Matching Information

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Motivation

- We explore the following matching problem:
  - A population of heterogeneous agents must be partitioned in groups of a given size (not necessarily two)
  - Agents differ in expertise in estimating an unknown variable that is relevant for the performance of the group
    - Expertise $\equiv$ Information technology that generates a signal about the unknown variable of interest
    - A better expert is one with a more informative technology
  - Within a group, agents take a joint action
  - Agents can make monetary transfers among themselves

Motivation

- We address the following standard matching question:
    - What is the optimal sorting of agents into groups?
    - Will *more informed* agents be paired with *more* or *less informed* ones?
- We also shed light on:
    - The role of correlated information on sorting patterns
    - Matching groups of experts with heterogeneous firms
    - Endogenous group size

Motivation

- Remarks:
  - Many Groups
  - Interpretations: Partnerships, groups that are paired with identical firms, groups within an organization
  - Matchmaker can be the planner, or can take place in a decentralized market
  - Applications: financial experts, R&D groups, composition of skilled workers across firms, etc.

Main Insight

- Diversification of expertise within groups is optimal
  - Even if in isolation more information is better
  - Matching $\Rightarrow$ information diversification
- In the canonical case of conditionally independent signals, we obtain a stronger result:
  1. Maximally balanced teams are optimal
  2. Strong form of diversification of expertise

Related Literature

Matching

- Becker (1973) theory of marriage (see also Legros and Newman (2007), and assignment games)
  - Matching problem among heterogeneous men and women
  - Match output depends on their attributes
  - Transferable utility
  - Positive (negative) sorting (PAM or NAM) if supermodular (submodular) payoff
  - Both centralized solution and competitive equilibrium
- Pycia (2012) matching with peer effects
- Kelso and Crawford (1982) labor market model
- Most applications assume attributes are scalars (ordered)
- More general interpretation: stochastic sorting

Related Literature

Theory of Teams

- Marschak-Radner's (1972) theory of teams (see also Cremer (1990), Prat (2002), Lamberson and Page (2011), etc.)
  - Information decentralization: decision makers have heterogeneous information within an organization
  - Need to make decision with common goal
  - What is the optimal decision function given information?
  - Compare different information structures
- Meyer (1984) on fractional assignment
- Olzewski and Vohra (2012) on optimal composition of a team
- Unlike this literature:
  - Many teams that form instead of a team in isolation
  - Matching problem

Related Literature

Value of Information

- The paper is related to three topics in this literature:
  1. Comparison of multivariate normal experiments: Hansen and Torgersen (1974), Shaked and Tong (1990, 1992)
     - In our model, each group runs a multivariate normal experiment
     - Correlation affects informativeness of the signals
  2. Substitute and complementary signals: Borgers, Hernando-Veciana, and Krämer (2010)
     - We provide results for normally distributed signals
     - We cast model as a matching problem
  3. Value of Information: Radner and Stiglitz (1984), Chade and Schlee (2002), Moscarini and Smith (2002)
     - We exploit concavity properties of the quadratic payoff/normal signals problem
     - Shed light on extent to which results generalize

Related Literature

Partitioning Problems

- There is a recent literature in discrete optimization on partitioning problems: Chakravarty, Orlin, Rothblum (1985), Anily and Federgruen (1991), Hwang and Rothblum (2012)
  - They focus on problems that deliver consecutive partitions (similar to PAM)
  - Our model does not fit their framework: harder to solve

Model

- Agents
  - Finite set I of agents, with $|I| = kN$
  - Set of 'types' $[\underline{x}, \overline{x}]$: function $x : I \to [\underline{x}, \overline{x}]$ assigns types to agents, where $x(i) \equiv x_i$, $i = 1, 2, ..., kN$
  - $\Upsilon = \{x_1, x_2, ..., x_{kN}\}$ (multiset), assume $x_1 \leq x_2 \leq \cdots \leq x_{kN}$
  - Each agent assigned to a group of size k; there are N groups

