Matching and Sorting in a Global Economy*

Gene M. Grossman
Princeton University

Elhanan Helpman
Harvard University and CIFAR

Philipp Kircher
University of Edinburgh and London School of Economics

April 14, 2013

Abstract

We develop a neoclassical trade model with heterogeneous factors of production. We consider a world with two factors, "managers" and "worker", each with a distribution of ability levels. Production combines a manager of some type with a group of workers. The output of a unit depends on the types of the two factors, with complementarity between them, while exhibiting diminishing returns to the number of workers. We examine the sorting of factors to sectors and the matching of factors within sectors, and we use the model to study the determinants of the trade pattern and the effects of trade on the wage and salary distributions and on measured productivity. Finally, we extend the model to include search frictions and consider the distribution of employment rates.

Keywords: heterogeneous labor, matching, sorting, productivity, wage distribution, international trade.

JEL Classification: F11, F16

*Part of this research was completed while Grossman was visiting STICERD at the London School of Economics and CREI at the University of Pampeu Fabra and while Helpman was visiting Yonsei University as SK Chaired Professor. They thank these institutions for their hospitality and support. The authors are grateful to Rohan Kekre, Ran Shorrer, and especially Kevin Lim for their research assistance. Finally, Grossman and Helpman thank the National Science Foundation and Kircher thanks the European Research Council for financial support.

1 Introduction

In this paper, we study how international trade affects the sorting of heterogenous workers and managers into industries and the matching of workers with managers in production units. It is by now well known that firms in the same industry differ in size, in the compositions of their workforces, in the technologies and capital goods they use, and in the wages they pay to their workers. Industries differ in factor intensities and in the marginal contributions of worker and managerial ability to firm productivity. Workers differ in physical attributes, in cognitive abilities, and in their education, training, and experience. Although some studies of international trade have examined the assignment of heterogeneous labor to different sectors and others have considered the matching of workers to heterogeneous teammates or technologies, relatively little is known about the general problem of how factors sort and match in the open economy when several of these factors are differentiated, when fixed quantities of one impart decreasing returns to the others, and when industries differ in their factor intensities and in the usefulness of factor "quality." Our paper addresses these more general, allocational issues and their implications for factor rewards. Because workers and managers are heterogeneous, our analysis sheds light on the impact of trade on the *distribution* of wages and managerial salaries, and thereby on the impact of trade on earnings inequality.

By allowing for worker, manager, and industry heterogeneity, we can better understand a number of issues concerning the pattern and consequences of international trade. First, we can study how countries' *distributions* of differentiated factors, in conjunction with their aggregate endowments of these factors, determine their comparative advantage in the various sectors. Bombardini et al. (2012) provide evidence, for example, that countries' skill dispersions have a quantitatively similar impact on trade flows as do their aggregate endowments of human capital. Second, we can investigate how trade influences factor returns across the entire income distribution, affecting more than just the relative compensation paid to one factor versus another or to workers employed in one industry versus another. These additional dimensions of inequality can be useful for understanding recent findings of substantial variation in wages that is not easily explained by observable worker characteristics. Helpman et al. (2012) show, for example, that within-industry wage variation accounts for a majority of wage inequality in Brazil even after controlling for workers' occupations. Finally, we can examine how globalization affects measured productivity in different sectors as a result of the altered patterns of sorting and matching that are induced by trade. The effect of trade liberalization on measured productivity has been the focus of much recent empirical research; see, for example, Pavcnik (2002), Trefler (2004), and De Loecker (2011).

The literature on the *sorting* of workers to industries includes recent work by Costinot (2009), Costinot and Vogel (2010), and Ohnsorge and Trefler (2007), as well as earlier work by Mussa

(1982) and Ruffin (1988).[1] All of these authors emphasize the comparative advantage that the various types of labor have when employed in different industries. They study the determinants of the trade pattern in countries that differ in the compositions of their labor forces and the impact that trade has on income inequality across the skill or ability spectrum. But most assume a linear relationship between labor input (of a given quality) and output or, what amounts to the same, an absence of interactions between quantities of labor and quantities of other factors of production. As emphasized by Eeckhout and Kircher (2012), models with one worker per firm or with a linear relationship between labor quantity and output cannot speak to the determinants of a firm's capital intensity or its manager's span of control.

The *matching* of workers to technologies within an industry is the focus of work by Yeaple (2005) and Sampson (2012). These authors also assume a production function with constant returns to labor and thus omit interactions between labor and any other factors of production.[2] Similarly, Grossman and Maggi (2000) study the pairing of workers who perform different production tasks, but in a context with exactly two workers per firm and therefore no scope for variation in factor intensity or firm size. The work of Antràs et al. (2006) does allow for endogenous span of control in a model with matching of workers and managers, but theirs is a one-sector model with international production teams and they assume a particular technology that tightly links the quality and the quantity of labor that a given manager can oversee.

Our analysis extends a familiar trade model with two sectors, two factors, and perfectly-competitive product markets. While most of our analysis assumes frictionless factor markets, we also consider an economy with search and matching frictions. We call one factor "labor" and assume throughout that workers are differentiated along a single dimension that we term "ability." Workers with greater ability are assumed to be more productive in both industries, but the contribution of ability to output may differ across uses. We refer to the second input as "managers." Similar to workers, managers generally differ in ability and more able managers contribute more to output in both sectors, albeit to an extent that may vary by industry. With this formulation, we can address how the economy matches a fixed but heterogenous supply of one input (managers) with a fixed but heterogeneous supply of another (labor) in a setting where the relative number of workers per manager is a matter for firms to decide.

In the next section, we lay out our basic model of an open economy with two countries, two competitive industries, and two heterogeneous factors of production. Section 3 considers trade between countries that have heterogeneous workers but homogeneous managers. Our analysis of this simpler setting aids in understanding the more general case discussed in Sections 4 and 5, where managers also are assumed to vary in ability. We show that, with homogeneous managers, the sorting of workers is guided by a cross-industry comparison of the ratio of the elasticity of

[1] We use the term "sorting" to refer to the allocation of heterogeneous factors to different industries and the term "matching" to refer to the combination of differentiated factors within an industry.

[2] Both of these authors assume, however, that firms produce differentiated products in a world of monopolistic competition, so that inputs of additional labor by a firm do generate decreasing returns in terms of *revenue*. Thus, these models do share some features with the ones that we study below.

output with respect to labor quality to the elasticity of output with respect to labor quantity. This can generate a simple sorting pattern in which all the best workers with ability above some threshold level are employed in one sector and the remaining workers are employed in the other. But it also can generate more complex patterns in which, for example, the most able and least able workers sort to one sector while workers with intermediate levels of ability are allocated to the other. Trade between countries with similar distributions of worker talent is determined by their aggregate factor endowments as in the Heckscher-Ohlin model, whereas trade between countries with similar relative endowments reveals a comparative advantage for a country with a superior distribution of labor quality (as reflected in a proportional rightward shift of its talent distribution) in the good produced by the industry in which worker ability contributes more elastically to productivity. With homogeneous managers, relative price movements do not affect within-sector relative wages and therefore have no effect on wage inequality *within industries*. Across industries, the impact of trade on wages reflects a blend of Stolper-Samuelson and Ricardo-Viner forces, as in models with imperfect factor mobility such as Mussa (1982) and Grossman (1983).

Section 4 addresses the sorting of heterogeneous workers and heterogeneous managers for the special case in which the elasticity of output with respect to any factor's ability is constant in both industries. In obvious analogy with production functions based on quantities alone, we refer to this as the Cobb-Douglas (productivity) case. In this setting, there is a unique sorting pattern for each factor—which again reflects the ratio of a sector's elasticity of output with respect to an input's ability and the elasticity of output with respect to the input's quantity—but the matching of workers and managers within an industry is not uniquely determined. Comparative advantage again reflects relative aggregate endowments and the distributions of ability. An abundance of managers per worker generates comparative advantage in the manager-intensive sector, whereas a "better" distribution of some factor generates comparative advantage in the sector that exhibits the greater elasticity of output with respect to that factor's quality. In the Cobb-Douglas case, the wages of workers and the salaries of managers in a given sector both rise with ability at constant rates. These rates, which differ by factor and industry, reflect technological considerations alone. It follows that trade has no impact on within-industry wage or salary inequality. Other dimensions of factor rewards again are driven by a mix of Stolper-Samuelson and Ricardo-Viner forces.

In Section 5 we turn to the most interesting case, which has heterogeneity of both inputs and productivity that is a strictly log supermodular function of the abilities of the production unit's manager and workers. Unlike the Cobb-Douglas case, the strong complementarities that are captured by strict log supermodularity induce positive assortative matching in each sector. That is, among the sets of workers and managers that sort to a given sector, the better workers are matched with the better managers. We provide sufficient conditions under which all of the workers with ability above some threshold level and all the managers with ability above some (different) threshold level sort to the same sector. We also provide conditions under which the high-ability workers sort to the same sector as the low-ability managers. More complex sorting patterns are possible as well. When countries share the same distributions of abilities and the sorting patterns do involve a single threshold for each factor, then the country endowed with more managers per worker must export the manager-intensive good.

When there are strong complementarities between the types of workers and managers, the effects of trade or trade liberalization on the wage distribution are subtle and interesting. An increase in the relative price of some good might worsen the matches for all workers and improve the matches for all managers, or vice versa. Alternatively, a change in relative price might improve the matches for workers in one industry while worsening those for workers in the other. We identify conditions for these various shifts in the matching functions and discuss their implications for factor rewards. In particular, we show that trade may cause within-industry income inequality to rise or fall and the impact of trade on an input's within-sector earnings inequality can differ from the changes that occur across sectors.

In all of these settings, if the calculation of total factor productivity (TFP) fails to account for factor heterogeneity, then trade will affect measured TFP in each industry and in the economy as a whole. Consider, for example, a setting where the more able workers and managers sort into the same sector in both countries, and the countries open to trade. The resulting price changes induce workers and managers to move from the import-competing sector to the export sector in each country. In the country where the import-competing sector employs the most able factors, the marginal workers and managers that relocate are more able than those they join in their new industry but less able than those that remain behind. Then average worker and manager quality rise in each sector, and with them, measured TFP in each sector and in the economy as a whole. Just the opposite happens in the other country, where average factor quality falls in both sectors. Accordingly, trade can generate a convergence or divergence of measured productivity, depending on the initial conditions and the patterns of comparative advantage, even if the underlying production functions do not change. The effects on measured productivity in our model are reminiscent of those in the seminal Roy (1951) model of labor sorting, except that here the changes in factor composition occur due to trade.

In Section 6, we extend the analysis to include economies with labor-market frictions by assuming that workers engage in directed search. In this setting, each potential worker seeks a job at a firm of his choosing and manages to be hired by that firm with a probability that depends on the number of applicants per vacancy. We show that, with these search frictions, wage and employment rates both vary with ability; more able workers not only earn higher wages but also enjoy better job prospects. Moreover, trade affects both wage and employment-rate inequality.

Section 7 contains some concluding remarks.

2 The Economic Environment

We examine a world economy comprising two countries, two industries, and two factors of production. We call one of the factors "labor" and refer to individuals as "workers." Each country is endowed with a continuum of workers with various abilities. The exogenous supply of workers of ability q_L in country c is $\bar{L}^c \phi_L^c(q_L)$ for $c = \{A, B\}$, where $\bar{L}^c$ is the aggregate endowment of labor and $\phi_L^c(q_L)$ is the density of workers with ability q_L. For ease of exposition, we assume throughout that $\phi_L^c(q_L)$ is continuous and strictly positive on its finite support $S_L^c = [q_{L\min}^c, q_{L\max}^c]$, where $0 < q_{L\min}^c < q_{L\max}^c < +\infty$. We refer to the second factor as "managers." Country c has a continuum of managers of measure $\bar{H}^c$. We begin in Section 3 by assuming that all managers are alike. Subsequently, we introduce manager heterogeneity and then denote the density of managers with ability q_H by $\phi_H^c(q_H)$, with $\phi_H^c(q_H)$ continuous and strictly positive on its finite support $S_H^c = [q_{H\min}^c, q_{H\max}^c]$.[3]

Firms in the two countries have access to the same constant-returns-to-scale technologies. Output per manager in an industry reflects the *number* of workers that is combined with a manager there and the *abilities* of the inputs that are used in the production process. Specifically, when a firm combines a manager with a group of workers, it must allocate a fraction of the manager's "time" to each of the workers. The greater is the fraction of managerial time that is devoted to a worker, the greater is his productivity, but with diminishing returns. This formulation, which is familiar from previous models of a manager's "span of control" such as Sattinger (1975), Lucas (1978) and Garicano (2000), implies that firms will combine a given manager with a group of workers of uniform type and will divide the manager's time evenly among them.[4] To conserve on notation, we invoke this implication of the firm's optimal combination of inputs and write the output in sector i of a manager of ability q_H who is teamed with ℓ workers of (a common) type q_L as[5]

$$x_i = \psi_i(q_H, q_L)\,\ell^{\gamma_i}, \;\; 0 < \gamma_i < 1, \tag{1}$$

where $\gamma_i < 1$ is a parameter that reflects the diminishing returns from dividing the manager's time more finely and $\psi_i(q_H, q_L)$ is a strictly increasing, twice continuously differentiable, log supermodular function that captures the complementarities between the types of the two factors. We assume that factor type contributes to productivity in qualitatively the same way in both sectors and, without further loss of generality, order the types so that $\partial\psi_i/\partial q_F > 0$ for $i = 1, 2$ and $F = H, L$. With this labeling convention, we refer to q_F as the "ability" of factor F. Note that the industries generally differ in the strength of the complementarities between factors, in the contributions of factor abilities to productivity, and in their factor intensities.

The rest of the model is familiar from neoclassical trade theory. Consumers worldwide share

[3] We focus on an environment where factor endowments are invariant to trade. This makes our results comparable to most previous studies. Future work might consider adjustments in factor endowments - e.g., taking the terminology of workers and managers literally one might study long-run skill acquisition that turns workers into managers.

[4] The key assumption here is that there is no teamwork or synergy between workers in a firm; they interact only in the sense that they compete for the time of the manager. See Eeckhout and Kircher (2012) for more discussion. In such circumstances, the primitive for technology gives output as a function of the type of the manager and types of all workers with which it is combined. But there is no need for us to develop notation for this more general formulation since we know that, in our setting, a firm will not gain (and typically will lose) by choosing to combine a given type of manager with a variety of types of workers.

[5] We adopt a Cobb-Douglas-in-quantities specification in order to simplify the analysis. Some of our results would remain the same with an arbitrary constant-returns-to-scale production technology provided that there are no factor intensity reversals.

identical and homothetic preferences. Firms hire workers and managers on frictionless national factor markets and engage in perfect competition on integrated world product markets. Countries trade freely, with balanced trade. Note that we neglect for now the search frictions that are a realistic and interesting feature of many markets with heterogeneous factors. We shall extend the analysis to incorporate such frictions in Section 6 below.

3 Homogeneous Managers

We are ultimately interested in the sorting and matching of two heterogeneous factors of production. However, before we get to that, we consider a simpler case in which there is no variation in the types of one of the factors. By examining a setting with homogeneous managers we can gain insight into the sorting of the heterogeneous workers into different sectors without needing to concern ourselves with the matching of managers and workers. We will introduce manager heterogeneity in Section 4 below.

Suppose that all managers are interchangeable and assume, without further loss of generality, that their common ability level is $q_H = 1$. Let $\tilde{\psi}_i(q_L) \equiv \psi_i(q_L, 1)$ be the productivity in sector i of workers of ability q_L when combined with any manager who might be employed there. Output per manager in sector i can now be written as $x_i = \tilde{\psi}_i(q_L)\ell^{\gamma_i}$, considering the diminishing returns to the manager's time.

A key variable in the analysis will be the ratio of two elasticities that describe a sector's production technology. One elasticity is $\varepsilon_{\tilde{\psi}_i}(q_L) \equiv q_L\tilde{\psi}_i'(q_L)/\tilde{\psi}(q_L)$, which reflects the responsiveness of output to worker *ability* in sector i, holding constant the number of workers per manager. The other elasticity is γ_i, which is the responsiveness of output to labor *quantity*, holding constant the ability of the workers. Let

$$s_L(q_L) \equiv \frac{\varepsilon_{\tilde{\psi}_1}(q_L)}{\gamma_1} - \frac{\varepsilon_{\tilde{\psi}_2}(q_L)}{\gamma_2}$$

be the difference across sectors in these ratios. We assume for now that $s_L(q_L)$ has a uniform sign for all q_L in the domain of the ability distribution and label the industries so that $s_L(q_L) > 0$. More formally, we adopt for now the following assumption:

Assumption 1 $S_H = \{1\}$ and $s_L(q_L) > 0$ for all $q_L \in S_L^A \cup S_L^B$.

A firm in sector i chooses the ability and number of its workers (per manager) to maximize $\pi_i(q_L, \ell) = p_i\tilde{\psi}_i(q_L)\ell^{\gamma_i} - w(q_L)\ell - r$, where p_i is the price of good i, $w(q_L)$ is the wage of a worker with ability q_L, and r is the salary of the representative manager.[6] We solve the firm's profit maximization problem in two stages. First, we calculate the optimal demand (per manager) for workers of ability q_L when the wage of such workers is $w(q_L)$, which yields

$$\ell_i(q_L) = \left[\frac{\gamma_i p_i\tilde{\psi}_i(q_L)}{w(q_L)}\right]^{\frac{1}{1-\gamma_i}}. \tag{2}$$

[6] We suppress for now the country superscript c, because we focus on firms' decisions in a single country.

Substituting this labor demand into the profit function gives an expression for profits that depends only on the ability of the workers, namely

$$\tilde{\pi}_i(q_L) = \bar{\gamma}_i p_i^{\frac{1}{1-\gamma_i}} \tilde{\psi}_i(q_L)^{\frac{1}{1-\gamma_i}} w(q_L)^{-\frac{\gamma_i}{1-\gamma_i}} - r, \tag{3}$$

where $\bar{\gamma}_i \equiv \gamma_i^{\frac{\gamma_i}{1-\gamma_i}}(1-\gamma_i)$. In the second stage, we choose q_L to maximize $\tilde{\pi}_i(q_L)$. To characterize this optimal choice, let Q_{Li} be the set of abilities of workers that sort into sector i and let Q_{Li}^{int} be the interior of this set. Since the equilibrium wage function must be continuous, strictly increasing, and differentiable at all points in Q_{Li}^{int}, $i = 1, 2$, the first-order condition of the second-stage maximization problem implies

$$\frac{\varepsilon_{\tilde{\psi}_i}(q_L)}{\gamma_i} = \varepsilon_w(q_L) \text{ for all } q_L \in Q_{Li}^{int}, \tag{4}$$

where $\varepsilon_w(q_L)$ is the elasticity of the wage schedule with respect to ability.[7]

Evidently, the firms in sector i choose workers so that the elasticity of output with respect to ability divided by the elasticity of output with respect to quantity is just equal to the elasticity of the wage schedule.[8] If (4) were to hold at only one value of q_L, then all firms in industry i would hire workers with the same ability level. Of course, such an outcome would not be consistent with full employment for all types of workers. Instead, (4) must hold for all $q_L \in Q_{Li}^{int}$. In such circumstances, the firms in sector i are indifferent among the various types of workers that are employed in the sector. This indifference incorporates not only the heterogeneous productivities of the different workers, but also the optimal adjustment in the number of workers that the firm would make were it to switch from one type of worker to another. The accompanying adjustment in quantity explains why it is the ratio of the two elasticities—and not just the responsiveness of output to ability—that firms take into account when they contemplate a change in the ability of their employees.

The requirement that the wage function has an elasticity $\varepsilon_{\tilde{\psi}_i}(q_L)/\gamma_i$ for all worker types that are hired in sector i is equivalent to the requirement that the wage function takes the form

$$w(q_L) = w_i \tilde{\psi}_i(q_L)^{1/\gamma_i} \text{ for } q_L \in Q_{Li}\ , \tag{5}$$

for some constant wage anchor, w_i. This wage function dictates the sorting pattern for labor. Consider any worker type, say q_L^*, that is hired in equilibrium by both sectors and is paid the same wage in both. Under Assumption 1, workers with ability greater than q_L^* can earn more in sector

[7] The wage function has to be strictly increasing because the productivity functions $\tilde{\psi}_i(q_L)$ are strictly increasing; that is, if wages were decreasing with ability no one would hire workers with lower ability in the declining range. The wage function also has to be continuous because if it had an upward jump no one would hire workers just to the right of the jump. In the appendix we prove differentiability of the wage function for the case in which managers are also heterogeneous and the same method can be used to prove differentiability for the case of homogeneous mangers considered in this section.

[8] Note that Costinot and Vogel (2010) derive a similar wage schedule, except that $\gamma_i = 1$ for all i for their economy with linear output.

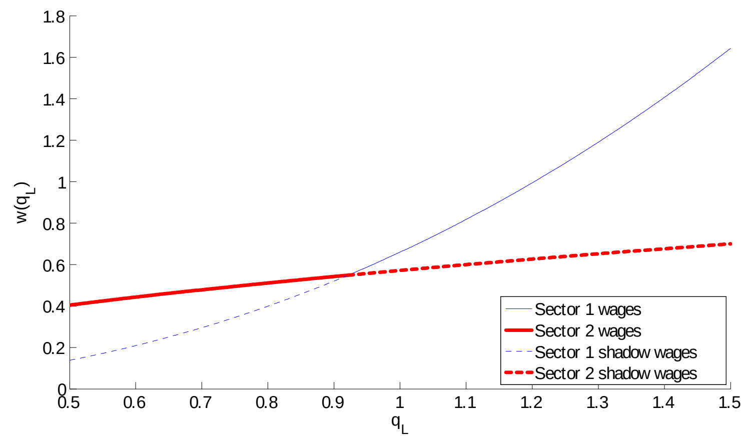

Figure 1: Wages of workers: homogeneous managers

1 than in sector 2, because the wage that makes firms indifferent between these more able workers and workers of ability q_L^* is higher there. Similarly, workers with ability less than q_L^* face better prospects in sector 2, because firms there are more willing to sacrifice ability after taking account of the optimal adjustment in quantity. It follows that the equilibrium sorting pattern has a single cutoff level q_L^* such that workers with ability above q_L^* are employed in sector 1 and those with ability below q_L^* are employed in sector 2.

Figure 1 shows the qualitative features of any equilibrium wage schedule. The solid curve depicts what workers of different abilities actually are paid, considering that those with ability $q_L \geq q_L^*$ are employed in sector 1 and those with ability $q_L \leq q_L^*$ are employed in sector 2. The broken curves show what the workers of different types would be paid if they moved to the opposite sector from their place of employment, considering that they would only be hired there if firms were indifferent between employing them and hiring the types that they actually employ in equilibrium. From now on, we will refer to these wage opportunities in the opposite sector as the "shadow wages." Notice that the shadow wages are less than the actual wages, as of course they must be. Notice too that the worker with the marginal ability q_L^* earns the same wage in either of his job opportunities.

We record our observations about the equilibrium sorting pattern in

Proposition 1 *Suppose that Assumption 1 holds. Then, in any competitive equilibrium with employment in both sectors, the more able workers with $q_L \geq q_L^*$ are employed in sector* 1 *and the less able workers with $q_L \leq q_L^*$ are employed in sector* 2*, for some $q_L^* \in S_L$.*

The intuition for this sorting pattern should be apparent by now. Sorting is determined by comparing across sectors the ratios $\varepsilon_{\tilde{\psi}_i}/\gamma_i$. On the one hand, when $\varepsilon_{\tilde{\psi}_i}$ is large, there is a big return to moving higher *ability* workers to sector i inasmuch as marginal ability contributes greatly to productivity there. On the other hand, when γ_i is large, output in sector i expands rapidly with the *number* of employed workers, irrespective of their ability. In such circumstances, it makes economic sense to deploy relatively large numbers of workers in the industry. The equilibrium sorting pattern reflects a trade-off between the returns to ability and the returns to quantity.

We can now write down the remaining equilibrium conditions by invoking labor-market clearing for the various types of workers, the aforementioned wage-continuity condition at q_L^*, and a requirement that all active firms must break even. Consider first the aggregate supply and demand for workers with ability greater than q_L^*. Define $e_i(q_L) = \tilde{\psi}_i(q_L)^{1/\gamma_i} \ell(q_L)$ as the "effective labor" hired per manager by a firm that employs workers with ability q_L. Such a firm produces $[e_i(q_L)]^{\gamma_i}$ units of good i for every manager it employs. Using the expression for labor demand (2) and considering the wage schedule (5), every firm operating in sector i combines the same amount of effective labor with any one of its managers, namely $e_i = (\gamma_i p_i/w_i)^{1/(1-\gamma_i)}$. It follows that the firms operating in sector i collectively demand $H_i e_i = H_i(\gamma_i p_i/w_i)^{1/(1-\gamma_i)}$ units of effective labor, where H_i is the measure of managers employed in sector i. The total supply of effective labor is simply the measure of effective units of labor among those that sort to sector i. Equating demand and supply gives

$$H_i\left(\frac{\gamma_i p_i}{w_i}\right)^{\frac{1}{1-\gamma_i}} = \bar{L}\int_{q_L\in Q_{Li}} \tilde{\psi}_i(q_L)^{1/\gamma_i}\phi_L(q_L)\,dq_L, \text{ for } i=1,2.$$

Proposition 1 tells us which workers are employed in which sectors, i.e., $Q_{L1} = [q_L^*, q_{L\max}]$ and $Q_{L2} = [q_{L\min}, q_L^*]$. So we can write

$$H_1\left(\frac{\gamma_1 p_1}{w_1}\right)^{\frac{1}{1-\gamma_1}} = \bar{L}\int_{q_L^*}^{q_{L\max}} \tilde{\psi}_1(q_L)^{1/\gamma_1}\phi_L(q_L)\,dq_L \tag{6}$$

and

$$\left(\bar{H}-H_1\right)\left(\frac{\gamma_2 p_2}{w_2}\right)^{\frac{1}{1-\gamma_2}} = \bar{L}\int_{q_{L\min}}^{q_L^*} \tilde{\psi}_2(q_L)^{1/\gamma_2}\phi_L(q_L)\,dq_L \tag{7}$$

where, in (7), we have used the market-clearing condition for managers, $H_1 + H_2 = \bar{H}$.

We have observed that the wage function must be continuous at q_L^*. Continuity of the wage schedule at q_L^* implies in turn that

$$w_1\tilde{\psi}_1(q_L^*)^{1/\gamma_1} = w_2\tilde{\psi}_2(q_L^*)^{1/\gamma_2}. \tag{8}$$

Finally, profits must be equal to zero for firms operating in both sectors, assuming that the economy is incompletely specialized (otherwise they are zero in the active sector and potentially negative in the other). These requirements together with (3) pin down the equilibrium salary for managers, $r = \bar{\gamma}_i p_i^{\frac{1}{1-\gamma_i}} \tilde{\psi}_i(q_L)^{\frac{1}{1-\gamma_i}} w(q_L)^{-\frac{\gamma_i}{1-\gamma_i}}$, and also ensure that

$$\bar{\gamma}_1 p_1^{\frac{1}{1-\gamma_1}} w_1^{-\frac{\gamma_1}{1-\gamma_1}} = \bar{\gamma}_2 p_2^{\frac{1}{1-\gamma_2}} w_2^{-\frac{\gamma_2}{1-\gamma_2}}. \tag{9}$$

Equations (6)-(9) jointly determine the marginal worker q_L^*, the wage anchors w_1 and w_2, and the measure of managers H_1 employed in sector 1 for any economy that produces positive amounts of both goods. The equilibrium salary of managers is given by

$$r = \bar{\gamma}_i p_i^{\frac{1}{1-\gamma_i}} w_i^{-\frac{\gamma_i}{1-\gamma_i}}, \ i = 1, 2. \tag{10}$$

In what follows, we are interested in the determinants of the trade pattern between countries that differ in their relative endowments of labor to managers and in their distributions of worker ability. We are also interested in how trade between such countries affects their distributions of income and measured TFP.

3.1 Determinants of the Trade Pattern

Consider two countries that trade freely at common world prices but that differ in some way in their factor supplies. Since consumers have identical and homothetic tastes worldwide, the trade pattern between them can be identified by examining the countries' relative outputs of the two goods at the common prices. Accordingly, we investigate how a change in parameters reflecting factor endowments affects relative outputs of the two goods at given prices.

In each country, a firm in industry i employs $e_i = (\gamma_i p_i / w_i)^{1/(1-\gamma_i)}$ units of effective labor per manager, thereby producing $e_i^{\gamma_i}$ units of good i. Thus, aggregate output in sector i is

$$X_i = H_i \left(\frac{\gamma_i p_i}{w_i} \right)^{\frac{\gamma_i}{1-\gamma_i}}, \quad i = 1, 2, \tag{11}$$

and so

$$\frac{X_1}{X_2} = \frac{H_1}{\left(\bar{H} - H_1\right)} \frac{(\gamma_1 p_1)^{\frac{\gamma_1}{1-\gamma_1}}}{(\gamma_2 p_2)^{\frac{\gamma_2}{1-\gamma_2}}} \frac{w_2^{\frac{\gamma_2}{1-\gamma_2}}}{w_1^{\frac{\gamma_1}{1-\gamma_1}}}.$$

We can substitute the equal-profit condition (9) into this expression to eliminate the wage anchors. This yields[9]

$$\frac{X_1}{X_2} = \frac{H_1}{\left(\bar{H} - H_1\right)} \frac{(1-\gamma_2) p_2}{(1-\gamma_1) p_1},$$

which implies that the relative output of good 1 is greater in whichever country allocates a greater share of its managers to producing that good.

