*** Working Draft ***

GENERIC VIRTUAL DETERMINACY FOR STATIONARY OVERLAPPING GENERATIONS

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We formulate and prove generic virtual determinacy for a class of stationary overlapping-generations models that allows a continuum of differentiated commodities in continuous time. Since that class subsumes a finite list of commodities in discrete time, we prove for many of the leading examples of indeterminacy in the literature there exists a generic (residual) perturbation of preferences that ensures virtually determinacy. That is, there may be a continuum of equilibria in a neighborhood of a reference equilibrium, but consumers are indifferent about moving among those alternative equilibria. Such findings contradict the common conclusion that indeterminacy in those examples is robust.

Keywords: generic, determinacy, overlapping generations.

1. INTRODUCTION

The general-equilibrium literature supports appraising models by their determinacy: If a model is determinate, then forecasting future consumption from fundamentals (preferences, endowments, and the underlying delineation of commodities and consumer lifespans) is conclusive, and rational expectations are model consistent, with agents inside the model using the model to determine their expectations. Debreu interprets any general-equilibrium model with at least one locally-unique equilibrium allocation as determinate insofar as consumers only consider small deviations from that equilibrium [2, Section 2]. Kehoe and Levine describe a stronger standard for determinacy in stationary overlapping-generations models. They suppose the economy attained a steady state in the past, and require for determinacy that staying on that steady-state path in the future be locally unique among all (stationary and non-stationary) equilibria [6]. One interpretation of that standard is agents inside the model only know the model in a neighborhood of their past experience, and so only consider small deviations from the steady-state path.

The literature finds many classes of models are generically determinate when there is a fixed, finite population of consumers for all time [2]; but many other models may be indeterminate when there is an infinite population of overlapping-generations of consumers spread through time [4, 7, 8]. To illustrate the latter possibility, consider stationary, pure-exchange overlapping-generations models with one consumer per generation, two (40-year) periods per lifetime, and a single type of consumption good (food) available only in the exact middle of each period (lunch). Under standard conditions, Gale shows almost every such model has two steady states (autarky and the Golden rule), and one steady state is locally unique and the other is not [4].[1] For the latter steady state, there are a continuum of nearby non-stationary equilibrium allocations of "lunch" consumption. Under Kehoe and Levine's standard, the model is thus determinate if exogenous, past consumption were the locally-unique steady state; but indeterminate if past consumption were the other state.

Standard analysis in the literature suggest such indeterminacy is robust since it remains after any suitably-small perturbation of preferences and endowments for lunch consumption. However, Burke selects one of Gale's models and finds indeterminacy is not robust to perturbing *all* fundamentals: preferences, endowments, *and* the underlying delineation of commodities. The perturbation of Gale's model increases the number of goods per period by judiciously splitting the single good (lunch) into two close substitutes (*lunch* and *dinner*) [1]; it allows a continuum of equilibrium allocations in a neighborhood of the past steady state, but ensures consumers are indifferent between staying on the steady-state path or moving to any one of those alternative equilibria. The perturbation thus changes the model from indeterminate to *virtually* determinate. The current paper advances that analysis in three ways that make determinacy among overlapping generations more like determinacy among a fixed, finite population of consumers.

First, the current paper makes the perturbation of the selected Gale model fit standard analysis by adapting to overlapping generations the commodity-differentiation model of Jones [5], then embedding both the original 1-good model and the perturbed 2-good model as special cases of a single model with a continuum of commodities differentiated by their moment of availability in continuous time. The perturbation that changes indeterminacy to virtual determinacy perturbs preferences and endowments, but not the delineation of commodities. The embedded 1-good model (our Example 1) has a total endowment of 3 units of lunch and 0 of dinner, and the embedded 2-good model (Example 2) divides that 3 units between lunch and dinner.

Second, the current paper develops similar perturbations that ensure virtual determinacy in a universe of economic models that accommodates a continuum

[1] In particular, having every steady state locally unique is exceptional; it only happens when there is a unique equilibrium. Overlapping-generations analysis thus differs from Debreu's analysis for a finite number of consumers and goods, where the sufficient conditions found for guaranteeing at least one equilibrium is locally unique are the same as the conditions guaranteeing every equilibrium is locally unique [2].

of differentiated commodities in continuous time. That includes, as embedded special cases, all examples in the literature based on a fixed, finite number of commodities available in discrete time [4, 7, 8].

Third, the current paper develops a completely metrizable space of economic models and shows virtual determinacy holds for a residual set of models. That perturbations that ensure virtual determinacy are thus generic. Further, the equilibrium correspondence is virtually continuous, meaning indeterminacy returns slowly if the economic model is perturbed out of the virtually-determinate residual set.

