z. On the other hand, w is in the convex hull of $A = \{x, y, z, v\}$ since every linear ordering ranks w above at least one of the elements in A.

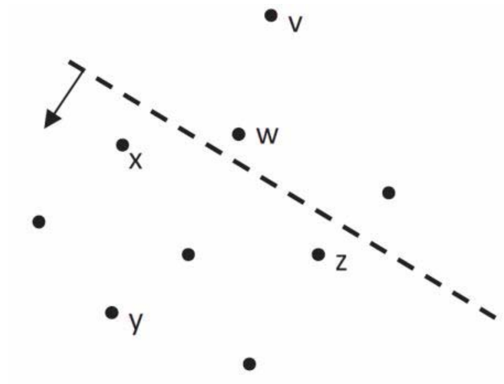

Figure 1

The representation theorem states that for any finite convex geometry there exists a set of orderings that play a role analogous to that of the algebraic linear orderings in the standard Euclidean setting. By an *ordering*, we mean a reflexive, complete, anti-symmetric and transitive binary relation. Note that in our terminology orderings are strict.

We say that the set of orderings $\{\geq_k\}$ *generates* K and that its members are *primitive orderings* for K, if for all A:

$$K(A) = \{x \mid \forall k, \exists a_k \in A \; s.t. \; x \geq_k a_k\}.$$

One interpretation of this representation is that agents have a set of criteria (orderings) in mind with which they reflect on alternatives. A set is convex if for any element outside the set, one of the criteria ranks it "inferior" to all elements in the set. To illustrate, in the case that X is a finite set in an Euclidean space with the standard convexity, a set of primitive orderings is given by the set of linear orderings that do not have any "ties" between elements of X.

Note that in the standard setting a coupling property holds: If an ordering is primitive, then so is its inverse. We do not make such an assumption here; some of the convex geometries we consider satisfy this property while others do not.

Claim 1 (Representation Theorem for Convex Geometries):

(a) For every (finite) convex geometry, there is a set of orderings that generates it.

(b) Any set of orderings generates a convex geometry.

Proof: (a) Let K be a convex geometry. Theorem 2.2 of Edelman and Jamison (1985) states that every maximal chain of convex sets $\emptyset \subset C_1 \subset C_2 \subset \ldots \subset X$ has length $|X|+1$. For completeness, here is a proof. Since $\emptyset$ and X are convex, it suffices to show that for any two convex sets $A \subset B$ where $|B - A| > 1$, there is a convex set C such that $A \subset C \subset B$. By A1 and A2, for every $x \in B \backslash A$ we have $A = K(A) \subset K(A \cup x) \subseteq K(B) = B$ and by A3, $K(A \cup x)$ is convex. Take two elements $a, b \in B \backslash A$. If neither $K(A \cup a)$ nor $K(A \cup b)$ is a proper subset of B, then $K(A \cup a) = K(A \cup b) = B$, violating A4.

For any maximal chain of convex sets $\emptyset = C_0 \subset C_1 \subset \ldots C_{|X|} = X$, define $c_l = C_l \backslash C_{l-1}$ and attach an ordering $c_1 > \ldots > c_{|X|-1} > c_{|X|}$. We now show that the set of the orderings $\{\geq_k\}$ attached to all maximal chains of convex sets generates K.

If $y \in K(A)$ and if there is an attached ordering $\geq_k$ such that $a >_k y$ for all $a \in A$, then by construction there is a convex set L containing A and $y \notin L$. However, $y \in K(A) \subseteq K(L) = L$, a contradiction. Thus, $y \in \{x \mid \forall k, \exists a_k \in A \text{ s.t. } x \geq_k a_k\}$.

If $y \notin K(A)$, take a maximal chain of convex sets that extends $\emptyset \subseteq K(A) \subseteq X$. By the corresponding ordering $\geq_k$, it must be that $a >_k y$ for all $a \in A$. Thus, $y \notin \{x \mid \forall k, \exists a_k \in A \text{ s.t. } x \geq_k a_k\}$.

(b) Let $\{\geq_k\}$ be a set of orderings. Define $K(A) = \{x \mid \forall k, \exists a_k \in A \text{ s.t. } x \geq_k a_k\}$. Clearly, K satisfies A1, A2 and A3. Regarding A4, assume that A is convex, $a, b \notin A$ and $a \in K(A \cup b)$. Since $a \notin K(A)$ there is an ordering $\geq_k$ such that $x >_k a$ for all $x \in A$. Since $a \in K(A \cup b)$ it must be that for the same ordering $a >_k b$. Thus, $x >_k b$ for any $x \in A \cup a$ and therefore $b \notin K(A \cup a)$. ■

Note that the strictness of the primitive orderings is essential for Claim 1(b) as otherwise the induced K may violate the anti-exchange property. We will return to this issue in Section 6C. It is beyond the scope of this paper, but one can prove the claim above also for infinite X using Zorn's Lemma.

Following the standard terminology, we say that a preference relation $\succsim$ is *convex* if for every $x^* \in X$ the *strict upper contour set* $U(\succsim, x^*) = \{x \mid x \succ x^*\}$ is convex. The first part of the next lemma states that primitive orderings are convex in the geometry that they induce. This is analogous to the Euclidean property that *the strict upper contour sets of linear functions, namely open half-spaces, are convex*. The second part shows that, as in Euclidean spaces, *the weak upper contour sets of convex preferences are convex* as well. The third part establishes another analogy with the Euclidean setting: *The convex hull of a set is the* *intersection of all open half-spaces which contain that set.*

Lemma 2: Let K be a convex geometry generated by $\{\geq_k\}$. Then:

(i) Each ordering $\geq_k$ is convex.

(ii) For any element x^* and convex preferences $\succsim$, the set $\{x \mid x \succsim x^*\}$ is convex.

(iii) $K(A)$ is the intersection of all strict upper contour sets that contain A.