Model

- Information
  - State of the world, prior belief $\tilde{s} \sim \mathcal{N}(\mu, \tau^{-1})$
  - Agent x_i observes signal $\tilde{\sigma}_i \sim f(\cdot|s, x_i) = \mathcal{N}(s, x_i^{-1})$
  - Informativeness: $x_i \uparrow \Rightarrow$ more informative signals (Blackwell)
  - Signals from different partners can be correlated, with pairwise covariance given by $\rho/(x_i x_j)^{0.5}$, for all i, j, $i \neq j$
  - $\rho \in (-(k-1)^{-1}, 1)$ to ensure that the covariance matrix is positive semi-definite

Model

- Actions and payoffs
  - Group observes signal realizations of all partners $\Rightarrow$ choose joint, signal-contingent action
  - Joint action $a \in \mathbb{R}$ to maximize the group's profit
  - Maximize expected value of $\pi - (a - s)^2$, where $\pi \geq 1/\tau$

Group Problem

- A group S with types $\vec{x}^S = (x_1^S, x_2^S, ..., x_k^S)$
- Choose $a : \mathbb{R}^k \to \mathbb{R}$ to maximize

$$V\left(\vec{x}^S\right) = \max_{a(\cdot)} \quad \pi - \int \cdots \int \left[a(\vec{\sigma}) - s\right]^2 f(\vec{\sigma}|s, \vec{x}^S, \rho) h(s) \prod_{i=1}^{k} d\sigma_i ds$$

  where $\vec{\sigma} = (\sigma_1, \sigma_2, ..., \sigma_k)$, and $f(\vec{\sigma}|s, \vec{x}^S, \rho)$ is the joint density of the signals generated by the members of the group (distributed according to a multivariate normal)
- Denote by $V\left(\vec{x}^S\right)$ the maximum expected payoff of group S

Matching

- Matching: partition of $\Upsilon = \{x_1, x_2, ..., x_{kN}\}$ in groups (sub-multisets) of size k
- N elements in each partition
- Transferable utility
- Optimal partition problem
    - Find partition that maximizes $\sum_S V(\vec{x}^S)$
- Remark:
    - Allocation can be decentralized if $k = 2$ (Becker (1973)) or if there is fractional assignment
    - No general result for integer assignment with $k > 2$ since Kelso and Crawford (1982) gross substitutes condition fails

Solution to the Group Problem

- After observing $\vec{\sigma}$, the posterior density function $h\left(\cdot\left|\vec{\sigma},\vec{x}^S,\rho\right.\right)$ is normally distributed
- Optimal action solves

$$\max_{a\in\mathbb{R}} \;\; \pi-\int(a-s)^2 h\left(s|\vec{\sigma},\vec{x}^S,\rho\right)ds$$

- From FOC,

$$a^\star(\vec{\sigma})=\int s\;\, h\left(s|\vec{\sigma},\vec{x}^S,\rho\right)ds=\mathbb{E}\left[\tilde{s}|\vec{\sigma},\vec{x}^S,\rho\right]$$

- Inserting $a^\star(\vec{\sigma})$ in objective function we obtain, after algebra,

$$V\left(\vec{x}^S\right)=\pi-\int\cdots\int \mathrm{Var}\left(s|\vec{\sigma},\vec{x}^S,\rho\right) f\left(\vec{\sigma}|\vec{x}^S,\rho\right)\prod_{i=1}^{k} d\sigma_i$$

where $f\left(\vec{\sigma}|\vec{x}^S,\rho\right)\equiv\int f(\vec{\sigma}|s,\vec{x}^S,\rho)h(s)ds$

Solution to the Group Problem

- Algebra is as follows: $V\left(\vec{x}^S\right)$ equals

$$
\begin{aligned}
&= \pi - \int \cdots \int \left(\mathbb{E}\left[\tilde{s}|\vec{\sigma},\vec{x}^S\right] - s\right)^2 f(\vec{\sigma}|s,\vec{x}^S,\rho)h(s)\prod_{i=1}^{k} d\sigma_i ds \\
&\overset{(2)}{=} \pi - \int \cdots \int \left(\int \left(s - \mathbb{E}\left[\tilde{s}|\vec{\sigma},\vec{x}^S\right]\right)^2 h(s|\vec{\sigma},\vec{x}^S,\rho)ds\right) f(\vec{\sigma}|\vec{x}^S,\rho)\prod_{i=1}^{k} d\sigma_i \\
&\overset{(3)}{=} \pi - \int \cdots \int \mathrm{Var}\left(s|\vec{\sigma},\vec{x}^S,\rho\right) f(\vec{\sigma}|\vec{x}^S,\rho)\prod_{i=1}^{k} d\sigma_i
\end{aligned}
$$