The current paper is restricted, however, to stationary models and assumed intertemporal local substitution, where consumption at nearby moments in time must be close substitutes. We thus do not address indeterminacy based on continuous-time preferences defined by an integral of instantaneous strictly-concave utility functions. For example, Demichelis and Polemarchakis analyze determinacy for preferences defined by logarithmic instantaneous utility functions [3]. Such preferences violate intertemporal local substitution because the integral is negative infinity if consumption were ever zero over any brief time interval, regardless of consumption at all other times. It may be that determinacy properties of models depends on the local substitution of their preferences.

Because of our restriction to preferences and endowments that are stationary across past, present and future generations, we offer the current paper as a prototype for the analysis of equilibrium in general overlapping-generations models satisfying intertemporal local substitution. Our positive generic determinacy result should provoke the study of determinacy for nearly stationary economies, such as appear in comparative-statics exercises.

Section 2 adapts to overlapping generations the commodity differentiation model of Jones [5], including the definition of an *economy* and a simple illustration of indeterminacy (Example 1). Section 3 defines *virtual determinacy* of an "economy", and finds a perturbation that changes Example 1 from indeterminate to virtually determinate (Example 2). Section 4 defines a completely metrizable space of economies and states the main result (Theorem 4.1) that generic economies are "virtually determinate". Section 5 generalizes results to subsume most indeterminacy examples in the literature based on a fixed, finite number of commodities in discrete time. Finally, proofs are in two appendices.

2. ECONOMIES AND EQUILIBRIA

This section adapts the commodity differentiation model of Jones [5] to an infinite sequence of overlapping generations. We carefully define economies and equilibria, then embed one of Gale's models [4] to illustrate indeterminate equilibria in a canonical case of a continuum of differentiated commodities.

Births and deaths occur at discrete time periods $t \in \mathbb{Z} := \{0, \pm 1, ...\}$. Throughout the paper, preferences and endowments are stationary across generations.

And until Section 5, there is only one consumer (one type of consumer) per generation, and are only two time periods per lifetime.

At each time period $t \in \mathbb{Z}$, there are a continuum of differentiated commodities indexed by an arbitrary compact metrizable space X. Fix any metric d_X inducing the topology on X. The canonical case is $X := [0,1]$, with either x in X specifying one of a continuum of types of commodities available at discrete period t, or x in X specifying one moment in continuous time within period t. In the latter case, only one type of commodity is available, $x = 0$ is the beginning of the time period; $x = .5$, the middle; and $x = 1$, the end.[2] Other cases include $X := \{1, ..., N\} \times [0,1]$ for N types of commodities in continuous time, and $X := [0,1] \times [0,1]$ for a continuum of types in continuous time.

At each time t, a **commodity** is an element x in X, and a commodity **bundle** is a finite signed Borel measure on X. Let $M(X)$ denote the vector space of such measures on X, and $M^+(X)$ the cone of positive measures. Under positive bundle μ, interpret $\mu(B)$ as the total amount of commodities having characteristics in the Borel subset $B \subset X$. In the canonical case, the Dirac measure δ_x ($\delta_x(B) = 1$ if $x \in B$, 0 otherwise) is a "gulp" of 1 unit at moment $x \in X$, and a measure μ that is absolutely continuous (with respect to Lebesgue measure λ) is a "sip".

Consumer $t \in \mathbb{Z}$ lives in time periods t and $t+1$. Consumer t chooses pairs $c = (\mu, \nu)$ of bundles, with $\mu \in M^+(X)$ at time t and $\nu \in M^+(X)$ at time $t+1$. Let $L := M(X) \times M(X)$ denote the **consumption set** of such lifetime bundles, and L^+ the cone of positive bundles. Represent preferences by a **utility** function u over L^+. Consumer t has an initial commodity **endowment** $e = (\alpha, \beta) \in L^+$. To keep notation straight, note: Greek letters $\mu, \nu, \alpha, \beta, ...$ denote commodity bundles in $M(X)$, and Roman letters $x, c, e, ...$ denote either pure commodities in X or pairs of bundles in L.

At each time t, prices are a continuous function on the commodity type space X. Let $\mathcal{C}(X)$ be the Banach space of such functions on X, and $\mathcal{C}^+(X)$ the cone of positive functions. Given price p in $\mathcal{C}(X)$ and bundle μ in $M(X)$, value is defined in the usual way as a bilinear product

$$p \cdot \mu = \int_X p(x) d\mu(x) \tag{1}$$

In particular, when consumption is a finite sum of gulps ($\mu = \sum_i m_i \delta_{x_i}$),

$$p \cdot \mu = \sum_i p(x_i) m_i$$

and when μ is absolutely continuous in the canonical case ($\mu(B) = \int_B f(x)dx$),

$$p \cdot \mu = \int_X p(x) f(x) dx$$

[2] The continuous time interpretation requires careful treatment of consumption since the last moment in period t is the same as the first moment in period $t+1$.