3.1.1 Relative Factor Endowments

First, suppose the two countries have the same distributions of worker ability but differ in their relative aggregate endowments, $\bar{H}/\bar{L}$. To find the pattern of trade, we totally differentiate the four-equation system comprising (6)-(9) with respect to $\bar{H}/\bar{L}$ and examine how a change in relative endowments affects the allocation of managers to sector 1. The algebra in the appendix establishes

[9] This condition can alternatively be derived from the observation that in sector i the fraction $1 - \gamma_i$ of revenue is paid to managers, i.e., $(1 - \gamma_i) p_i X_i = r H_i$.

the following proposition.

Proposition 2 *Suppose that Assumption 1 holds and that $\phi_L^A(q_L) = \phi_L^B(q_L)$ for $q_L \in S_L^A = S_L^B$. Then country A exports the manager-intensive good if and only if $\bar{H}^A/\bar{L}^A > \bar{H}^B/\bar{L}^B$.*

Proposition 2 represents, of course, an extension of the Heckscher-Ohlin theorem. When worker talent is distributed similarly in the two countries, the sorting of workers to sectors generates no comparative advantages and so has no independent bearing on the trade pattern. Comparative advantage is governed instead by relative quantities of the factors, just as in the case of homogeneous labor.

3.1.2 Distributions of Labor Ability

Now suppose that the relative number of managers and workers is the same in the two countries, but that country A has relatively better workers in the sense that the density function for worker ability in country A is a rightward shift (RS) of the similar density function in country B. That is,

$$\phi_L^B(q_L/\lambda) = \phi_L^A(q_L) \quad \text{for all } q_L \in S_L^A, \text{ for some } \lambda > 1, \tag{12}$$

which has the interpretation that every worker in country A is λ times as productive as his counterpart in the talent distribution in country B. Again, we need to totally differentiate the system of equations (6)-(9) in order to identify the impact of a rightward shift in the talent distribution on employment of managers in sector 1. The algebra in the appendix supports the following conclusion.

Proposition 3 *Suppose that Assumption 1 holds, that $\bar{H}^A/\bar{L}^A = \bar{H}^B/\bar{L}^B$, and that $\phi_L^A(q_L)$ is a rightward shift of $\phi_L^B(q_L)$ for some $\lambda > 1$. If $\varepsilon_{\tilde{\psi}_i}(q'_L) > \varepsilon_{\tilde{\psi}_j}(q''_L)$ for all $q'_L, q''_L \in S_L^A \cup S_L^B$, $i \neq j$, $i, j \in \{1, 2\}$, then country A exports good i.*

The proposition states that the country that has the superior labor force exports the good produced in the industry where worker ability contributes more elastically to productivity. Notice that this need not be the good produced by the country's most able workers inasmuch as sorting reflects the ranking of $\varepsilon_{\tilde{\psi}_1}(q_L)/\gamma_1$ versus $\varepsilon_{\tilde{\psi}_2}(q_L)/\gamma_2$, whereas the trade pattern depends only on the ranking of $\varepsilon_{\tilde{\psi}_1}(q_L)$ versus $\varepsilon_{\tilde{\psi}_2}(q_L)$. This result can be understood by thinking about the sources of comparative advantage in this setting. With $\bar{H}^A/\bar{L}^A = \bar{H}^B/\bar{L}^B$, the cross-sectoral difference in factor intensity is not a source of comparative advantage for either country. Meanwhile, with $\varepsilon_{\tilde{\psi}_1}(q_L)$ different from $\varepsilon_{\tilde{\psi}_2}(q_L)$, worker ability contributes differently to productivity in the two sectors. Country A, which is relatively better endowed with more able workers, enjoys a comparative advantage in the industry in which ability matters more for output.[10]

[10] In the special case in which $\tilde{\psi}_i(q_L)$ is a power function for $i = 1, 2$, i.e., $\tilde{\psi}_i(q_L) = a_i q_L^{\alpha_i}$ for some $a_i, \alpha_i > 0$, $\varepsilon_{\tilde{\psi}_i}(q'_L) > \varepsilon_{\tilde{\psi}_j}(q''_L)$ for all q'_L and q''_L if and only if $\alpha_i > \alpha_j$. Moreover, in this case, $s_L(q_L) > 0$ for all q_L if and only if $\alpha_1/\gamma_1 > \alpha_2/\gamma_2$. Evidently, the conditions of Proposition 3 are easily satisfied. When $\tilde{\psi}_i(q_L)$ is not a power function for $i = 1, 2$, the requirement that $\varepsilon_{\tilde{\psi}_i}(q'_L) > \varepsilon_{\tilde{\psi}_j}(q''_L)$ for all $q'_L, q''_L \in S_L^A \cup S_L^B$, $i \neq j$, $i, j \in \{1, 2\}$ is not trivial, but it can be weakened into a comparison of the average elasticities of productivity with respect to ability in the two sectors. See the proof of Proposition 3 in the appendix.

We should emphasize, however, that RS puts a great deal of structure on the sense in which Country A is better endowed with high ability workers than Country B. We might ask, for example, whether an analogous result to Proposition 3 applies when the distributions of worker talent in the two countries satisfy the monotone likelihood ratio property (MLRP). The answer is that it does not. Under MLRP, the country that has the more talented work force will be especially well endowed with workers that sort to industry 1 even though ability might contribute more to productivity in industry 2. In such circumstances, the differences in relative supplies of the various qualities could offset the difference in the contribution of ability to productivity. The structure imposed by RS ensures that this cannot happen.

3.2 The Effects of Trade on Income Distribution and Measured Productivity

We study next the effect of trade on the income distribution and on measured total factor productivity (TFP) by examining the comparative statics of the equilibrium with respect to a change in the relative price of the traded goods.

3.2.1 The Wage Distribution and Managers' Salaries

Suppose that country A exports good 1, the good that is produced with the country's most able workers. This might be because the countries have similar distributions of talent but differ in their relative numbers of workers versus managers, or because the countries have similar relative factor endowments but differ in their distributions of talent, or for some combination of these reasons. In any case, we consider the effects on factor returns of an increase in the price of good 1, which corresponds to an improvement in country A's terms of trade. When integrated over the range of prices between the autarky price and the free-trade price, it also reveals the effects in country A of an opening of international trade.

Note first that the wage function (5) pins down the relative wages of the various workers employed in either of the two sectors. A small change in the relative price alters the relative pay only of workers employed in different industries. The calculations in the appendix establish the following findings.[11]

Proposition 4 *Suppose that Assumption 1 holds. Then when* $\hat{p}_1 > 0$, *(i)* $\hat{w}_1 > \hat{w}_2$; *(ii) if* $\gamma_1 \approx \gamma_2$, *then* $\hat{w}_1 > \hat{p}_1 > \hat{r} > 0 > \hat{w}_2$; *(iii) if* $\gamma_1 > \gamma_2$ *and* $s_L(q_L^*) \approx 0$, *then* $\hat{w}_1 \approx \hat{w}_2 > \hat{p}_1 > 0 > \hat{r}$; *and (iv) if* $\gamma_1 < \gamma_2$ *and* $s_L(q_L^*) \approx 0$, *then* $\hat{r} > \hat{p}_1 > 0 > \hat{w}_1 \approx \hat{w}_2$.

Proposition 4 captures the two distinct influences on factor returns in an economy with heterogeneous labor. The cross-sectoral difference in factor intensities introduces a force akin to that in the standard Heckscher-Ohlin model with homogeneous labor, whereby real wages tend to rise and real managerial salaries tend to fall if the sector experiencing the increase in relative price is the more labor intensive of the two. But the heterogeneity of labor implies that different workers are

[11] In what follows, we use a "hat" over a variable to indicate an incremental, proportional change; i.e., $\hat{z} = dz/z$.

not equally proficient as potential employees in the two sectors, which introduces a force akin to that in a specific-factors model (see, e.g., Jones, 1971). Indeed, our result is reminiscent of findings in a model with "imperfect factor mobility" (Mussa, 1982) or "partially mobile capital" (Grossman, 1983). That is, if the factor intensity differences across industries is large (i.e., $\gamma_1 \neq \gamma_2$) and the forces for inter-industry sorting of the different worker types are muted (i.e., $s_L(q_L^*) \approx 0$), then all types of the factor used intensively in sector 1 must gain, while all types of the factor used intensively in sector 2 must lose (parts (iii) and (iv) of the proposition). On the other hand, if the factor intensity difference is small (i.e., $\gamma_1 \approx \gamma_2$) and the different types of worker are imperfect substitutes in the two sectors (i.e., $s_L(q_L) > 0$), then all workers initially employed in the expanding sector will gain, all workers that continue to be employed in the contracting sector will lose, and the effect on the well being of managers will depend on their consumption pattern (part (ii) of the proposition). Finally, note from the wage equation (5) that the relative wages of two workers with different abilities that are employed in the same sector do not depend on prices. Therefore an increase in the price of good 1 does not change wage inequality *within* sectors. Meanwhile, an increase in the price of good 1 raises the wage anchor in sector 1 relative to the wage anchor in sector 2 (see part (i) of the proposition). And since the higher-ability, higher-wage workers are employed in sector 1, this implies that by raising wages in sector 1 relative to wages in sector 2 an increase in the price of good 1 increases overall wage inequality, while reducing wage inequality in the other country.

3.2.2 Measured TFP

In our setting, trade affects productivity by altering the composition of factors employed in each industry. Of course, if factor heterogeneity were properly taken into account in any measurement exercise, there could be no productivity gains or losses here inasmuch as all firms in an industry use the same production technology and technologies do not change as a result of trade. But productivity measures often do not account for fine differences in worker or managerial ability. Rather, they consider productivity gains as a residual after accounting for changes in output that can be associated with changes in input quantities in broad factor categories. Accordingly, it seems interesting to ask what our model has to say about the effects of trade on measured TFP when we take a stylized representation of the way that productivity typically is measured.

With our specification of the production functions, output in each industry is a Cobb-Douglas function of the quantities of capital and labor, with productivity determined by the abilities of the workers employed there. Let us write aggregate output in sector i as

$$X_i = A_i L_i^{\gamma_i} H_i^{1-\gamma_i},$$

where $L_i = \bar{L} \int_{q_L \in Q_{Li}} \phi_L(q_L)\, dq_L$ is the aggregate employment of labor in sector i and $H_i = \bar{L} \int_{q_L \in Q_{Li}} [\phi(q_L)/\ell(q_L)]\, dq_L$ is the aggregate number of managers hired there. We can view A_i as a measure of TFP in industry i when the abilities of different workers are not observed by the analyst. This measure of productivity is close to what is used in most empirical studies. We ask, how does trade affect A_i?

When the relative price of good 1 increases, additional workers are drawn to industry 1. The marginal workers that join the sector are less productive than those employed there beforehand, since $s_L(q_L) > 0$ implies that the the industry initially attracts all workers with ability above the threshold, q_L^*. Firms match these marginal workers with appropriate numbers of homogeneous managers. It follows that measured TFP in industry 1 falls. Meanwhile, industry 2 sheds its most able workers. So measured TFP in that sector falls as well. In short, the country that exports good 1 sees a fall in measured productivity in both sectors as the result of an opening of trade or after any increase in the price of its export good. Just the opposite is true in the other country, where an expansion of the export sector means that the marginal workers are more talented than any who were previously employed there and the contraction of the import-competing sector means that this sector loses its least able workers.

Formally, we show in the appendix that

$$A_1^{1/\gamma_1} = \mathbb{E}\left[\tilde{\psi}_1(q_L)^{1/\gamma_1} \,|\, q_L \geq q_L^*\right]$$

and

$$A_2^{1/\gamma_2} = \mathbb{E}\left[\tilde{\psi}_2(q_L)^{1/\gamma_2} \,|\, q_L \leq q_L^*\right],$$

where $\mathbb{E}$ is the expectations operator. Apparently, both A_1 and A_2 are increasing functions of q_L^*. As p_1 increases and sector 1 expands, the ability of the marginal worker q_L^* declines in the country that exports good 1 and measured TFP falls in both sectors. The opposite is true in the country that imports good 1; as p_1 declines there, q_L^* grows, and measured TFP rises in both sectors. We have therefore established

Proposition 5 *Suppose that Assumption 1 holds. Then international trade reduces measured TFP in both sectors in the country that exports good* 1 *and raises measured TFP in both sectors in the country that imports this good.*

Here, trade has opposite implications for measured productivity in the two countries. If, for example, the country that has a comparative advantage in good 1 also has access to superior technologies for producing the two goods, then the opening of trade will generate a convergence in measured TFP. Such convergence would reflect only the induced changes in factor composition in the various sectors and not any international diffusion of technology.

3.3 Sorting Reversal

So far, we have used Assumption 1 to characterize the sorting of heterogeneous workers and the resulting trade structure. In this final part of the section on homogeneous managers we clarify what can happen when $s_L(q_L)$ switches sign.

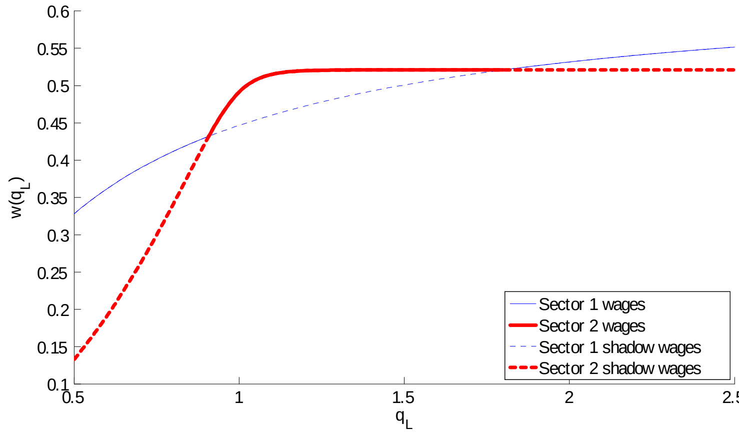

Figure 2: Wages with a reversal of sorting

First note that if $\tilde{\psi}_i(q_L)$ is a power function for $i = 1, 2$, the function $s_L(q_L)$ does not depend on q_L inasmuch as the elasticities of productivity with respect to ability then are constants. In such circumstances, $s_L(q_L)$ is either always positive or always negative, and we can assume $s_L(q_L) > 0$ without loss of generality, because this only amounts to a particular labeling of the sectors. However, when $\tilde{\psi}_i(q_L)$ is not a power function for some i, the assumption that $s_L(q_L)$ has a uniform sign for all $q_L \in S_L$ imposes meaningful restrictions on the forms of the productivity functions and the support of the distribution of worker talent. Without these restrictions, we cannot be sure that the most able workers sort into one sector and the least able workers sort into the other.

To illustrate what can happen when $s_L(q_L)$ changes signs, suppose that the productivity of a firm in sector i that hires workers of ability q_L is given by

$$\tilde{\psi}_i(q_L) = \left(\alpha_i q_L^{\rho_i} + 1\right)^{1/\rho_i}, \ \alpha_i > 0, \ \rho_i < 0 \ \text{ for } i = 1, 2. \tag{13}$$

This specification implies a constant elasticity of substitution between the ability of workers and the ability of managers in generating the productivity of the firm, and that worker and managerial ability are, in fact, complements. Of course, with homogeneous managers, firms have no possibility to adjust manager type in order to take advantage of this complementarity. Nonetheless, the CES specification for productivity represents a legitimate and even a plausible functional form.

When productivity takes the form indicated in (13), the elasticity of productivity with respect to worker ability in sector i is given by $\varepsilon_{\tilde{\psi}_i}(q_L) = \alpha_i q_L^{\rho_i} / \left(\alpha_i q_L^{\rho_i} + 1\right)$. If $\rho_1 \neq \rho_2$ then $\varepsilon_{\tilde{\psi}_1}(q_L) - \varepsilon_{\tilde{\psi}_2}(q_L)$ necessarily switches signs on $q_L \in [0, +\infty)$ and therefore $s_L(q_L)$ may switch signs on the support of the distribution of worker ability, depending on the industry factor intensities and the range of the talent distribution.

Figure 2 depicts an equilibrium wage schedule for an economy in which $s_L(q_L) < 0$ for low values of q_L and $s_L(q_L) > 0$ for high values of q_L.[12] In this economy, the most and least able workers sort to sector 1 while a middle range of workers is hired into sector 2. The thin solid curves

[12] See Lim (2013) for the functional forms and parameter values that were used to generate this figure.

in the figure depict the wages of workers employed in sector 1 as a function of their ability, while the thick solid curve depicts the wages of workers employed in sector 2. The broken thin curve depicts the shadow wage for workers in sector 2, i.e., the wage offers they could garner were they to seek jobs in sector 1. Similarly, the broken thick curve depicts the shadow wages available in sector 2 for workers actually employed in sector 1. Clearly, each worker sorts into the industry that offers him the highest wage.

Figure 2 represents an economy in which $\gamma_1 = \gamma_2 = 0.5$, i.e., the industries have similar factor intensities. However, $\rho_1 \neq \rho_2$, which generates the different elasticities of productivity at different levels of ability. The comparative statics reveal an interesting response of wages to relative price changes for these parameter values. Inasmuch as the factor intensities are common to the two industries, there are no Stolper-Samuelson forces at work. But the workers that sort to sector 1 are better suited for employment there than their counterparts working in sector 2. The forces akin to those in a specific-factors model imply that when p_1 rises, the real wages of all workers employed in sector 1 also rise, while the real wages of all workers employed in sector 2 decline. In short, an increase in the relative price of good 1 generates income gains for workers with high or low wages but income losses for those in the middle of the wage distribution.[13]

When the two sectors differ in their factor intensities, the Stolper-Samuelson forces will again play a role in determining the effects of trade on the wage distribution. Take, for example, a case in which $\gamma_1 = 0.9$ and $\gamma_2 = 0.1$, so that sector 1 is much more labor intensive than sector 2. We have solved this example numerically for various sets of the other parameter values.[14] In all such cases, we found that an increase in the price of good 1 raises both wage anchors more than in proportion to the price change, so that all workers gain in real income. Meanwhile, the salary of managers falls. These results are familiar from the Stolper-Samuelson theorem, and they are similar to what we found with great disparities in factor intensities for economies that satisfy Assumption 1. We find as well that an increase in p_1 benefits workers employed in sector 1 more than those employed in sector 2, in keeping with our observations that workers are partially specific to their industry of employment due to comparative productivity differences.[15] Price changes do not affect relative wages for workers employed in the same industry, even if those workers are at opposite tails of the talent distribution as is the case for some pairs of workers that sort to sector 1.

[13] For this example, we calculate that a 5% increase in p_1 raises the wage anchor w_1 by 5.7%, while depressing the wage anchor w_2 by 4.2%. Managers' salaries rise by 4.3%, which is proportionately less than the increase in price.

[14] As one example, we have solved the model for the case in which world prices are $(p_1, p_2) = (1, 1)$ and the economy has an aggregate endowment of $(\bar{H}, \bar{L}) = (1, 1)$. In this example, we assumed that worker ability is drawn from a truncated Pareto distribution on the support $S_L = [0.8, 1.8]$ with the shape parameter 3, and that the technological parameters are given by $(\gamma_1, \alpha_1, \rho_1) = (0.9, 0.7, -1)$ and $(\gamma_2, \alpha_2, \rho_2) = (0.1, 0.3, -20)$. In the computed equilibrium, sector 2 employes workers with $q_L \in [1.0346, 1.2116]$ and 0.9532 managers. The wage anchors are $w_1 = 0.7179$ and $w_2 = 0.4339$ while the managers earn a salary of $r = 0.7646$.

[15] Using the parameter values detailed in the previous footnote, we find that a 5% increase in the price p_1 generates a wage hike of 5.6% for workers employed in sector 1, a wage hike of 5.4% for workers employed in sector 2, and a slary reduction of 0.6% for all managers.

4 Heterogeneous Managers with Cobb-Douglas Productivity

We now introduce manager heterogeneity. We begin with a special case in which managerial ability and worker ability make multiplicatively separable contributions to the productivity of the unit and take a Cobb-Douglas (i.e., constant elasticity) form. In particular, we shall assume in this section that

$$\psi_i(q_H, q_L) = q_H^{\beta_i} q_L^{\alpha_i} \text{ for } i = 1, 2;\ \alpha_i, \beta_i > 0. \tag{14}$$

Note that, in this case, productivity is a *weakly* log supermodular function of the two ability levels. As such, the complementarity between the talent of workers and that of the manager is somewhat muted compared to what arises with *strict* log supermodularity, which means the forces for positive assortative matching within a sector are correspondingly weaker. The Cobb-Douglas case is simpler to analyze than the case with stronger complementarities, so we postpone the latter in order to shed light on some of the economic forces at works.

In this section and what follows, we model the diversity of manager types in parallel to that for workers. In particular, there is a mass $\bar{H}^c$ of managers in country c and a probability density $\phi_H^c(q_H)$ of managers with ability q_H for $q_H \in S_H^c = [q_{H\min}^c, q_{H\max}^c]$. We take the supply of managers and their ability distribution as given throughout the analysis.

There is no need to go through all the steps of a firm's profit maximization problem, because the derivation proceeds much as for the case with homogeneous managers in Section 3. Suffice it to say that the demand per manager for workers of ability q_L by a firm in industry i that pairs these workers with a manager of ability q_H is given by

$$\ell(q_L, q_H) = \left[\frac{\gamma_i p_i q_H^{\beta_i} q_L^{\alpha_i}}{w(q_L)}\right]^{\frac{1}{1-\gamma_i}}. \tag{15}$$

Substituting (15) into the expression for profits yields

$$\tilde{\pi}_i(q_L, q_H) = \bar{\gamma}_i p_i^{\frac{1}{1-\gamma_i}} \left(q_H^{\beta_i} q_L^{\alpha_i}\right)^{\frac{1}{1-\gamma_i}} w(q_L)^{-\frac{\gamma_i}{1-\gamma_i}} - r(q_H), \tag{16}$$

where $r(q_H)$ is the salary of a manager with ability q_H and $\bar{\gamma}_i \equiv \gamma_i^{\frac{\gamma_i}{1-\gamma_i}}(1-\gamma_i)$. Every firm chooses the ability of its workers and the ability of its manager so as to maximize profits, yet free entry dictates that these profits must be equal to zero in equilibrium. Let M_i be the set of all matches that maximize profits in sector i. For each pairing (q_L, q_H) in M_i,

$$r(q_H) = \bar{\gamma}_i p_i^{\frac{1}{1-\gamma_i}} \left(q_H^{\beta_i} q_L^{\alpha_i}\right)^{\frac{1}{1-\gamma_i}} w(q_L)^{-\frac{\gamma_i}{1-\gamma_i}},\ \ i = 1, 2, \tag{17}$$

by dint of the zero-profit condition. Profit maximization with respect to the choice of types, evaluated for pairings that achieve zero profits in accordance with (17), yields the first-order conditions,

$$\frac{\alpha_i}{\gamma_i} = e_w(q_L) \text{ for } q_L \in Q_{Li}^{int} \tag{18}$$

and

$$\frac{\beta_i}{1-\gamma_i} = e_r(q_H) \text{ for } q_H \in Q_{Hi}^{int}. \tag{19}$$

Equation (18) is the analog to (4) and equates the ratio of the elasticities of output with respect to worker ability and labor quantity to the elasticity of the wage schedule. Equation (19) has a similar interpretation regarding a firm's choice of manager type.

In equilibrium, all worker types must be employed, which means that firms in some sector (or both) must demand the full range of workers. Equation (18) can be satisfied for a range of workers only if the wage schedule has a constant elasticity over this range. Therefore, the equilibrium wage schedule must take the form

$$w(q_L) = w_i q_L^{\alpha_i/\gamma_i} \text{ for } q_L \in Q_{Li}^{int}. \tag{20}$$

The salary schedule for managers must have a similar form, namely

$$r(q_H) = r_i q_H^{\beta_i/(1-\gamma_i)} \text{ for } q_H \in Q_{Hi}^{int}, \tag{21}$$

where r_i is a "salary anchor" analogous to w_i.

When the wage function has a constant elasticity equal to α_i/γ_i for a range of worker types, a firm in sector i is indifferent as to its choice of employees among workers in this range, irrespective of the ability of its manager. And when the salary function has an elasticity equal to $\beta_i/(1-\gamma_i)$, the firm is indifferent to the ability of its managers. Accordingly, the matching of workers and managers among those that sort to sector i is *indeterminate* in the Cobb-Douglas case. This indeterminacy reflects the fact that the productivity function in (14) is only weakly log supermodular and thus provides no clear incentives for positive (or negative) assortative matching.

Although the matching of workers and managers in a sector is not determined in the Cobb-Douglas case, the sorting of these factors to the two sectors follows a familiar pattern. The elasticity of the wage schedule must be greater along its upper segment than along its lower segment, or else firms that hire the less able workers would all prefer to upgrade their employees. Similarly, the elasticity of the salary schedule must be greater along its upper segment than its lower segment. We designate as sector 1 whichever industry has the greater ratio of the output elasticity with respect to worker ability to the output elasticity with respect to labor quantity. With this labeling convention, $s_L = \alpha_1/\gamma_1 - \alpha_2/\gamma_2 > 0$. Then, in any equilibrium in which a country produces both goods, sector 1 attracts the workers with ability q_L above some cutoff q_L^*. If $s_H = \beta_1/(1-\gamma_1) - \beta_2/(1-\gamma_2) > 0$, then sector 1 also attracts the more able managers with $q_H > q_H^*$; otherwise, the sorting of managers is opposite to that for workers.

For precision, we state more formally the environment we consider throughout this section and the sorting pattern that results.

Assumption 2 (i) $S_H = [q_{H\min}, q_{H\max}]$, $0 < q_{H\min} < q_{H\max} < +\infty$; (ii) $\psi_i(q_H, q_L) = q_H^{\beta_i} q_L^{\alpha_i}$, $\alpha_i, \beta_i > 0$, for $i = 1, 2$; and (iii) $s_L \equiv \alpha_1/\gamma_1 - \alpha_2/\gamma_2 > 0$.

Proposition 6 *Suppose that Assumption 2 holds. Then, in any competitive equilibrium with employment in both sectors, the more able workers with $q_L \geq q_L^*$ are employed in sector 1 and the less able workers with $q_L \leq q_L^*$ are employed in sector 2, for some $q_L^* \in S_L$. If $s_H > 0$ ($s_H < 0$), the more able managers with $q_H \geq q_H^*$ are employed in sector 1 (sector 2) and the less able managers with $q_H \leq q_H^*$ are employed in sector 2 (sector 1), for some $q_H^* \in S_H$.*

To describe the equilibrium once the sorting pattern has been settled, we invoke factor-market clearing, continuity of worker wages, continuity of managerial salaries, and the zero-profit conditions. For concreteness, let us focus on the case in which $s_H > 0$ so that the more able managers sort to industry 1; the opposite case can be handled similarly.

It proves convenient to define $e_{Hi}(q_H) = q_H^{\beta_i/(1-\gamma_i)}$ as the effective managerial input of a manager with ability q_H who works in sector i. Then the aggregate supplies of effective managerial input in sectors 1 and 2 are

$$H_1 = \bar{H} \int_{q_H^*}^{q_{H\max}} q_H^{\frac{\beta_1}{1-\gamma_1}} \phi_H(q_H)\, dq_H, \tag{22}$$

and

$$H_2 = \bar{H} \int_{q_{H\min}}^{q_H^*} q_H^{\frac{\beta_2}{1-\gamma_2}} \phi_H(q_H)\, dq_H\ , \tag{23}$$

respectively. Note that $H_1/\bar{H}$ depends only on q_H^* and is a monotonically decreasing function, and $H_2/\bar{H}$ also depends only on q_H^* and is monotonically increasing.

Consider now the supply and demand for effective labor in sector 1, where we define $e_{Li}(q_L) = q_L^{\alpha_i/\gamma_i}$ as the effective labor provided by a worker of ability q_L in sector i. From the labor demand equation (15), a firm in sector 1 combines a manager with e_{Hi} units of effective managerial input with $e_{Hi}(\gamma_i p_i/w_i)^{1/(1-\gamma_i)}$ units of effective labor. Therefore, the H_1 units of effective managerial input that are hired into sector 1 are combined with $H_1(\gamma_1 p_1/w_1)^{1/(1-\gamma_1)}$ units of effective labor. Noting the definition of H_1 and equating the demand for effective labor in sector 1 with the supply of effective labor among those with ability above q_L^*, we have

$$\bar{H}\left(\frac{\gamma_1 p_1}{w_1}\right)^{\frac{1}{1-\gamma_1}} \int_{q_H^*}^{q_{H\max}} q_H^{\frac{\beta_1}{1-\gamma_1}} \phi_H(q_H)\, dq_H = \bar{L} \int_{q_L^*}^{q_{L\max}} q_L^{\frac{\alpha_1}{\gamma_1}} \phi_L dq_L. \tag{24}$$

A similar condition applies in sector 2, where labor-market clearing requires

$$\bar{H}\left(\frac{\gamma_2 p_2}{w_2}\right)^{\frac{1}{1-\gamma_2}} \int_{q_{H\min}}^{q_H^*} q_H^{\frac{\beta_2}{1-\gamma_2}} \phi_H(q_H)\, dq_H = \bar{L} \int_{q_{L\min}}^{q_L^*} q_L^{\frac{\alpha_2}{\gamma_2}} \phi_L dq_L. \tag{25}$$

Continuity of the wage schedule at q_L^* requires that

$$w_1 (q_L^*)^{\frac{\alpha_1}{\gamma_1}} = w_2 (q_L^*)^{\frac{\alpha_2}{\gamma_2}}\ . \tag{26}$$

The salary function for managers must also be continuous and firms that hire managers with ability q_H^* must earn zero profits in either sector. Together, these considerations imply

$$\bar{\gamma}_1 p_1^{\frac{1}{1-\gamma_1}} w_1^{-\frac{\gamma_1}{1-\gamma_1}} \left(q_H^*\right)^{\frac{\beta_1}{1-\gamma_1}} = \bar{\gamma}_2 p_2^{\frac{1}{1-\gamma_2}} w_2^{-\frac{\gamma_2}{1-\gamma_2}} \left(q_H^*\right)^{\frac{\beta_2}{1-\gamma_2}} . \tag{27}$$

Equations (24)-(27) comprise four equations that can be used to solve for the two wage anchors, w_1 and w_2, and the two cutoffs, q_L^* and q_H^*. The effective supply of managers in sectors 1 and 2, H_1 and H_2, can then be solved from (22) and (23). Finally, the salary anchors for the managers can be computed from the zero-profit conditions, which imply

$$r_i = \bar{\gamma}_i p_i^{\frac{1}{1-\gamma_i}} w_i^{-\frac{\gamma_i}{1-\gamma_i}} \quad \text{for } i = 1, 2. \tag{28}$$

This completes our characterization of the supply-side equilibrium for an economy that faces prices p_1 and p_2.