- where:
  - (2) from $h(s|\vec{\sigma},\vec{x}^S,\rho)f(\vec{\sigma}|\vec{x}^S,\rho) = h(s)f(\vec{\sigma}|s,\vec{x}^S,\rho)$
  - (3) replacing the expression for the variance of posterior density

Solution to the Group Problem

- Easy to compute in the conditionally independent case ($\rho = 0$)
- After observing $\vec{\sigma}$, the posterior density function:

$$h\left(\cdot \middle| \vec{\sigma}, \vec{x}^S\right) \quad \sim \quad \mathcal{N}\left(\frac{\mu\tau + \sum_{i=1}^{k} \sigma_i x_i^S}{\tau + \sum_{i=1}^{k} x_i^S}, \frac{1}{\tau + \sum_{i=1}^{k} x_i^S}\right)$$

- Notice that the variance of the posterior is independent of $\vec{\sigma}$
- Therefore

$$V\left(\vec{x}^S\right) = \pi - \left(\frac{1}{\tau + \sum_{i=1}^{k} x_i^S}\right)$$

Solution to the Group Problem

- More generally, we have the following result:

Proposition

The value function of the group problem is

$$V\left(\vec{x}^S\right) = \pi - \left(\frac{1}{\tau + \mathcal{B}(\vec{x}^S, \rho)}\right)$$

where

$$\mathcal{B}(\vec{x}^S, \rho) = \frac{(1 + (k-2)\rho)\sum_{i=1}^{k} x_i^S - 2\rho \sum_{i=1}^{k-1}\sum_{j=i+1}^{k}(x_i^S x_j^S)^{0.5}}{(1-\rho)(1+(k-1)\rho)}$$

- $\mathcal{B}(\vec{x}^S, \rho)$ is the index of informativeness
  - Higher $\mathcal{B}(\vec{x}^S, \rho)$ implies Blackwell-more-informative signals
  - Notice that $\mathcal{B}(\vec{x}^S, 0) = \sum x_i^S$

Solution to the Group Problem

- Some special cases of

$$V\left(\vec{x}^S\right) = \pi - \left(\frac{1}{\tau + \frac{(1+(k-2)\rho)\sum_{i=1}^{k} x_i^S - 2\rho\sum_{i=1}^{k-1}\sum_{j=i+1}^{k}(x_i^S x_j^S)^{0.5}}{(1-\rho)(1+(k-1)\rho)}}\right)$$

  - $\rho = 0$ yields the conditionally independent case
  - $k = 2$ yields

$$V\left(\vec{x}^S\right) = \pi - \left(\frac{1}{\tau + \frac{x_1^S + x_2^S - 2\rho(x_1^S x_2^S)^{0.5}}{(1-\rho^2)}}\right)$$

  - $x_1^S = x_2^S = \ldots = x_k^S = x$ yields

$$V\left(\vec{x}^S\right) = \pi - \left(\frac{1}{\tau + \frac{kx}{1+(k-1)\rho}}\right)$$

Solution to the Group Problem

- The proof is by induction after obtaining the general functional form of the inverse of the covariance matrix
- Sketch:
  - Start with $s \sim N(\mu, 1/\tau)$, $\sigma_1 \sim N(s, 1/x_1)$ and $s|_{\sigma_1} \sim N(\mu_1, 1/\tau_1)$
  - Show that formula holds for $k = 1$ (trivial)
  - Assume true for $k = n - 1$
  - Find $\sigma_n|_{\sigma_1,\ldots,\sigma_{n-1},s}$ and compute $s|_{\sigma_1,\ldots,\sigma_n}$
  - Show formula holds for $k = n$

Solution to the Group Problem

- A generalization of objective function:
  - Same value function if $\pi - (a - s)^n$, n even (since all odd centered moments of normal are zero)
- A generalization to a class of distributions:
  - Same value function if:
    - Restriction to actions that are weighted averages of signals
    - Joint signal distribution has mean and covariance assumed