A **price system** is a sequence $\{p_t\}_{t\in\mathbb{Z}}$ of positive within-period price functions $p_t \in \mathcal{C}^+(X)$. A "price system" is **stationary** when $p_t = R^t \bar{p}$ for some discount factor $R > 0$.

Given an initial endowment $e = (\alpha, \beta)$, an **allocation** is a sequence $\{c_t\}_{t\in\mathbb{Z}}$ of consumption $c_t = (\mu_t, \nu_t) \in L^+$ that balances materials in each period t, when the old Consumer $(t-1)$ trades with the young Consumer t:

$$\nu_{t-1} + \mu_t = \beta + \alpha \qquad (t \in \mathbb{Z}) \tag{2}$$

Given a utility function and an endowment, a **steady state** is an allocation that is stationary ($c_t = \bar{c}$) and supported by a stationary price system in the sense that lifetime consumption $\bar{c} = (\bar{\mu}, \bar{\nu})$ by each consumer maximizes utility $u(c)$ over all consumption $c = (\mu, \nu) \in L^+$ satisfying the budget constraint

$$\bar{p} \cdot \mu + R\,\bar{p} \cdot \nu \leq \bar{p} \cdot \alpha + R\,\bar{p} \cdot \beta \tag{3}$$

The fundamental requirement for determinacy is whether an economy that attained a steady state in the past necessarily stays on that path in the future. Hence, define an **economy** as a triple $(u, e, \bar{c})$ of the utility function u over L^+ and endowment $e = (\alpha, \beta) \in L^+$ of each consumer, and of the past steady state $\bar{c}$ (3) for (u, e). Call an allocation $\{c_t\}$ for an economy $(u, e, \bar{c})$ an **equilibrium** if it is supported by a price system in the sense that

E.1: Consumption is at the steady state before period 1: ($\mu_t = \bar{\mu}$ for $t \leq 0$, and $\nu_t = \bar{\nu}$ for $t \leq -1$)

E.2: Consumption ν_0 by old Consumer 0 maximizes utility $u(\bar{\mu}, \nu_0)$ over all consumption $\nu_0 \in M^+(X)$ satisfying the budget constraint

$$p_1 \cdot \nu_0 \leq p_1 \cdot \beta + S$$

where S is a free endogenous parameter that measures the (positive or negative) value of Consumer 0's net savings.[3]

E.3: Lifetime consumption $c_t = (\mu_t, \nu_t)$ by Consumer $t \geq 1$ maximizes utility $u(c_t)$ over all consumption $c_t \in L^+$ satisfying the budget constraint

$$p_t \cdot \mu_t + p_{t+1} \cdot \nu_t \leq p_t \cdot \alpha + p_{t+1} \cdot \beta$$

All overlapping-generations models in the literature with a fixed, finite number of types of commodities in discrete time can be embedded into our model as special cases. One embedding simply divides each type of commodity in discrete time into a continuum of perfect substitutes across moments in continuous time. In particular, embedding Gale's models provides simple examples of indeterminacy in the canonical case of a continuum $X = [0, 1]$ of differentiated commodities [4]:

[3] Since our results are that equilibria are determinate, results are stronger by leaving S free.

Example 1: Consider the stationary economy with canonical index $X = [0, 1]$, and logarithmic utility

$$u(\mu, \nu) = \ln(\mu(X)) + \ln(\nu(X)) \tag{4}$$

(Genericity analysis in Section 4 excludes such logarithmic utility because it's not finite valued, but that can be fixed with an innocuous perturbation.) Note $\ln(\mu(X)) = \ln(\int_X d\mu(x))$ is a logarithm of an integral, rather than the more common integral of logarithms. Evidently, the former utility satisfy an extreme form of intertemporal local substitution because consumption across moments $x \in X$ within a time period are perfect substitutes. In contrast, an integral of logarithms would violate intertemporal local substitution because the integral is negative infinity if consumption were ever zero over any non-degenerate subinterval of $[0, 1]$.