4.1 Pattern of Trade

As before, we need an expression for an economy's relative outputs in order to conduct the comparative static analysis that reveals the pattern of trade between countries that differ in their relative factor endowments or in their distributions of factor types. The H_i units of effective managers employed in sector i collectively produce $X_i = H_i \left(\gamma_i p_i\right)^{\gamma_i/(1-\gamma_i)} w_i^{-\gamma_i/(1-\gamma_i)}$ units of good i. Each effective unit of managerial input is paid a salary of r_i in sector i and—by continuity of the salary function—$r_1/r_2 = \left(q_H^*\right)^{-s_H}$ (see (21)). Using this condition together with (24)-(25) and (27)-(28), we can write

$$\begin{aligned} \frac{X_1}{X_2} &= \frac{r_1 H_1}{r_2 H_2} \frac{\left(1-\gamma_2\right) p_2}{\left(1-\gamma_1\right) p_1} \\ &= \frac{\left(1-\gamma_2\right) p_2 \int_{q_H^*}^{q_{H\max}} q_H^{\frac{\beta_1}{1-\gamma_1}} \phi_H\left(q_H\right) dq_H}{\left(1-\gamma_1\right) p_1 \int_{q_{H\min}}^{q_H^*} q_H^{\frac{\beta_2}{1-\gamma_2}} \phi_H\left(q_H\right) dq_H} \left(q_H^*\right)^{-s_H} . \end{aligned} \tag{29}$$

Similar to the case of homogeneous managers, the first line of (29) reflects the fact that the aggregate salaries of all managers in sector i absorb a fraction $1-\gamma_i$ of revenue. And the second line implies that, since $s_H > 0$ in the case under consideration, X_1/X_2 is a decreasing function of q_H^*. Therefore, to identify the pattern of trade, we need only find which country allocates more effective managerial input to sector 1 relative to its aggregate endowment of managers; that is, how q_H^* varies with factor endowments.[16]

The system of equations (24)-(27) that applies with Cobb-Douglas productivity is quite similar to the system (6)-(9) that applies when managers are homogeneous, except that now we need

[16] Note that in the opposite case, when $s_H < 0$, managers with $q_H \geq q_H^*$ sort into sector 2 while managers with $q_H \leq q_H^*$ sort into sector 1. As a result, X_1/X_2 is an increasing function of q_H^*.

to use the effective managerial input in a sector in place of the pure number of managers. In other words, the multiplicative separability of the productivity function allows us to construct an aggregate measure of managerial input that plays the same role as does the number of managers when managers are equally productive. We can do so, because there are no forces present in the Cobb-Douglas case to induce any particular pattern of matching within either sector. It stands to reason that the determinants of the trade pattern with heterogeneous managers but Cobb-Douglas productivity are analogous to those we found for the case of homogeneous managers. In the appendix, we prove

Proposition 7 *Suppose that Assumption 2 holds. Then if* $\phi_L^A(q_L) = \phi_L^B(q_L)$ *for all* $q_L \in S_L^A = S_L^B$*,* $\phi_H^A(q_H) = \phi_H^B(q_H)$ *for all* $q_H \in S_H^A = S_H^B$*, and* $\bar{H}^A/\bar{L}^A > \bar{H}^B/\bar{L}^B$*, country A exports the manager-intensive good.*

Proposition 8 *Suppose that Assumption 2 holds and* $\bar{H}^A/\bar{L}^A = \bar{H}^B/\bar{L}^B$*. Then, (i) if* $\phi_H^A(q_H) = \phi_H^B(q_H)$ *for all* $q_H \in S_H^A = S_H^B$ *and* $\phi_L^A(q_L)$ *is a rightward shift of* $\phi_L^B(q_L)$ *for some* $\lambda > 1$*, then country A exports good 1 if and only if* $\alpha_1 > \alpha_2$*; (ii) if* $\phi_L^A(q_L) = \phi_L^B(q_L)$ *for all* $q_L \in S_L^A = S_L^B$ *and* $\phi_H^A(q_H)$ *is a rightward shift of* $\phi_H^B(q_H)$ *for some* $\lambda > 1$*, then country A exports good 1 if and only if* $\beta_1 > \beta_2$*.*

In short, the Heckscher-Ohlin theorem applies when countries have similar distributions of factor types but differ in their relative aggregate endowments of managers versus workers. Alternatively, if the relative factor endowments are the same in the two countries but they differ in their distributions of one of the factors, then the country with the rightward-shifted distribution of a factor exports the good produced by the industry in which productivity responds more elastically to that factor's ability.

4.2 Effects of Trade on Income Distribution and Measured Productivity

Our results on income distribution also carry over straightforwardly from the case with homogeneous managers to that with manager heterogeneity but Cobb-Douglas productivity. First note that within-industry income distribution is not affected by world trade inasmuch as the elasticity of the wage schedule for workers employed in a given industry is constant. As a result, (20) implies that $w(q_L')/w(q_L'') = (q_L'/q_L'')^{\alpha_i/\gamma_i}$ for $q_L', q_L'' \in Q_{Li}$ and (21) implies that $r(q_H')/r(q_H'') = (q_H'/q_H'')^{\beta_i/(1-\gamma_i)}$ for $q_H', q_H'' \in Q_{Hi}$. Second, relative rewards of workers and managers that are employed in different industries do change with trade, inasmuch as the wage and salary anchors w_i and r_i change. In the appendix we prove

Proposition 9 *Suppose that Assumption 2 holds and* $s_H \approx 0$*. When* $\hat{p}_1 > 0$*, (i)* $\hat{w}_1 > \hat{w}_2$*; (ii) if* $\gamma_1 \approx \gamma_2$*, then* $\hat{w}_1 > \hat{p}_1 > \hat{r}_1 \approx \hat{r}_2 > 0 > \hat{w}_2$*; (iii) if* $\gamma_1 > \gamma_2$ *and* $s_L \approx 0$*, then* $\hat{w}_1 \approx \hat{w}_2 > \hat{p}_1 > 0 > \hat{r}_1 \approx \hat{r}_2$*; (iv) if* $\gamma_1 < \gamma_2$ *and* $s_L \approx 0$*, then* $\hat{r}_1 \approx \hat{r}_2 > \hat{p}_1 > 0 > \hat{w}_1 \approx \hat{w}_2$*.*

Proposition 9 can be understood by recognizing that the model with heterogeneous workers and managers contains a blend of Stolper-Samuelson and Ricardo-Viner forces. When $s_H \approx 0$, there is no difference in the suitability of the various managers for employment in one sector versus the other, because the comparative advantage associated with greater ability of the input just offsets the comparative advantage associated with greater quantity. Then, it is as if managers are a perfectly mobile, homogeneous factor. When s_L also is small, the Stolper-Samuelson forces will dominate, and workers in both industries will see a gain in real income if the relative price of the labor-intensive good rises and will see a loss in real income if the relative price of the labor-intensive good falls. In contrast, if factor intensities are approximately the same in the two industries, the Stolper-Samuelson forces will be muted, and the partial specificity of workers arising from the comparative advantage of ability in sector 1 will govern the income responses. Then, workers will benefit in real terms when the relative price of the good they produce rises and will lose in real terms if the relative price of this good falls. Also note that similar considerations imply that if $s_H > 0$ but $s_L \approx 0$ and $\gamma_1 \approx \gamma_2$, the economy behaves like one with sector-specific managers and perfectly mobile labor. Then $\hat{r}_1 > \hat{p}_1 > \hat{w}_1 \approx \hat{w}_2 > 0 > \hat{r}_2$, i.e., managers in the expanding sector gain, managers in the contracting sector lose, and workers may gain or lose in real terms depending on their consumption pattern. Finally, similarly to Proposition 4, an increase in the price of good 1 raises overall wage inequality, because it does not change relative wages within sectors and it increases wages of the more able, better-paid workers employed in sector 1 relative to the less able, lower-paid workers in sector 2.

Turning to the effects of trade on measured productivity, our conclusions also are reminiscent of those we have seen before. Recalling that

$$X_i = \bar{H}\left(\frac{\gamma_i p_i}{w_i}\right)^{\frac{\gamma_i}{1-\gamma_i}} \int_{q_H \in Q_{Hi}} q_H^{\beta_i/(1-\gamma_i)} \phi_H(q_H)\, dq_H, \quad \text{for } i = 1, 2,$$

we can substitute the labor market clearing conditions (24) and (25) to write output in sector i as

$$X_i = \bar{L}^{\gamma_i} \bar{H}^{1-\gamma_i} \left(\int_{q_L \in Q_{Li}} q_L^{\alpha_i/\gamma_i} \phi_L(q_L)\, dq_L\right)^{\gamma_i} \left(\int_{q_H \in Q_{Hi}} q_H^{\beta_i/(1-\gamma_i)} \phi_H(q_H)\, dq_H\right)^{1-\gamma_i}.$$

Since the aggregate factor inputs in sector i are $L_i = \bar{L} \int_{q_L \in Q_{Li}} \phi_L(q_L)\, dq_L$ and $H_i = \bar{H} \int_{q_H \in Q_{Hi}} \phi_H(q_H)\, dq_H$, we can write measured TFP as

$$\begin{aligned} A_i &= \frac{\left(\int_{q_L \in Q_{Li}} q_L^{\alpha_i/\gamma_i} \phi_L(q_L)\, dq_L\right)^{\gamma_i} \left(\int_{q_H \in Q_{Hi}} q_H^{\beta_i/(1-\gamma_i)} \phi_H(q_H)\, dq_H\right)^{1-\gamma_i}}{\left(\int_{q_L \in Q_{Li}} \phi_L(q_L)\, dq_L\right)^{\gamma_i} \left(\int_{q_H \in Q_{Hi}} \phi_H(q_H)\, dq_H\right)^{1-\gamma_i}} \\ &= \left(\mathbb{E}\left[q_L^{\alpha_i/\gamma_i} \mid q_L \in Q_{Li}\right]\right)^{\gamma_i} \left(\mathbb{E}\left[q_H^{\beta_i/(1-\gamma_i)} \mid q_H \in Q_{Hi}\right]\right)^{1-\gamma_i}. \end{aligned}$$

Now take the case in which $s_H > 0$. Then an increase in p_1 causes sector 1 to expand by attracting both more workers and more managers; i.e., both q_L^* and q_H^* decline. The movement of marginal factors from sector 2 to sector 1 reduces the average ability of both factors in both industries. As a result, measured productivity falls in both sectors. However, if $s_H < 0$, sector 1 attracts the best workers but the worst managers. As this sector expands, average worker ability declines but average manager ability grows. In this case, TFP can rise or fall in either industry and possibly can move in opposite directions in the two industries.

5 Strong Complementarities between Heterogeneous Factors

The Cobb-Douglas case is special, because when alternative worker teams are paired with a given manager, their relative productivity is independent of the ability of that manager.[17] In such circumstances, the matching of workers and managers is not determined by the requirements for a competitive equilibrium. We depart now from multiplicative separability in order to study productivity functions that induce a determinate pattern of matching in each industry. In particular, we adopt

Assumption 3 (i) $S_H = [q_{H\min}, q_{H\max}]$, $0 < q_{H\min} < q_{H\max} < +\infty$; (ii) $\psi_i(q_H, q_L)$ is strictly increasing, twice continuously differentiable, and *strictly* log supermodular for $i = 1, 2$.

This assumption implies that $\psi_{iH}(q_H, q_L) / \psi_i(q_H, q_L)$ is increasing in q_L and $\psi_{iL}(q_H, q_L) / \psi_i(q_H, q_L)$ is increasing in q_H, where $\psi_{iF}(q_H, q_L)$ is the partial derivative of $\psi_i(q_H, q_L)$ with respect to q_F, $F = H, L$.

Proceeding as before, we first find the labor demand per manager by a firm in sector i, taking as given the common ability of the team of workers and the ability of the manager. We substitute the optimal labor demand $\ell(q_H, q_L)$ into the expression for profits to derive the profit function,

$$\tilde{\pi}_i(q_H, q_L) = \bar{\gamma}_i p_i^{\frac{1}{1-\gamma_i}} \psi_i(q_H, q_L)^{\frac{1}{1-\gamma_i}} w(q_L)^{-\frac{\gamma_i}{1-\gamma_i}} - r(q_H). \quad (30)$$

Each firm chooses the ability of its workers and the ability of its manager so as to maximize profits taking the wage and salary schedules as given, while free entry dictates that realized profits for active firms are zero. The wage schedule $w(q_L)$ is continuous and strictly increasing for all $q_L \in S_L$ and the salary schedule $r(q_H)$ is continuous and strictly increasing for all $q_H \in S_H$.[18]

We solve the firm's profit-maximization problem in two stages. First, given q_H, the firm chooses the most suitable workers, deriving thereby the profits

$$\Pi_i(q_H) = \max_{q_L \in S_L} \tilde{\pi}_i(q_H, q_L), \text{ for } q_H \in S_H, \ i = 1, 2. \quad (31)$$

[17] In fact, this property is shared by any productivity function that is multiplicatively separable in the ability levels of the two factors.

[18] The productivity function $\psi_i(\cdot)$ is strictly increasing for $i = 1, 2$. Therefore if $w(\cdot)$ were discontinuous at some q_L, then there would be no demand for workers with abilities just above or just below q_L. Moreover, if the wage function were not strictly increasing, there would be no demand for some positive measure of workers. If, for example, $w(\cdot)$ were flat or declining over some interval beginning at q_L, there would be no demand for workers in an interval bounded below by q_L. Analogous arguments apply to the salary schedule $r(\cdot)$.

Second, it chooses q_H to maximize $\Pi_i(q_H)$. We show in the appendix that the solution to this problem results in equilibrium allocation sets Q_{Li} and Q_{Hi} that must be unions of closed intervals, where Q_{Fi} is the set of types of factor F that sorts to industry i, for $F = H, L$ and $i = 1, 2$. Moreover, there is positive assortative matching (PAM) within each sector; that is, in each industry the better workers are matched with the better managers (see Eeckhout and Kircher, 2012). It can happen, however, that when comparing a more able manager employed in sector 2 and a less able manager employed in sector 1, the latter oversees better workers than the former. In other words, PAM may fail *across sectors*, as we shall see in several examples below.

Let $m_i(q_H)$ denote the solution to (31). Then

$$m(q_H) = \begin{cases} m_1(q_H) & \text{for } q_H \in Q_{H1} \\ m_2(q_H) & \text{for } q_H \in Q_{H2} \end{cases}.$$

The equilibrium pairings in sector i are

$$M_i = \left[\{q_H, q_L\} \mid q_L \in m_i(q_H) \text{ for all } q_H \in Q_{Hi}\right],$$

where M_i is a closed graph consisting of a union of connected sets M_i^n such that $m_i(q_H)$ is continuous and strictly increasing in each set but may jump discontinuously between them.

Now consider an equilibrium with incomplete specialization, so that both Q_{H1} and Q_{H2} are of positive measure. Then $\Pi_i(q_H) = 0$ for all $q_H \in Q_{Hi}$, $i = 1, 2$, which implies

$$r(q_H) = \bar{\gamma}_i p_i^{\frac{1}{1-\gamma_i}} \psi_i[q_H, m_i(q_H)]^{\frac{1}{1-\gamma_i}} w[m_i(q_H)]^{-\frac{\gamma_i}{1-\gamma_i}} \quad \text{for all } q_H \in Q_{Hi},\ i = 1, 2. \tag{32}$$

Continuity of the wage and salary schedules implies that both functions are differentiable almost everywhere. Moreover, profit maximization and (32) imply that, at all interior points of a connected subset M_i^n of M_i, the salary function $r(\cdot)$ and the wage function $w(\cdot)$ are differentiable; see the appendix for proof. It follows that the solution to (31) must satisfy the first-order condition

$$\frac{m(q_H)\psi_{iL}[q_H, m(q_H)]}{\gamma_i \psi_i[q_H, m(q_H)]} = \varepsilon_w(m(q_H)) \text{ for all } \{q_H, m(q_H)\} \in M_i^{n,int}, \quad n \in N_i, \quad i = 1, 2, \tag{33}$$

where $M_i^{n,int}$ is the interior of M_i^n. Also, (32) and (33) imply that

$$\frac{q_H \psi_{iH}[q_H, m(q_H)]}{(1-\gamma_i)\psi_i[q_H, m(q_H)]} = \varepsilon_r(q_H) \text{ for all } \{q_H, m(q_H)\} \in M_i^{n,int}, \quad n \in N_i, \quad i = 1, 2. \tag{34}$$

Note the similarity between these equations and (18) and (19), which apply in the Cobb-Douglas case. The difference is that now the elasticities of productivity with respect to a factor's ability depend on the worker-manager combinations that occur in equilibrium.

It remains to describe the sorting conditions at boundary points between some M_1^n and some $M_2^{n'}$. Let $q_L^\dagger$ be some such boundary point, so that workers with ability just above $q_L^\dagger$ sort to one sector while workers with ability just below $q_L^\dagger$ sort to the other. For this, we require the wage function $w(q_L)$ to be at least as steep to the right of $q_L^\dagger$ as to the left; otherwise the firms that employ workers with abilities just below $q_L^\dagger$ could earn positive profits by hiring slightly more able workers and likewise firms that hire workers with abilities above $q_L^\dagger$ could earn profits by hiring slightly less able workers. By a similar argument, the salary function $r(q_H)$ must be (weakly) steeper just to the right of any boundary point $q_H^\dagger$ than just to the left of such a point.

We turn next to the factor-market clearing conditions. To this end, define $\boldsymbol{Q}_{Hi}(q_H)$ as the set of all managers that sort to sector i whose ability does not exceed q_H. Similarly, define $\boldsymbol{Q}_{Li}(q_L)$ as the set of workers that sort to sector i whose ability does not exceed q_L. A profit-maximizing firm in sector i that hires workers of ability q_L and managers of ability q_H demands $\ell(q_H, q_L) = [\gamma_i r(q_H)] / [(1-\gamma_i) w(q_L)]$ workers per manager. Since the matching function is everywhere increasing, it follows that

$$\begin{aligned}
&\bar{H} \int_{q \in \boldsymbol{Q}_{Hi}\left(q_{Hi}^{\min}\right)} \frac{\gamma_i r(q)}{(1-\gamma_i) w[m(q)]} \phi_H(q)\, dq + \bar{H} \int_{q_{Hi}^{\min}}^{q_H} \frac{\gamma_i r(q)}{(1-\gamma_i) w[m(q)]} \phi_H(q)\, dq \\
&= \bar{L} \int_{q \in \boldsymbol{Q}_{Li}\left[m\left(q_{Hi}^{\min}\right)\right]} \phi_L(q)\, dq + \bar{L} \int_{m\left(q_{Hi}^{\min}\right)}^{m(q_H)} \phi_L(q)\, dq \quad \text{for all } q_H \in \left(q_{Hi}^{\min}, q_{Hi}^{\max}\right), \quad i = 1, 2,
\end{aligned}$$

where the left-hand side represents the labor demanded by all firms in sector i hiring managers with ability not exceeding q_H and the right-hand side represents the measure of workers available to be teamed with those managers. Since the left-hand side is differentiable in q_H, this equation implies that the matching function $m(q_H)$ also is differentiable at points in $\left(q_{Hi}^{\min}, q_{Hi}^{\max}\right)$. That being the case, we can differentiate the labor-market clearing condition with respect to q_H to derive a differential equation for the matching function, namely

$$\begin{aligned}
\bar{H} \frac{\gamma_i r(q_H)}{(1-\gamma_i) w[m(q_H)]} \phi_H(q_H) \quad &= \quad \bar{L} \phi_L[m(q_H)]\, m'(q_H) \\
\text{for } \{q_H, m(q_H)\} \quad &\in \quad M_i^{n,int},\ n \in N_i,\ i = 1, 2.
\end{aligned} \tag{35}$$

Equations (33), (34) and (35) comprise three differential equations that are satisfied in any competitive equilibrium by the wage schedule $w(q_L)$, the salary schedule $r(q_H)$, and the matching function $m(q_H)$. Together with the zero-profit condition and a set of boundary conditions, these equations can be used to characterize an equilibrium allocation.

Let us consider first the possibility that the set of workers that sorts to each sector comprises a single, connected interval, and similarly for managers. That is, we consider equilibria that are characterized by two thresholds, q_L^* and q_H^*, such that all workers with ability less than q_L^* sort to some sector while all workers with ability greater than q_L^* sort to the other, and all managers with ability less than q_H^* sort to some sector while all managers with ability greater than q_H^* sort to the other. Note that we do not insist that the better workers and better managers sort to the same sector, nor do we claim that all competitive equilibria are characterized by such a simple sorting pattern.

When the set of workers employed in sector i comprises a single, connected interval, (33) implies

$$\ln w_i(q_L) - \ln w_i(q_{L0}) = \int_{q_{L0}}^{q_L} \frac{\psi_{iL}[\mu(x), x]}{\gamma_i \psi_i[\mu(x), x]} dx, \quad \text{for all } q_L, q_{L0} \in Q_{Li} \ , \tag{36}$$

where $\mu(\cdot)$ is the inverse of $m(\cdot)$ (and the latter function is invertible in sector i due to strict log supermodularity of $\psi_i(\cdot)$). Similarly, when the set of managers employed in sector i comprises a single connected interval, (34) implies

$$\ln r_i(q_H) - \ln r_i(q_{H0}) = \int_{q_{H0}}^{q_H} \frac{\psi_{iH}[x, m(x)]}{(1-\gamma_i)\psi_i[x, m(x)]} dx, \quad \text{for all } q_H, q_{H0} \in Q_{Hi}. \tag{37}$$

We see from (36) that the relative wage of the more able of any pair of workers employed in a given sector rises if all workers with abilities between the two are rematched with better managers than before. Similarly, from (37), the relative salary of the better manager in a pair that is employed in the same sector rises if the matches improve for all managers with abilities intermediate between the two. These observations reflect the complementarity between worker and manager ability that is implied by (strict) log supermodularity of the productivity functions.

Our next task is to describe sufficient conditions for the existence of a threshold equilibrium. The following proposition provides such conditions.

Proposition 10 *Suppose that Assumption 3 holds.*
(i) If

$$\frac{\psi_{iH}(q_H, q_{L\min})}{(1-\gamma_i)\psi_i(q_H, q_{L\min})} > \frac{\psi_{jH}(q_H, q_{L\max})}{(1-\gamma_j)\psi_j(q_H, q_{L\max})} \quad \textit{for all } q_H \in S_H, \quad i \neq j, \quad i = 1, 2,$$

then in any competitive equilibrium with employment of managers in both sectors, the more able managers with $q_H \geq q_H^*$ *are employed in sector* i *and the less able managers with* $q_H \leq q_H^*$ *are employed in sector* j*, for some* $q_H^* \in S_H$*.*
(ii) If

$$\frac{\psi_{iL}(q_{H\min}, q_L)}{\gamma_i \psi_i(q_{H\min}, q_L)} > \frac{\psi_{jL}(q_{H\max}, q_L)}{\gamma_j \psi_j(q_{H\max}, q_L)} \quad \textit{for all } q_L \in S_L, \quad i \neq j, \quad i = 1, 2,$$

then in any competitive equilibrium with employment of workers in both sectors, the more able workers with $q_L \geq q_L^*$ *are employed in sector* i *and the less able workers with* $q_H \leq q_H^*$ *are employed in sector* j*, for some* $q_L^* \in S_L$*.*

Part (i) of the proposition states that all high-ability managers—those with indexes above some threshold—will sort to sector i if the ratio of the elasticity of productivity with respect to manager ability to the elasticity of output with respect to managerial time is higher in that sector when a given manager is teamed with the economy's *least able* workers than the similar elasticity ratio that applies for sector j when the manager instead is teamed with the economy's *most able* workers. In such circumstances, the combinations of workers and managers cannot overturn the forces that we have previously identified that indicate sorting of the best managers to sector i.[19] Part (ii) of the proposition has a similar interpretation for labor sorting; the condition ensures that the ranking of sectors by elasticity ratio cannot be overturned even after allowing for the workers' most favorable pairing in one sector compared to their least favorable pairing in the other.

If the conditions for Proposition 10 are satisfied, then the top tier of managers sorts to some sector as does the top tier of workers, although the sector chosen by the best workers need not be the same as that chosen by the best managers. We refer to a sorting pattern that has both top managers and top workers employed in the same sector as an HH/LL equilibrium (for "high-high" and "low-low") and one that has the more able managers employed in the same sector as the less able workers as an HL/LH equilibrium (for "high-low" and "low-high"). We will see examples of both types of equilibrium in what follows.

Our next proposition provides sufficient conditions for the existence of an equilibrium with an HH/LL sorting pattern. These conditions impose less severe requirements on the productivity function than those in Proposition 10, although we do not mean to imply by this that an HH/LL equilibrium is in any sense more "likely" than an HL/LH equilibrium.[20] In the appendix we prove

Proposition 11 *Suppose that Assumption 3 holds. If*

$$\frac{\psi_{1H}(q_H,q_L)}{(1-\gamma_1)\psi_1(q_H,q_L)} > \frac{\psi_{2H}(q_H,q_L)}{(1-\gamma_2)\psi_2(q_H,q_L)} \quad \textit{for all } q_H \in S_H, \quad q_L \in S_L,$$

and

$$\frac{\psi_{1L}(q_H,q_L)}{\gamma_1\psi_1(q_H,q_L)} > \frac{\psi_{2L}(q_H,q_L)}{\gamma_2\psi_2(q_H,q_L)} \quad \textit{for all } q_H \in S_H, \quad q_L \in S_L,$$

then in any competitive equilibrium with employment of managers and workers in both sectors, the more able managers with $q_H \geq q_H^*$ are employed in sector 1 and the less able managers with $q_H \leq q_H^*$ are employed in sector 2, for some $q_H^* \in S_H$; the more able workers with $q_L \geq q_L^*$ are employed in sector 1 and the less able workers with $q_H \leq q_H^*$ are employed in sector 2, for some $q_L^* \in S_L$.

The difference in the antecedents in Proposition 10 and 11 is that, in the former we compare the elasticity ratio for each factor when it is combined with the least able type of the other factor in one sector versus the most able type in the other sector, whereas in the latter we compare the

[19] The strict log supermodularity of $\psi_i(\cdot)$ implies that $\psi_{iH}(q_H,q_L)/\psi_i(q_H,q_L)$ is increasing in q_L for every value of q_H. Therefore, if the inequality condition in part (i) of the proposition holds, we must have

$$\frac{\psi_{iH}(q_H,q_L)}{(1-\gamma_i)\psi_i(q_H,q_L)} > \frac{\psi_{jH}(q_H,q_L')}{(1-\gamma_j)\psi_j(q_H,q_L')} \quad \text{for all } q_H \in S_H \text{ and all } q_L, q_L' \in S_L, \quad i \neq j.$$

Then, the ratio of elasticities for a given manager is greater in sector i than in sector j for a given manager irrespective of the matches that form in one sector or the other. In this case, the most able managers sort to the sector where the ratio of elasticity of productivity with respect to managerial ability to the elasticity of output with respect to manager quantity is (unambiguously) highest. Under the condition of part (ii) of the proposition, an analagous argument can be made regarding the workers.

[20] The sufficient conditions in the two propositions also impose restrictions on the factor-intensity parameters γ_1 and γ_2, which are in general easier to satisfy for an HL/LH equilibrium than for an HH/LL equilibrium.

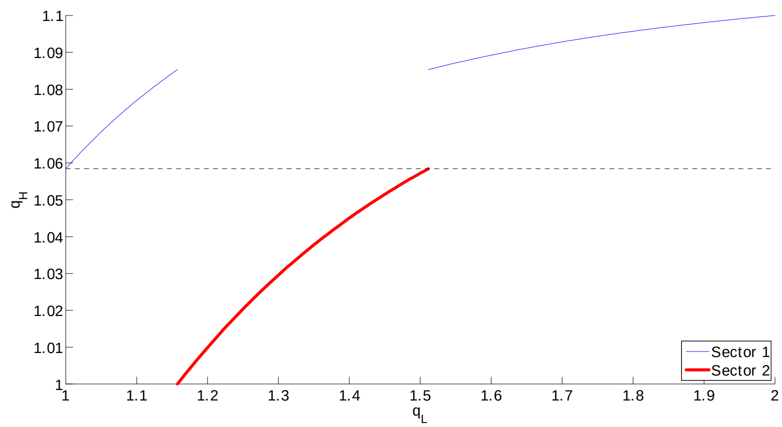

Figure 3: Matching: The most and least able workers and the most able mangers sort into sector 1

elasticity ratios for common partners in the two sectors. The difference arises, because an HH/LL equilibrium has PAM within *and across* industries, while an HL/LH equilibrium has PAM only within industries. In an HL/LH equilibrium, an able manager in sector i might be tempted to move to sector j despite a generally greater responsiveness of productivity to ability in i, because the better workers have incentive to sort to j, and with log supermodularity of $\psi_j(\cdot)$, the able manager stands to gain most from this superior match. In contrast, in an HH/LL equilibrium, the able manager in sector i would find less able workers to match with were she to move to sector j, so the temptation to switch sectors in order to upgrade partners is not present.