Proof: (i) Let $U = \{x \mid x >_k a\}$ and take $y \in K(U)$. As $\{\geq_k\}$ generates K, there is an element $z_k \in U$ such that $y \geq_k z_k$. Thus, $y \in U$ and U is convex.

(ii) If x^* is $\succsim$-minimal, then $\{x \mid x \succsim x^*\} = X$, which is a convex set. Otherwise $\{x \mid x \succsim x^*\} = \cap_{\{z \mid x^* \succ z\}} \{x \mid x \succ z\}$ and in general the intersection of a collection of convex sets $\{A_i\}$ is convex. To see this, notice that by A2 for all j, $K(\cap A_i) \subseteq K(A_j) = A_j$ and therefore $K(\cap A_i) \subseteq \cap A_i$. Combined with $A1$, we obtain $K(\cap A_i) = \cap A_i$.

(iii) We now show that $K(A)$ is the intersection of all sets $U(\geq_k, b)$ that contain A: by (i), any $U(\geq_k, b)$ is convex and thus, by A2, if $A \subseteq U(\geq_k, b)$, then $K(A) \subseteq U(\geq_k, b)$. In the other direction, assume $y \notin K(A)$. Then, there is an ordering $\geq_k$ s.t. for every $a \in A$, $a >_k y$, and therefore $A \subseteq U(\geq_k, y)$, but $y \notin U(\geq_k, y)$. ■

Note that the set of primitive orderings generating a convexity is not unique since any convex ordering can be added to the set of primitives and still generates the same convexity. On the other hand, if a non-convex ordering is added to a set of primitives that generates K, then this larger set of primitives generates a different convexity K'. The convexities are related since for every A, $K'(A) \subseteq K(A)$ and thus every convex set according to K is also a convex set according to K'. However, K' will have additional convex sets: The non-convex ordering has a strict upper contour set that is not convex in K, but this set must be convex in K' (as shown in Lemma 2(i)).

Here are some examples of convex geometries:

Example 1: Degenerate Convexity

The convex geometry $K(A) \equiv A$ captures the degenerate case in which no element is between any combination of other elements.

The set of all orderings on X generates this convexity. For a minimal set of primitive orderings, list all elements $\{z_1, .., z_L\}$ and take the L orderings

$z_l >_l z_{l+1} .. >_l z_L >_l z_1 >_l \ldots >_l z_{l-1}$.

Example 2: Box Convexity

The box convexity is defined based on a set of orderings $\{\geq_k\}$ as
$K(A) = \{x \mid \forall k,\ \exists a_k, b_k \in A \text{ s.t. } b_k \geq_k x \geq_k a_k\}$.

An element belongs to $K(A)$ if according to each criterion it is sandwiched between some pair of elements in A. This geometry is generated by the set of all orderings $\geq_k$ and their reversals.

An example that this type of convexity fits is the case where the elements are characterized by a vector of attribute values and an element is included in $K(A)$ if its value for each attribute is not an extreme with respect to A.

Example 3: **Monotonic Convexity**

Let R be a reflexive anti-symmetric partial ordering of a set X and define
$K(A) = \{x \mid \exists a \in A \text{ s.t. } xRa\}$.

By Theorem 3.2 in Edelman and Jamison (1985), every convex geometry K that satisfies the additional *union property* $K(A \cup B) = K(A) \cup K(B)$ can be represented as such a convex geometry. A set of primitive orderings for this geometry is the set of all completions of the partial ordering R. However, there are often smaller sets of primitives which are more natural.

A setting that fits this convexity: X is a set of products and aRb means that a can be produced from b. The set $K(A)$ contains all the products that can be produced from any single element of A. Another setting which fits this convexity: X is the set of members of a hierarchical organization and aRb means that a is a superior of b. The expression $x \in K(A)$ means that x is superior to at least one member of A.

Note that this convexity differs from the others in the sense that $a \in K(A)$ depends solely on the presence of certain other individual elements in A while in the other examples it depends on the presence of certain combinations of elements in A. Thus, the notion of convexity here has the flavor of a monotonicity condition rather than of betweenness.

Example 4: **Set Union Convexity**

Let Z be a set of elements and X be the set of all menus (non-empty subsets of Z). Define $K(A)$ as the set of all menus that are unions of menus in A. This type of convexity fits a setting where a set stands for the support of a distribution. For the support of one distribution to be between the supports of two others, it must be that it is the union of those two supports. This type of convexity relates to the standard algebraic convexity for distributions in that if a distribution can be expressed as an algebraic combination of a set of distributions, then its support is equal to the union of the supports of the combined distributions. Thus, for distributions, this is a weaker notion of betweenness than the standard algebraic one.

We now show that K is generated by the set of all extensions of the strict partial orderings $\{R_z\}_{z \in Z}$ defined by bR_za if $b \supset a$ and $z \notin b - a$.

On the one hand, take a menu $a \in K(A) \backslash A$. Then a is a union of its strict subsets in A. For $z \in a$, there is a menu $c_z \in A$ such that $z \in c_z \subset a$ and thus aR_zc_z. For $z \notin a$, take $c_z \in A$ such that $c_z \subset a$ and thus aR_zc_z Therefore, a is not minimal for any extension of any R_z.

On the other hand, take a menu a that is not minimal in $A \cup \{a\}$ for every extension of every R_z. Then, for every z there must be a menu b_z such that aR_zb_z. This implies that for every $z \in a$, $z \in b_z \subseteq a$. Thus, $a = \cup_{z \in a} b_z$ and $a \in K(A)$.

Comments on the use of abstract geometry in Economics:

(1) Koshevoy (1999) pointed out a connection between the literature on convex geometry and that of choice theory for finite sets. He compared the properties of choice correspondences to those of the operator $ext(A)$, defined as the set of all $x \in A$ such that $x \notin K(A - x)$ (an element is *extreme* in A if it is not between other elements of A). The operator *ext* satisfies two familiar properties in the choice theory literature:

Heritage: If $A' \subseteq A$, then $ext(A') \supseteq ext(A) \cap A'$.