Initial endowments are the Dirac measures $(\alpha, \beta) = (2\delta_{.5}, \delta_{.5})$; that is, 2 units are initially available to be "gulped" midway through the first period of life, and 1 unit midway through the second period. Material balance (2) requires $c_t = (\mu_t, \nu_t) = (m_t\delta_{.5}, n_t\delta_{.5})$ and $n_{t-1} + m_t = 3$. Budget-constrained utility maximization (E.3) yields first-order conditions and a budget constraint satisfied by $c_t = (m_t\delta_{.5}, n_t\delta_{.5})$, for some price system, when

$$2 = \frac{2}{m_t} + \frac{1}{n_t}$$

Hence, material balance (2) reduces equilibrium equations to

$$\frac{2}{3 - n_{t-1}} + \frac{1}{n_t} = 2, \quad n_t \in [0, 3] \tag{5}$$

That first-order difference equation has two stationary solutions: $n_t = 3/2$ and $n_t = 1$. The latter is the autarkic steady state, $(m_t, n_t) = (2, 1)$. The stable set of initial values that converge to autarky is $[0, 3/2)$.

Putting it all together, supposing the economy (4) attained the autarkic steady state in the past, the equilibrium in the future need not stay on that path but can jump at time 1 to the solution to the difference equation (5) corresponding to any initial old-age gulp of consumption $\nu_0 = n_0\delta_{.5}$ with $n_0 \in [0, 3/2)$ but $n_0 \neq 1$. *Q.E.D.*

3. Virtual Determinacy

This section defines virtual determinacy of an economy, and finds a perturbation that changes Example 1 from indeterminate to virtually determinate (Example 2). Compared with the usual notion of local determinacy, virtual determinacy is weaker in that only utility levels are determinate, but the virtual determinacy we establish can be stronger in that the neighborhood of determinacy can be larger that just an arbitrary open neighborhood. In Example 2, the neighborhood is open and dense in the set of feasible consumption bundles.

To be precise requires defining a topology and a norm over commodity bundles. The Riesz Representation Theorem implies the commodity space $M(X)$ is the topological dual of the Banach space $\mathcal{C}(X)$ of continuous functions (endowed with the supremum norm). Throughout this paper, endow the commodity spaces $M(X)$ and $L = M(X) \times M(X)$ with the **weak-star** topology, and the positive cone L^+ with the relative "weak-star" topology. Note: the topological space L^+ is then metrizable, but L or $M(X)$ might not be. Convergence $\mu_n \to \mu$ of a sequence or net in $M(X)$ means $p \cdot \mu_n \to p \cdot \mu$ for each p in $\mathcal{C}(X)$. That topology is coarser (weaker) than the topology generated by the dual norm, or **variation** norm. The "variation" norm of a pair positive measures $c = (\mu, \nu) \in L^+$ is simply the total amount of all commodities, $|c| = \mu(X) + \nu(X)$.

Consider any economy $(u, e, \bar{c})$ and any subset $C \subset L(e)$ containing $\bar{c}$, let

$$\Psi(u, e, \bar{c}, C) = \left\{ \{u(c_t)\} : \{c_t\} \in \prod_{t \in \mathbb{Z}} C \text{ is an equilibrium for } (u, e, \bar{c}) \right\} \tag{6}$$

denote the set of utility sequences generated by some equilibrium (E.1,E.2,E.3) with each consumption $c_t \in C$. Material balance (2) implies all such utility sequences are bounded, $\Psi(u, e, \bar{c}, C) \subset \ell^\infty$.

D: $(u, e, \bar{c}, C)$ is **virtually** determinate if the equilibrium utility set $\Psi(u, e, \bar{c}, C)$ is a singleton.

Evidently, the strength of "virtual determincy" depends on the size of the neighborhood C relative to $L(e)$. In the following analysis, C is open-dense (generic) in $L(e)$.

The indeterminacy in Example 1 is evidently robust to perturbing preferences but keeping endowments fixed so there is only one commodity available each period; and to perturbing endowments but keeping preferences fixed. The main result of this paper (Theorem 4.1) implies that virtual indeterminacy is not robust to perturbing preferences when there are many commodities available. Finding and analyzing that perturbation for Example 1 requires the following constructions:

- The **feasible** set of an economy is the set $L(e) = \{c \in L^+ : c \leq (\beta + \alpha, \beta + \alpha)\}$ of all lifetime consumption bundles that are part of some allocation.
- For any positive continuous function $m \in \mathcal{C}(X)$ and any positive measure $\mu \in M(X)$, define the **scaled** measure $m\mu$ by $m\mu(B) = \int_B m(x) d\mu(x)$. Notice $m\mu(X) = m \cdot \mu$.

Example 2: Perturb Example 1's indeterminacy into virtual determinacy in five steps.