Propositions 10 and 11 provide sufficient conditions for the existence of a threshold equilibrium in which the allocation set for each factor and industry comprises a single, connected interval. These conditions are not necessary, however, so a threshold equilibrium can arise even if they are not satisfied. Nonetheless, not all parameter configurations give rise to equilibria with such a simple sorting pattern. An example of a more complex sorting pattern is illustrated in Figure 3.[21] In this example, the most able and least able workers sort to sector 1 while an intermediate interval of worker types sort to sector 2. The firms in sector 1 hire the economy's most able managers whereas those in sector 2 hire those with ability below some threshold level. Notice that graphs M_1 and M_2 display the general properties that we described above; they are unions of connected sets, with a matching function $m(q_H)$ that is continuous and increasing within any such set. The figure reflects a "sorting reversal" for workers that arises because the elasticity ratio for labor is higher in sector 1 when worker ability is low or high, but higher in sector 2 for a middle range of abilities. Of course, other sorting patterns besides that depicted in Figure 3 also are possible.

Armed with an understanding of the forces that drive factor sorting, we will turn shortly to the relationship between factor endowments and trade and the effects of trade on the wage and salary distributions. But before that, it will prove helpful to examine how matching and factor prices are determined for some connected intervals of worker and manager types employed in a given sector.

[21] The functional forms and parameter values underlying this example are presented in Lim (2013).

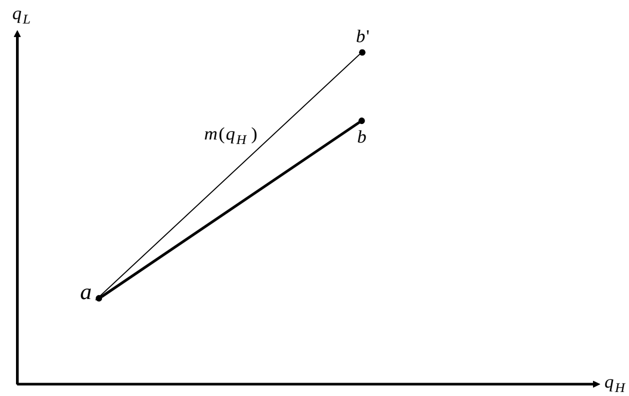

Figure 4: Shift in the matching function when q_L^b rises to $q_L^{b'}$

5.1 Matching and Factor Price Determination in an Allocation Set

Consider the subset of factors employed and matched in some sector comprising the interval of managers $Q_H = [q_{Ha}, q_{Hb}]$ and the interval of workers $Q_L = [q_{La}, q_{Lb}]$.[22] Matching between these factors and all wages and salaries are determined by a system of differential equations together with the relevant boundary conditions.[23] Our aim is to characterize a solution to the system comprising (33)-(35) for $q_H \in Q_H$ and $q_L = m(q_H) \in Q_L$ that also satisfies the zero-profit condition (32) and the boundary conditions, $m(q_{Hz}) = q_{Lz},\ \ z = a, b$. The solution to this system, which is unique, is developed in more detail in the appendix.

The solution has several notable properties. First, when the price of final output increases by some proportion, all wages for workers in Q_L and all salaries for managers in Q_H rise by this same proportion, while the matching of workers and managers in M_i^n remains the same. Second, when the ratio of the number of managers to workers increases by some proportion $\hat{\eta}$, the wages of all workers in Q_L rise by the proportion $(1-\gamma)\hat{\eta}$, while the salaries of all managers in Q_H fall by the proportion $\gamma\hat{\eta}$. This too has no effect on the matching of workers and managers in M_i^n. See Lemma 1 in the appendix for a formal statement and proof of these results.

Next consider how changes in the boundary points affect matching and factor rewards. Figure 4 illustrates how the matching function shifts, for example, when the uppermost boundary of the interval of workers rises from q_{Lb} to $q_{Lb'}$. Lemma 2 in the appendix establishes that, when (32)-(35) are satisfied for a given productivity function $\psi(\cdot)$ and given parameters $p, \gamma, \bar{H}$ and $\bar{L}$ but different boundary points, then the corresponding matching functions can intersect at most once. Moreover, if such an intersection exists, the solution with the steeper matching function at the

[22] We omit for now the subscripts that identify the sector of employment, because we will be examining only this single group of workers and managers.

[23] With Cobb-Douglas productivity, as in Section 4, matching between workers and managers is indeterminate and all wages and salaries dictated by the conditions for full employment, which require constant elasticities of the two factor-price schedules. Now, optimal matching depends on factor prices and factor productivities depend on the matches, which generates the system of interdependent, differential equations.

point of intersection also has lower wages and higher salaries for all ability levels that are common to the two settings; see Lemma 6 in the appendix. In the figure, the matching functions that apply before and after the increase in the upper boundary of worker ability necessarily intersect at (q_{Ha}, q_{La}). By Lemma 2, we know that this can be the only intersection of the two curves, and then the fact that a manager with ability q_{Hb} initially matches with a group of workers with ability q_{Lb} but ultimately matches with a group of ability $q_{Lb'}$ implies that the matching function shifts upward everywhere in the interior of M_i^n, as shown. Finally, Lemma 6 implies that wages fall for all workers with $q_L \in [q_{La}, q_{Lb}]$ as a result of the addition of workers at the upper end of the interval.

The rematching depicted in Figure 4 has implications for within-industry wage and salary inequality. Using (36) and (37) with $q_{L0} = q_{La}$ and $q_{H0} = q_{Ha}$, we see that the wage schedule rises with ability more slowly after the upper bound on worker ability increases to $q_{Lb'}$. This is so, because the original worker types are matched with less able managers after the expansion in the interval of workers and, while the downgrades are detrimental to the productivity of all workers, they are especially so for those with greater ability. Consequently, wage inequality among workers with $q_L \in [q_{La}, q_{Lb}]$ narrows. Meanwhile, the managers all find better matches than before, which raises their productivity, but especially so for the most able among them. Therefore, salary inequality grows.

Similar reasoning can be used to find the shift in the matching function—and the wage and salary responses—for changes in the other boundary points. For example, if the lower boundary of the interval of managers rises from q_{Ha} to $q_{Ha'}$, the matching function shifts downward (thereby connecting a point to the right of a in Figure 4 with point b), and thus the manager types that remain in the sector find themselves teamed with less able workers while all workers in Q_L find improved matches with managers. Such rematching narrows the salary distribution while exacerbating wage inequality. The key intuition is that, when the matches improve for some set of types of a factor, the marginal products rise proportionally more for those types that are more able, in view of the complementarities that are present.

We are ready to turn our attention to the sources of comparative advantage and the impact of trade on wages and salaries.

5.2 Pattern of Trade

Consider the pattern of trade in an environment with sorting and matching. We note first that two countries that share identical and homothetic preferences and similar distributions of factor types but different relative factor endowments will not engage in trade unless the two industries have different factor intensities. More formally, we state

Proposition 12 *Suppose that Assumption 3 holds,* $\phi_H^A(q_H) = \phi_H^B(q_H) > 0$ *for all* $q_H \in S_H^A = S_H^B = S_H$*,* $\phi_L^A(q_L) = \phi_L^B(q_L) > 0$ *for all* $q_L \in S_L^A = S_L^B = S_L$*, and* $\gamma_1 = \gamma_2 = \gamma$*. Then* $X_1^A/X_2^A = X_1^B/X_2^B$ *for all* $\bar{H}^A/\bar{L}^A$ *and* $\bar{H}^B/\bar{L}^B$*.*

We present the proof of this proposition here in the main text, because it helps to clarify the economics of the result and what follows.

Proof. To prove the result, we examine the equilibrium response to an increase in the endowment ratio $\bar{H}/\bar{L}$ at given relative prices. The initial equilibrium is characterized by sets Q_{Li} and Q_{Hi} for $i = 1, 2$, a matching function $m(q_H)$ that is strictly increasing in each of Q_{H1} and Q_{H2}, and wage and salary functions $w(q_L)$ and $r(q_H)$ that are continuous and strictly increasing in S_L and S_H, respectively. These various functions satisfy (32)-(35) and an appropriate set of boundary conditions. Now suppose that the endowment ratio $\bar{H}/\bar{L}$ increases by some proportion $\hat{\eta}$. Let us conjecture that the sets Q_{Li} and Q_{Hi} for $i = 1, 2$ and the matching function $m(q_H)$ remain unchanged. Meanwhile, let the wage schedule rise by the proportion $(1-\gamma)\hat{\eta}$ and let the salary schedule fall by the proportion $\gamma\hat{\eta}$, so that the factor-price ratio $w[m(q_H)]/r(q_H)$ increases by the proportion $\hat{\eta}$ for all $q_H \in S_H$. With these changes in factor prices, every firm increases its labor demand (per manager) by the proportion $\hat{\eta}$, irrespective of the ability of its managers. Thus, the labor-market clearing condition (35) continues to be satisfied. Clearly, the new wage and salary schedules are continuous and strictly increasing and they satisfy the first-order conditions, (33) and (34), and the zero-profit condition (32). So, the new factor prices and the original matching function and allocation sets indeed constitute an equilibrium after the increase in $\bar{H}/\bar{L}$. Output grows in both sectors by the same proportion, $\gamma\hat{\eta}$, and thus relative outputs do not change. ■

When the industries differ in their factor intensities, the above construction—with equiproportionate growth in both sectors, more workers per manager everywhere, and no change in matching—does not work. Then a change in relative factor endowments does, in general, necessitate a change in the composition of output. We focus on the case in which two countries that differ (only) in relative factor endowments both display threshold equilibria; that is, in each country an interval of the more able workers sorts to one sector while the remaining workers sort to the other, and similarly for managers. We do not require that the more able workers sort to the same sector as the more able managers, so we allow here for either an HH/LL equilibrium or an HL/LH equilibrium.

Let us begin with the latter. Suppose, for concreteness, that country A is relatively well endowed with managers compared to country B ($\bar{H}^A/\bar{L}^A > \bar{H}^B/\bar{L}^B$) and that industry 1 is relatively manager intensive compared to industry 2 ($\gamma_1 < \gamma_2$). Figure 5 depicts the qualitative features of the inverse matching functions in such circumstances. In the figure, the solid curves depict the matches that occur in country A when the more able workers with abilities $q_L \in [q_L^*, q_{L\max}]$ sort to industry 1 and match there with the less able managers with abilities $q_H \in [q_{H\min}, q_H^*]$. As previously noted, the equilibrium features PAM within each sector but not across sectors. The broken curves in the figure represent the matches that occur in country B. We show in the appendix that the threshold q_L^* always is smaller in the country that is relatively abundant in managers and the threshold q_H^* is larger in that country if and only if industry 1 is manager intensive; i.e., the country with more managers per worker employs a greater fraction of its managers and a greater fraction of its workers in the manager-intensive sector. As is apparent from the figure and Lemma 3 in the appendix (that allows at most one crossing within a sector), the inverse matching function

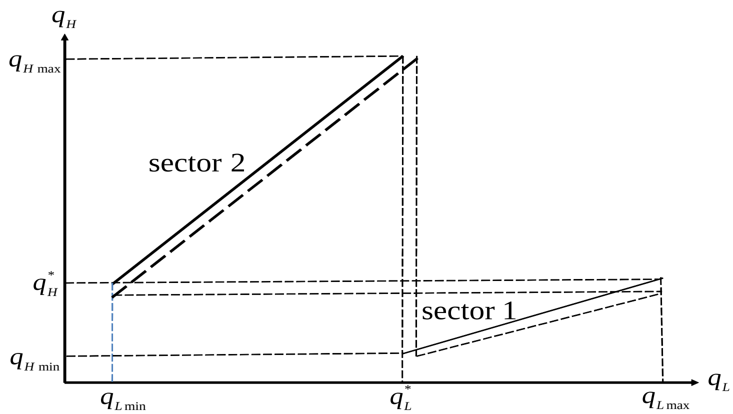

Figure 5: Sorting and matching: HL/LH equilibrium

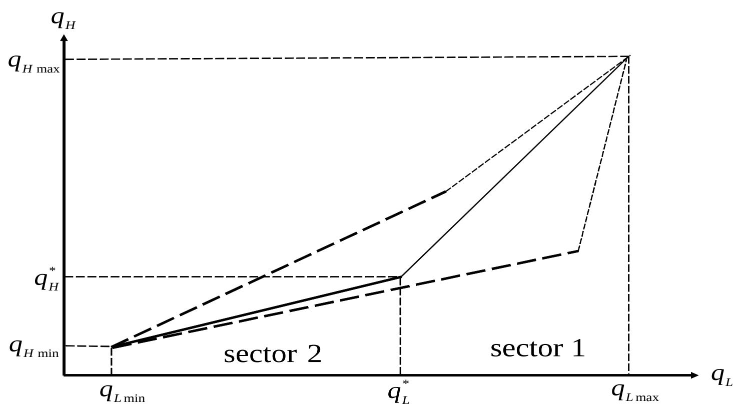

Figure 6: Sorting and matching: HH/LL equilibrium

for country B must lie below that for country A, both for the set of worker and manager types that are employed in sector 1 in both countries and for the set of worker and manager types that are employed in sector 2 in both countries. Among these types, the managers in country B achieve better matches than their counterparts of similar ability in country A, whereas the workers in country B achieve worse matches than their counterparts of similar ability in country A. Just the opposite is true about the relative positions of the matching functions and the comparisons of the matches when sector 2 is the more manager intensive. In either case, country A—with its relative abundance of managers—always exports the manager-intensive good.

Now consider an HH/LL equilibrium in which the best workers and the best managers sort to sector 1. The (inverse) matching function for such an equilibrium is continuous, monotonically increasing, and has a slope that rises at the threshold q_L^*, such as the one depicted for country A by the solid curve in Figure 6. We show in the appendix that if $\bar{H}^A/\bar{L}^A > \bar{H}^B/\bar{L}^B$ then the threshold ability levels q_L^* and q_H^* both are greater in country A than in country B if and only if industry 2

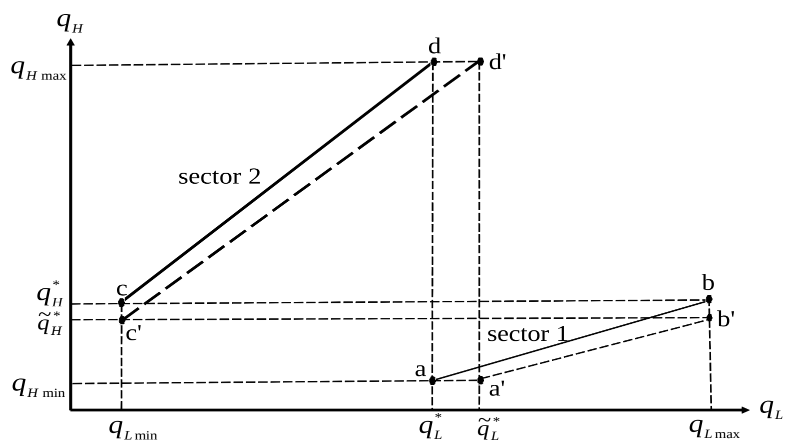

Figure 7: Effects of a rise in p_2 on matching: HL/LH equilibrium

is the labor-intensive sector. Again, the country that is relatively abundant in managers devotes greater fractions of its managers and workers to production in the manager-intensive sector. It is not clear whether managers of a given quality find better matches in country A or in country B, or whether workers do so; the figure shows with broken curves the two possible outcomes when $\bar{H}^A/\bar{L}^A > \bar{H}^B/\bar{L}^B$ and $\gamma_1 > \gamma_2$. In any case, the manager-abundant country exports the manager-intensive good.

We summarize in

Proposition 13 *Suppose that: (i) Assumption 3 holds; (ii) countries A and B are identical except for $\bar{H}^A/\bar{L}^A > \bar{H}^B/\bar{L}^B$; (iii) both countries are characterized by threshold equilibria with a single cutoff for workers q_L^* and for managers q_H^*; and (iv) $\gamma_1 \neq \gamma_2$. Then country A exports the manager-intensive good.*

5.3 Effects of Trade on Income Distribution

The strong complementarities between factors that are implied by strict log supermodularity of the productivity function induce PAM within sectors, as we have seen. The matches are fully determined in the general equilibrium, unlike what occurs for Cobb-Douglas productivity, and so changes in relative price generated by the opening of trade affect within-sector matching and the within-sector income distribution. We turn now to the question of how trade affects these outcomes.

The opening of trade elevates the relative price of a country's export good. For concreteness, consider the country that exports good 2. In Figure 7, the solid curves *cd* and *ab* depict the (inverse) matching function prior to the opening of trade for the case of an HL/LH equilibrium in which the more able workers sort to industry 1. Now let p_2 rise as a result of trade. This draws managers and workers into sector 2, so that q_H^* falls and q_L^* rises.[24] The new boundary points are represented by

[24] Before any factor reallocation, the increase in p_2 raises the value marginal product of the marginal workers and

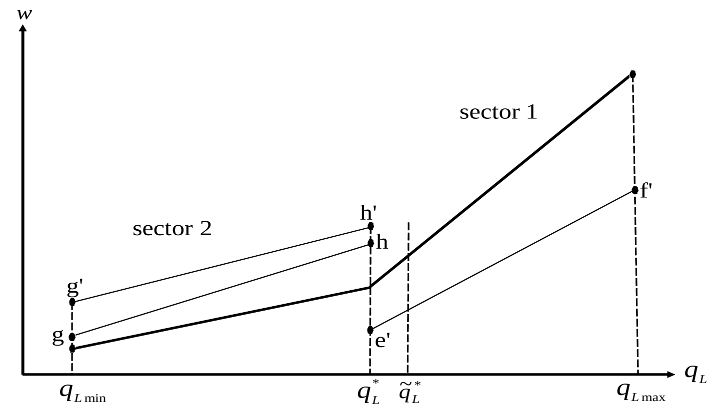

Figure 8: Effects of a rise in p_2 on wages: HL/LH equilibrium

c', d', a' and b'. As is evident from the figure, the new inverse matching function (represented by the broken curves) lies below the old for all worker and manager types that remain in their original industry of employment after the opening of trade. As a result, the opening of trade allows all managers except those that switch sectors to achieve better matches than before, while causing all workers except those that switch sectors to realize worse matches than before.

Proposition 14 summarizes these effects of trade on matching for the case of an HL/LH equilibrium and reports the implications for wage and salary inequality.

Proposition 14 *Suppose that: (i) Assumption 3 holds and (ii) the initial equilibrium is a threshold equilibrium with an HL/LH sorting pattern. Then an increase in p_2 (a) raises the labor cutoff q_L^* and reduces the manager cutoff q_H^* so that more workers and more managers are employed in sector 2; (b) worsens the matches for all workers except those that switch from sector 1 to sector 2; (c) improves the matches for all managers except those that switch from sector 1 to sector 2; (d) reduces within-industry wage inequality in both sectors and overall wage inequality in the economy; and (e) increases within-industry salary inequality in both sectors and overall salary inequality.*

In what follows, we discuss the effects of an increase in p_2 on the wage distribution; the effects on the salary distribution can be understood similarly.

Consider Figure 8, where the unlabeled thick curve represents the wage schedule in an initial equilibrium. On impact—that is, prior to any resource reallocation—wages for workers with $q_L \in [q_{L\min}, q_L^*)$ rise in proportion to the increase in p_2. These higher wages are depicted by the thin curve gh in the figure. Were matching in each sector to remain the same despite the movement of workers and managers from sector 1 to sector 2, we could trace the shadow wage schedule for sector 2 beyond gh and find the intersection with the wage schedule for sector 1 in order to identify the

managers in sector 2 relative to those in sector 1. As factors reallocate, marginal products change and rematching occurs. But we show in the appendix that these secondary effects cannot overturn the impact effects, so that q_H^* must fall and q_L^* must rises in the setting described by the figure.

new cutoff ability level. However, the matching functions in each sector do not remain the same in the wake of a price change, as we have already seen.

Let us refer back to Figure 7 and suppose, counterfactually, that as the cutoff for managers declines to its new equilibrium level at $\tilde{q}_H^*$ there is no change in the cutoff for workers. Were this to be so, the new inverse matching function would comprise a curve connecting points c' and d in sector 2, along with a curve connecting points a with b' in sector 1. Such a shift would imply a flatter relationship between wages and ability in each sector, considering the strict log supermodularity of the productivity functions. Moreover, the new inverse matching function would be flatter at point a than the old. By Lemma 6 in the appendix (as discussed in Section 5.1), the wage of a worker with ability q_L^* employed in sector 1 would fall. We indicate this drop in wage by the point e' in Figure 8 and draw the curve $e'f'$ to represent the slower rise of wages as a function of ability. Meanwhile, in sector 2, the inverse matching function is steeper at point d of Figure 7, where workers of ability q_L^* match with managers of ability $q_{H\max}$. So the wage of a worker with ability q_L^* employed in sector 2 would be at a point such as h' in Figure 8, higher than before. Since wages rise at a slower pace in this sector too, the hypothetical wage curve for sector 2 must be above gh, such as at $g'h'$ in the figure.

We see that our counterfactual assumption of no change in the cutoff for workers cannot be sustained. The gap in wages between points e' and h' induces movement of workers from sector 1 to sector 2. This generates an additional rotation of the two segments of the matching function in Figure 7 to curves between points a' and b' for sector 1 and between points c' and d' in sector 2. Compared to the matching that would occur without a change in q_L^*, there is a further worsening of matches for workers, so that wages rise even more slowly than along $e'f'$ and $g'h'$ in Figure 8. The movement of workers from sector 1 to sector 2 makes the inverse matching function steeper in sector 2 and flatter in sector 1 for the manager with ability $\tilde{q}_H^*$. The former implies a decline in the wage of the worker with ability $q_{L\min}$ to a point below g' and a flattening of the wage schedule for workers in sector 1. The latter implies a rise in the wage of the worker with ability $q_{L\max}$ and a steepening of the wage schedule relative to $e'f'$. Together, these shifts eliminate the gap in wages for the (new) marginal worker with ability $\tilde{q}_L^*$.

Evidently, wage inequality falls among workers originally in industry 2 and among those remaining in industry 1. Take for example any two workers q_L' and q_L'' such that $q_{L\min} \leq q_L' < q_L'' \leq q_L^*$. Both workers see their match deteriorate as a result of the increase in the price of good 2, but the rematching harms the worker with ability q_L'' by relatively more due to the presence of strong complementarities between factor types. The same is true for any pair of workers with abilities between $\tilde{q}_L^*$ and $q_{L\max}$. Finally, consider a pair of workers that switch sectors; i.e., those that have ability levels between q_L^* and $\tilde{q}_L^*$. The relative wage of the less able worker in this pair must rise, because the elasticity of the wage schedule in (33) is determined after the price change by the elasticity of the productivity function in sector 2, whereas before it was determined by the elasticity of the productivity function in sector 1. Since the more able workers sort to sector 1, it must be that the former elasticity is smaller than the latter. It follows that wage inequality declines also

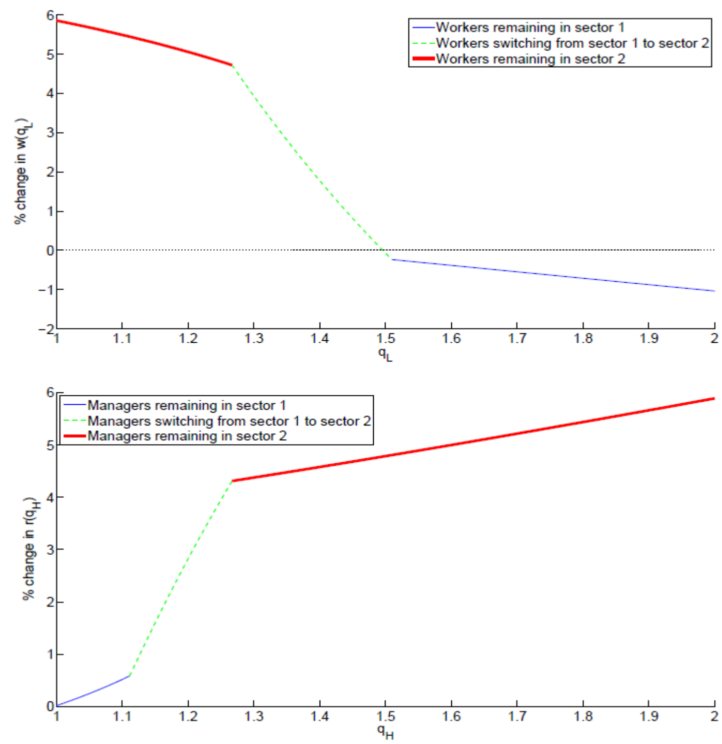

Figure 9: Effects of a 5% increase in p_2 on wages and salaries in an HL/LH equilibrium

among workers that switch sectors and therefore among all workers in the economy; see Figure 9 for an example.

What is the overall effect of the price change on the welfare of the various workers? There are several possibilities that can emerge, as can be seen in the numerical simulations presented by Lim (2013). First, if sector 1 is labor intensive and the difference in factor intensities across sectors is large relative to the specificity of the heterogeneous factors, then the Stolper-Samuelson forces dominate. In such circumstances, real wages decline for all workers while real salaries increase for all managers. Of course, if sector 2 is the labor-intensive industry, then the opposite outcomes are possible, with real gains for all workers and losses for all managers.

Figure 9 depicts the wage and salary responses for a less extreme case.[25] Here, sector 2 is labor intensive and p_2 rises by 5%. All workers initially in sector 2 see their wages rise and those at the bottom end of the ability distribution enjoy a wage hike in excess of 5%. Meanwhile, the workers who remain in sector 1 suffer a decline in wages despite the rise in the price of the labor-intensive goods. These workers suffer from their comparative disadvantage in the expanding sector. As for managers, those at the top end of the ability distribution gain the most and some see salary improvements in excess of 5%. Those at the bottom of the ability distribution enjoy welfare gains only if they devote little of their income to the export good. The figure shows the widening of salary inequality among managers.

A host of other possible configurations can emerge, but all can be understood similarly with

[25] See Lim (2013) for the parameter values and functional forms that underlie this figure.

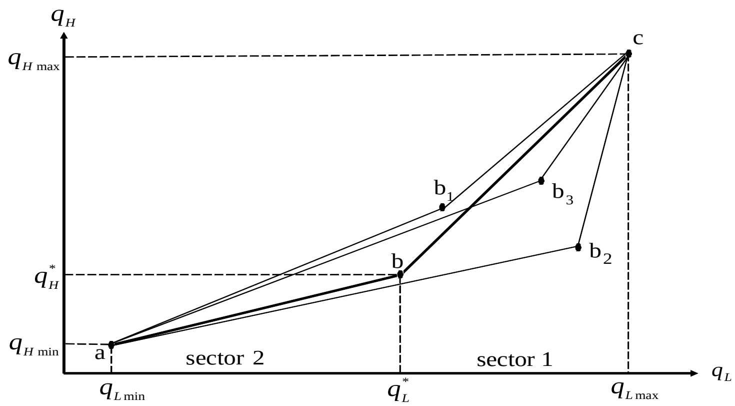

Figure 10: Impact of a rise in p_2 on matching: HH/LL equilibrium

reference to the relevant factor intensities and sector specificities; see Lim (2013) for examples. Rather than dwell on these cases, we turn now to the wage and salary effects of trade in an HH/LL equilibrium. Recall the matching and sorting patterns for such an equilibrium that were displayed in Figure 6. We show in the appendix that, when the price of good 2 rises in such a setting, sector 2 expands by attracting both additional workers and additional managers. It follows that both q_L^* and q_H^* rise. In this case, the implications for matching vary according to whether the movement of workers or the movement of managers dominates.

Figure 10 illustrates the various possibilities.[26] The thick curve abc represents the initial inverse matching function. Now suppose that q_L^* rises only modestly, while q_H^* rises more dramatically.[27] Then the new equilibrium would be represented by an inverse matching function such as ab_1c. In the event, all workers' matches improve following the price increase, whereas all managers see their matches deteriorate. Alternatively, the inflow of workers to sector 2 can be large relative to that for managers, in which case q_L^* could expand greatly compared to the expansion in q_H^*. This possibility is illustrated by the inverse matching function ab_2c in the figure, and it implies a deterioration in match quality for all workers and an improvement for all managers. Finally, the inverse matching function ab_3c depicts an intermediate case. Notice that the matches improve for all workers initially in sector 2 but deteriorate for all those remaining in sector 1.

Let us focus on the case where the outcome is an inverse matching function such as ab_1c to discuss the implied wage and salary responses. Since workers' matches improve, wages rise faster with ability than before. Since managers' matches deteriorate, the opposite is true of managerial salaries. Notice that the inverse matching function has a steeper slope at point a in the new equilibrium than before the price change. It follows from Lemma 6 that the wage of the least able workers must rise. These workers benefit directly from the increase in p_2 and indirectly from the improvement in their matches. The direct benefit alone matches the proportional increase in price,

[26] Lim (2013) provides numerical examples of each along with the underlying parameter values.

[27] This outcome plausibly arises when sector 2 is considerably more manager intensive than sector 1.