Outcast: If $ext(A) \subseteq A' \subseteq A$, then $ext(A') = ext(A)$.

Heritage is actually the α property in Sen (1970) and Outcast is Postulate 5* in Chernoff (1954). The representation $ext(A) = \{x \in A \mid x$ is the $\geq_k$ -minimum in A for some $\geq_k \}$ can be derived from Claim 1. Our representation could also have been proved using Koshevoy's observations and Aizerman and Malishevski (1981)'s result: A choice correspondence C satisfies Heritage and Outcast if and only if there is a finite number of orderings over X, such that $C(A)$ is the set of the unique maximums of these orderings in A.

(2) Baldwin and Klemperer (2013) study the existence and properties of standard competitive equilibria in an Euclidean setting with indivisibilities using *tropical geometry*, a new concept related to algebraic geometry. The mathematical concepts they used and the economic issues involved are not closely related to those studied in this paper.

3. The convex economy

We now turn to define the abstract economic environment. Let $N = \{1,..,n\}$ be a set of agents. Each agent chooses an *element* from an abstract set X endowed with a convex geometry K generated by a set of primitive orderings $\{\geq_k\}$. No further structure is imposed on X. A profile assigns one element to each agent. Not all profiles are feasible. The feasibility constraint is given by a set $F \subset X^N$. We assume that F is closed under all permutations but the definitions are not contingent upon this assumption. We have in mind (conventional and unconventional) examples like the following:

The Exchange Economy: $X = R_+^L$ and an element of X is interpreted as a bundle in a world with L commodities. The set F is the set of all allocations of a total endowment ω among the agents.

The Housing Economy: The set X contains n houses. A feasible allocation assigns a distinct house to each of the n agents.

The Sequential Production Economy: There is a set X of alternatives. One element of X is denoted as w and stands for the starting point of the sequential production process. A production process orders the agents and each agent's production is restricted by his predecessor's production. A profile is feasible if there is a listing of the agents $i_1,\ldots,i_n$ such that $x^{i_0} = w$ and $x^{i_m} \in f(x^{i_{m-1}})$ where f is a correspondence from X to itself.

The Paired Matching Economy: $X = N$ and each agent chooses a partner. The feasibility constraint is that if i chooses j, then j must choose i.

The Paired Exchange Economy: $X = R_+^L$ and each agent has an endowment. A feasible profile is an outcome of some partition of the agents into pairs such that trade takes place only within pairs.

Each agent i possesses a preference relation $\succsim^i$ on X (an upper index always indicates the agent). Thus, unlike in a typical game setting, he is interested only in the element he himself chooses, independent of other agents' choices. We assume that each $\succsim^i$ is *convex*. Given the representation theorem in Claim 1, the convexity of preferences has the following novel interpretation: All agents have a set of criteria $\{\geq_k\}$ in mind. An agent who ranks all elements in a set Y to be superior to x^* must also rank z to be superior to x^*, if for each criterion $\geq_k$ there is $y^k \in Y$ such that $z >_k y^k$.

We will refer to the tuple $< N, X, \{\succsim^i\}_{i \in N}, F, \{\geq_k\} >$ as a *convex economy*.

Note that we included a set of primitive orderings in the definition of a convex economy rather than just a convexity notion. As mentioned earlier, there are many different sets of primitive orderings that generate the same convexity. The specification of a particular set of primitive orderings is intended to capture some prominent aspect of the economy (as do algebraic linear orderings in the case of the standard economy) and will play an important role throughout the paper.

Notice also that our concept of an economy lacks "initial endowments". We will discuss this point in the Final Comments.

4. Abstract competitive equilibrium

The constraints imposed by the set *F* will typically introduce conflicts between the agents. An equilibrium concept provides a method of resolving such conflicts in a way that produces some form of stability:

Definition: An *unrestricted competitive equilibrium (UCE)* is a pair $< (x^i)_{i \in N}, P >$ where $(x^i)_{i \in N}$ is a profile and *P* is an ordering on *X*, such that: (i) the profile is in *F* and (ii) for each *i*, the element x^i is $\succsim^i$ -optimal in the set $B(P, x^i) = \{z | \, x^i P z \text{ or } z = x^i\}$.

We refer to the ordering *P* as a *public ordering* and use the letter *B* to emphasize the analogy to the familiar term "budget set". We call $< (x^i)_{i \in N}, P >$ an unrestricted competitive equilibrium in order to emphasize that the definition of UCE imposes no restrictions on *P* (such as convexity) apart from being a strict ordering. The requirement that *P* be strict is without loss of generality since any equilibrium profile supported by an ordering that contains indifferences will also be an equilibrium profile with arbitrary breaking of the indifferences.

Our main interpretation of this concept of equilibrium views *P* as a social ordering that reflects the elements' worth or prestige. The term *aPb* means that *a is more expensive than b* or that *a is more prestigious than b*. For a profile to be a competitive equilibrium, there must exist a public ordering such that each agent is satisfied with his assigned element given his ability to replace it only with an element that is considered less expensive or less prestigious according to the public ordering. An equilibrium ordering stabilizes the equilibrium profile in the sense that each agent is happy with his assignment *given* the "worth" of his assigned element.

Our second interpretation is that *P* is a socially agreed-upon or imposed motive that systematically affects the agents' preference relations. The relation *aPb* means that "*a* is *less* *socially desirable* than b" (or "a is *less prestigious* than b"). An agent can choose any alternative in X, but must rationalize his choice as furthering society's goals (or alternatively he can't bear to suffer a loss of prestige). Thus, the ordering P systematically affects the agents' preferences: given P and an assigned element x^i, the agent's lexicographical first priority is to "not move up the P ordering" while his second priority is to maximize his original preference. An equilibrium is a profile and a social ordering such that no agent wishes to deviate from his assigned element (given his personal preferences) and is able to justify it as furthering the social goal (or alternatively as not decreasing his prestige).