Step 1: Start with Example 1's logarithmic utility and initial endowment of *food in the midpoint of youth and in the midpoint of old age. Disaggregate endowments so that there is food in at least two disjoint time intervals, lunch and dinner.*

Formally, consider the canonical index $X = [0, 1]$ and logarithmic utility (4) from Example 1. Perturb endowments from $e = (2\delta_{.5}, \delta_{.5})$ to any weak-star nearby positive measures (α, β) with the same total amounts, $(\alpha(X), \beta(X)) = (2, 1)$, and with the support of the total endowment $(\alpha + \beta)$ containing at least 2 commodities x and y in X (rather than just the single commodity .5 in the support of the Dirac measure $\delta_{.5}$).

There are now a continuum of steady states $\bar{c} = (\bar{\mu}, \bar{\nu})$ corresponding to autarky in the perturbed Example 1: such steady states need only satisfy material balance $\bar{\nu} + \bar{\mu} = \beta + \alpha$ and utility maximization $(\bar{\mu}(X), \bar{\nu}(X)) = (2, 1)$. The economy (u, e) is not virtually determinate if any one of those steady states is attained in the past. But given any one of those steady states $\bar{c}$, preferences will be perturbed to make the economy $(u^k, e, \bar{c})$ virtually determinate.

Step 2: For the chosen steady state, either the young eat some "lunch" and the old eat some "dinner", or the young eat some dinner and the old eat some lunch. It's enough to just consider just the former case. The neighborhood C of the chosen steady state to be used to demonstrate virtual determinacy consists of consumption where the young eat some lunch and the old eat some dinner.

Formally, for the chosen steady state $\bar{c} = (\bar{\nu}, \bar{\mu})$ satisfying material balance $\bar{\nu} + \bar{\mu} = \beta + \alpha$ and utility maximization $(\bar{\mu}(X), \bar{\nu}(X)) = (2, 1)$, material balance and the restriction on the disaggregation of endowments implies the support of the total consumption $(\bar{\mu} + \bar{\nu})$ contains at least 2 commodities. And utility maximization implies $\bar{\mu}(X) > 0$ and $\bar{\nu}(X) > 0$. Hence, for some disjoint pair of compact subsets of commodities, $\bar{\mu}(A) > 0$ and $\bar{\nu}(B) > 0$. (Think of A as lunch time, and B as dinner.) Hence, the Urysohn Lemma yields a pair of continuous functions $m : X \to [0, 1]$ and $n : X \to [0, 1]$ with
(a) $m \cdot \bar{\mu} > 0$ and $n \cdot \bar{\nu} > 0$
(b) $\emptyset = \text{supp } m \cap \text{supp } n$

Hence, consider the (weak-star) open-dense neighborhood $C \subset L(e)$ of $\bar{c}$ consisting of all bundles (μ, ν) in $L(e)$ preserving a variation of inequality (a):
(a′) $m \cdot \mu > 0$ and $n \cdot \nu > 0$

Step 3: Approximate the original utility $u(\mu, \nu) = \ln(\mu(X)) + \ln(\nu(X))$, which treats lunch and dinner as perfect substitutes, by subtracting a small multiple of a function φ so that the old now may have a slight preference for lunch over dinner. That preference depends on lifetime consumption.

Formally, approximate u by the sequence of functions

$$u_k = u - \frac{1}{k}\,\varphi, \quad \text{for } \varphi(\mu, \nu) := (\mu(X) - 2)^2 \left(49 + (1 + n \cdot \nu)^2\right) \tag{7}$$

One can show each function u_k is concave and strictly increasing over the feasible set $L(e)$, and that u_k can be extended to be concave and strictly increasing over all of the lifetime consumption set L^+.

Step 4: For any equilibrium, consider a net trade γ that increases lunch and decreases dinner. The young may sell γ (increasing lunch and decreasing dinner), and the old may buy γ (decreasing lunch and increasing dinner).

Formally, consider any equilibrium $\{c_t\}$ of economy $(u_k, e, \bar{c})$ with each consumption c_t in the neighborhood C of the chosen steady state (so that Property (a′) is satisfied). Let $\{p_t\}$ be a supporting price system. Consider the period $t \geq 2$ where old Consumer $t-1$ consumes ν_{t-1} and young Consumer t consumes μ_t. Hence, consider the signed measure

$$\gamma := \begin{cases} \frac{n\cdot\nu_{t-1}}{m\cdot\mu_t} m\mu_t - n\nu_{t-1} & \text{if } n\cdot\nu_{t-1} \leq m\cdot\mu_t \\ m\mu_t - \frac{n\cdot\mu_{t-1}}{m\cdot\nu_t} n\nu_{t-1} & \text{if } m\cdot\mu_t < n\cdot\nu_{t-1} \end{cases}$$