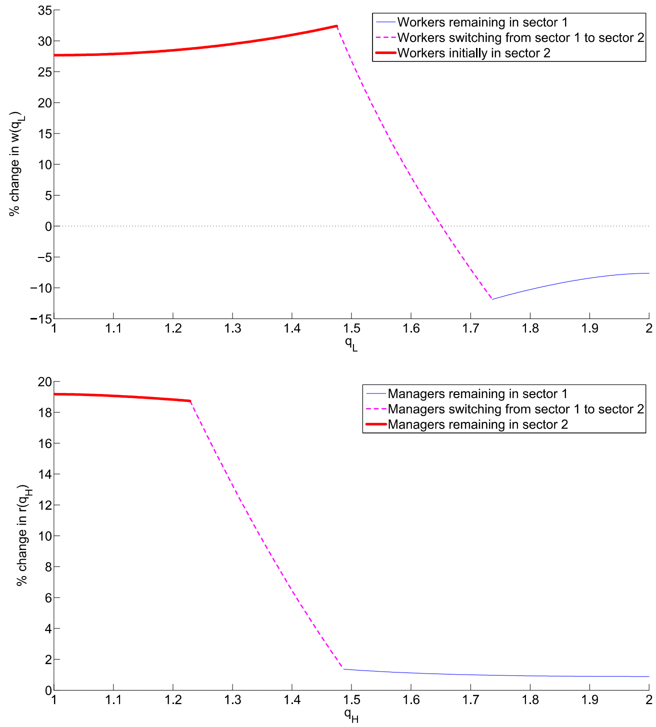

Figure 11: Effects of a 20% increase in p_2 on wages and salaries in an HH/LL equilibrium

so these workers enjoy real income gains. At the opposite end of the spectrum, the most able workers must lose. The change in p_2 has no direct effect on their value marginal product. Since the new inverse matching function is flatter at point c than the initial function, Lemma 6 implies that these workers suffer a decline in nominal wages. The gain in real income for the least able workers and the loss for the most able workers represents a narrowing of wage inequality across sectors, whereas the improved matching implies that wages are more unequal within each sector.

Figure 11 presents another example drawn from Lim (2013). Notice that the least able workers enjoy real income gains, though not as large as for those more able than themselves who initially were employed in the same sector. Meanwhile, the most able workers lose, but not as much as those less able than themselves who remain in sector 1. The figure also shows the effect on managerial salaries. In this example, all managers realize income gains in terms of good 2 but losses in terms of good 1. These gains are smaller and the losses larger as we move up the salary distribution. A decline in $r\left(q_{H\min}\right)/p_2$ is guaranteed in this case, because the direct effect for the least able managers is a salary increase proportional to the rise in p_2, but the steepening of the inverse matching function at a implies that their salaries must fall relative to the price of what they produce. The rise in $r\left(q_{H\max}\right)/p_1$ also is guaranteed, because the inverse matching function is flatter at point c than before. Finally, we know that the new salary function is flatter than the old both for managers initially in sector 2 and for those that remain in sector 1, because the deterioration in match quality hits especially hard for the more able managers in any sector.

If the inverse matching function instead is qualitatively like that depicted by ab_2c in Figure 10, then the outcomes are just the opposite. Low-ability managers gain from an increase in p_2, because their value marginal product rises in proportion to the price hike and rises further as a result of the rematching. High-ability managers lose in real terms, because $r\left(q_{H\max}\right)/p_1$ falls. All wages rise, albeit less than in proportion to the price increase. The wage hikes are proportionally greatest for those at the bottom end of the ability distribution. As a result of these factor price responses, wage inequality declines both within and between sectors, whereas salaries become more unequal within sectors, but those at the bottom who are employed in sector 2 gain relative to those at the top who are employed in sector 1.

Finally, if the inverse matching function is like that depicted by ab_3c, then the outcomes are a mix of those described above. In this case, all workers initially employed in sector 2 must benefit from the price increase, while all managers initially employed in sector 1 must lose. The low-ability managers and the high-ability workers both gain in compensation relative to the price of good 1, but lose relative to the price of good 2. Lim (2013) provides numerical examples.

Clearly, by allowing for worker and manager heterogeneity and strong complementarities between these factors, we can accommodate a rich set of possible effects of globalization on the wage and salary distributions. Some forces are familiar. For example, trade tends to benefit the factor (managers or workers) used intensively in the export industry. And trade tends to benefit those types of each factor that have a comparative advantage in the export sector. But other forces are new. Trade can improve the matches for some factor in one sector or in both. If it does so, the productivity of the factor will rise beyond what is predicted by the usual forces, and especially so for the more able types. Predictions about which types will gain or lose—and about whether the income distribution will widen or narrow in response to an opening of trade—may require detailed information about technologies, factor intensities, and distributions of talent and know-how.

5.4 Effects of Trade on Measured Productivity

We conclude this section with a brief discussion of the effects of trade on measured productivity. We shall see that the subtle implications of rematching introduce ambiguities here, just as they do for the links between trade and factor prices.

As before, we measure productivity using factor quantities and the Cobb-Douglas nature of the production technology. In particular, we write

$$X_i = A_i L_i^{\gamma_i} H_i^{1-\gamma_i},$$

where L_i and H_i are aggregate employment of workers and managers, respectively, in sector i, and A_i captures TFP. The firms in sector i devote a fraction γ_i of their revenues to wages and the remaining fraction to salaries. It follows that $\gamma_i p_i X_i = \int_{q\in Q_{Li}} w\left(q\right)\phi_L\left(q\right)dq$ and $\left(1-\gamma_i\right)p_i X_i = \int_{q\in Q_{Hi}} r\left(q\right)\phi_H\left(q\right)dq$. Using these expressions together with the expressions for aggregate employment of workers and managers in each sector, we can write

$$A_i = \gamma_i^{-\gamma_i} (1-\gamma_i)^{-(1-\gamma_i)} \left(\mathbb{E}\left[\frac{w(q_L)}{p_i} \middle| q_L \in Q_{Li} \right] \right)^{\gamma_i} \left(\mathbb{E}\left[\frac{r(q_H)}{p_i} \middle| q_H \in Q_{Hi} \right] \right)^{1-\gamma_i}.$$

Evidently, measured productivity in a sector varies with the average own-product real wage paid to the workers employed there and the average own-product real salary paid to managers. These averages reflect, of course, the average marginal products of the various factor types.

Consider, for example, a country that has an HH/LL sorting pattern, such as that depicted in Figure 6. We know that an increase in p_2 draws more of both factors into sector 2. The workers and managers that change sector raise the average marginal products everywhere, because these marginal workers and managers have higher abilities than those initially employed in sector 2 and lower abilities than those that remain employed in sector 1. If matching were to remain as before, then measured TFP would rise in both sectors. This is much the same as for the case of Cobb-Douglas productivity, which we considered previously in Section 4.[28]

But, as we know, the matches do not remain the same, and the rematching impacts the marginal productivity of every manager and worker. Consider further the example of factor-price responses that is depicted in Figure 11. In sector 2, own-product real wages rise and own-product real salaries decline. This reflects an improvement in match quality for the workers initially employed in sector 2 and a deterioration in match quality for the managers there. The former raises measured TFP and the latter lowers it, with the net effect depending in a complex way on factor intensities and the densities of the factor distributions. Meanwhile, the own-product real wages fall in sector 1 and the own-product real salaries rise there, as a result of the rematching that occurs. Again, the net effects are ambiguous. More generally, the rematching of factors in a sector raises the productivity of one factor while reducing it for the other. The effects of trade on measured TFP are thus bound to be an empirical matter.

6 Labor Market Frictions

Until now, we have assumed that labor markets flawlessly and costlessly allocate the various types of labor to their most efficient uses. Of course, the smooth functioning of labor markets is notoriously suspect and worker heterogeneity would only seem to exacerbate the potential difficulties. In this section, we show how a simple form of search frictions can be incorporated into the analysis. The extension allows us to discuss the distribution of unemployment rates across the ability spectrum alongside the distribution of wages.

To keep matters simple, we continue to assume a frictionless market for managers. In other words, firms can hire managers of whatever ability and in whatever numbers they wish by offering

[28] Note, however, that if a country has an HL/LH sorting pattern, the expansion of sector 2 leads to an improvement in average worker ability but a decline in average manager ability in both sectors. Then, even without considering the effects of rematching, the implications for measured TFP are ambiguous.

a competitive salary.[29] But firms must search for their workers and workers for jobs. We follow Peters (1991, 2000), Acemoglu and Shimer (1999), Burdett et al. (2001), Eeckhout and Kircher (2010a), and others in modeling labor-market frictions with "directed search," whereby firms post costly "vacancies" and workers and firms meet randomly. We extend this approach to allow for worker heterogeneity and multiple hires per firm.

Suppose, as before, that the output in industry i of a production unit comprising a manager of ability q_H and ℓ workers of ability q_L is given by (1). A firm (or entrepreneurial manager) hires workers by posting vacancies. Each posting costs c_i units of the the firm's final output. A posting lists the ability level q_L that the firm targets and the wage ω that it will pay to any employee of this type. We assume that the firm can commit to these job attributes, in the sense that it will not hire workers with ability different from the posted level nor attempt to renegotiate its wage offer after it meets with a job applicant.[30] The firm chooses v, the number of its vacancies, to maximize profits.

Workers are risk neutral. Each worker applies for a single job of his choosing.[31] Workers consider only the jobs for which they are qualified, because firms will not hire types different from those targeted in their announcements. Among relevant jobs, each worker applies for the position that offers the greatest expected income. In equilibrium, workers must be indifferent among the range of openings posted for their type.

Let s be the number of workers seeking jobs at a firm that has posted v vacancies. We assume that the search process results in the consummation of $M(s, v)$ jobs, where

$$M(s,v) = Bs^{\tau}v^{1-\tau}, \tag{38}$$

$B > 0$ and $0 < \tau < 1$.[32] For a firm, the probability of filling any given vacancy is $\delta_v(s/v) = B(s/v)^{\tau}$, whereas for a worker the probability of a successful application is $\delta_s(s/v) = B(s/v)^{-(1-\tau)}$. The former is increasing in s/v, while the latter is decreasing in s/v; i.e., a firm's chances of filling a vacancy improve and a worker's chances of landing a job decline with the number of applicants per posting.

[29] Perhaps the best way to justify this assumption is to imagine the manager as an entrepreneur, as in Lucas (1978). Then it is the manager that searches for employees and her salary amounts to the residual profits after wages and hiring costs are paid. Alternatively, one might think of the second factor as being *capital*, instead of managers, in which case an assumption that firms can readily find machines of the quality they desire is not so hard to swallow.

[30] Alternatively, we could allow a firm to post a wage schedule and to hire any worker it happens to meet at the wage specified by the schedule. If each vacancy generates at most one meeting with a job applicant, then it is never optimal for the firm to induce applications from more than one type of worker; see Eeckhout and Kircher (2010a, 2010b) for proof of this assertion in related environments. In such circumstances, there is no loss of generality in assuming that the firm targets only one type of worker. Shimer (2005) studies a setting in which one vacancy can result in multiple meetings with potential employees. Then, in the general, it is optimal for any firm to induce applications from several different types. We do not explore this possibility here.

[31] This assumption is common in the literature on direct search. Galenianos and Kircher (2009) describe settings in which the restriction to one application per worker does not change the qualitative predictions of the model.

[32] The job-search literature refers to $M(s, v)$ as a "matching function" but we eschew that terminology so as to avoid confusion with the function that "matches" workers and managers, $q_L = m(q_H)$. The Cobb-Douglas form for $M(\cdot)$ is common in the literature, and is implicitly coupled with the usual restriction that B is sufficiently small to imply meeting probabilities below unity for both vacancies and workers.

Now let $w(q_L)$ be the *expected wage* that workers of type q_L obtain in equilibrium, which each firm takes as given. A firm must offer at least this expected wage or it will find itself without applicants; and it has no reason to offer more. In equilibrium, a firm with v vacancies that offers a wage ω targeted to workers with ability q_L attracts s applicants, where s is such as to make the applicants indifferent between the firm's openings and their other opportunities; i.e., s solves $\delta_s(s/v)\omega = w(q_L)$. Using (38), this can be rewritten as

$$\frac{s}{v} = \left[\frac{B\omega}{w(q_L)}\right]^{\frac{1}{1-\tau}}. \tag{39}$$

Equation (39) is the main building block in a model with directed search; it ties the wage announcement ω to the endogenous number of applications per vacancy s/v, which in turn determines the firm's fill rate, $\delta_v(s/v)$.[33] Given the expected wage $w(q_L)$, the firm can use (39) to compute the number of workers that will seek its employment and thus the number of workers $\ell = M(s, v)$ that it will succeed in hiring. Again using (38), together with (39), we see that a firm that posts v vacancies targeted at workers with ability q_L and that offers a wage of ω manages to hire ℓ workers, where

$$\ell = B^{\frac{1}{1-\tau}} \left[\frac{\omega}{w(q_L)}\right]^{\frac{\tau}{1-\tau}} v. \tag{40}$$

Evidently, hires are proportional to the number of vacancies and rise with the ratio of the firm's wage offer to the workers' outside option.

Now consider the profit-maximization problem facing a firm with a manager of ability q_H that chooses to operate in industry i. The firm pays $p_i c_i v$ to post v vacancies and pays ω to each of the ℓ workers that it eventually hires. Its profits are given by

$$\pi_i = p_i \psi_i(q_H, q_L)\ell^{\gamma_i} - \omega\ell - p_i c_i v - r(q_H),$$

where $r(q_H)$ as before represents the manager's salary. Then, using (39), (23), and the first-order condition for the firm's optimal choice of wage offer, we can re-express its profits as

$$\pi_i = p_i \varphi_i(q_H, q_L) s^{\zeta_i} - w(q_L)s - r(q_H),$$

where

$$\varphi_i(q_H, q_L) \equiv [1-(1-\tau)\gamma_i]\left[\frac{(1-\tau)\gamma_i}{c_i}\right]^{\frac{(1-\tau)\gamma_i}{1-(1-\tau)\gamma_i}} B^{\frac{\gamma_i}{1-(1-\tau)\gamma_i}} \psi_i(q_H, q_L)^{\frac{1}{1-(1-\tau)\gamma_i}}$$

[33] Peters (1991, 2000) and Burdett et al. (2001) provide microfoundations for a relationship similar to (39). They begin by assuming a finite number of jobs and vacancies and then allow the economy to grow large without bound. This generates a balls–and-urns type function for applicants per vacancy, rather than the Cobb-Douglas form that is more commonly assumed. Galenianos and Kircher (2012) extends their setup to generate CES and Cobb-Douglas matching functions. With but a few exceptions, the literature on directed search specifies the matching function individually for each vacancy, and we follow in this tradition.

and

$$0 < \zeta_i \equiv \frac{\tau \gamma_i}{1 - (1 - \tau)\,\gamma_i} < 1.$$

Notice that this expression for profits has the same mathematical properties as the profit function $\pi_i = p_i \psi_i(q_H, q_L)\ell^{\gamma_i} - w(q_L)\ell - r\,(q_H)$ that we encountered in Section 5, because if $\psi_i(q_H, q_L)$ satisfies part (ii) of Assumption 3 (i.e., it is strictly increasing, continuously differentiable, and strictly log supermodular) so too does $\varphi_i(q_H, q_L)$, and ζ_i like γ_i is between zero and one.[34] In other words, the firm's choice about the number of job applications to invite in a setting with search frictions is much like its choice about the number of workers to hire in a setting without them. The first-order condition for s implies

$$s = \left[\frac{\delta_i p_i \varphi_i\,(q_H, q_L)}{w(q_L)}\right]^{\frac{1}{1-\delta_i}}, \tag{41}$$

which generates the profit function

$$\pi_i\,(q_H, q_L) = \bar{\zeta}_i p_i^{\frac{1}{1-\delta_i}} \varphi_i\,(q_H, q_L)^{\frac{1}{1-\zeta_i}}\, w\,(q_L)^{-\frac{\zeta_i}{1-\zeta_i}} - r\,(q_H)\,,$$

where $\bar{\zeta}_i \equiv \zeta_i^{\frac{\zeta_i}{1-\zeta_i}}\,(1 - \zeta_i)$. This expression has much the same form as (30), which applies in the absence of search frictions. Finally, the analog to the labor-market clearing condition from before is the requirement that the aggregate number of applications induced by firms operating in industry i and targeting workers of ability q_L must equal the number of workers with that ability level that sort to the sector in search of a job. With these observations, we conclude that the equilibrium expected wage function $w\,(q_L)$, salary function $r\,(q_H)$ and matching function $q_L = m\,(q_H)$ can be characterized as the solution to three differential equations analogous to (33)-(35), a zero profit condition analogous to (32), and a set of boundary conditions. Evidently, comparative advantage again derives from a country's relative factor endowments and its distributions of worker and manager ability. Moreover, since $\zeta_1 > \zeta_2$ if and only if $\gamma_1 > \gamma_2$, the cross-sectoral differences in factor intensities interact with differences in factor endowments to determine the pattern of trade in much the same way as before. The search frictions themselves are not an independent source of comparative advantage so long as these frictions are similar in the two sectors.[35]

The model with search frictions features different employment rates across the range of ability levels. In order to discuss the impact of trade on employment, we combine the optimal choice of wage offer with a firm's desired number of applications per manager to derive

$$\omega\,(q_L) = B^{-\gamma_i} \left(\frac{1-\tau}{\tau p_i c_i}\right)^{-(1-\tau)\gamma_i} w(q_L)^{1-(1-\tau)\gamma_i}.$$

[34] Note too that if $\psi_i(q_H, q_L)$ is a product of power functions, as in Section 4, so too is $\varphi_i(q_H, q_L)$. And if $\psi_i(q_H, q_L)$ has a constant elasticity of substitution between q_H and q_L, so too does $\varphi_i(q_H, q_L)$.

[35] If the number of meetings in (38) varies by sector, then it is immediate from the definition $\zeta_i \equiv \tau_i\gamma_i/(1 - (1 - \tau_i)\,\gamma_i)$ that the search process constitutes an additional source of comparative advantage.

The expected wage $w(q_L)$ must be an increasing function of ability. It follows that, among workers that seek employment in a given industry i, those with greater ability see higher posted wages for the jobs they pursue. Next, we substitute this expression for $\omega(q_L)$ into (40) to derive an expression for the employment rate for workers of ability q_L, namely

$$\frac{\ell}{s} = B^{\gamma_i} \left(\frac{1-\tau}{\tau c_i} \right)^{(1-\tau)\gamma_i} \left[\frac{w(q_L)}{p_i} \right]^{(1-\tau)\gamma_i} . \tag{42}$$

Since the expected wage on the right-hand side is an increasing function of ability, we conclude that so too is the employment rate among workers seeking jobs in a given industry. We record our findings in

Proposition 15 *Suppose that Assumption 3 holds. Let $q_L', q_L'' \in Q_i$, with $q_L' > q_L''$. Then the job listings targeted to workers with ability q_L' offer a higher expected wage and a greater probability of employment than those targeted to q_L''. The opening of trade causes within-sector wage inequality and employment inequality to move in the same direction.*

In a setting with search frictions, the opening of trade affects differently the employment rates at different ability levels. Let us consider just one example to illustrate how the analysis can be performed. Suppose a country has an HL/LH sorting pattern such as that depicted in Figure 5 and that the country exports good 2. The opening of trade generates an increase in p_2. Figure 7 shows the effects of such a price change on the matching of worker and manager types in each sector. As we have seen, the workers who do not switch sectors find themselves teamed with a less able manager than before. Now, Figure 9 can be interpreted as illustrating the predicted impact on *expected* wages. The figure shows an increase in $w(q_L)/p_2$ for some of the least able workers, who sort to sector 2, a decline in $w(q_L)/p_2$ for some moderately able workers that sort to sector 2, and a decline in $w(q_L)/p_1$ for the most able workers, who sort to sector 1.

We refer now to equation (42), which applies in the presence of search frictions. The equation implies that the employment rate rises for the aforementioned group of least able workers while it falls for those with moderate and high ability. Overall, the distribution of employment rates becomes more equal across the worker population. Of course, the effects of trade on the distribution of employment would be just the opposite if the country instead imported good 2. Evidently, trade can widen or narrow the inequality in employment rates across the ability distribution according to the sorting pattern that is realized and the comparative advantage of the country. The determinants of these outcomes in an economy with directed search are similar to the determinants of wage inequality in an economy that has frictionless labor markets.

7 Concluding Remarks

In this paper, we have extended the familiar two-sector, two-factor model of international trade to include heterogeneous factors of production. In a model with factor heterogeneity, we can examine the determinants of factor sorting to industries and the determinants of factor matching within industries. When the productivity of a production unit depends on both the manager's and workers' abilities—and particularly when there are strong complementarities between the two—the forces that guide sorting and matching become inextricably linked. The economy-wide pattern of factor assignments can be subtle and complex even in the presence of strong complementarities that dictate positive assortative matching within every sector.

A model with heterogeneous factors allows a more complete analysis of the distributional effects of trade than is possible in one with homogeneous factors. In particular, we can ask how the opening of trade or trade liberalization affects the wage and salary distributions over the entire range of compensation levels. In general, there are three considerations that determine the effects of trade on the income of a particular individual. First, as in the standard Heckscher-Ohlin world with homogeneous factors, there is the question of whether the export sector is intensive in the use of workers or managers. Second, as in the standard Ricardo-Viner world with factor specificity, there is the question of whether an individual's type generates a personal comparative advantage in the export sector or the import-competing sector. Finally, and most novel, there is the question of how trade affects the individual's match with other factors of production. If a change in trade conditions causes a worker to rematch with a better manager than before, then his productivity will improve and his wage will receive an upward boost. If instead a worker's match deteriorates, then his wage may suffer. Interestingly, the effects of trade on wage or salary inequality across sectors may run counter to the effects on inequality within a sector.

We have shown that the Heckscher-Ohlin theorem extends to a setting with heterogeneous factors provided that the countries share similar distributions of worker and managerial talent. But we have also noted how differences in the distributions of talent can be an independent source of comparative advantage. A country that has more able workers than another—in the sense of a rightward shift in the talent distribution—will produce relatively more of the good for which productivity responds more elastically to ability. We have also seen how trade affects measured TFP in settings where individuals' talents are not fully observable to the analyst. The equilibrium sorting pattern dictates whether the marginal factors that enter or exit an industry as the result of trade are more or less productive than the average.

Finally, we have incorporated search frictions. In a simple setting with directed search, firms create vacancies and make wage offers to workers of a targeted type. In such a setting, trade affects not only the distribution of wages but also the distribution of employment rates across the different types of workers. We provide an example in which the main insights from the earlier analysis carry over without modification to an environment with unemployment. But much work remains to the elucidate the connection between trade and the efficiency of matching and to understand how globalization affects equilibrium unemployment rates for different types of workers.

References

[1] Acemoglu, Daron and Shimer, Robert, 1999. "Holdups and Efficiency With Search Frictions," *International Economic Review 40*(4), 827–849. "

[2] Antràs, Pol, Garicano, Luis and Rossi-Hansberg, Esteban, 2006. "Offshoring in a Knowledge Economy," *Quarterly Journal of Economics 121(1)*, 31-77.

[3] Bombardini, Matilde, Gallipoli, Giovanni and Pupato, Germán, 2012. "Skill Dispersion and Trade Flows," *American Economic Review 102(5)*, 2327-48.

[4] Burdett, Kenneth, Shi, Shouyong. and Wright, Randall, 2001."Pricing and Matching With Frictions," *Journal of Political Economy 109*(5), 1060–1085.

[5] Costinot, Arnaud, 2009. "An Elementary Theory of Comparative Advantage," *Econometrica 77(4)*, 1165-92.

[6] Costinot, Arnaud and Vogel, Jonathan, 2010. "Matching and Inequality in the World Economy," *Journal of Political Economy 118(4)*, 747-86.

[7] De Loecker, Jan, 2011. "Product Differentiation, Multiproduct Firms and Estimating the Impact of Trade Liberalization on Productivity," *Econometrica 79*(5), 1407-51.

[8] Eeckhout, Jan and Kircher, Philipp, 2010a. "Sorting and Decentralized Price Competition," *Econometrica 78*(2), 439-74.

[9] Eeckhout, Jan and Kircher, Philipp, 2010b. "Sorting vs Screening - Search Frictions and Competing Mechanisms," *Journal of Economic Theory 145*(4), 1354-85.

[10] Eeckhout, Jan and Kircher, Philipp, 2012. "Assortative Matching with Large Firms: Span of Control over More versus Better Workers." Downloadable at http://personal.lse.ac.uk/kircher/Papers/sorting-and-factor-intensity.pdf

[11] Galenianos, Manolis and Kircher, Philipp, 2009."Directed Search with Multiple Job Applications," *Journal of Economic Theory 114*(2), 445-71.

[12] Galenianos, Manolis and Kircher, Philipp, 2012. "On the Game-theoretic Foundations of Competitive Search Equilibrium," *International Economic Review 53*(1), 1-21.

[13] Garicano, Luis, 2000. "Hierarchies and the Organization of Knowledge in Production," *Journal of Political Economy 108(5)*, 874-904.

[14] Grossman, Gene M., 1983. "Partially-Mobile Capital: A General Approach to Two-Sector Trade Theory," *Journal of International Economics 15(1-2)*, 1-17.

[15] Grossman, Gene M. and Maggi, Giovanni, 2000. "Diversity and Trade," *American Economic Review 90(5)*, 1255-75.

[16] Helpman, Elhanan, Itskhoki, Oleg, Muendler, Marc-Andreas and Redding, Stephen, 2012. "Trade and Inequality: From Theory to Estimation," NBER Working Paper No. 17991.

[17] Jones, Ronald, 1971. "A Three-Factor Model in Theory, Trade and History," J. Bhagwati, R. Jones, R. Mundell and J. Vanek, eds., *Trade, Balance of Payments and Growth*, Amsterdam: North-Holland.

[18] Lim, Kevin, 2013. "Numerical Simulations for 'Matching and Sorting in the Global Economy'." Downloadable at http://www.princeton.edu/~grossman/grossman_working_papers.htm.

[19] Lucas, Robert E., Jr., 1978. "On the Size Distribution of Business Firms," *The Bell Journal of Economics 9*(2), 508-23.

[20] Mussa, Michael, 1982. "Imperfect Factor Mobility and the Distribution of Income," *Journal of International Economics 12(1-2)*, 125-141.

[21] Ohnsorge, Franziska and Trefler, Daniel, 2007. "Sorting it Out: International Trade with Heterogeneous Workers," *Journal of Political Economy 115(5)*, 868-92.

[22] Pavcnik, Nina, 2002. "Trade Liberalization, Exit, and Productivity Improvement: Evidence from Chilean Plants," *Review of Economic Studies 69*(1), 245-76.

[23] Peters, Michael, 1991. "Ex ante Price Offers in Matching Games: Non Steady States," *Econometrica 59*(5), 1425–54.

[24] Peters, Michael, 2000. "Limits of Exact Equilibria for Capacity Constrained Sellers With Costly Search," *Journal of Economic Theory 95*(2), 139–68.

[25] Roy, Andrew D., 1951. "Some Thoughts on the Distribution of Earnings," *Oxford Economic Papers 3(2)*, 135-46.

[26] Ruffin, Roy, 1988. "The Missing Link: The Ricardian Approach to the Factor Endowment Theory of International Trade," *American Economic Review 78(4)*, 759-72.

[27] Sampson, Thomas, 2012. "Selection into Trade and Wage Inequality," CEP Discussion Papers No. 1152, Centre for Economic Policy Performance.

[28] Sattinger, Michael, 1975. "Comparative Advantage and the Distributions of Earnings and Abilities," *Econometrica 43*(3), 455-68.

[29] Shimer, Robert, 2005. "The Assignment of Workers to jobs in an Economy with Coordination Frictions," *Journal of Political Economy 113(5)*, 996-1025.

[30] Trefler, Daniel, 2004. "The Long and Short of the Canada-U. S. Free Trade Agreement," *American Economic Review 94*(4), 870-95.

[31] Yeaple, Stephen R, 2005. "A Simple Model of Firm Heterogeneity, International Trade, and Wages," *Journal of International Economics 65(1),* 1-20.

Appendix

This appendix provides proofs of results stated in the main text.

Proofs for Section 3

First, note that in the system comprising (6)-(9), a proportional increase in the number of managers and workers, $\bar{H}$ and $\bar{L}$, raises H_1 by the same factor of proportionality and leaves q_L^* and the wage anchors w_1 and w_2 unchanged. Therefore, in this case, the output of each good rises proportionately, so that the ratio X_1/X_2 does not change. It follows that in order to find the impact of $\bar{H}/\bar{L}$ on X_1/X_2, it suffices to find the impact of $\bar{L}$ on the relative supply.

Differentiating the equilibrium system (6)-(9), we obtain

$$\begin{pmatrix} 1 & -1 & s_L(q_L^*) & 0 \\ -\frac{\gamma_1}{1-\gamma_1} & \frac{\gamma_2}{1-\gamma_2} & 0 & 0 \\ 0 & \frac{E_2}{1-\gamma_2} & \bar{L}\tilde{\psi}_2(q_L^*)^{1/\gamma_2}\phi_L(q_L^*)q_L^* & E_2H_1/H_2 \\ \frac{E_1}{1-\gamma_1} & 0 & -\bar{L}\tilde{\psi}_1(q_L^*)^{1/\gamma_1}\phi_L(q_L^*)q_L^* & -E_1 \end{pmatrix} \begin{pmatrix} \hat{w}_1 \\ \hat{w}_2 \\ \hat{q}_L^* \\ \hat{H}_1 \end{pmatrix} \tag{43}$$

$$= \begin{pmatrix} 0 \\ 0 \\ -E_2 \\ -E_1 \end{pmatrix} \hat{\bar{L}} + \begin{pmatrix} 0 \\ -\frac{1}{1-\gamma_1} \\ 0 \\ \frac{E_1}{1-\gamma_1} \end{pmatrix} \hat{p}_1,$$

where

$$E_i = H_i \left(\frac{\gamma_i p_i}{w_i}\right)^{\frac{1}{1-\gamma_i}}, \quad i = 1, 2; \quad H_2 = \bar{H} - H_1.$$

Here E_i represents the aggregate measure of effective labor units (defined as above) allocated to sector i.