The two interpretations are dual: according to the main interpretation, agents consider only the elements further down the public ordering to be feasible, while according to the alternative interpretation, all elements are feasible, but agents find only elements further down the P ordering (that is, higher in prestige) to be socially acceptable.

The existence of an UCE does not require any further assumptions and follows from the following version of the Second Fundamental Welfare Theorem (SWT):

Claim 3 (SWT-UCE): Any Pareto-efficient profile is an unrestricted competitive equilibrium profile.

Proof: Let $(a^i)_{i \in N}$ be a Pareto-efficient profile. Define the relation R by xRy if (i) there are i and j such that $x = a^i \succ^j a^j = y$ or (ii) $x \notin \{a^1, \ldots, a^n\}$ and $y \in \{a^1, \ldots, a^n\}$. The first condition guarantees that if j envies the element assigned to i he will not be able to replace his element with i's. The second condition guarantees that agents will find any unassigned element "unaffordable".

Since the profile is Pareto-efficient and F is closed to permutations, R does not have cycles and thus can be extended to a complete ordering P. According to this ordering, a^j is optimal in $B(P, a^j)$ for each agent j. ■

According to the public ordering built in the proof of Claim 3, any alternative not assigned to one of the agents is too "expensive" to be considered. An agent is only left to consider exchanging his alternative for one that is assigned to another agent. The equilibrium public ordering on the set of assigned alternatives is defined as in the housing economy where the houses are the set of assigned elements.

Note that any Pareto-inefficient profile can also be an UCE profile as long as there are no cycles in the envy relation. In particular, any feasible profile that assigns the same element to all agents is supported by an *unrestricted* public ordering that makes it the cheapest good.

Furthermore, note that the public ordering constructed in Claim 3 does not relate to the convexity notion of the economy. In particular, if $X - \{a^1, \ldots, a^N\}$ is not convex, then the constructed public ordering necessarily fails to be convex.

5. Primitive Competitive Equilibrium

In standard economic models, where the set X is a subset of an Euclidean space, the following hold:

I) The set of algebraic linear orderings generates the convex geometry.

II) The public ordering (price system) is required to be an algebraic linear ordering.

Thus, a natural analogy is to require that the equilibrium public ordering is one of the primitive orderings generating the geometry (and thus is convex, as shown in Claim 2). This brings us to the central definition of the paper:

Definition: Let $< N, X, \{\succsim^i\}_{i \in N}, F, \{\geq_k\} >$ be a convex economy. A *primitive competitive equilibrium (PCE)* is an UCE $< (x^i)_{i \in N}, P >$ where P is one of the primitive orderings.

We focus on PCE, but occasionally also refer to the weaker notion of UCE *with a convex public ordering (CCE)*. Convexity of the public ordering means that any set of the form $U(P, x^*) = \{x \mid xPx^*\}$ is convex. The set $U(P, x^*)$ is the set of elements that an agent who holds x^* finds unaffordable. The convexity of those sets is a reasonable requirement if the market is managed by a market maker who declares all exchanges he is willing to make. The term yPx^* means that the market maker is not willing to exchange y for x^*. Accordingly, $U(P, x^*)$ is the set of all elements preferred by the market maker to x^*. The requirement that any set $U(P, x^*)$ be convex is equivalent to the assumption that the market maker's preferences are convex. Another context in which the price ordering is naturally convex is an exchange market with non-linear prices where quantity discounts are available. In such a market, if two bundles are unaffordable, then so is any algebraic convex combination of the bundles.

Note that the notion of CCE depends only on the convexity induced by the set of primitive orderings and, unlike the notion of PCE, does not depend on the particular set of primitive orderings.

The existence of a PCE is not guaranteed. The convex economy in the following example does not even have a CCE:

Example: Consider the housing economy with four houses arranged on a line $a - b - c - d$, with the standard convexity. Two of the agents, 1 and 2, are "leftish" and hold the convex preferences $a \succ^i b \succ^i c \succ^i d$, while the other two, 3 and 4, are "rightish" and hold the convex preferences $a \prec^i b \prec^i c \prec^i d$. Claim 3 guarantees the existence of an UCE (consisting of the profile (a, b, c, d) and the public ordering $aPdPbPc$).

We now show that there is no CCE. In equilibrium, it is impossible that a leftish agent will be assigned a rightish house $x \in \{c, d\}$, since then one of the rightish agents will be assigned a leftish house $y \in \{a, b\}$, and it must be that both xPy and yPx. Thus, without loss of generality, the only possible equilibrium profile is (a, b, c, d). For the ordering P to support this profile, it must be that aPb and dPc. Therefore, b or c is the P-minimal element and $U(P, b) = \{a, c, d\}$ or $U(P, c) = \{a, b, d\}$, both of which are not convex.

What is special about the standard exchange economy that makes every Pareto-efficient allocation a PCE profile? It can be attributed to the following Richness property (illustrated in Figure 2):

We say that the convex economy $< N, X, \{\succsim^i\}_{i \in N}, F, \{\geq_k\} >$ satisfies *Richness* if the following holds: Let $\succsim$ and $\succsim'$ be two convex preferences over X and let a and a' be two elements in X. Let $\geq$ and $\geq'$ be two different primitive orderings such that a is $\succsim$-maximal in $B(\geq, a)$ but not in $B(\geq', a)$ and a' is $\succsim'$-maximal in $B(\geq', a')$ but not in $B(\geq, a')$. Then, there is a pair (b, b') such that (i) whenever $(a, a', a^3, .., a^n) \in F$ it is also the case that $(b, b', a^3, .., a^n) \in F$ and (ii) $b \succ a$ and $b' \succ' a'$.