In that definition, the products $n\cdot\nu_{t-1}$ and $m\cdot\mu_t$ are scalars, and $m\mu_t$ and $n\nu_{t-1}$ are positive measures. That signed measure has these properties:
(i) $\mu_t - h\gamma \geq 0$ for $h \in [0,1]$, since $0 < n\cdot\nu_{t-1} \leq m\cdot\mu_t$ and $m : X \to [0,1]$
(ii) $\nu_{t-1} + h\gamma \geq 0$ for $h \in [0,1]$, since $0 < m\cdot\mu_t < n\cdot\nu_{t-1}$ and $n : X \to [0,1]$
(iii) $\gamma(X) = 0$, since $m\mu_t(X) = m\cdot\mu_t$ and $n\nu_{t-1}(X) = n\cdot\nu_{t-1}$
(iv) $n\cdot\gamma < 0$, since $\emptyset = \text{supp } m \cap \text{supp } n$ and $0 < n\cdot\nu_{t-1}$

Step 5: Since the young eat some lunch and the old eat some dinner and, under the approximated utility, the young treat lunch and dinner as perfect substitutes, then the old can not have any marginal preference for lunch over dinner. The nature of the approximation to utility then restricts lifetime consumption so the young eat a total of 2 units, and the old 1 unit. And that those totals, utility is the same as at the steady state.

Formally, for Consumer t, property (i) implies it's feasible for Consumer t to change consumption from the equilibrium choice of (μ_t, ν_t) to $(\mu_t - h\gamma, \nu_t)$ for $h \in (0,1]$. But property (iii) implies that change does not affect total young-age consumption $(\mu_t - h\gamma)(X) = \mu_t(X)$, and so does not affect utility (7): $u_k(\mu_t - h\gamma, \nu_t)$ is constant in $h \in [0,1]$. Hence, Consumer t's utility maximization at equilibrium (E.3) implies $p_t \cdot \gamma \leq 0$.

For Consumer $t-1$, property (iii) implies

$$u_k(\mu_{t-1}, \nu_{t-1} + h\gamma) = u_k(\mu_{t-1}, \nu_{t-1}) - \frac{1}{k}\left(\mu_{t-1}(X) - 2\right)^2 \left(49 + \left(1 + n\cdot\nu_{t-1} + h(n\cdot\gamma)\right)^2\right)$$

Hence

$$\frac{\partial}{\partial h^+} u_k(\mu_{t-1}, \nu_{t-1} + h\gamma)|_{h=0} = -\frac{2}{k}\left(\mu_{t-1}(X) - 2\right)^2 \left(1 + n\cdot\nu_{t-1}\right) n\cdot\gamma$$

Hence, $p_t\cdot\gamma \leq 0$ from the previous paragraph and Consumer $(t-1)$'s utility maximization (E.3) implies the left hand directional derivative must not be positive; hence,

$$0 \;\leq\; \left(\mu_{t-1}(X) - 2\right)^2 \left(1 + n\cdot\nu_{t-1}\right) n\cdot\gamma$$

Hence, $n \cdot \nu_{t-1} > 0$ and property (iv) $n \cdot \gamma < 0$ imply $\mu_{t-1}(X) = 2$. That holds for each $t \geq 2$ fixed in Step 4. Hence, $(\alpha(X), \beta(X)) = (2, 1)$ and material balance (2) imply $\nu_{t-2}(X) = 1$.

Putting it all together, for each Consumer t, the previous paragraph and setting initial consumption equal to the autarkic steady state (E.1) imply $(\mu_t(X), \nu_t(X)) = (2, 1)$, which implies $u_k(\mu_t, \nu_t) = u_k(e)$, which holding for every equilibrium $\{c_t\}$ of $(u, e, \bar{c})$ with consumption in C implies the equilibrium utility set $\Psi(u_k, e, \bar{c}, C)$ is a singleton (D). *Q.E.D.*

What happened to the indeterminate equilibria of Example 1, where the initially old ate any amount of food from 0 to 3/2 except 1?

Each of those equilibrium paths translates into equilibria of the perturbed economy in Example 2. Each path falls into one three groups according to the total endowment A of lunch: If the old eat more than A total along a path, then those equilibria have the old eating all the lunch and some of the dinner. If the old eat exactly A total along a path, then those equilibria have the old eating all the lunch but none of the dinner. And if the old eat less than A total along a path, then those equilibria have the old eating some the lunch but none of the dinner. In all cases, consumption for each consumer $t \geq 0$ in each path falls outside the neighborhood C chosen in Step 2, where the young eat some lunch and the old eat some dinner, and so is contained in a closed nowhere dense (negligible) subset of feasible consumption. Hence, the perturbed pairing (u_k, e, C) can be virtually determinate despite having that continuum of equilibria.