Let D_{ho} be the determinant of the matrix on the left-hand side of (43). Then

$$D_{ho} = \bar{L}\phi_L(q_L^*)q_L^*\frac{w(q_L^*)}{w_1w_2}H_1\frac{(\gamma_2-\gamma_1)^2}{(1-\gamma_1)^2(1-\gamma_2)^2} + \frac{s_L(q_L^*)}{(1-\gamma_1)(1-\gamma_2)}\left(\gamma_1+\gamma_2\frac{H_1}{H_2}\right)E_1E_2,$$

because (8), (9) and the definition of E_i imply that

$$\tilde{\psi}_1(q_L^*)^{1/\gamma_1}E_2\frac{H_1}{H_2} - \tilde{\psi}_2(q_L^*)^{1/\gamma_2}E_1 = H_1\left[\tilde{\psi}_1(q_L^*)^{1/\gamma_1}\left(\frac{\gamma_2p_2}{w_2}\right)^{\frac{1}{1-\gamma_2}} - \tilde{\psi}_2(q_L^*)^{1/\gamma_2}\left(\frac{\gamma_1p_1}{w_1}\right)^{\frac{1}{1-\gamma_1}}\right] \tag{44}$$

$$= \frac{w(q_L^*)}{w_1w_2}H_1\frac{\gamma_2-\gamma_1}{(1-\gamma_1)(1-\gamma_2)}. \tag{45}$$

It follows that $s_L(q_L^*) > 0 \Rightarrow D_{ho} > 0$.

We now use (43) to calculate the response of H_1 to an increase in labor supply $\bar{L}$, which yields

$$\hat{H}_1D_{ho} = \frac{\gamma_1-\gamma_2}{(1-\gamma_1)(1-\gamma_2)}\left[E_1\bar{L}\tilde{\psi}_2(q_L^*)^{1/\gamma_2}\phi_L(q_L^*)q_L^* + E_2\bar{L}\tilde{\psi}_1(q_L^*)^{1/\gamma_1}\phi_L(q_L^*)q_L^* + E_1E_2s_L(q_L^*)\right]\hat{\bar{L}}.$$

Therefore, given $s_L(q_L^*) > 0$, an increase in labor supply raises the number of managers in sector 1 if and only if sector 1 is labor intensive, i.e., $\gamma_1 > \gamma_2$; otherwise it raises the number of managers in sector 2. When H_1 increases, the relative supply X_1/X_2 increases as well. It follows that the country with relatively more workers produces relatively more of the labor-intensive good and exports that good. This proves Proposition 2.

Next we calculate the response of the two wage anchors to changes in the price of good 1. From (43) and (44), we obtain

$$\hat{w}_1 D_{ho} = \bar{L}\phi_L\left(q_L^*\right) q_L^* \frac{1}{1-\gamma_1}\frac{w\left(q_L^*\right)}{w_1 w_2} H_1 \frac{\gamma_1-\gamma_2}{\left(1-\gamma_1\right)\left(1-\gamma_2\right)}\hat{p}_1 + s_L\left(q_L^*\right)\frac{E_1 E_2}{\left(1-\gamma_1\right)\left(1-\gamma_2\right)}\left(1+\gamma_2\frac{H_1}{H_2}\right)\hat{p}_1,$$

$$\hat{w}_2 D_{ho} = \bar{L}\phi_L\left(q_L^*\right) q_L^* \frac{1}{1-\gamma_1}\frac{w\left(q_L^*\right)}{w_1 w_2} H_1 \frac{\gamma_1-\gamma_2}{\left(1-\gamma_1\right)\left(1-\gamma_2\right)}\hat{p}_1 - s_L\left(q_L^*\right)\frac{E_1 E_2}{1-\gamma_1}\frac{H_1}{H_2}\hat{p}_1.$$

It follows that $\left(\hat{w}_1-\hat{w}_2\right)/\hat{p}_1 > 0$ when $s_L\left(q_L^*\right) > 0$, which implies that $\hat{w}_1 > \hat{w}_2$ when $\hat{p}_1 > 0$, as argued in part (i) of Proposition 4. We also calculate the response of the salary of managers, given in (10), to a price hike in sector 1. Using $i = 2$, a manager's salary can be expressed as

$$r = \bar{\gamma}_2 p_2^{\frac{1}{1-\gamma_2}} w_2^{-\frac{\gamma_2}{1-\gamma_2}}.$$

Therefore, when the price of good 1 rises, the salary response is

$$\hat{r} = -\frac{\gamma_2}{1-\gamma_2}\hat{w}_2.$$

Evidently, the managers' salary moves in the opposite direction to the wage anchor in sector 2.

Now consider the cases discussed in Proposition 4. In case (ii) we have $\gamma_1 \approx \gamma_2$ and therefore

$$\hat{w}_1 \approx \frac{\gamma_1^{-1}+\frac{H_1}{H_2}}{1+\frac{H_1}{H_2}}\hat{p}_1,$$

$$\hat{w}_2 \approx -\frac{\left(1-\gamma_1\right)\frac{H_1}{H_2}}{\gamma_1\left(1+\frac{H_1}{H_2}\right)}\hat{p}_1,$$

$$\hat{r} \approx \frac{\frac{H_1}{H_2}}{1+\frac{H_1}{H_2}}\hat{p}_1.$$

It follows that, in this case, $\hat{w}_1 > \hat{p}_1 > \hat{r} > 0 > \hat{w}_2$, which proves part (ii) of the proposition. In cases (iii) and (iv), we have $s_L\left(q_L^*\right) \approx 0$, which implies

$$\hat{w}_1 \approx \hat{w}_2 \approx \frac{1-\gamma_2}{\gamma_1-\gamma_2}\hat{p}_1,$$

$$\hat{r} = -\frac{\gamma_2}{\gamma_1-\gamma_2}\hat{p}_1.$$

Therefore, if $\gamma_1 > \gamma_2$ then $\hat{w}_1 \approx \hat{w}_2 > \hat{p}_1 > 0 > \hat{r}$ and if $\gamma_1 < \gamma_2$ then $\hat{r} > \hat{p}_1 > 0 > \hat{w}_1 \approx \hat{w}_2$, which proves parts (iii) and (iv).

In order to evaluate the impact of trade on sectoral productivity levels, we need to compute the impact of p_1 on the cutoff q_L^*. From (8) we obtain $\hat{q}_L^* = \left(\hat{w}_2-\hat{w}_1\right)/s_L\left(q_L^*\right)$. Since we have shown above that an increase in the price of good 1 raises w_1 proportionately more than it does w_2, it follows that an increase in the price of good 1 reduces the cutoff q_L^*, which implies that more workers are attracted to sector 1.

Now consider measured productivity A_i, $i = 1, 2$. By definition,

$$A_i = \frac{X_i}{L_i^{\gamma_i} H_i^{1-\gamma_i}}, \quad i = 1, 2,$$

where L_i is the measure of workers employed in sector i and H_i is the measure of managers employed in the sector. The measure of workers employed in each one of the sectors are given by

$$\begin{aligned} L_1 &= \bar{L} \int_{q_L^*}^{q_{L\max}} \phi_L(q_L)\, dq_L, \\ L_2 &= \bar{L} \int_{q_{L\min}}^{q_L^*} \phi_L(q_L)\, dq_L, \end{aligned}$$

while (6) and (7) imply that the measures of managers employed in each one of the sectors are:

$$\begin{aligned} H_1 &= \left(\frac{\gamma_1 p_1}{w_1}\right)^{-\frac{1}{1-\gamma_1}} \bar{L} \int_{q_L^*}^{q_{L\max}} \psi_1(q_L)^{1/\gamma_1} \phi_L(q_L)\, dq_L, \\ H_2 &= \left(\frac{\gamma_2 p_2}{w_2}\right)^{-\frac{1}{1-\gamma_2}} \bar{L} \int_{q_{L\min}}^{q_L^*} \psi_2(q_L)^{1/\gamma_2} \phi_L(q_L)\, dq_L. \end{aligned}$$

Substituting these measures of inputs into the expressions for output (11), we obtain

$$A_1^{1/\gamma_1} = \frac{\int_{q_L^*}^{q_{L\max}} \psi_1(q_L)^{1/\gamma_1} \phi_L(q_L)\, dq_L}{\int_{q_L^*}^{q_{L\max}} \phi_L(q_L)\, dq_L} = E[\psi_1(q_L)^{1/\gamma_1} \mid q_L \geq q_L^*],$$

$$A_2^{1/\gamma_2} = \frac{\int_{q_{L\min}}^{q_L^*} \psi_2(q_L)^{1/\gamma_2} \phi_L(q_L)\, dq_L}{\int_{q_{L\min}}^{q_L^*} \phi_L(q_L)\, dq_L} = E[\psi_2(q_L)^{1/\gamma_2} \mid q_L \leq q_L^*].$$

Evidently, A_1 and A_2 are both increasing in q_L^*.

We now consider the impact of a rightward shift of the density function $\phi(q_L)$, as defined in (12). To perform these comparative statics, can equivalently hold the distribution of types constant but endow a worker of type q_L with λq_L units of ability. In each sector, the demand for efficiency units of labor must equal the supply. A worker in sector i of type q_L provides $\tilde{\psi}_i(\lambda q_L)^{1/\gamma_i}$ units of efficiency labor. The labor-market clearing conditions should now be written as

$$H_1 \left(\frac{\gamma_1 p_1}{w_1}\right)^{\frac{1}{1-\gamma_1}} = \bar{L} \int_{q_L^*}^{q_{L\max}} \tilde{\psi}_1(\lambda q)^{1/\gamma_1} \phi_L(q)\, dq \tag{46}$$

and

$$\left(\bar{H} - H_1\right) \left(\frac{\gamma_2 p_2}{w_2}\right)^{\frac{1}{1-\gamma_2}} = \bar{L} \int_{q_{L\min}}^{q_L^*} \tilde{\psi}_2(\lambda q)^{1/\gamma_2} \phi_L(q)\, dq. \tag{47}$$

A worker of type q_L working in sector i earns the salary

$$w(q_L) = w_i \tilde{\psi}_i(\lambda q_L)^{1/\gamma_i} \quad ,$$

so wage continuity at the marginal worker q_L^* requires

$$w_1\tilde{\psi}_1\left(\lambda q_L^*\right)^{1/\gamma_1} = w_2\tilde{\psi}_2\left(\lambda q_L^*\right)^{1/\gamma_2}. \tag{48}$$

Finally, profits for a firm in industry i that hires workers with index q_L are

$$\tilde{\pi}_i\left(q_L\right) = \bar{\gamma}_i p_i^{\frac{1}{1-\gamma_i}}\left[\tilde{\psi}_i\left(\lambda q_L\right)\right]^{\frac{1}{1-\gamma_i}} w\left(q_L\right)^{-\frac{\gamma_i}{1-\gamma_i}} - r$$

and zero profits in both sectors implies

$$\bar{\gamma}_1 p_1^{\frac{1}{1-\gamma_1}} w_1^{-\frac{\gamma_1}{1-\gamma_1}} = \bar{\gamma}_2 p_2^{\frac{1}{1-\gamma_2}} w_2^{-\frac{\gamma_2}{1-\gamma_2}}. \tag{49}$$

Equations (46) - (49) determine q_L^*, H_1, w_1 and w_2.

Now we define $Q_L^* = \lambda q_L^*$ and totally differentiate the equilibrium system (evaluated at $\lambda = 1$), which yields

$$\begin{pmatrix} 1 & -1 & s_L\left(q_L^*\right) & 0 \\ -\frac{\gamma_1}{1-\gamma_1} & \frac{\gamma_2}{1-\gamma_2} & 0 & 0 \\ 0 & \frac{E_2}{1-\gamma_2} & \bar{L}\tilde{\psi}_2\left(q_L^*\right)^{1/\gamma_2}\phi_L\left(q_L^*\right)q_L^* & E_2H_1/H_2 \\ \frac{E_1}{1-\gamma_1} & 0 & -\bar{L}\tilde{\psi}_1\left(\lambda q_L^*\right)^{1/\gamma_1}\phi_L\left(q_L^*\right)q_L^* & -E_1 \end{pmatrix}\begin{pmatrix} \hat{w}_1 \\ \hat{w}_2 \\ \hat{Q}_L^* \\ \hat{H}_1 \end{pmatrix}$$
$$= \begin{pmatrix} 0 \\ 0 \\ -\frac{E_2}{\gamma_2}\bar{\varepsilon}_{\tilde{\psi}_2}\left(q_L^*\right) + \bar{L}\tilde{\psi}_2\left(q_L^*\right)^{1/\gamma_2}\phi_L\left(q_L^*\right)q_L^* \\ -\frac{E_1}{\gamma_1}\bar{\varepsilon}_{\tilde{\psi}_1}\left(q_L^*\right) - \bar{L}\tilde{\psi}_1\left(q_L^*\right)^{1/\gamma_1}\phi_L\left(q_L^*\right)q_L^* \end{pmatrix}\hat{\lambda},$$

where $\bar{\varepsilon}_{\tilde{\psi}_i}\left(q_L^*\right)$ is a weighted average of the elasticities $\varepsilon_{\tilde{\psi}_i}\left(q_L\right)$ in sector i, with weights

$$v_i\left(q_L\right) = \frac{\tilde{\psi}_i\left(q_L\right)^{1/\gamma_i}\phi_L\left(q_L\right)}{\int_{q_L\in Q_{Li}}\tilde{\psi}_i\left(q_L\right)^{1/\gamma_i}\phi_L\left(q_L\right)dq_L}, \quad i = 1, 2.$$

Therefore,

$$\begin{aligned}\hat{H}_1 D_{ho}\left(1-\gamma_1\right)\left(1-\gamma_2\right) &= E_1E_2 s_L\left(q_L^*\right)\left[\bar{\varepsilon}_{\tilde{\psi}_1}\left(q_L^*\right) - \bar{\varepsilon}_{\tilde{\psi}_2}\left(q_L^*\right)\right]\hat{\lambda} \\ &+ \bar{L}\phi_L\left(q_L^*\right)q_L^*\left[\tilde{\psi}_1\left(q_L^*\right)^{1/\gamma_1}E_2 + \tilde{\psi}_2\left(q_L^*\right)^{1/\gamma_2}E_1\right]\left[\bar{\varepsilon}_{\tilde{\psi}_1}\left(q_L^*\right) - \bar{\varepsilon}_{\tilde{\psi}_2}\left(q_L^*\right)\right]\hat{\lambda}.\end{aligned}$$

It follows that, given $s_L\left(q_L^*\right) > 0$, an increase in λ raises H_1 if and only if $\bar{\varepsilon}_{\tilde{\psi}_1}\left(q_L^*\right) > \bar{\varepsilon}_{\tilde{\psi}_2}\left(q_L^*\right)$. Moreover, $\bar{\varepsilon}_{\tilde{\psi}_1}\left(q_L^*\right) > \bar{\varepsilon}_{\tilde{\psi}_2}\left(q_L^*\right)$ if $\varepsilon_{\tilde{\psi}_1}\left(q_L'\right) > \varepsilon_{\tilde{\psi}_2}\left(q_L''\right)$ for all $q_L', q_L'' \in S_L^A \cup S_L^B$ and $\bar{\varepsilon}_{\tilde{\psi}_1}\left(q_L^*\right) < \bar{\varepsilon}_{\tilde{\psi}_2}\left(q_L^*\right)$ if $\varepsilon_{\tilde{\psi}_1}\left(q_L'\right) < \varepsilon_{\tilde{\psi}_2}\left(q_L''\right)$ for all $q_L', q_L'' \in S_L^A \cup S_L^B$. This proves Proposition 3.

Proofs for Section 4

First note that, in the system comprising (24)-(27), a proportional increase in the number of managers and workers has no effect on the wage anchors w_1 and w_2 or on the ability cutoffs q_L^* and q_H^*. Therefore, it does not change the output ratio X_1/X_2 (see (29)). It follows that if countries A and B differ only in size, with $\bar{H}$ and $\bar{L}$ being proportionately larger in one of the countries, they will have the same relative demand for the two goods and the same relative supply and they will not trade with one another. Accordingly, we can find the impact of $\bar{H}/\bar{L}$ on the pattern of trade by analyzing the impact of $\bar{L}$ on q_H^*, which will tell us how the relative supply X_1/X_2 is affected.

Differentiating the equilibrium system (24)-(27), we obtain

$$\begin{pmatrix} 1 & -1 & s_L & 0 \\ -\frac{\gamma_1}{1-\gamma_1} & \frac{\gamma_2}{1-\gamma_2} & 0 & s_H \\ 0 & \frac{E_2}{1-\gamma_2} & \Lambda_2 & -\Theta_2 \\ \frac{E_1}{1-\gamma_1} & 0 & -\Lambda_1 & \Theta_1 \end{pmatrix} \begin{pmatrix} \hat{w}_1 \\ \hat{w}_2 \\ \hat{q}_L^* \\ \hat{q}_H^* \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ -E_2 \\ -E_1 \end{pmatrix} \widehat{\bar{L}} + \begin{pmatrix} 0 \\ -\frac{1}{1-\gamma_1} \\ 0 \\ \frac{E_1}{1-\gamma_1} \end{pmatrix} \hat{p}_1,$$

where E_i is effective labor in sector i, defined as

$$E_1 = \bar{H} \left(\frac{\gamma_1 p_1}{w_1} \right)^{\frac{1}{1-\gamma_1}} \int_{q_H^*}^{q_{H\max}} q_H^{\frac{\beta_1}{1-\gamma_1}} \phi_H(q_H)\, dq_H,$$

$$E_2 = \bar{H} \left(\frac{\gamma_2 p_2}{w_2} \right)^{\frac{1}{1-\gamma_2}} \int_{q_{H\min}}^{q_H^*} q_H^{\frac{\beta_2}{1-\gamma_2}} \phi_H(q_H)\, dq_H,$$

and

$$\Lambda_1 = \bar{L} (q_L^*)^{\frac{\alpha_1}{\gamma_1}+1} \phi_L(q_L^*),$$

$$\Lambda_2 = \bar{L} (q_L^*)^{\frac{\alpha_2}{\gamma_2}+1} \phi_L(q_L^*),$$

$$\Theta_1 = \bar{H} \left(\frac{\gamma_1 p_1}{w_1} \right)^{\frac{1}{1-\gamma_1}} (q_H^*)^{\frac{\beta_1}{1-\gamma_1}+1} \phi_H(q_H^*),$$

$$\Theta_2 = \bar{H} \left(\frac{\gamma_2 p_2}{w_2} \right)^{\frac{1}{1-\gamma_2}} (q_H^*)^{\frac{\beta_2}{1-\gamma_2}+1} \phi_H(q_H^*).$$

The determinant of the matrix on the left-hand side of this system, D_{CD}, satisfies

$$\begin{aligned} (1-\gamma_2)(1-\gamma_1)(-D_{CD}) &= (\Theta_1\Lambda_2 - \Theta_2\Lambda_1)(\gamma_1-\gamma_2) + s_H \left[\Lambda_1 E_2 (1-\gamma_1) + \Lambda_2 E_1 (1-\gamma_2)\right] \\ &\quad + s_L (\Theta_1 \gamma_1 E_2 + \Theta_2 \gamma_2 E_1) + E_1 E_2 s_H s_L. \end{aligned}$$

Using the equilibrium conditions (26) and (27), we find that

$$(\Theta_1\Lambda_2 - \Theta_2\Lambda_1)(\gamma_1-\gamma_2) = \Theta_2\Lambda_1 \frac{(\gamma_1-\gamma_2)^2}{\gamma_2(1-\gamma_1)} > 0.$$

Therefore $D_{CD} < 0$. We also compute

$$\hat{q}_H^* D_{CD} = (\Lambda_1 E_2 + \Lambda_2 E_1 + E_1 E_2 s_L) \frac{\gamma_1 - \gamma_2}{(1-\gamma_1)(1-\gamma_2)} \widehat{\bar{L}}.$$

Since $D_{CD} < 0$, an increase in $\bar{L}$ reduces q_H^* if and only if $\gamma_1 > \gamma_2$. So, the output of good 1 rises relative to that of good 2 if and only if sector 2 is more labor intensive than sector 1. This proves Proposition 7.

Next, we calculate the impact of p_1 on the wage anchors:

$$\begin{aligned}\hat{w}_1(1-\gamma_2)(1-\gamma_1)(-D_{CD}) &= (\Theta_1\Lambda_2-\Theta_2\Lambda_1+\Lambda_2E_1s_H)(1-\gamma_2)\hat{p}_1\\ &\quad+[(\Theta_1E_2+\Theta_2\gamma_2E_1)s_L+E_1E_2s_Hs_L]\hat{p}_1,\\ \hat{w}_2(1-\gamma_1)(-D_{CD}) &= (\Theta_1\Lambda_2-\Theta_2\Lambda_1+\Lambda_2E_1s_H-\Theta_2E_1s_L)\hat{p}_1.\end{aligned}$$

Therefore,

$$(\hat{w}_1-\hat{w}_2)(1-\gamma_1)(-D_{CD})=[(\Theta_1E_2+\Theta_2\gamma_2E_1)s_L+E_1E_2s_Hs_L+\Theta_2E_1s_L(1-\gamma_2)]\hat{p}_1.$$

Since $D_{CD}<0$, it follows that an increase in the price of good 1 results in $\hat{w}_1>\hat{w}_2$, which proves part (i) of Proposition 9.

Next consider the case in which $s_H\approx 0$ and $\gamma_1\approx\gamma_2$. In this case,

$$(1-\gamma_2)(1-\gamma_1)(-D_{CD})\approx s_L(\Theta_1\gamma_1E_2+\Theta_2\gamma_2E_1).$$

Then

$$\hat{w}_1\approx\hat{w}_2\approx\frac{\Theta_1E_2+\Theta_2\gamma_2E_1}{\Theta_1\gamma_1E_2+\Theta_2\gamma_2E_1}\hat{p}_1,$$

because $\gamma_1\approx\gamma_2$ implies $\Theta_1\Lambda_2-\Theta_2\Lambda_1\approx 0$. Evidently, in this case, $\hat{w}_1>\hat{p}_1>0>\hat{w}_2$. To complete the proof of part (ii) of Proposition 9, we need to calculate the response of the anchors r_1 and r_2 for the managers' salaries. When p_1 rises, (28) yields $\hat{r}_1=(1-\gamma_1)^{-1}\hat{p}_1-\gamma_1(1-\gamma_1)^{-1}\hat{w}_1$ and $\hat{r}_2=-\gamma_2(1-\gamma_2)^{-1}\hat{w}_2$. In case (ii) of Proposition 9, with $s_H\approx 0$ and $\gamma_1\approx\gamma_2$, these imply

$$\hat{r}_1\approx\hat{r}_2\approx\frac{\Theta_2\gamma_2E_1}{\Theta_1\gamma_1E_2+\Theta_2\gamma_2E_1}\hat{p}_1.$$

It follows that $\hat{p}_1>\hat{r}_1\approx\hat{r}_2>0$. So, part (ii) of the proposition is proved.

We turn now to parts (iii) and (iv) of Proposition 9. The antecedents $s_H\approx 0$ and $s_L\approx 0$ imply

$$\begin{aligned}(1-\gamma_2)(1-\gamma_1)(-D_{CD}) &\approx (\Theta_1\Lambda_2-\Theta_2\Lambda_1)(\gamma_1-\gamma_2),\\ \hat{w}_1(1-\gamma_1)(-D_{CD}) &\approx (\Theta_1\Lambda_2-\Theta_2\Lambda_1)\hat{p}_1,\\ \hat{w}_2(1-\gamma_1)(-D_{CD}) &\approx (\Theta_1\Lambda_2-\Theta_2\Lambda_1)\hat{p}_1.\end{aligned}$$

It follows that

$$\hat{w}_1\approx\hat{w}_2\approx\frac{1-\gamma_2}{\gamma_1-\gamma_2}\hat{p}_1,$$

which implies that $\hat{w}_1\approx\hat{w}_2>\hat{p}_1>0$ for $\gamma_1>\gamma_2$ and $\hat{w}_1\approx\hat{w}_2<0<\hat{p}_1$ for $\gamma_1<\gamma_2$. Moreover, since $\hat{r}_1=(1-\gamma_1)^{-1}\hat{p}_1-\gamma_1(1-\gamma_1)^{-1}\hat{w}_1$ and $\hat{r}_2=-\gamma_2(1-\gamma_2)^{-1}\hat{w}_2$, we have

$$\hat{r}_1\approx\hat{r}_2\approx-\frac{\gamma_2}{\gamma_1-\gamma_2}\hat{p}_1.$$

Evidently, in this case, $\hat{r}_1\approx\hat{r}_2<0<\hat{p}_1$ when $\gamma_1>\gamma_2$ and $\hat{r}_1\approx\hat{r}_2>\hat{p}_1>0$ when $\gamma_1<\gamma_2$. This completes the proof of Proposition 9.

We next consider the impact of a rightward shift of the density function $\phi(q_L)$, as defined in (12).

To perform these comparative statics, we follow the procedure from the previous section; that is, we hold the distribution of types constant but endow a worker of type q_L with λq_L units of ability, and we define $Q_L^* = \lambda q_L^*$. Differentiating the equilibrium system (24)-(27), we now obtain

$$\begin{pmatrix} 1 & -1 & s_L & 0 \\ -\frac{\gamma_1}{1-\gamma_1} & \frac{\gamma_2}{1-\gamma_2} & 0 & s_H \\ 0 & \frac{E_2}{1-\gamma_2} & \Lambda_2 & -\Theta_2 \\ \frac{E_1}{1-\gamma_1} & 0 & -\Lambda_1 & \Theta_1 \end{pmatrix} \begin{pmatrix} \hat{w}_1 \\ \hat{w}_2 \\ \hat{Q}_L^* \\ \hat{q}_H^* \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ -\frac{\alpha_2}{\gamma_2} E_2 + \Lambda_2 \\ -\frac{\alpha_1}{\gamma_1} E_1 - \Lambda_1 \end{pmatrix} \widehat{\lambda}.$$

Using (29), it follows that for $s_H > 0$ an increase in λ raises the relative output of good 1 if it reduces q_H^*. However, from the above system of equations we obtain:

$$\hat{q}_H^* (1-\gamma_2)(1-\gamma_1)(-D_{CD}) = -(\Lambda_1 E_2 + \Lambda_2 E_1 + s_L E_1 E_2)(\alpha_1 - \alpha_2)\hat{\lambda}.$$

Therefore a rightward shift of the density function $\phi(q_L)$ raises the relative output of good 1 if and only if $\alpha_1 > \alpha_2$. This proves the first part of Proposition 8. The second part is proved in similar fashion.

Proofs for Section 5

Denote by $m_i(q_H)$ the solution set to problem (31). Because: (i) S_L and S_H are compact; (ii) $m_i(q_H)$ is upper hemicontinuous (because $\tilde{\pi}_i(q_L, q_H)$ is a continuous function); and (iii) $m_i(q_H)$ is closed-valued; the graph

$$G_i = [\{q_H, q_L\} \mid q_L \in m_i(q_H) \text{ for all } q_H \in S_H]$$

is closed. The matching correspondence satisfies

$$m(q_H) = \begin{cases} m_1(q_H) & \text{for } q_H \in Q_{H1} \\ m_2(q_H) & \text{for } q_H \in Q_{H2} \end{cases}$$

and the equilibrium allocation graph in sector i is

$$M_i = [\{q_H, q_L\} \mid q_L \in m_i(q_H) \text{ for all } q_H \in Q_{Hi}] \subseteq G_i.$$

Since $Q_{Hi} \subseteq S_H$, the graph M_i is also closed.

Now consider a connected subset $M_i^n \subseteq M_i$:

$$M_i^n = [\{q_H, q_L\} \mid q_L \in m_i(q_H) \text{ for all } q_H \in [q_{H1}, q_{H2}] \subseteq Q_{Hi}].$$

Since M_i is a closed graph, such a subset exists and there exists an interval $[q_{L1}, q_{L2}]$, $q_{L2} > q_{L1}$, satisfying: (a) $m_i(q_H) \in [q_{L1}, q_{L2}]$ for all $q_H \in [q_{H1}, q_{H2}]$; and (b) for every point $q_L \in [q_{L1}, q_{L2}]$ there exists a managerial ability level $q_H \in [q_{H1}, q_{H2}]$ satisfying $q_L \in m_i(q_H)$. This means that, in M_i^n, workers of ability $[q_{L1}, q_{L2}]$ are matched with managers of ability $[q_{H1}, q_{H2}]$ and all workers and managers have matches. Then, as Eeckhout and Kircher (2012) have shown, strict log supermodularity of $\psi_i(\cdot)$ ensures strict positive assortative matching (PAM) between such groups in sector i. It follows that $m_i(q_H)$ is a continuous and strictly increasing function in the interior of $[q_{H1}, q_{H2}]$. M_i consists of a union of connected sets, $M_i = \cup_{n \in N_i} M_i^n$, such that $m_i(q_H)$ is continuous and strictly increasing in each one of them and $m_i(q_H)$ jumps upwards between them.