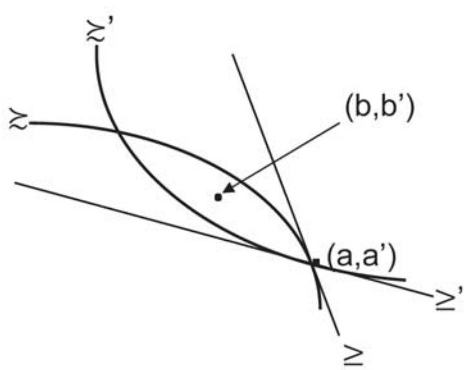

Figure 2

Claim 4 (SWT-PCE): In a convex economy that satisfies Richness, any Pareto-efficient profile is a PCE profile.

Proof: Let $\{x^i\}$ be a Pareto-efficient feasible profile. By the convexity of preferences, the set $U^i = \{z \mid z \succ^i x^i\}$ is convex and $x^i \notin U^i$ for each agent i. Therefore, there is at least one primitive ordering that ranks x^i below all members of U^i. Let O^i be the (non-empty) set of all such primitive orderings.

The intersection $\cap_i O^i$ is not empty since otherwise there would be two agents i and j such that O^i and O^j are non-nested sets. Take $\geq_i \in O^i \backslash O^j$ and $\geq_j \in O^j \backslash O^i$. The element x^i is $\succsim^i$ -maximal in the set $B(\geq_i, x^i)$ but not in $B(\geq_j, x^i)$, and likewise for agent j. Richness then implies that there is a pair of elements (y^i, y^j) such that the profile obtained by replacing the pair (x^i, x^j) with (y^i, y^j) in the profile $\{x^i\}$ is feasible and Pareto-dominating. Thus, there exists $\geq_k \in \cap_i O^i$ and then $< \{x^i\}, \geq_k >$ is a PCE. ■

In the case of a one-agent convex economy, no richness condition is necessary and every feasible element (and not just the agent's preference-maximal elements) is an outcome of some PCE.

Claim 5: In a one-agent convex economy, for every feasible element x^*, there exists a PCE $< (x^*), P >$.

Proof: Let x^* be a feasible element. By convexity of the agent's preferences, the set $\{x \mid x \succ^1 x^*\}$ is convex. Since x^* is not a member of this convex set, there is a primitive ordering $\geq_k$ such that $x >_k x^*$ for every $x \succ^1 x^*$. Thus, x^* is $\succsim^1$ -optimal in $\{x \mid x^* \geq_k x\}$. Taking P to be $\geq_k$, we conclude that $< (x^*), P >$ is a PCE. ■

With regard to the First Fundamental Welfare Theorem (FWT), Claim 5 demonstrates that a PCE profile need not be Pareto-efficient. Furthermore, in multi-agent settings, if all agents have identical convex preferences equal to one of the primitive orderings, then every feasible profile combined with that primitive ordering is a PCE.

The following claim provides a condition under which the FWT holds for PCE. The condition states that there are no two feasible profiles that Pareto-dominate one another according to some primitive ordering. This condition is satisfied by two prominent convex economies: i) the standard exchange economy with the standard convexity and ii) the housing economy with any convexity.

Claim 6 (First Fundamental Welfare Theorem): Consider a convex economy such that there are no two feasible profiles (a^i) and (b^i) and a primitive ordering $\geq_k$ such that for all i either $b^i = a^i$ or $b^i >_k a^i$. Then, any PCE profile (a^i) is weak Pareto-efficient (in the sense that there is no other feasible (b^i) such that for all i either $b^i = a^i$ or $b^i \succ^i a^i$).

Proof: Assume that $< (a^i), \geq_k >$ is a PCE. If (a^i) is not weak Pareto-efficient, then there is another feasible profile (b^i) such that for all i either $b^i = a^i$ or $b^i \succ^i a^i$. Then, for all i, either $b^i = a^i$ or $b^i >_k a^i$, a contradiction. ■

6. Economic Examples I

In this section we analyze the various concepts of competitive equilibrium in the context of several simple convex economies. In particular, we examine the relationships between the equilibrium concepts and Pareto efficiency.

A. The "give and take" Economy

Consider a society in which agents either give to or take from a voluntary public fund. The fund must be balanced. An agent's preferences embody his attitude towards the tradeoff between being egalitarian and being selfish. Each agent has in mind an ideal amount that he wishes to either contribute or withdraw. The problem is that agents' ideals may not match in the sense that the sum of the ideal contributions of the people who wish to give, is not equal to the sum of the ideal withdrawals of those who wish to take.

In this economy, there are two natural orderings: one values more giving and less taking; the other is the opposite. The standard convexity is generated by these two orderings. A competitive equilibrium provides order in the society by assigning a contribution or withdrawal to each of the agents along with a common social ladder. In equilibrium, no agent wishes to deviate from his assignment in the socially acceptable direction. In a PCE, the common social ladder is not arbitrary but rather is one of the two primitive orderings that generate the convexity.

The economy: Let $X = [-1, 1]$; a positive number represents a contribution to the social fund and a negative number represents a withdrawal from the social fund. We take the primitive orderings to be the increasing and decreasing orderings which generate the standard convexity. Assume that each agent i has convex preferences with a single peak denoted by $peak^i$. Let F be the set of all profiles that sum up to 0. The interesting case arises when

$\sum peak^i \neq 0$. Without loss of generality, we assume that $\sum peak^i < 0$.

Only one of the primitive orderings is an equilibrium public ordering: If there is a PCE with the increasing ordering, then all agents must be at or to the left of their peak (otherwise, he desires to and can afford to move to his peak), thus violating the feasibility constraint. On the other hand, any feasible profile that assigns to each agent an element at or to the right of his peak, together with the decreasing ordering, is a PCE.