Correlation and Informativeness

- Let $k = 2$ and $x_1^S = x_2^S = x$
  - Then $\mathcal{B}(\vec{x}^S, \rho) = \frac{2x}{(1+\rho)} \gtrless 2x = \mathcal{B}(\vec{x}^S, 0)$ if $\rho \lesseqgtr 0$
  - Negatively (positively) correlated signals are more (less) informative than conditionally independent ones
- Some 'intuition'
  - Consider first extreme cases of $\rho = \pm 1$
  - More generally, $\sigma_2|_{\sigma_1, s} \sim N\left((1-\rho)s + \rho\sigma_1, \frac{1-\rho^2}{x}\right)$
  - Correlation reduces variance of second signal
  - Negative correlation makes mean 'more sensitive' to s
- In the general case we have the following result:

Proposition (Correlation and Team Precision)

(i) If $\rho < 0$, then $\mathcal{B}(\vec{x}^S, \rho) > \mathcal{B}(\vec{x}^S, 0)$ $(\forall \vec{x}^S)$
(ii) There is a $\hat{\rho}$ s.t. if $0 < \rho < \hat{\rho}$, then $\mathcal{B}(\vec{x}^S, \rho) < \mathcal{B}(\vec{x}^S, 0)$ $(\forall \vec{x}^S)$

Optimal Matching Properties

- The main sorting properties follow from this result:

Lemma (Value Function Properties)

Consider any group S with $\vec{x}^S$:

(i) There exists an interval $(-r, r)$, where r depends on $(\underline{x}, \overline{x}, \tau, k)$ such that if $\rho \in (-r, r)$ then $V(\vec{x}^S)$ is strictly submodular in $\vec{x}^S$;

(ii) If $\rho > r$, then $V(\vec{x}^S)$ cannot be supermodular in $\vec{x}^S$, and it is strictly submodular if τ is sufficiently large;

(iii) If $\rho < -r$, then $V(\vec{x}^S)$ cannot be supermodular in $\vec{x}^S$ unless τ is sufficiently large.

- This result reveals that the value function is:
  - Submodular in many cases
  - Not supermodular in most cases

Optimal Matching Properties

- The properties of the team value function yield:

Proposition (Optimality of Diversification)

(i) Diversification within teams is always optimal for values of ρ *in a neighborhood of 0.*
(ii) Diversification is optimal on an open subset of $[\underline{x}, \overline{x}]^k$ *when* ρ *is positive, and it is always optimal if* τ *is large enough.*
(iii) Diversification is optimal when ρ *is negative, so long as* τ *is not too large.*

- This result reveals that the optimal matching:
  - Cannot be PAM except in 'rare' cases
  - Cannot exhibit two 'ordered' teams (except in 'rare' cases)
  - Exhibits 'balanced' expertise assignment across teams

Optimal Matching Properties

- Consider part (i) of the proposition:
  - V strictly submodular on $[\underline{x}, \overline{x}]^k$ for $\rho \in (-r, r)$
  - Thus, PAM does not maximize $\sum V(\vec{x}^S)$
    - If not, swap (e.g.) the best expert in one group with the worst expert in the other group
    - By strict submodularity, the objective function increases
  - Implication $\rightarrow$ the optimal matching will consist of teams with diversified composition of expertise
    - No team can have all members with uniformly higher types than any other team

Optimal Matching Properties

- Consider part (*ii*) of the proposition:
  - V cannot be supermodular on $[\underline{x}, \overline{x}]^k$ if $\rho > 0$
  - There is an open set around $x_1 = x_2 = ... = x_k$ such that V is submodular in that set
  - Thus, in that region one can do some profitable swapping if types belong to it
  - Diversification can 'sometimes' occurs (but PAM cannot occur for all multisets Υ, or for $0 < \rho < r$)
  - For each $\rho > 0$, diversification occurs if τ is large enough