4. GENERIC DETERMINACY STATEMENT

This section defines a completely metrizable space of economies and states the main result (Theorem 4.1) that generic economies are virtually determinate. For the rest of the paper, fix the exogenous endowment e and steady state $\bar{c}$. Assume material balance $\bar{\nu}+\bar{\mu} = \beta+\alpha$ (2). Hence, parameterize economies by their utility functions. One utility function that makes $(u, e, \bar{c})$ an economy (that is, $\bar{c}$ is a steady state for (u, e)) is the continuous linear function $u(\mu, \nu) := \mu(X) + \nu(X)$. We only consider utility functions that are **continuously twice differentiable**, or C^2, although Theorem 4.1 is also true for C^r functions and the C^r metric $(1 \leq r \leq \infty)$.

- Call a utility function C^2 if $u : L^+ \to \mathbb{R}$ is (weak-star) continuous and if there are continuous functions $Du : L^+ \times L \to \mathbb{R}$ and $D^2u : L^+ \times L \to \mathbb{R}$ such that

a) at each bundle $c \in L^+$, the partial functions $Du(c; \cdot) : L \to L$ and $D^2u(c; \cdot) : L \to L$ are linear;

b) at each pair $(c; \hat{c}) \in L^+ \times L$ where $c + h\hat{c} \geq 0$ for sufficiently small $h > 0$, $Du(c; \hat{c})$ is the Gâteaux right-hand first derivative

$$(8) \qquad Du(c; \hat{c}) \quad := \quad \lim_{h \to 0^+} \frac{u(c + h\hat{c}) - u(c)}{h} \quad = \quad \frac{d}{dh^+} u(c + h\hat{c}) \Big|_{h=0}$$

and $D^2u(c;\hat{c})$ is the Gâteaux right-hand second derivative

$$(9)\qquad D^2u(c;\hat{c}) \;:=\; \frac{d}{dh^+}\left(\frac{d}{dh}u(c+h\hat{c})\right)\bigg|_{h=0}$$

The functions Du and D^2u are uniquely defined for a C^2 function u because linearity implies $Du(c;\hat{c}) = Du(c;\hat{c}^+) - Du(c;\hat{c}^-)$ for the positive and negative parts of $\hat{c}$, and so equation (8) defines both $Du(c;\hat{c}^+)$ and $Du(c;\hat{c}^-)$. D^2u is likewise unique.

- The commodity space L is (weak-star) sigma-compact since it is the union of the sequence of closed balls $L_m := \{c \in L : |c| \leq m\}$ for the variation norm, and Alaoglu's theorem implies each ball L_m is compact. The feasible set $L(e) = \{c \in L^+ : c \leq (\beta+\alpha, \beta+\alpha)\}$ is likewise compact.

Lemma 4.1 *The set $C^2(L^+)$ of C^2 functions over L^+ is a completely metrizable topological vector space when endowed with the topology of* C^2**-uniform convergence on compacta**. *Specifically, C^2-convergence $u_n \to u$ means $\|u - u_n\|_m \to 0$ for each semi-norm*

$$(10)\qquad \|u\|_m \;:=\; \max_{(c;\hat{c})\in L_m^+\times L_m} |u(c)| \;+\; |Du(c;\hat{c})| \;+\; \left|D^2u(c;\hat{c})\right|$$

The subset $\mathcal{U}$ of concave and strictly-increasing functions u that make $(u, e, \bar{c})$ an economy is a completely metrizable subspace of $C^2(L^+)$.

For the exogenous steady state $\bar{c} = (\bar{\mu}, \bar{\nu})$, assume

A: Either some commodity $\bar{x}$ in supp $\bar{\mu}$ is a limit point of supp $\bar{\nu}$, or some commodity $\bar{x}$ in supp $\bar{\nu}$ is a limit point of supp $\bar{\mu}$.

In the canonical case $X = [0,1]$ with each commodity in positive supply $X = \operatorname{supp}\bar{\mu} \cup \operatorname{supp}\bar{\nu}$, Assumption A merely requires that both the young and old consume something, $\operatorname{supp}\bar{\mu} \neq \emptyset$ and $\operatorname{supp}\bar{\nu} \neq \emptyset$.

To construct the sets used in the definition of virtual determinacy (D), for each positive integer n, define the open ball

$$(11)\qquad B_n \;:=\; \left\{\, x \in X : d(x,\bar{x}) < \frac{1}{n} \,\right\}$$

with diameter $1/n$ and centered on the reference commodity $\bar{x}$ from Assumption A, and the set

$$(12)\qquad C_n \;:=\; \left\{\, (\mu,\nu) \in L(e) : \mu \mid B_n \geq \frac{1}{n}\bar{\mu} \mid B_n, \quad \nu \mid B_n \geq \frac{1}{n}\bar{\nu} \mid B_n \,\right\}$$

There, $(\mu \mid B_n) \in M^+(X)$ denotes the restriction of μ to B_n, so that $(\mu \mid B_n)(B) = \mu(B \cap B_n)$ for each Borel set $B \subset X$. Likewise for the other measures. The sets C_n were defined to contain the steady state $\bar{c}$ and be as big as possible for our statement of virtual determinacy.