The decreasing public ordering embodies the social norm that only allows an agent to consider giving more or taking less than his assigned element. This is a sound norm to govern a voluntary public fund in a society where the "average" tendency of agents, as reflected by their preferences, is to take rather than to give. According to the anti-prestige interpretation, giving more or taking less is more prestigious. This again brings order in a society where agents would ideally prefer to take more than is assigned to them but do not do so, not because of a feasibility restriction, but rather because they don't want to lose prestige.

FWT: The condition in Claim 6 holds and therefore any PCE outcome is weak Pareto-efficient. However, there can be a CCE profile that is not weak Pareto-efficient. Consider a case with an even number of agents, half of them "leftish" with negative peaks and half of them "rightish" with positive peaks. The feasible Pareto-inefficient profile in which each "leftish" agent is assigned -1 and each "rightish" agent is assigned 1 is supported by the convex public ordering P, where xPy if $|x| \leq |y|$. The social norm expressed by this ordering allows an agent to move from an assigned element only to a "more extreme" one.

SWT: Given the assumption that $\sum peak^i < 0$, in any Pareto-efficient feasible profile, all agents are at or to the right of their peak, which is supported by the decreasing public ordering. (All agents being to the left of their peak violates feasibility. If one agent is to the left of his peak and another is to the right of his peak, there is a Pareto improvement where each agent moves closer to his peak.) Alternatively, note that the Richness condition in Claim 4 holds.

B. An Economy with Production

This example suggests a possible expansion of the convex economy model to a world with production. It highlights the fundamental difference in the way we commonly model consumers as opposed to producers: while consumers' preferences are taken to be exogenous, producers' preferences are determined only by the public ordering.

The Economy: Let X consist of four locations $a - b - c - d$ arranged on a line with the natural convexity generated by the increasing and decreasing orderings.

There are six consumers, each of whom can consume one unit at one location. The consumers hold preferences over the locations where they will consume the product. Consumers 1, 2 and 3 hold the convex preferences $a \succ b \succ c \succ d$ while consumers 4, 5 and 6 hold the convex preferences $a \prec b \prec c \prec d$.

There are six producers, each of whom can produce one unit at one location. The producers are divided into three pairs: $\{1,2\}$, $\{3,4\}$ and $\{5,6\}$, who can produce the good at the convex sets of locations: $\{a,b\}$, $\{b,c\}$ and $\{c,d\}$, respectively. Producers are "profit maximizers" in the sense that they rank locations according to the public ordering.

Each of the producers and consumers chooses a location. The feasibility constraint requires that "supply equals demand", that is, at each location, the number of producers is equal to the number of consumers.

A competitive equilibrium is given by a public ordering on X (possibly with indifferences) and a feasible production and consumption profile, such that no consumer wishes to switch from his assigned location to any (weakly) cheaper one and no producer can move to a strictly higher-ranked location.

The economy has a competitive equilibrium with a non-convex public ordering: The feasible profile with the production vector (a,a,b,c,d,d) and the consumption vector (a,a,b,c,d,d) is supported by the public ordering $aIdPbIc$ (where xIy means that x and y are "equally ranked").

Ties in the equilibrium public ordering cannot be arbitrarily broken, unlike in a "consumers-only economy", since producers' preferences depend on the public ordering. In particular, breaking the tie between b and c will result in either producer 3 or 4 no longer being assigned an optimal feasible location.

A CCE does not exist: If there were a CCE, then either a or d would be minimal with respect to the public ordering. In such an equilibrium, this location must be assigned to the three consumers who rank it highest, but only two producers can locate there.

C. An Almost Standard Exchange Economy

The Economy: Let $X = \Pi_{l=1}^{L}[0,z_l]$ be a set of bundles in an L-commodity world with total endowment z. Feasibility is given by $\sum_{i=1}^{n} x^i = z$. All agents hold monotonic and convex preference relations. Agents have in mind a non-empty (finite or infinite) set of linear orderings $\Lambda = \{\geq_k\}$ *(with indifferences),* each of which is characterized by a non-negative vector $v_k \neq 0$ such that $x \geq_k y$ if $v_k \bullet x \geq v_k \bullet y$. Define the operator K as:

$$K(A) = \{x \mid \forall k,\ \exists a_k \in A,\ x \geq_k a_k\}.$$

This operator is defined as in our representation theorem but with a key difference that the underlying orderings have indifferences. The operator satisfies A1, A2 and A3, but not A4. However, it does satisfy a slightly weaker version of A4: If A is convex and x and y are two points outside A and if x is in the interior of $K(A \cup y)$, then y is not in the interior of $K(A \cup x)$.

When Λ is a singleton, all agents hold the unique convex preference relation represented by the utility function $v_1 \bullet x$. In the case that $\Lambda = \{(1,0),(0,1)\}$, each indifference curve must be "right-angled" and every efficient allocation has the feature that either no agent has a surplus of good 1 or no agent has a surplus of good 2.

FWT: The economy satisfies the conditions of Claim 6 and the argument there holds and thus any competitive equilibrium with a public ordering from Λ is weak Pareto-efficient.

SWT: The Richness property used in Claim 4 holds and thus any Pareto-efficient allocation is a competitive equilibrium profile with a public ordering from Λ. The standard textbook SWT only guarantees the existence of a competitive equilibrium with some algebraic linear ordering but not necessarily one from Λ.

7. Economic Examples II

In each of the examples in this section we use a convexity of the form:

$$K(A) = \{x \mid \text{there exists } y \in A \text{ s.t. } xRy\}$$

where R is a partial ordering (reflexive, transitive and anti-symmetric, but not necessarily complete). As mentioned earlier (example 3), such convexities are precisely those that satisfy the union property $K(A \cup B) = K(A) \cup K(B)$.