Optimal Matching Properties

- Consider part (*iii*) of the proposition:
  - If $\rho < 0$, then V supermodular if τ is large enough
  - Partition generates $(\mathcal{B}_1, ..., \mathcal{B}_N)$ with 'mean' and 'variance'
  - $\mathcal{B}(\vec{x}_S, \rho)$ is supermodular in $\vec{x}_S$ when $\rho < 0$
  - Hence, PAM maximizes $\sum \mathcal{B}_S$
  - As $\tau \to \infty$, $V = \pi - \frac{1}{\tau + \mathcal{B}} \Rightarrow -V_{\mathcal{B}\mathcal{B}}/V_{\mathcal{B}} \to 0$
  - Thus planner behaves as if he maximizes $\sum \mathcal{B}_S$ when τ is large enough, i.e., PAM is optimal
  - (Similar intuition applies to $\rho > 0$ and τ large when $k = 2$)
- Remark:
  - How large should τ be for PAM?
  - When $k = 2$, for each $\rho > 0$, τ should be strictly bigger than $16\overline{x}$ (eight times the precision of the best team possible)
    - Thus, information about s is 'very' precise to begin with
  - If $\tau \leq 16\overline{x}$, V is strictly submodular and diversification ensues

Optimal Matching Properties

- We can say more if $k = 2$
  - $V(\cdot)$ submodular $\Rightarrow$ Negative Assortative Matching
  - If types $x_1 > x_2 \geq x_3 > x_4$, then total payoff maximized if $\{x_1, x_4\}$ and $\{x_2, x_3\}$
  - x_4 can outbid x_2 and x_3 when competing for x_1
  - For $k = 2$, optimal matching is straightforward:
    1. x_1 with x_{2N}
    2. x_2 with x_{2N-1}

       ...

    N. x_N with x_{N+1}
  - Similarly for PAM and supermodularity
  - Optimal matching can be decentralized as outcome of Walrasian Equilibrium

The Conditional Independent Case (CIC)

- From now on we focus on the canonical case with conditionally independent signals
- Recall that group value functions in this case is

$$V\left(\vec{x}^S\right) = \pi - \left(\frac{1}{\tau + \sum_{i=1}^{k} x_i^S}\right)$$

- Consistent with previous result, it is strictly submodular in $\vec{x}^S$
- Since $V(\cdot)$ depends on $\vec{x}^S$ only through the sum, we define

$$v\left(\sum x_i^S\right) \equiv V\left(\vec{x}^S\right)$$

CIC: Discrete Assignment

- Given the value of a group $v\left(\sum x_i^S\right)$, what is the optimal sorting of agents into groups?
- That is, we want the partition that maximizes $\sum_S v\left(\sum x_i^S\right)$
  - Sorting with $k > 2$ is much more complex
  - $v\left(\sum x_i^S\right)$ submodular $\Rightarrow$ PAM not optimal
  - But exact sorting pattern is not obvious
  - Clear: optimal matching entails diversification within groups $\Rightarrow$ balanced teams

CIC: Discrete Assignment

- Notice that:
  - Every partition has the same sum $\sum_{S=1}^{N}\sum_{i=1}^{k} x_i^S = X$
  - Objective function is (strictly) Schur-concave on vector of groups precision (sums of members' precisions) partially ordered by majorization
    - $x = (x_1, x_2, ..., x_N)$ majorizes $x' = (x_1', x_2', ..., x_N')$ $(x \succ x')$ if $\sum_{\ell=1}^{m} x_{[\ell]} \geq \sum_{\ell=1}^{m} x'_{[\ell]}$ for all m, with $\sum_{\ell=1}^{N} x_{[\ell]} = \sum_{\ell=1}^{N} x'_{[\ell]}$
    - $x_{[\ell]}$ is the ℓ-th largest coordinate of the vector x
    - Majorization can be thought of as a notion of similarity/dispersion of vectors
    - $f : \mathbb{R}^k \to \mathbb{R}$ is Schur concave if $x \succ x'$ implies $f(x') \geq f(x)$
    - $f(x_1, x_2, ..., x_N) = \sum_{i=1}^{N} g(x_i)$ is strictly Schur concave if g is strictly concave

CIC: Discrete Assignment

- Optimal matching in CIC:

Proposition (Maximally Balanced Teams)

Assume conditionally independent signals.