We next prove differentiability of $w(\cdot)$ in $M_i^{n,int}$.[36] Let $m^{-1}(\cdot)$ be the inverse of the sectoral matching function in $M_i^{n,int}$. Since $m(\cdot)$ is continuous and strictly increasing in $M_i^{n,int}$, this inverse exists. Now consider an interval $[q_L', q_L' + dq_L) \in M_i^{n,int}$. The zero-profit condition (32) implies

$$w(q_L') = \bar{\gamma}_i^{\frac{1-\gamma_i}{\gamma_i}} p_i^{\frac{1}{\gamma_i}} \psi_i \left[m^{-1}(q_L'), q_L'\right]^{\frac{1}{\gamma_i}} r\left[m^{-1}(q_L')\right]^{-\frac{1-\gamma_i}{\gamma_i}}$$

and profit maximization implies

$$w(q_L' + dq_L) \geq \bar{\gamma}_i^{\frac{1-\gamma_i}{\gamma_i}} p_i^{\frac{1}{\gamma_i}} \psi_i \left[m^{-1}(q_L'), q_L' + dq_L\right]^{\frac{1}{\gamma_i}} r\left[m^{-1}(q_L')\right]^{-\frac{1-\gamma_i}{\gamma_i}}.$$

Together, these expressions imply

$$w(q_L' + dq_L) \geq w(q_L') \left\{ \frac{\psi_i \left[m^{-1}(q_L'), q_L' + dq_L\right]}{\psi_i \left[m^{-1}(q_L'), q_L'\right]} \right\}^{\frac{1}{\gamma_i}}. \tag{50}$$

Similarly, (32) implies $w(q_L' + dq_L) = \bar{\gamma}_i^{\frac{1-\gamma_i}{\gamma_i}} p_i^{\frac{1}{\gamma_i}} \psi_i \left[m^{-1}(q_L' + dq_L), q_L' + dq_L\right]^{\frac{1}{\gamma_i}} r\left[m^{-1}(q_L' + dq_L)\right]^{-\frac{1-\gamma_i}{\gamma_i}}$ and profit maximization implies

$$w(q_L') \geq \bar{\gamma}_i^{\frac{1-\gamma_i}{\gamma_i}} p_i^{\frac{1}{\gamma_i}} \psi_i \left[m^{-1}(q_L' + dq_L), q_L'\right]^{\frac{1}{\gamma_i}} r\left[m^{-1}(q_L' + dq_L)\right]^{-\frac{1-\gamma_i}{\gamma_i}}.$$

Together, these expressions imply

$$w(q_L') \geq w(q_L' + dq_L) \left\{ \frac{\psi_i \left[m^{-1}(q_L' + dq_L), q_L'\right]}{\psi_i \left[m^{-1}(q_L' + dq_L), q_L' + dq_L\right]} \right\}^{\frac{1}{\gamma_i}}. \tag{51}$$

Inequalities (50) and (51) jointly imply

$$\begin{aligned}
&\frac{w(q_L')}{\psi_i \left[m^{-1}(q_L'), q_L'\right]^{\frac{1}{\gamma_i}}} \frac{\psi_i \left[m^{-1}(q_L'), q_L' + dq_L\right]^{\frac{1}{\gamma_i}} - \psi_i \left[m^{-1}(q_L'), q_L'\right]^{\frac{1}{\gamma_i}}}{dq_L} \\
&\leq \frac{w(q_L' + dq_L) - w(q_L')}{dq_L} \\
&\leq \frac{w(q_L')}{\psi_i \left[m^{-1}(q_L' + dq_L), q_L'\right]^{\frac{1}{\gamma_i}}} \frac{\psi_i \left[m^{-1}(q_L' + dq_L), q_L' + dq_L\right]^{\frac{1}{\gamma_i}} - \psi_i \left[m^{-1}(q_L' + dq_L), q_L'\right]^{\frac{1}{\gamma_i}}}{dq_L}.
\end{aligned}$$

Since, by Assumption 3, the productivity function is continuous, strictly increasing, and differentiable, and since the inverse of the sectoral matching function is continuous and strictly increasing in this range, taking the limit as $dq_L \to 0$ implies that the derivative of $w(\cdot)$ at q_L' exists and

$$\frac{dw(q_L')}{dq_L} = \frac{w(q_L')}{\psi_i \left[m^{-1}(q_L'), q_L'\right]^{\frac{1}{\gamma_i}}} \frac{\partial \psi_i \left[m^{-1}(q_L'), q_L'\right]^{\frac{1}{\gamma_i}}}{\partial q_L}.$$

Similar arguments can be used to show that the salary function is differentiable.

[36] This proof is similar to the proof of differentiability of the wage function in Sampson (2012).

We now prove Proposition 10. We prove part (i) by contradiction. To this end, suppose that the inequality condition holds, but the equilibrium is such that there are managers employed in sector j who are of higher ability than some managers employed in sector i. In such circumstances, there exists an ability level $\tilde{q}_H$ at one of the boundaries between Q_{Hi} and Q_{Hj} such that managers with ability $\tilde{q}_H - \varepsilon_i \in Q_{Hi}^{int}$ are employed in sector i and managers with ability $\tilde{q}_H + \varepsilon_j \in Q_{Hj}^{int}$ are employed in sector j for $\varepsilon_i > 0$ and $\varepsilon_j > 0$ small enough. Moreover, the equilibrium conditions (32)-(34) are satisfied, the matching function $m(q_H)$ is continuous at Q_{Hi}^{int} and Q_{Hj}^{int} close to $\tilde{q}_H$ (but can be discontinuous at the boundary point between these sets), the wage function $w(q_L)$ is continuous and increasing in S_L and differentiable in Q_{Li}^{int} and Q_{Lj}^{int}, and the salary function $r(q_H)$ is continuous and increasing in S_H and differentiable in Q_{Hi}^{int} and Q_{Hj}^{int}.

Now recall the continuous profit function $\Pi_i(q_H)$ defined in (31). In equilibrium, $\Pi_i(q_H) = 0$ for all $q_H \in Q_{Hi}$, but the maximal profits $\Pi_i(q_H)$ may differ from zero for $q_H \notin Q_{Hi}$. Therefore $\Pi_i(q_H) = 0$ for all $q_H \in (\tilde{q}_H - \varepsilon_i, \tilde{q}_H)$ and, by continuity, $\lim_{q_H \nearrow \tilde{q}_H} \Pi_i(q_H) = 0$.

Next consider the profits that accrue to an entrepreneur that hires a manager with ability $\tilde{q}_H + \varepsilon$ in order to produce good i, where $\varepsilon < \varepsilon_j$. Choosing workers so as to maximize profits, this entrepreneur earns $\Pi_i(\tilde{q}_H + \varepsilon) \geq \pi_i\left[\tilde{q}_H + \varepsilon, m\left(\tilde{q}_H^-\right)\right]$, where $m\left(\tilde{q}_H^-\right) = \lim_{\varepsilon \searrow 0} m(\tilde{q}_H - \varepsilon)$ and $\lim_{\varepsilon \searrow 0} \Pi_i(\tilde{q}_H + \varepsilon) = \lim_{\varepsilon \searrow 0} \pi_i\left[\tilde{q}_H + \varepsilon, m\left(\tilde{q}_H^-\right)\right] = 0$. The first-order approximation to $\pi_i\left[\tilde{q}_H + \varepsilon, m\left(\tilde{q}_H^-\right)\right]$ is therefore

$$\pi_i\left[\tilde{q}_H + \varepsilon, m\left(\tilde{q}_H^-\right)\right] \approx \varepsilon \pi_{iH}\left[\tilde{q}_H + \varepsilon, m\left(\tilde{q}_H^-\right)\right],$$

where $\pi_{iH}(\cdot)$ is the partial derivative of $\pi_i(\cdot)$ with respect to q_H. This derivative exists because the salary function is differentiable in Q_{Hj}^{int}, and

$$\begin{aligned}
&\pi_{iH}\left[\tilde{q}_H + \varepsilon, m\left(\tilde{q}_H^-\right)\right] \\
&= \bar{\gamma}_i p_i^{\frac{1}{1-\gamma_i}} \psi_i\left[\tilde{q}_H + \varepsilon, m\left(\tilde{q}_H^-\right)\right]^{\frac{1}{1-\gamma_i}} w\left[m\left(\tilde{q}_H^-\right)\right]^{-\frac{\gamma_i}{1-\gamma_i}} \frac{\psi_{iH}\left[\tilde{q}_H + \varepsilon, m\left(\tilde{q}_H^-\right)\right]}{(1-\gamma_i)\psi_i\left[\tilde{q}_H + \varepsilon, m\left(\tilde{q}_H^-\right)\right]} - r'(\tilde{q}_H + \varepsilon) \\
&= \left\{\frac{\psi_i\left[\tilde{q}_H + \varepsilon, m\left(\tilde{q}_H^-\right)\right]}{\psi_i\left[\tilde{q}_H, m\left(\tilde{q}_H^-\right)\right]}\right\}^{\frac{1}{1-\gamma_i}} r\left(\tilde{q}_H^-\right) \frac{\psi_{iH}\left[\tilde{q}_H + \varepsilon, m\left(\tilde{q}_H^-\right)\right]}{(1-\gamma_i)\psi_i\left[\tilde{q}_H + \varepsilon, m\left(\tilde{q}_H^-\right)\right]} - r'(\tilde{q}_H + \varepsilon),
\end{aligned}$$

where the last equality uses the free-entry condition (32), which applies to sector 1 at points in Q_{Hi}^{int} in the conjectured equilibrium, and $r\left(\tilde{q}_H^-\right) = r(\tilde{q}_H)$ due to the continuity of the salary function. Since $\tilde{q}_H + \varepsilon \in Q_{Hj}^{int}$, condition (34) implies

$$\lim_{\varepsilon \searrow 0} \pi_{iH}\left[\tilde{q}_H + \varepsilon, m_i\left(\tilde{q}_H^-\right)\right] = r(\tilde{q}_H)\left\{\frac{\psi_{iH}\left[q_H, m_i\left(\tilde{q}_H^-\right)\right]}{(1-\gamma_i)\psi_i\left[q_H, m_i\left(\tilde{q}_H^-\right)\right]} - \frac{\psi_{jH}\left[q_H, m\left(\tilde{q}_H^+\right)\right]}{(1-\gamma_j)\psi_j\left[q_H, m\left(\tilde{q}_H^+\right)\right]}\right\},$$

where $m\left(\tilde{q}_H^+\right) = \lim_{\varepsilon \searrow 0} m(\tilde{q}_H + \varepsilon)$. It now follows from part (i) in Proposition 10 that the right-hand side of this equation is strictly positive irrespective of the values of $m_i\left(\tilde{q}_H^-\right)$ and $m\left(\tilde{q}_H^+\right)$, and therefore that $\pi_{iH}\left[\tilde{q}_H + \varepsilon, m_i\left(\tilde{q}_H^-\right)\right] > 0$ for ε small enough, which contradicts the zero profit condition. This therefore contradicts the supposition that in equilibrium there are managers employed in sector j who are more able than some managers employed in sector i. Consequently, every manager in sector i has greater ability than any manager employed in sector j. This completes the proof.

Next we prove Proposition 11. Suppose that the inequality conditions in Proposition 11 hold but the

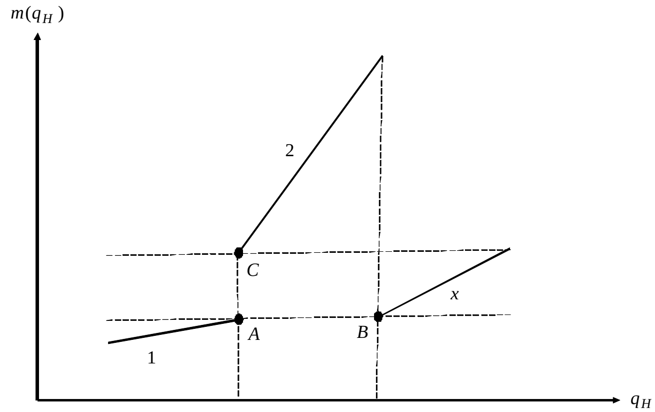

Figure 12: Matching function with discontinuity

equilibrium is such that there exist managers in sector 2 who are more able than some managers in sector 1. In such circumstances, there exists an ability $\tilde{q}_H$ at one of the boundary points between Q_{H1} and Q_{H2} such that managers of ability $\tilde{q}_H - \varepsilon_1$ are employed in sector 1 and managers of ability $\tilde{q}_H + \varepsilon_2$ are employed in sector 2 for $\varepsilon_1 > 0$ and $\varepsilon_2 > 0$ small enough. Let $m(\tilde{q}_H^-) = \lim_{q_H \nearrow \tilde{q}_H} m(q_H)$ and $m(\tilde{q}_H^+) = \lim_{q_H \searrow \tilde{q}_H} m(q_H)$ Then

$$\lim_{\varepsilon \to 0} \pi_{iH} \left[\tilde{q}_H + \varepsilon, m(q_H^-)\right] = r\left(\tilde{q}_H\right) \left[\frac{\psi_{1H}\left[\tilde{q}_H, m(\tilde{q}_H^-)\right]}{(1-\gamma_1)\,\psi_1\left[\tilde{q}_H, m(\tilde{q}_H^-)\right]} - \frac{\psi_{2H}\left[\tilde{q}_H, m\left(\tilde{q}_H^+\right)\right]}{(1-\gamma_2)\,\psi_2\left[\tilde{q}_H, m\left(\tilde{q}_H^+\right)\right]} \right], \qquad (52)$$

which we derive in the same way as in the proof of Proposition 10. Under the supposition that the managers to the left of $\tilde{q}_H$ sort into sector 1 and those to the right of $\tilde{q}_H$ sort into sector 2 the partial derivative in (52) cannot be positive and therefore

$$\frac{\psi_{1H}\left[\tilde{q}_H, m(\tilde{q}_H^-)\right]}{(1-\gamma_1)\,\psi_1\left[\tilde{q}_H, m(\tilde{q}_H^-)\right]} \le \frac{\psi_{2H}\left[\tilde{q}_H, m\left(\tilde{q}_H^+\right)\right]}{(1-\gamma_2)\,\psi_2\left[\tilde{q}_H, m\left(\tilde{q}_H^+\right)\right]}.$$

In view of the first inequality in Proposition 11 and the strict log supermodularity of the productivity function, this inequality implies $m\left(\tilde{q}_H^+\right) > m\left(\tilde{q}_H^-\right)$. That is, the matching function is discontinuous at $\tilde{q}_H$ and it jumps upwards there. As a result, there must exist an ability level for workers $\check{q}_L \in \left[m\left(\tilde{q}_H^-\right), m\left(\tilde{q}_H^+\right)\right]$ such that workers in the range $(\check{q}_L - \check{\varepsilon}_1, \check{q}_L)$ are employed in sector 1 and workers in the range $(\check{q}_L, \check{q}_L + \check{\varepsilon}_2)$ are employed in sector 2, for $\check{\varepsilon}_1$ and $\check{\varepsilon}_2$ small enough. Due to the upward jump of the matching function and due to PAM in each sector, in this range of worker abilities the ability of managers matched with workers in sector 1 must be strictly greater than the ability of managers matched with workers in sector 2. This is illustrated in Figure 12. At point A, we have $q_H = \tilde{q}_H$ and the matching function exhibits an upward jump from point A to C. The supposition is that managers to the left of A sort into sector 1 and managers to the right of A sort into sector 2, as illustrated in the figure. Clearly, workers with ability between points A and C must be matched with managers in some sector. Sector x illustrates a possible matching of these workers with high-ability managers. It is not possible for x to be sector 2, however, because this will imply non-monotonic matching in sector 2, which is not possible with strictly log supermodular productivity functions. So x must be sector 1. In this case, $\check{q}_L$ is the ability of workers at point C. Workers with ability just below

C work in sector 1 and workers with ability just above C work in sector 2. Evidently, the ability of managers with whom these workers are matched in sector 1 is higher than the ability of managers with whom their slightly better peers are matched in sector 2. It can be seen from this figure that a similar outcome obtains if the matching along x is to the left of point A, except that in this case x stands for sector 2 and $\check{q}_L$ is the ability of workers at point A. Evidently, in this case too, at points around $\check{q}_L$ the ability of managers matched with workers in sector 1 is higher than the ability of managers matched with workers in sector 2.

In short, consider the inverse function $m_1^{-1}(q_L)$ for $q_L \in (\check{q}_L - \check{\varepsilon}_1, \check{q}_L)$; this inverse exists in the specified range because $m_1(q_H)$ is continuous and strictly increasing at points in $(\tilde{q}_H - \varepsilon, \tilde{q}_H)$ for ε small enough. Similarly, consider the inverse function $m_2^{-1}(q_L)$ for $q_L \in (\check{q}_L, \check{q}_L + \check{\varepsilon}_2)$; this inverse also exists in the specified range because $m_2(q_H)$ is continuous and strictly increasing at points in $(\tilde{q}_H, \tilde{q}_H + \varepsilon)$ for ε small enough. Moreover, under the supposition of our sorting pattern $m^{-1}(q_L) = m_1^{-1}(q_L)$ for $q_L \in (\check{q}_L - \check{\varepsilon}_1, \check{q}_L)$ and $m^{-1}(q_L) = m_2^{-1}(q_L)$ for $q_L \in (\check{q}_L, \check{q}_L + \check{\varepsilon}_2)$ and the argument in the previous paragraph showed that $m^{-1}(q_L) = m_1^{-1}(q_L) > m^{-1}(q'_L) = m_2^{-1}(q'_L)$ for $q_L \in (\check{q}_L - \check{\varepsilon}_1, \check{q}_L)$ and $q'_L \in (\check{q}_L, \check{q}_L + \check{\varepsilon}_2)$. Taking limits as $\check{\varepsilon}_1, \check{\varepsilon}_2 \searrow 0$, this implies that $m^{-1}\left(\check{q}_L^-\right) > m^{-1}\left(\check{q}_L^+\right)$.

Next, following steps similar to those in the proof of Proposition 10, which considered the response of profits to variations in the ability of managers at points around $\tilde{q}_H$, an analysis of the response of profits to variations in the ability of workers at points around $\check{q}_L$ establishes that a necessary condition for optimality is

$$\frac{\psi_{1L}\left[m^{-1}\left(\check{q}_L^-\right), \check{q}_L\right]}{\gamma_1\psi_1\left[m^{-1}\left(\check{q}_L^-\right), \check{q}_L\right]} \leq \frac{\psi_{2L}\left[m^{-1}\left(\check{q}_L^+\right), \check{q}_L\right]}{\gamma_2\psi_2\left[m^{-1}\left(\check{q}_L^+\right), \check{q}_L\right]}.$$

In view of the second inequality in Proposition 11 and the strict log supermodularity of the productivity function, this inequality implies $m^{-1}\left(\check{q}_L^+\right) = m_2^{-1}\left(\check{q}_L^+\right) > m_1^{-1}\left(\check{q}_L^-\right) = m^{-1}\left(\check{q}_L^-\right)$, which contradicts the above established result that $m_1^{-1}\left(\check{q}_L^-\right) > m_2^{-1}\left(\check{q}_L^+\right)$. It follows that the best managers sort into sector 1. By symmetrical arguments the best workers also sort into sector 1.

Matching and Factor Prices in an Allocation Set

In order to prove the remaining propositions in the main text, we need to understand how matching within an allocation set and the wages and salaries for workers and managers within the set respond to changes in factor endowments, the price of the output produced by these factors, and the boundaries of workers' and managers' abilities.

Suppose that some sector employs workers and managers whose abilities form the intervals $S_L = [q_{La}, q_{Lb}]$ and $S_H = [q_{Ha}, q_{Hb}]$. To simplify notation, we drop the sectoral index i and change variables to $q = q_H$, and we consider the following industry equilibrium conditions:

$$r(q) = \bar{\gamma} p^{\frac{1}{1-\gamma}} \psi[q, m(q)]^{\frac{1}{1-\gamma}} w[m(q)]^{-\frac{\gamma}{1-\gamma}}, \quad \bar{\gamma} = \gamma^{\frac{\gamma}{1-\gamma}}(1-\gamma) \tag{53}$$

$$\frac{\psi_L[q, m(q)]}{\gamma\psi[q, m(q)]} = \frac{w'[m(q)]}{w[m(q)]}, \tag{54}$$

$$\bar{H}\frac{\gamma r(q)}{(1-\gamma)w[m(q)]}\phi_H(q) = \bar{L}\phi_L[m(q)]m'(q), \tag{55}$$

and the boundary conditions,

$$\begin{aligned} m\left(q_{Hz}\right) &= q_{Lz}, \quad z=a,b; \\ q_{Lb} &> q_{La}>0, \quad q_{Hb}>q_{Ha}>0. \end{aligned} \tag{56}$$

Equation (53) is taken from (32), (54) is taken from (33) and (55) is taken from (35). We seek to characterize the solution for the three functions, $w(\cdot)$, $r(\cdot)$ and $m(\cdot)$.

We use (53) and (54) to obtain

$$\ln r\left(q_H\right)-\ln r\left(q_{H0}\right)=\int_{q_{H0}}^{q_H} \frac{\psi_H\left[x,m\left(x\right)\right]}{\left(1-\gamma\right)\psi\left[x,m\left(x\right)\right]}dx, \quad \text{for } q_H,q_{H0}\in S_H, \tag{57}$$

$$\ln w\left(q_L\right)-\ln w\left(q_{L0}\right)=\int_{q_{L0}}^{q_L} \frac{\psi_L\left[\mu\left(x\right),x\right]}{\gamma\psi\left[\mu\left(x\right),x\right]}dx, \quad \text{for } q_L,q_{L0}\in S_L, \tag{58}$$

where $\mu(\cdot)$ is the inverse of $m(\cdot)$. We substitute (53) into (55) to obtain

$$\begin{aligned} \frac{1}{1-\gamma}\ln w\left[m\left(q\right)\right] &= \frac{1}{1-\gamma}\ln\gamma+\ln\left(\frac{\bar{H}}{\bar{L}}\right)+\frac{1}{1-\gamma}\ln p \\ &+\frac{1}{1-\gamma}\ln\psi\left[q,m\left(q\right)\right]+\log\phi_H\left(q\right)-\log\phi_L\left[m\left(q\right)\right]-\log m'\left(q\right). \end{aligned} \tag{59}$$

The differential equations (54) and (59) together with the boundary conditions (56) uniquely determine the solution of $w(\cdot)$ and $m(\cdot)$ when the productivity function $\psi(\cdot)$ is twice continuously differentiable and the density functions $\phi_F(\cdot)$, $F=H,L$, are continuously differentiable.

By differentiating (59) and substituting (54) into the result, we generate a second-order differential equation for the matching function,

$$\frac{m''\left(q\right)}{m'\left(q\right)}=\frac{\psi_H\left[q,m\left(q\right)\right]}{\left(1-\gamma\right)\psi_L\left[q,m\left(q\right)\right]}-\frac{\psi_L\left[q,m\left(q\right)\right]m'\left(q\right)}{\gamma\psi\left[q,m\left(q\right)\right]}+\frac{\phi_H'\left(q\right)}{\phi_H\left(q\right)}-\frac{\phi_L'\left[m\left(q\right)\right]m'\left(q\right)}{\phi_L\left[m\left(q\right)\right]}. \tag{60}$$

Given boundary conditions $m\left(q_0\right)=q_{L0}>0$, $m'\left(q_0\right)=t_0>0$, this differential equation has a unique solution, which may or may not satisfy the boundary conditions (56). The solution to the original matching problem is found by finding a value t_a such that $m\left(q_{Ha}\right)=q_{La}$ and $m'\left(q_{Ha}\right)=t_a$ yield a solution that satisfies the second boundary condition $m\left(q_{Hb}\right)=q_{Lb}$. Note that this solution depends neither on the price p nor on the factor endowments $\bar{H}$ and $\bar{L}$. Therefore, changes in these variables do no affect the matching function, but they change proportionately wages, as can be seen from (59), and salaries, as can be seen from (53). We have

Lemma 1 *(i) The matching function $m(\cdot)$ does not depend on $\left(p,\bar{H},\bar{L}\right)$. (ii) An increase in the price p, $\hat{p}>0$, raises the wage and salary schedules proportionately by $\hat{p}$. (iii) An increase in $\bar{H}/\bar{L}$, $\hat{\eta}=\hat{H}-\hat{L}>0$, raises the wage schedule proportionately by $(1-\gamma)\hat{\eta}$ and reduces the salary schedule proportionately by $\gamma\hat{\eta}$.*

We now prove several lemmas that will be used in the main analysis.

Lemma 2 *Let $\left[m_\varkappa\left(q\right),w_\varkappa\left(q_L\right)\right]$ and $\left[m_\varrho\left(q\right),w_\varrho\left(q_L\right)\right]$ be solutions to the differential equations (54) and (59), each for different boundary conditions (56), such that $m_\varkappa\left(q_0\right)=m_\varrho\left(q_0\right)=q_{L0}$ and $m_\varrho'\left(q_0\right)>m_\varkappa'\left(q_0\right)$*

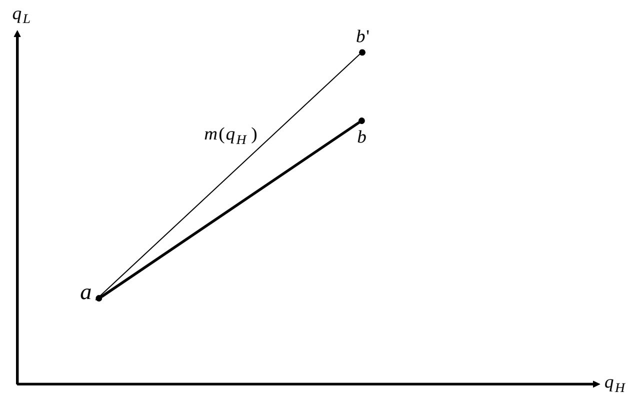

Figure 13: Shift in the matching function when q_L^b rises to $q_L^{b'}$

for $q_0 \in S_{H\varkappa} \cap S_{H\varrho}$. *Then* $m_\varrho(q) > m_\varkappa(q)$ *for all* $q > q_0$ *and* $m_\varrho(q) < m_\varkappa(q)$ *for all* $q < q_0$ *in the overlapping range of abilities.*

Proof. Consider $q > q_0$ and suppose that contrary to the claim there exists a $q_1 > q_0$ such that $m_\varrho(q_1) \leq m_\varkappa(q_1)$. Then differentiability of $m_\iota(\cdot)$, $\iota = \varkappa, \varrho$, implies that there exists $q_2 > q_0$ such that $m_\varrho(q_2) = m_\varkappa(q_2)$, $m_\varrho(q) > m_\varkappa(q)$ for all $q \in (q_0, q_2)$ and $m'_\varrho(q_2) < m'_\varkappa(q_2)$. This also implies $\mu_\varrho(x) < \mu_\varkappa(x)$ for all $x \in (m_\varrho(q_0), m_\varrho(q_2))$, where $\mu_\iota(\cdot)$ is the inverse of $m_\iota(\cdot)$. Under these conditions (59) implies $w_\varrho[m_\varrho(q_0)] < w_\varkappa[m_\varrho(q_0)]$ and $w_\varrho[m_\varrho(q_2)] > w_\varkappa[m_\varrho(q_2)]$, and therefore

$$w_\varkappa[m_\varrho(q_2)] - w_\varkappa[m_\varrho(q_0)] < w_\varrho[m_\varrho(q_2)] - w_\varrho[m_\varrho(q_0)].$$

On the other hand, (58) implies

$$\ln w_\iota[m_\varrho(q_2)] - \ln w_\iota[m_\varrho(q_0)] = \int_{m_\varrho(q_0)}^{m_\varrho(q_2)} \frac{\psi_L[\mu_\iota(x), x]}{\gamma\psi[\mu_\iota(x), x]} dx, \quad \iota = \varkappa, \varrho.$$

Together with the previous inequality, this gives

$$\int_{m_\varrho(q_0)}^{m_\varrho(q_2)} \frac{\psi_L[\mu_\varkappa(x), x]}{\psi[\mu_\varkappa(x), x]} dx < \int_{m_\varrho(q_0)}^{m_\varrho(q_2)} \frac{\psi_L[\mu_\varrho(x), x]}{\psi[\mu_\varrho(x), x]} dx.$$

Note, however, that strict log supermodularity of $\psi(\cdot)$ and $\mu_\varrho(x) < \mu_\varkappa(x)$ for all $x \in (m_\varrho(q_0), m_\varrho(q_2))$ imply the reverse inequality, a contradiction. It follows that $m_\varrho(q) > m_\varkappa(q)$ for all $q > q_0$. A similar argument shows that $m_\varrho(q) < m_\varkappa(q)$ for all $q < q_0$. ■

The key implication of this lemma is that changes in the boundary conditions (56) shift the matching function in such a way as to generate at most one point in common with the original matching function. We next show how the matching function and wage function respond to the boundary conditions. To this end, consider Figure 13. Let the thick curve between points a and b represent the solution to the matching function when points a and b are the boundary points (56). Now consider the shift of the equilibrium matching function in response to a decline in q_{Lb}; that is, the end point b shifts upward to b'. Since point a is common to the old and new matching function, Lemma 2 implies that they can have no additional points in common, which implies that the new inverse matching function—represented by the thin curve between points a and b'—is everywhere above the old one. It follows that an increase in q_{Lb} increases the ability of workers matched with every manager except for the least able manager. Other shifts in the boundary points can be analyzed in similar fashion to establish

Lemma 3 *(i) $dm(q_H)/dq_{La} > 0$ for all $q_H < q_{Hb}$ and $d\mu(q_L)/dq_{La} < 0$ for all $q_L < q_{Lb}$; (ii) $dm(q_H)/dq_{Lb} > 0$ for all $q_H > q_{Ha}$ and $d\mu(q_L)/dq_{Lb} < 0$ for all $q_L > q_{La}$; (iii) $d\mu(q_L)/dq_{Ha} > 0$ for all $q_L < q_{Lb}$ and $dm(q_H)/dq_{Ha} < 0$ for all $q_H < q_{Hb}$; and (iv) $d\mu(q_L)/dq_{Hb} > 0$ for all $q_L > q_{La}$ and $dm(q_H)/dq_{Hb} < 0$ for all $q_H > q_{Ha}$.*

The rule that emerges from this lemma is that an improvement in the ability of workers at a boundary of S_L improves the quality of the matches for all the managers (except those at the other boundary) and deteriorates the quality of the matches for all the workers (except those at the other boundary). Similarly, an improvement in the ability of managers at a boundary of S_H improves the quality of the matches for all workers (except those at the other boundary) and deteriorates the quality of the matches for all the managers (except those at the other boundary).