Here, a preference $\succsim$ is convex if and only if it is a completion of R. To verify, suppose that $\succsim$ is convex and xRy. The set $\{z \mid z \succsim y\}$ is convex and contains y and therefore contains x and thus $x \succsim y$. On the other hand, if $\succsim$ is an extension of R, any set $\{z \mid z \succsim y\}$ is convex since if wRz and $z \succeq y$, then $w \succeq y$ as $\succsim$ extends R. Thus, requiring that agents' preferences are convex is the same as requiring that if aRb then all agents must weakly prefer a to b. Thus, convexity of preferences here has the flavor of a monotonicity property.

The following claim is a version of the SWT and, in particular, shows the existence of CCE for these convexities:

Claim 7 (SWT-CCE): Consider a convex economy in which the convexity satisfies the union property. Then, any Pareto-efficient profile is a CCE profile.

Proof: Let (x^i) be a Pareto-efficient profile. Define the binary relation B as yBz if either i) yRz or ii) for some agent i, $z = x^i$ and $y \succ^i x^i$.

The binary relation B has no cycles. Assume the contrary and take a minimal cycle $z^1Bz^2B\ldots Bz^mBz^1$. By the acyclicity of R, there is some l and i for which $z^l \succ^i z^{l+1}$. By the Pareto-optimality of (x^i), there is some other l for which z^lRz^{l+1}. Thus, there are three consecutive elements in the cycle for which $z^lRz^{l+1} \succ^i z^{l+2}$. However, by its convexity, $\succ^i$ is an extension of R and therefore $z^l \succ^i z^{l+2}$ and thus the cycle can be shortened.

Since the binary relation B is strict and has no cycles, it has a transitive completion to an ordering P. This ordering extends B and thus extends R and therefore is convex. In addition, for all i, if $y \succ^i x^i$, then yBx^i and so yPx^i and therefore $< (x^i), P >$ is a CCE. ■

D. The Partially Ordered Housing Economy

The Economy: We return to the standard housing economy and take the set of primitive orderings to be the set of all extensions of a partial ordering R. This is the maximum set of primitive orderings as it includes all convex orderings. Assume that all agents have convex strict preferences. The standard housing model is obtained when R is the empty relation, which implies that all orderings are convex.

As mentioned earlier, the convexity of agents' preferences means here that if aRb then all agents prefer a to b. For example, the houses may have some objective attributes and aRb could mean that a is superior to b according to all of the attributes.

Any UCE is a PCE: To demonstrate this special feature, let $< (x^i), P >$ be a UCE. We prove that bRa implies bPa and as such P is an extension of R and thus is a primitive ordering. Let i be the agent who is assigned a. By the convexity of i's preferences, the set $\{x \mid x \succsim^i a\}$ is convex and by A2 it includes $K(\{a\})$, which by the definition of K includes b. Thus, by the strictness of i's preferences, $b \succ^i a$ and therefore bPa.

FWT: The structure of the feasibility constraint (consisting of all the permutations of the set of houses) implies the following property: for any strict ordering on the set of houses, there are no two feasible profiles such that one "dominates" the other. This property is stronger than the one used in Claim 6 and therefore any PCE profile, and thus any UCE profile, is Pareto-efficient.

SWT: By Claim 3, any Pareto-efficient profile is supported by some public ordering (also shown in Piccione and Rubinstein (2007)). As shown above, it must be a primitive ordering. Thus, in this economy any Pareto-efficient profile is a PCE profile.

E. The "*Kosher* food" Exchange Economy

We now consider an exchange economy consisting of two goods: meat and dairy. The goods are divisible and a consumer can either consume a quantity of meat or a quantity of dairy but not both.

The Economy: Let $X = M \cup D$ where $M = \{(a,0) \mid 0 < a \leq 3\}$ and $D = \left\{(0,b) \mid 0 < b \leq 3\right\}$ (for simplicity, we exclude the zero vector from X). The feasibility constraint is $\sum_i x^i = (3,3)$. Convexity is induced by xRy if $x \geq y$. Convexity of preferences is equivalent to monotonicity in both goods.

We take the set of primitive orderings to contain only the two orderings $\geq_M$ and $\geq_D$. The ordering $\geq_M$ is the increasing ordering that places all the elements of M above D and $\geq_D$ is the increasing ordering that places all elements of D above M.

FWT: Every PCE profile is Pareto-efficient as this economy satisfies the condition in Claim 6: Let (x^i) and (y^i) be two feasible profiles such that for every i, $x^i \geq_M y^i$ (an identical argument applies to $\geq_D$). For each agent i, $x_1^i \geq y_1^i$. Feasibility requires that $\sum x_1^i = \sum y_1^i = 3$ and therefore, for each agent, $x_1^i = y_1^i$. Thus, for any agent i who is assigned $x_1^i = y_1^i > 0$, it must be that $x^i = y^i$. For any agent who is assigned $x_1^i = y_1^i = 0$, it must be that $x_2^i \geq y_2^i > 0$. Feasibility again implies that for all i, $x_2^i = y_2^i$ and thus $x^i = y^i$.

SWT: Claim 7 applies and thus any Pareto-efficient profile is a CCE profile.

However, a Pareto-efficient profile may not be part of a PCE. Suppose, for example, that there are four agents with identical convex preferences satisfying $(2,0) \succ^i (0,3) \succ^i (0,1) \succ^i (1,0)$. The profile $((2,0),(0,2),(0,1),(1,0))$ is not a PCE profile although it is Pareto-efficient. (In any Pareto-dominating profile, 1 gets more than two units of meat and at least one of the agents 2 or 3 gets more than one unit of meat, thus violating feasibility.)

Note that, unlike the standard exchange economy, the Kosher food exchange economy allows for Pareto-efficient allocations that are not supported by any algebraic linear public ordering. If the above profile were supported by an equilibrium public ordering of the form $p_1x_1 + p_2x_2$, then either: $p_1 \le p_2$ and agent 2 would deviate from $(0,2)$ to $(2,0)$, or, $p_1 > p_2$ and agent 4 would deviate from $(1,0)$ to $(0,1)$.