(i) *The optimal matching must be an element of the set of partitions whose team precision vectors* $(X_1, X_2, ..., X_N)$ *are majorized by those generated by all the remaining partitions.*

(ii) *If a team precision vector is majorized by the precision vectors of all the feasible partitions of the agents, then its associated partition is the optimal matching.*

CIC: Discrete Assignment

- If there is a partition that is majorized by all the other ones, then it is optimal
  - *Solution is a partition with 'lowest spread' in group precision* $\rightarrow$ *maximum diversification*
  - Clearly, if there is a partition with $\sum x^S = X/N$ for all S, this is the solution
  - Still need to prove that there is a 'minorizing' partition
  - True if $N = 2$ or $k = 2$
- Weaker: Find partitions majorized by the remaining ones
  - Solution is in the set containing those partitions
- Proposition holds for a class of matching problems with $v(\sum_i x_i)$ strictly Schur concave

CIC: Discrete Assignment

- Is there an algorithm to find the optimal partition?
  - Hard problem in general (except for $k = 2$)
  - With $N = 2$, equivalent to NP-hard 'number partitioning problem' (Garey and Johnson (1978), Mertens (2006))
  - With $N = 3$, equivalent to strong NP-complete 3-partition problem (Garey and Johnson (1978))
  - Related: problem is a variant of submodular welfare maximization problem (Vondrak (2007)), which is NP-hard
    - Vondrak (2007) describe an approximation algorithm that captures $1 - 1/e$ of optimal value
- Example of failure of a greedy algorithm:
  - 8 agents with types are $1, 3, 6, 10, 12, 15, 20, 23$; $N = 2$
  - Greedy: $\{23, 12, 10, 1\}$, $X_1 = 46$; $\{20, 15, 6, 3\}$, $X_2 = 44$
  - Optimal: $\{23, 15, 6, 1\}$, $X_1 = 45$: $\{20, 12, 10, 3\}$, $X_2 = 45$
  - Flexible k does not solve the problem

CIC: Fractional Assignment

- No integer restriction
- Agents can be fractionally assigned to multiple groups
  - Agent's precision proportionally re-scaled according to fraction assigned to the group
- Interpretation: Time dedication to a group given time 'budget'
- Another interpretation: Approximates discrete solution with large but finite teams
- Assumption: s independent across teams (to avoid information 'spillovers')

CIC: Fractional Assignment

- $x(I) = \{x_1, ..., x_J\}$ set of *distinct* types; m_j # type x_j
- Let $X = \sum_{j=1}^{J} m_j x_j$ and $\sum_{j=1}^{J} m_j = kN$
- $\mu_{jn} \geq 0$ the fractional assignment of type-j agents to group n
- The optimal fractional assignment problem solves:

$$
\begin{aligned}
\max_{\{\mu_{jn}\}_{j,n}} \quad & \sum_{n=1}^{N} v\left(\sum_{j=1}^{J} \mu_{jn} x_j\right) \\
\text{s.t.} \quad & \sum_{n=1}^{N} \mu_{jn} = m_j \quad \forall j \\
& \sum_{j=1}^{J} \mu_{jn} = k \quad \forall n \\
& \mu_{jn} \geq 0 \quad \forall j, n
\end{aligned}
$$

CIC: Fractional Assignment

- Let $X_n \equiv \sum_{j=1}^{J} \mu_{jn} x_j \Rightarrow \sum_{n=1}^{N} X_n = X$.
- The 'relaxed problem':

$$\max_{\{X_n\}_{n=1}^{N}} \quad \sum_{n=1}^{N} v(X_n)$$
$$\text{s.t.} \quad \sum_{n=1}^{N} X_n = X$$

- Solution is $X_n = \frac{X}{N}, \forall n$
- Maximal balance achieved: Equal precision teams

CIC: Fractional Assignment

- If a feasible $\{\mu_{jn}\}$ implements the solution to the relaxed problem, then it solves the original problem.
- We have the following result:

Proposition (Perfect Diversification)

Any solution to the fractional assignment problem equalizes X_n across all groups, i.e., entails maximum diversification.
It can be implemented using $\mu_{jn} = m_j/N$ for all j, n.