Next consider changes in a boundary (q_{Hz}, q_{Lz}), $z = a, b$. For concreteness, suppose that (q_{Hb}, q_{Lb}) changes. Then the new matching function coincides with the old one at the other boundary point, (q_{Ha}, q_{La}), that has not changed. In this case, Lemma 2 implies that either the two matching functions coincide in the overlapping range of abilities or one is above the other everywhere except for at (q_{Ha}, q_{La}). A similar argument applies to changes in (q_{Ha}, q_{La}). We therefore have:

Lemma 4 *In response to a shift in a single boundary (q_{Hz}, q_{Lz}), $z = a, b$, either the new matching functions coincide with the old matching function in the overlapping range of abilities or one is above the other everywhere except for the other boundary point.*

We next discuss the impact of boundaries on wages and salaries. We focus the analysis on wages, but note that if a shift in boundaries raises the wage of workers with ability q_L then this shift reduces the salary of managers paired with these workers, and the opposite if the shift reduces the wage. This is seen from (53) by noting that a change in boundaries has no impact on $r(\cdot)$ through an induced shift in the matching function due to the first-order condition (54) (a version of the Envelope Theorem). Therefore the change in salary $r(q)$ is driven by the change in wages of workers matched with managers of ability q. This is summarized in

Lemma 5 *Suppose that the boundaries (q_{Hz}, q_{Lz}), $z = a, b$, change and that as a result $w(q_L)$ rises for some q_L such that q_L and $q = m^{-1}(q_L)$ are in the overlapping range of abilities of the old and new boundaries. Then $r(q)$ declines.*

For the subsequent analysis the following lemma is useful:

Lemma 6 *Let* $[m_{\varkappa}(q), w_{\varkappa}(q_L)]$ *and* $[m_{\varrho}(q), w_{\varrho}(q_L)]$ *be solutions to the differential equations (54) and (59), each for different boundary conditions (56), such that* $m_{\varkappa}(q_0) = m_{\varrho}(q_0) = q_{L0}$ *and* $m'_{\varrho}(q_0) > m'_{\varkappa}(q_0)$ *for some* $q_0 \in S_{L\varkappa} \cap S_{L\varrho}$*, and let* $r_{\varrho}(q)$ *and* $r_{\varkappa}(q)$ *be the corresponding solutions to (53). Then* $w_{\varrho}(q_L) < w_{\varkappa}(q_L)$ *and* $r_{\varrho}(q) > r_{\varkappa}(q)$ *in the overlapping range of abilities.*

Proof. From Lemma 2 we know that $m_{\varrho}(q) > m_{\varkappa}(q)$ for all $q > q_0$ and $m_{\varrho}(q) < m_{\varkappa}(q)$ for all $q < q_0$ in the overlapping range of abilities and $\mu_{\varrho}(x) < \mu_{\varkappa}(x)$ for all $x > q_{L0}$ and $\mu_{\varrho}(x) > \mu_{\varkappa}(x)$ for all $x < q_{L0}$ in the overlapping range of abilities. Moreover, $m'_{\varrho}(q_0) > m'_{\varkappa}(q_0)$ and (59) imply

$$\ln w_{\varkappa}(q_{L0}) > \ln w_{\varrho}(q_{L0})$$

while (58) implies

$$\ln w_{\iota}(q_L) - \ln w_{\iota}(q_{L0}) = \int_{q_{L0}}^{q_L} \frac{\psi_L[\mu_{\iota}(x), x]}{\gamma\psi[\mu_{\iota}(x), x]} dx, \quad \iota = \varkappa, \varrho.$$

Together, they imply

$$\begin{aligned} \ln w_{\varkappa}(q_L) - \ln w_{\varrho}(q_L) &> \int_{q_{L0}}^{q_L} \frac{\psi_L[\mu_{\varkappa}(x), x]}{\gamma\psi[\mu_{\varkappa}(x), x]} dx - \int_{q_{L0}}^{q_L} \frac{\psi_L[\mu_{\varrho}(x), x]}{\gamma\psi[\mu_{\varrho}(x), x]} dx \\ &= \int_{q_L}^{q_{L0}} \frac{\psi_L[\mu_{\varrho}(x), x]}{\gamma\psi[\mu_{\varrho}(x), x]} dx - \int_{q_L}^{q_{L0}} \frac{\psi_L[\mu_{\varkappa}(x), x]}{\gamma\psi[\mu_{\varkappa}(x), x]} dx. \end{aligned}$$

For $q_L > q_{L0}$ the right-hand side of the first line is positive due to the strict log supermodularity of the productivity function and $\mu_{\varrho}(x) < \mu_{\varkappa}(x)$ for all $x > q_{L0}$ and the second line is also positive for $q_L < q_{L0}$ due to the strict log supermodularity of the productivity function and $\mu_{\varrho}(x) > \mu_{\varkappa}(x)$ for all $x < q_{L0}$. It follows that $w_{\varkappa}(q_L) > w_{\varrho}(q_L)$ for all q_L in the overlapping range of abilities. A similar argument establishes that $r_{\varkappa}(q) < r_{\varrho}(q)$ for all q in the overlapping range of abilities. ■

This lemma, together with Lemma 4, have straightforward implications for the impact of boundary points on the wage and salary functions.

Corollary 1 *Suppose that the lower boundary* (q_{Ha}, q_{La}) *changes and the matching function shifts upwards as a result. Then salaries decline and wages rise in the overlapping range of abilities. The converse holds when the matching function shifts downwards.*

Corollary 2 *Suppose that the upper boundary* (q_{Hb}, q_{Lb}) *changes and the matching function shifts upwards as a result. Then salaries rise and wages decline in the overlapping range of abilities. The converse holds when the matching function shifts downwards.*

Not only do wages and salaries shift in a predictable way in response to a shift of a boundary point, the inequalities of wages and salaries also change in a predictable way. From (58) we see that a change in boundaries that shifts upwards the matching function induces less inequality of wages, because for every two ability levels the ratio of the wage of a high-ability worker to the wage of a low-ability worker declines for all abilities in the overlapping range. For salaries it is the opposite, as one can see from (57). We therefore have

Lemma 7 *Suppose that the matching function shifts upwards in response to a shift in the boundaries (56). Then wage inequality narrows and salary inequality expands. The opposite changes in inequality occur when the matching function shifts downwards.*

General Equilibrium

Consider a two-sector economy in which the most-able workers are employed in one sector and the least-able workers are employed in the other sector, and similarly for managers. In such circumstances, the equilibrium can take one of two forms: either the highest-ability workers and highest-ability managers are employed in the same sector and the lowest-ability workers and lowest-ability managers are employed in the other, which we describe as an HH/LL equilibrium, or the highest-ability workers and lowest-ability managers are employed in one sector and the lowest-ability workers and highest-ability managers are employed in the other, which we describe as an HL/LH equilibrium. Our first result is

Lemma 8 *Suppose that the economy has a threshold equilibrium either of the HH/LL or HL/LH type. Then: (i) if the best workers sort into the labor-intensive sector then an increase in $\bar{H}/\bar{L}$ raises the cutoff q_L^* and if the best workers sort into the manager-intensive sector then an increase in $\bar{H}/\bar{L}$ reduces the cutoff q_L^*; and (ii) if the best managers sort into the labor-intensive sector, then an increase in $\bar{H}/\bar{L}$ raises the cutoff q_H^* and if the best managers sort into the manager-intensive sector then an increase in $\bar{H}/\bar{L}$ reduces the cutoff q_H^*.*

To prove this lemma, label sectors so that the best workers sort into sector 1. We first prove the result for an HH/LL equilibrium followed by a proof for an HL/LH equilibrium.

HH/LL Equilibrium

In an HH/LL equilibrium the cutoffs $\{q_H^*, q_L^*\}$ satisfy:

$$w_1(q_L^*) = w_2(q_L^*), \tag{61}$$

$$r_1(q_H^*) = r_2(q_H^*), \tag{62}$$

where $[w_i(\cdot), r_i(\cdot), m_i(\cdot)]$ is a solution to the single sector differential equations (54) and (59) for $i = 1, 2$ with the boundary conditions

$$m_2(q_{H\min}) = q_{L\min}, \quad m_2(q_H^*) = q_L^*, \tag{63}$$

$$m_1(q_H^*) = q_L^*, \quad m_1(q_{H\max}) = q_{L\max}. \tag{64}$$

Evidently, the solutions to the wage, salary and matching functions depend on the parameters of the model, such as prices and factor endowments, as do the equilibrium cutoffs $\{q_H^*, q_L^*\}$. We denote by $dw_i(q_L)/d\vartheta$ the derivative of the wage function in sector i with respect to a parameter ϑ, where this derivative accounts for the endogenous adjustments of the wage, salary and matching functions. This derivative contrasts with $w_i'(q_L)$, which is the slope of the wage function for given parameters. And we use similar notation for the derivatives of the salary function.

For now, we are interested in $\eta = \bar{H}/\bar{L}$ and we shall use the following elasticities

$$\varepsilon^*_{w_i,\eta} = \left.\frac{dw_i(q_L)}{d\left(\bar{H}/\bar{L}\right)} \cdot \frac{\bar{H}/\bar{L}}{q_L}\right|_{q_L=q_L^*}, \quad \varepsilon^*_{r_i,\eta} = \left.\frac{dr_i(q_H)}{d\left(\bar{H}/\bar{L}\right)} \cdot \frac{\bar{H}/\bar{L}}{q_H}\right|_{q_H=q_H^*}.$$

Differentiating (61)-(62) with respect to $\eta \equiv \bar{H}/\bar{L}$ then yields

$$\left[\frac{w_1'(q_L^*)}{w_1(q_L^*)} - \frac{w_2'(q_L^*)}{w_2(q_L^*)}\right] dq_L^* = \varepsilon^*_{w_2,\eta} - \varepsilon^*_{w_1,\eta}, \tag{65}$$

$$\left[\frac{r_1'(q_H^*)}{r_1(q_H^*)} - \frac{r_2'(q_H^*)}{r_2(q_H^*)}\right] dq_H^* = \varepsilon^*_{r_2,\eta} - \varepsilon^*_{r_1,\eta}. \tag{66}$$

The assumptions that the equilibrium is of the HH/LL type and that the best workers and managers sort into sector 1 imply that the expressions in the square brackets are positive in both equations; that is, at the boundary $\{q_H^*, q_L^*\}$ between the two sectors the slopes of the wage and salary functions have to be steeper in sector 1 into which the more able employees sort. It follows that q_L^* rises in response to an increase in the ratio of managers to workers if and only if $\varepsilon^*_{w_2,\eta} > \varepsilon^*_{w_1,\eta}$ and the cutoff q_H^* rises if and only if $\varepsilon^*_{r_2,\eta} > \varepsilon^*_{r_1,\eta}$.

To understand the elasticities $\varepsilon^*_{w_i,\eta}$ and $\varepsilon^*_{r_i,\eta}$, note that a shift in $\bar{H}/\bar{L}$ impacts wages and salaries through two channels. First, there is the direct effect described in part (iii) of Lemma 1, which adds $1-\gamma_i$ to $\varepsilon^*_{w_i,\eta}$ and $-\gamma_i$ to $\varepsilon^*_{r_i,\eta}$. This stems from the fact that with constant boundaries factor endowments do not affect the matching functions. But given factor intensity differences across sectors, equations (65) and (66) imply that with no changes in matching the right-hand side of each one of these equations equals $\gamma_1 - \gamma_2$, which induce an increase in q_L^* and q_H^* if and only if $\gamma_1 - \gamma_2 > 0$. These shifts in the cutoffs trigger an adjustment in matching in each sector, which impacts in turn the wage and salary functions, as shown in Lemmas 3-6 and Corollaries 1 and 2 to Lemma 6. In other words, the impact effect of a hike in $\bar{H}/\bar{L}$ raises the cutoffs for both workers and managers, but we also have to account for the induced change in matching in order to obtain the full effect. In other words, we need to find a fixed point of $\{dq_H^*, dq_L^*\}$. To this end, we now express the elasticities $\varepsilon^*_{w_i,\eta}$ and $\varepsilon^*_{r_i,\eta}$ as follows:

$$\varepsilon^*_{w_i,\eta} = (1-\gamma_i)\,\hat{\eta} + \varepsilon^*_{w_iL}\hat{q}_L^* + \varepsilon^*_{w_iH}\hat{q}_H^*, \quad i = 1,2, \tag{67}$$

$$\varepsilon^*_{r_i,\eta} = -\gamma_i\hat{\eta} + \varepsilon^*_{r_iL}\hat{q}_L^* + \varepsilon^*_{r_iH}\hat{q}_H^*, \quad i = 1,2, \tag{68}$$

where $1-\gamma_i$ and $-\gamma_i$ represent the direct impacts of $\bar{H}/\bar{L}$, $\varepsilon^*_{w_iL}$ is the elasticity of $w_i(\cdot)$ with respect to the boundary q_L^* through the induced change in matching, evaluated at q_L^*, and $\varepsilon^*_{w_iH}$ is the elasticity of $w_i(\cdot)$ with respect to the boundary q_H^* through the induced change in matching, evaluated at q_L^*. From (53) and (54) we also have

$$\varepsilon^*_{r_iF} = -\frac{\gamma_i}{1-\gamma_i}\varepsilon^*_{w_iF}, \quad F = H, L; \quad i = 1,2. \tag{69}$$

Now substitute these equations into (65)-(66) to obtain

$$M_h^{HH/LL}\begin{pmatrix} \hat{q}_L^* \\ \hat{q}_H^* \end{pmatrix} = \begin{pmatrix} \gamma_1 - \gamma_2 \\ \gamma_1 - \gamma_2 \end{pmatrix}\hat{\eta}, \tag{70}$$

where

$$M_h^{HH/LL} = \begin{pmatrix} q_L^* \left[\frac{w_1'(q_L^*)}{w_1(q_L^*)} - \frac{w_2'(q_L^*)}{w_2(q_L^*)}\right] + \varepsilon_{w_1L}^* - \varepsilon_{w_2L}^* & \varepsilon_{w_1H}^* - \varepsilon_{w_2H}^* \\ \frac{\gamma_2\varepsilon_{w_2L}^*}{1-\gamma_2} - \frac{\gamma_1\varepsilon_{w_1L}^*}{1-\gamma_1} & q_H^* \left[\frac{r_1'(q_H^*)}{r_1(q_H^*)} - \frac{r_2'(q_H^*)}{r_2(q_H^*)}\right] + \frac{\gamma_2\varepsilon_{w_2H}^*}{1-\gamma_2} - \frac{\gamma_1\varepsilon_{w_1H}^*}{1-\gamma_1} \end{pmatrix}.$$

From Lemmas 3-6 we have

$$\varepsilon_{w_1L}^* > 0, \quad \varepsilon_{w_2L}^* < 0, \quad \varepsilon_{w_1H}^* < 0, \quad \varepsilon_{w_2H}^* > 0.$$

These equations provide a solution to $\hat{q}_L^*$ and $\hat{q}_H^*$.

The determinant of the matrix $M_h^{HH/LL}$ is

$$\begin{aligned} D_{M_h^{HH/LL}} &= \left\{ q_L^* \left[\frac{w_1'(q_L^*)}{w_1(q_L^*)} - \frac{w_2'(q_L^*)}{w_2(q_L^*)}\right] + \varepsilon_{w_1L}^* - \varepsilon_{w_2L}^* \right\} q_H^* \left[\frac{r_1'(q_H^*)}{r_1(q_H^*)} - \frac{r_2'(q_H^*)}{r_2(q_H^*)}\right] \\ &\quad + \left(\frac{\gamma_2\varepsilon_{w_2H}^*}{1-\gamma_2} - \frac{\gamma_1\varepsilon_{w_1H}^*}{1-\gamma_1}\right) q_L^* \left[\frac{w_1'(q_L^*)}{w_1(q_L^*)} - \frac{w_2'(q_L^*)}{w_2(q_L^*)}\right] \\ &\quad - \frac{\gamma_1-\gamma_2}{(1-\gamma_1)(1-\gamma_2)} \left(\varepsilon_{w_2H}^*\varepsilon_{w_1L}^* - \varepsilon_{w_1H}^*\varepsilon_{w_2L}^*\right). \end{aligned}$$

The expressions in the first two lines are positive. We now show that the expression in the third line also is positive. To this end, note from Lemma 2 that if we change a single boundary and the new boundary is on the original matching function then the new matching function coincides with the old one in the overlapping range of abilities. Therefore, if we choose $dq_L^* = m_i'(q_H^*)\, dq_H^*$, where $m_i(\cdot)$ is the solution of matching in sector i, then a change in the boundary (dq_H^*, dq_L^*) does not change the wage $w_i(q_L^*)$. In other words,

$$\varepsilon_{w_iH}^* + \varepsilon_{w_iL}^*\varepsilon_{m_i}^* = 0,$$

where $\varepsilon_{m_i}^*$ is the elasticity of $m_i(\cdot)$ evaluated at q_H^*. On the other hand, (55) implies for the HH/LL case that

$$\varepsilon_{m_i}^* = \frac{\kappa_m\gamma_i}{1-\gamma_i},$$

where

$$\kappa_m = \frac{\bar{H} r(q_H^*)\, \phi_H(q_H^*)\, q_H^*}{\bar{L} w(q_L^*)\, \phi_L(q_L^*)\, q_L^*}.$$

Therefore,

$$\varepsilon_{w_iH}^* = -\frac{\kappa_m\gamma_i}{1-\gamma_i}\varepsilon_{w_iL}^*.$$

Using this expression, we obtain

$$-\frac{\gamma_1-\gamma_2}{(1-\gamma_1)(1-\gamma_2)} \left(\varepsilon_{w_2H}^*\varepsilon_{w_1L}^* - \varepsilon_{w_1H}^*\varepsilon_{w_2L}^*\right) = -\frac{(\gamma_1-\gamma_2)^2 \kappa_m \varepsilon_{w_1L}^*\varepsilon_{w_2L}^*}{(1-\gamma_1)^2(1-\gamma_2)^2} > 0,$$

which proves that $D_{M_h^{HH/LL}} > 0$.

Solving (70) implies that $\hat{q}_L^* > 0$ and $\hat{q}_H^* > 0$ if and only if $(\gamma_1 - \gamma_2)\,\hat{h} > 0$. In other words, an increase in the $\bar{H}/\bar{L}$ ratio raises both cutoffs if and only if sector 1 is labor intensive.

Next consider price changes. An increase in the price of good i raises on impact wages and salaries in sector i by $\hat{p}_i$ and has no direct impact on wages and salaries in the other sector. Following the previous arguments, the change in equilibrium cutoffs can be found as the solution to

$$M_h^{HH/LL}\left(\begin{array}{c}\hat{q}_L^*\\ \hat{q}_H^*\end{array}\right)=\left(\begin{array}{c}\hat{p}_2-\hat{p}_1\\ \hat{p}_2-\hat{p}_1\end{array}\right), \tag{71}$$

where the matrix $M_h^{HH/LL}$ is the same as in (70). It follows from this system that $\hat{q}_L^*>0$ and $\hat{q}_H^*>0$ if and only if $\hat{p}_2>\hat{p}_1$. That is, an increase in the relative price of good 2 raises both cutoffs and therefore raises output in sector 2 and reduces output in sector 1.

HL/LH Equilibrium

In an HL/LH equilibrium, the cutoffs $\{q_H^*,q_L^*\}$ also satisfy the continuity conditions (61)-(62), but the boundary conditions are different. Assuming as before that the best workers sort into sector 1, this means that in an HL/LH equilibrium the best managers sort into sector 2 and the boundary conditions are:

$$m_1\left(q_{H\min}\right)=q_L^*,\quad m_1\left(q_H^*\right)=q_{L\max},$$

$$m_2\left(q_H^*\right)=q_{L\min},\quad m_2\left(q_{H\max}\right)=q_L^*.$$

Figure 5 depicts the pattern of sorting and matching in this type of equilibrium. For the more able workers to sort into sector 1 we require that

$$\frac{w_1'\left(q_L^*\right)}{w_1\left(q_L^*\right)}>\frac{w_2'\left(q_L^*\right)}{w_2\left(q_L^*\right)}$$

and for the more able managers to sort into sector 2 we require that

$$\frac{r_1'\left(q_H^*\right)}{r_1\left(q_H^*\right)}<\frac{r_2'\left(q_H^*\right)}{r_2\left(q_H^*\right)}.$$

To derive the required comparative statics results, we use as before conditions (65) and (66), which apply in this case too. We also can use the decomposition of elasticities (67) and (68), which still applies. Now, however, the relationship between the elasticities of the salary and wage functions, described in (69), does not apply, because workers of ability q_L^* do not pair with managers of ability q_H^*, as is evident from Figure 5. Instead, from (53) and (54) we now obtain

$$\varepsilon_{r_1F}^*=-\frac{\gamma_1}{1-\gamma_1}\varepsilon_{w_1F}^{\max},\quad F=H,L,$$

$$\varepsilon_{r_2F}^*=-\frac{\gamma_2}{1-\gamma_2}\varepsilon_{w_2F}^{\min},\quad F=H,L,$$

where $\varepsilon_{r_iF}^*$ is defined in the same way as before, $\varepsilon_{w_1F}^{\max}$ is the elasticity of $w_1(\cdot)$ with respect to the boundary q_F^* through the induced change in matching in sector 1, evaluated at $q_{L\max}$, and $\varepsilon_{w_2F}^{\min}$ is the elasticity of $w_2(\cdot)$ with respect to the boundary q_F^* through the induced change in matching in sector 2, evaluated at $q_{L\min}$. Using these results the systems of equations (70) and (71) are replaced by

$$M_h^{HL/LH}\left(\begin{array}{c}\hat{q}_L^*\\ \hat{q}_H^*\end{array}\right)=\left(\begin{array}{c}\gamma_1-\gamma_2\\ \gamma_1-\gamma_2\end{array}\right)\hat{\eta}, \tag{72}$$

$$M_h^{HL/LH}\left(\begin{array}{c}\hat{q}_L^*\\ \hat{q}_H^*\end{array}\right)=\left(\begin{array}{c}\hat{p}_2-\hat{p}_1\\ \hat{p}_2-\hat{p}_1\end{array}\right), \tag{73}$$

where

$$M_h^{HL/LH}=\left(\begin{array}{cc} q_L^*\left[\frac{w_1'(q_L^*)}{w_1(q_L^*)}-\frac{w_2'(q_L^*)}{w_2(q_L^*)}\right]+\varepsilon_{w_1L}^*-\varepsilon_{w_2L}^* & \varepsilon_{w_1H}^*-\varepsilon_{w_2H}^* \\ \frac{\gamma_2\varepsilon_{w_2L}^{\min}}{1-\gamma_2}-\frac{\gamma_1\varepsilon_{w_1L}^{\max}}{1-\gamma_1} & q_H^*\left[\frac{r_1'(q_H^*)}{r_1(q_H^*)}-\frac{r_2'(q_H^*)}{r_2(q_H^*)}\right]+\frac{\gamma_2\varepsilon_{w_2H}^{\min}}{1-\gamma_2}-\frac{\gamma_1\varepsilon_{w_1H}^{\max}}{1-\gamma_1}\end{array}\right). \tag{74}$$

From Lemmas 3-6, we have $\varepsilon_{w_1L}^*>0>\varepsilon_{w_2L}^*$, $\varepsilon_{w_1H}^*>0>\varepsilon_{w_2H}^*$, $\varepsilon_{r_1H}^*<0<\varepsilon_{r_2H}^*$, $\varepsilon_{r_1L}^*<0<\varepsilon_{r_2L}^*$. This implies that both entries in the top row in (74) are strictly positive and both entries in the bottom row are strictly negative.

Consider system (73) first. The previous observations imply that a positive term $\hat{p}_2-\hat{p}_1$ either raises q_L^* and reduces q_H^*, or it reduces q_L^* and raises q_H^*. The cutoffs cannot both move in the same direction as the effect in the top row on the left hand side of (73) would be opposite those in the bottom row, while on the right hand side both effects have the same sign. We will show that only a rise in q_L^* and a reduction q_H^* is consistent with equilibrium play, which implies that the determinant of $M_h^{HL/LH}$ must be negative ($D_{M_h^{HL/LH}}<0$). To prove this, consider an increase in the price p_2 to $p_2'>p_2$ while the price p_1 stays constant. Let X_1 and X_2 denote the output in each sector prior to the price change, and let X_1' and X_2' denote the corresponding output after the price change. Since only prices have changed (and not endowments), under each set of prices both the outputs (X_1,X_2) and (X_1',X_2') are feasible. Since the competitive equilibrium is efficient, the value of output is maximized given prices, which implies that

$$\begin{aligned} p_1X_1+p_2X_2 &\geq p_1X_1'+p_2X_2', \\ p_1X_1+p_2'X_2 &\leq p_1X_1'+p_2'X_2, \end{aligned}$$

where the first inequality states that prior to the price change the value of output is higher under production bundle (X_1,X_2) than under (X_1',X_2'), while the opposite holds after the price change. Subtracting and rearranging gives

$$(p_2-p_2')(X_2-X_2')\geq 0,$$

which implies that $X_2\leq X_2'$. An increase in output in sector two cannot be achieved with a fall in q_L^* and a rise q_H^*, because in this case there would be less worker types and less manager types in sector 2. Therefore, an increase in the relative price of good 2 leads to a rise in q_L^* and a reduction q_H^*. Moreover, this requires $D_{M_h^{HL/LH}}<0$.

Now consider system (72). Since $D_{M_h^{HL/LH}}<0$, a rise in the relative endowment $\eta\equiv\bar{H}/\bar{L}$ of managers raises q_L^* and reduces q_H^*. Finally, we must determine the effect of a change in the relative endowment of managers on the relative output levels, which are affected both by the change in matching and the change in endowments. Sector i pays managers a fraction $1-\gamma_i$ of revenue. Therefore, in an HL/LH equilibrium, we have

$$(1-\gamma_1)\,p_1X_1=\bar{H}\int_{q_{H\min}}^{q_H^*} r(q_H)\,\phi_H(q_H)\,dq_H,$$

$$(1-\gamma_2)\,p_2X_2\equiv\bar{H}\int_{q_H^*}^{q_{H\max}} r(q_H)\,\phi_H(q_H)\,dq_H,$$

which implies

$$\frac{(1-\gamma_2)\,p_2X_2}{(1-\gamma_1)\,p_1X_1}=\frac{\int_{q_H^*}^{q_{H\max}} r(q_H)\,\phi_H(q_H)\,dq_H}{\int_{q_{H\min}}^{q_H^*} r(q_H)\,\phi_H(q_H)\,dq_H}.$$

Using (37), this equation can be expressed as

$$\frac{X_2}{X_1}=\frac{(1-\gamma_1)\,p_1\int_{q_H^*}^{q_{H\max}}\exp\left[\int_{q_H^*}^{q_H}\frac{\psi_{2H}[q,m(q)]}{(1-\gamma_2)\psi_2[q,m(q)]}dq\right]\phi_H(q_H)\,dq_H}{(1-\gamma_2)\,p_2\int_{q_{H\min}}^{q_H^*}\exp\left[-\int_{q_H}^{q_H^*}\frac{\psi_{1H}[q,m(q)]}{(1-\gamma_1)\psi_1[q,m(q)]}dq\right]\phi_H(q_H)\,dq_H},\tag{75}$$

where we have used the property that $r(\cdot)$ is a continuous function. When sector 2 is manager-intensive, i.e., $\gamma_1>\gamma_2$, q_H^* is lower in country A, which has more managers per worker. We have shown above that, under these circumstances, managers are paired with higher-ability workers in country A. Due to the strict log supermodularity of the productivity functions this implies that $\psi_{iH}[q,m(q)]/\psi_i[q,m(q)]$ is higher in country A in both sectors. It follows that the impact of a higher $\bar{H}/\bar{L}$ on matching raises the relative output of good 2. In the opposite case„ when $\gamma_1<\gamma_2$, the shift in matching reduces the relative output of good 2. In short, the shift in matching raises the relative output of the manager-intensive good. To complete the analysis of the impact of factor endowments on relative outputs, we need to asses the direct impact of the cutoff q_H^* on the relative outputs in (75).

First note that q_H^* impacts relative outputs through the boundaries of four integrals. When $\gamma_1>\gamma_2$ and q_H^* declines in response to an increase in $\bar{H}/\bar{L}$, the shifts in the boundaries of the outer integrals in the numerator and denominator raise the relative output of good 2. In the opposite case, when $\gamma_1<\gamma_2$ and q_H^* rises, the relative output of good 1 increases. A shift in the boundaries of the two inner integrals in the numerator and denominator have opposite effects from each other. For this reason we need to evaluate their relative strength. Differentiation with respect to these boundaries yields:

$$-\frac{X_2}{X_1}\left\{\lim_{q\searrow q_H^*}\frac{\psi_{2H}[q,m(q)]}{(1-\gamma_2)\,\psi_2[q,m(q)]}-\lim_{q\nearrow q_H^*}\frac{\psi_{1H}[q,m(q)]}{(1-\gamma_1)\,\psi_1[q,m(q)]}\right\}.$$

Since the best managers sort into sector 2, this requires the slop of the salary function $r(\cdot)$ to be steeper at q_H^* in sector 2, or, using (34), it requires the term in the tilted bracket to be positive. It follows that a decline in q_H^* raises the relative output of good 2. In the case in which good 2 is labor intensive q_H^* rises in response to an increase in $\bar{H}/\bar{L}$, which raises the relative output of good 1. In either case, country A produces relatively more of the manager-intensive good.