F. The Set Allocation Economy

We now consider an economy with a set of indivisible goods in which (unlike in the housing economy) each agent can be allocated more than one good.

The Economy: Let Z be a collection of items and X be the set of all of its subsets (menus). We will use lower case letters for elements of Z and Greek symbols for menus. All agents have strict and convex preferences. The set F contains all profiles that allocate each item to one agent.

The convexity is induced by taking R to be the inclusion relation, that is, $K(A) = \{\Theta \mid \text{there exists } \Lambda \in A \text{ s.t. } \Theta \supseteq \Lambda\}$. This convexity is generated by the set of all strict orderings $\{\ge_v\}$ where v is a positive-valued function on Z and $\ge_v$ is represented by the utility function $\sum_{z\in\Theta} v(z)$. To see this, we show that for all A, $K(A) = L(A)$ where

$$L(A) = \{\Theta \mid \text{for every } \ge_v \text{, there exists } \Lambda_v \in A \text{ s.t. } \sum_{z\in\Theta} v(z) \ge \sum_{z\in\Lambda_v} v(z)\}.$$

If $\Theta \in K(A)$, then there exists $\Lambda \in A$ such that $\Lambda \subseteq \Theta$ and therefore $\sum_{z\in\Theta} v(z) \ge \sum_{z\in\Lambda_v} v(z)$ for every positive-valued function v. Thus, $\Theta \in L(A)$.

If $\Theta \notin K(A)$, then $\Lambda \not\subseteq \Theta$ for all $\Lambda \in A$. In that case, take a strict ordering $\ge_v$ induced by a function v for which $0 < v(z) < \frac{1}{2N}$ for all $z \in \Theta$ and $v(z) > 1$ for all $z \notin \Theta$. For this function v, $\sum_{z\in\Lambda} v(z) > 1 > 1/2 > \sum_{z\in\Theta} v(z)$, and therefore $\Theta \notin L(A)$.

FWT: All PCE are Pareto-efficient: The condition of Claim 6 holds. To see this, take a primitive ordering $\ge_v$. Any two feasible profiles, (Θ^i) and (Λ^i) satisfy $\sum_i \sum_{z\in\Theta^i} v(z) = \sum_{z\in Z} v(z) = \sum_i \sum_{z\in\Lambda^i} v(z)$. Therefore, if $\sum_{z\in\Theta^i} v(z) \ge \sum_{z\in\Lambda^i} v(z)$ for all i, then equality holds for all i and by the strictness of $\ge_v$, $\Theta^i = \Lambda^i$ for all i.

There are CCE that are not Pareto-efficient: Let $Z = \{a,b,c,d\}$ and $n = 2$. Both agents have identical preferences that rank any cardinally larger set higher and therefore are convex. Additionally, $ac \succ^i ab$ and $bd \succ^i cd$. The profile (ab, cd) together with the convex public ordering being identical to their preferences is a CCE which is Pareto-dominated by (ac, bd). Notice that it is not possible for a primitive ordering $\ge_v$ to rank both $acPab$ (so $v(c) > v(b)$)

and $bdPcd$ (so $v(b) > v(c)$) .

SWT: Claim 7 applies and thus every Pareto-efficient profile is a CCE profile.

There are Pareto-efficient profiles that are not PCE: Let $Z = \{a,b,c,d\}$. There are two agents, both of whom have preferences that rank any cardinally larger set higher and therefore are convex. Agent 1 ranks bd, ac above ab and ab above any other two-element set. Agent 2 ranks ad, bc above cd and cd above any other two-element set. The Pareto-efficient profile (ab, cd) is not supported by a PCE since a primitive public ordering P must rank $abPac$ which implies that $v(c) < v(b)$. Similarly, we conclude that $v(c) < v(b) < v(d) < v(a) < v(c)$, a contradiction.

8. Final comments

There were two motivations for this paper. The first was to suggest an abstract extension of the concept of competitive equilibrium that is suited to environments where agents make individualistic decisions and there is a feasibility constraint on the profiles of choices.

The second motivation was more methodological. Abstracting the concept of a competitive equilibrium helps to identify the properties that make it what it is and to sharpen the distinction between competitive and Nash equilibrium. The concept of a public ordering was defined and provided a decentralization method parallel to that of prices in the standard economic setting. Claims analogous to the first and second fundamental welfare theorems were proved and applied to several simple convex economies.

In this paper, we have not proven a general existence theorem for PCE (the existence of UCE was trivially proved). Often, we have shown the existence of PCE through the second welfare theorem, but we have also considered settings where existence is not guaranteed.

As mentioned earlier, our notion of a convex economy does not include initial endowments. This reflects our view of competitive equilibrium analysis as taking place in two stages. In the first stage, one defines an equilibrium concept. In the second stage, one looks for methods of selecting an equilibrium, perhaps given additional information such as an initial endowment profile.

We have focused only on the first stage. One approach for the second stage would be to add an initial profile to the definition of a convex economy and use the following definition: an equilibrium is a public ordering and a feasible profile such that each agent’s assigned alternative is best given the "budget set" defined by his initial assignment and the public ordering. Within this approach, for the housing economy, David Gale provided a beautiful existence proof cited in Shapley and Scarf (1974). His proof could be extended to show the existence of an unrestricted competitive equilibrium for our model with an initial profile. To see it, take a public ordering that orders those in the initial profile according to Gale's top-trading cycle argument and places all initially unassigned elements above them. A more intricate task would be to show the existence of a primitive competitive equilibrium in our model with an initial profile. For such an existence theorem, one would need to augment the model with additional topological structure.

It is a cliché to say so, but it is our belief that this paper should be the first stage of a deeper investigation into the abstract definition of competitive equilibrium presented here.

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