Theorem 4.1 *For each utility function u in some residual (generic) subset of $\mathcal{U}$, the utility set $\Psi(u, e, \bar{c}, \bigcup_n C_n)$ is a singleton (D), and each function* dia $\Psi(\cdot, e, \bar{c}, C_n) : \mathcal{U} \to \mathbb{R}$ *is zero and continuous at u.*

There, the diameter "dia" is defined with respect to any given metric d for the sup-norm topology on ℓ^∞. The diameter dia $\Psi(u, e, \bar{c}, C_n)$ of the utility set (6) thus measures the degree of virtual indeterminacy. The continuity of dia $\Psi(\cdot, e, \bar{c}, C_n)$ means indeterminacy returns slowly if the economic model is perturbed out of the residual set.[4]

The strength of the determinacy in Theorem 4.1 depends on the size of the sets C_n and of their union $\bigcup_n C_n$, relative to the entire feasible set $L(e)$. The union $\bigcup_n C_n$ is "big" in the sense that it includes the set

$$\left\{ (\mu, \nu) \in L(e) : \mu \geq \frac{1}{n}\bar{\mu} \text{ and } \nu \geq \frac{1}{n}\bar{\nu} \text{ for some } n > 0 \right\}$$

which is an open-dense (generic) subset of $L(e)$ for the topology induced by the norm

$$\|(\mu, \nu)\|_e := \inf\left\{\lambda > 0 : \mu^+ + \mu^- \leq \lambda(\alpha + \beta), \quad \nu^+ + \nu^- \leq \lambda(\alpha + \beta)\right\} \tag{13}$$

over L, where μ^+, ν^+ and μ^-, ν^- are the positive and negative parts of the measures. (That topology on $L(e)$ is only used to show the union $\bigcup_n C_n$ is "big"; it is not used to prove Theorem 4.1.)

The first part of Theorem 4.1 evidently follows from the second part. But for any function f defined over any topological space $\mathcal{U}$, the set

$$\{ u \in \mathcal{U} \ : \ f(u) = 0 \text{ and } f(\cdot) \text{ is continuous at } u \} \tag{14}$$

is G_δ in $\mathcal{U}$, being the countable intersection, over the positive integers n, of the open sets

$$\left\{ u \in \mathcal{U} \ : \ \sup_{v \in N_u} |f(v)| < 1/n \text{ for some open neighborhood } N_u \text{ of } u \right\}$$

Hence, for Theorem 4.1 it is sufficient to prove, for each utility function u in some residual subset of $\mathcal{U}$, each function dia $\Psi(\cdot, e, \bar{c}, C_n) : \mathcal{U} \to \mathbb{R}$ is zero and continuous at u.

5. GENERALIZATIONS

Example 1 embedded an economy with 1 good per period using type space $X = [0, 1]$. Any economy with n goods per period can likewise be embedded, but using type space $X = \{1, ..., n\} \times [0, 1]$:

[4] The function $\Psi(\cdot, e, \bar{c}, C_n)$ is like Thomae's popcorn function, which is zero and continuous at each irrational number (a residual set), and positive and discontinuous at each rational (a meagre set).

- Translate n-good endowment $(a_1, ..., a_n) \in \mathbb{R}^n_+$ into any measure $\alpha \in M^+(X)$ with the same total for each good, $a_i = \alpha(\{i\} \times [0,1])$
- Translate n-good utility function $v(x_1, ..., x_n; y_1, ..., y_n)$ over $\mathbb{R}^n_+ \times \mathbb{R}^n_+$ into a function u over $M^+(X) \times M^+(X)$ by splitting goods into perfect substitutes,

$$u(\mu, \nu) = v\,(\ \mu(\{1\}\times[0,1]), ..., \mu(\{n\}\times[0,1])\ ;\ \nu(\{1\}\times[0,1]), ..., \nu(\{n\}\times[0,1])\ )$$

There, $\mu(\{i\}\times[0,1])$ is the total youth consumption of the commodities in $\{i\}\times[0,1] \subset X$.
*** To make some recent additions that incorporate utility gradients into the definition of virtual determinacy, the rest of the paper will be presented in a final version, expected before the end of June 2014.

6. APPENDIX: PROOF OF GENERIC DETERMINACY

7. APPENDIX: PROOF OF LEMMAS

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