- Solution is not unique except when $J = 2$
- Example: $1, 2, 3, 4, 5, 5$
    1. $\left\{\frac{1}{2}1, \frac{1}{2}2, \frac{1}{2}3, \frac{1}{2}4, \frac{2}{2}5\right\} \Rightarrow$ both: $\pi - \frac{1}{\tau+10}$
    2. $\{1, 4, 5\}, \{2, 3, 5\} \Rightarrow$ both: $\pi - \frac{1}{\tau+10}$
- Unique symmetric solution

CIC: Firm Heterogeneity

- N heterogeneous firms $y_1 \leq y_2 \leq ... \leq y_N$
- Match payoff when firm y_S matches with team $\sum x_i^S$ is

$$y_S \cdot v\left(\sum x_i^S\right)$$

- Notice that match payoff is supermodular in $(y_S, \sum x_i^S)$
- Optimal matching properties are:
    - PAM between firms and groups (better firms match with better teams)
    - Diversity within groups (heterogeneous experts in each group)

CIC: Firm Heterogeneity

- We can pin down exact solution with fractional assignment
- Optimal assignment of firms to groups solves

$$\max_{\{X_n\}_{n=1}^{N}} \sum_{n=1}^{N} y_n v\left(X_n\right)$$

$$\text{s.t.} \sum_{n} X_n = X$$

- From FOC, we obtain

$$X_n = \frac{y_n^{0.5}}{\sum_{n=1}^{N} y_n^{0.5}} \left(\tau N + X\right) - \tau$$

CIC: Firm Heterogeneity

- We have the following result:

Proposition (Heterogeneous Firms)

Firms with higher types are matched with higher precision teams ($X_1 \leq X_2 \leq ... \leq X_N$).

An increase in $\tau \uparrow X_n$ iff $n \geq n^*$, and $\uparrow X_n - X_{n-1}$ for all n

An increase in $X \uparrow X_n$ and $\uparrow X_n - X_{n-1}$ for all n

An increase in the spread of $(y_1, y_2, ..., y_N)$ increases (decreases) the precision of teams above (below) $1 \leq \hat{n} \leq N$

There is a fractional assignment rule $\{\mu_{jn}\}$ that implements the optimal $(X_1, X_2, ..., X_N)$

CIC: Firm Heterogeneity

- Example:
  - $N = 2$, $y_1 = 1$, $y_2 = y \geq 1$, $\tau = 0$, $\Upsilon = \{5, 5, 20, 20\}$
  - Then

$$X_2 = \frac{y^{0.5}}{y^{0.5} + 1} X$$

$$X_1 = \frac{1}{y^{0.5} + 1} X$$

  - Easy to show that
    - If $y = 1$ then $X_2 = X_1$ and NAM, i.e. $\{5, 20\}, \{5, 20\}$
    - If $y = 16$ then $X_2 = 40$, $X_1 = 10$, and PAM, i.e., $\{5, 5\}, \{20, 20\}$
    - If $1 \leq y < 16$ there is diversification within groups
    - Fractional assignment rule: E.g., $\mu_{20,2} = 1 + (1/15)(X_2 - 25)$

CIC: Fractional Assignment

Endogenous k

- One way to endogeneize k is to assume a cost function $c(N)$ that is strictly increasing and convex in N
  - FOC that determines optimal N is

$$\pi - c'(N) = N(N\tau + X)^{-1} + NX(N\tau + X)^{-2}$$

  - This yields N; since I is fixed, $k = |I|/N$ is determined
- Another way to endogenize k is as follows:
  - Assume that identical firms are locations where groups of size k form (reinterpretation) under symmetric solution
  - Free entry of firms, entry cost $F > 0$, wages w_j
  - For any N, $w_j = v'(X_n)x_j$, where $v'(X_n) = (\tau + X_n)^{-2}$
  - Zero profit condition, $X_n = X/N$, and $\sum_j m_j x_j = X$ yield

$$\pi - F = N(N\tau + X)^{-1} + NX(N\tau + X)^{-2}$$

CIC: Fractional Assignment

Endogenous k

- Either way of endogeneizing k yields the following result

Proposition (Endogenous Group Size)

There exists a unique value of N and thus of $k = |I|/N$.

The equilibrium group size $k \downarrow$ in τ, and $\downarrow$ in X

CIC: Decentralized Market Solution

- Thus far, we have focused almost exclusively on the optimal matching problem
- If $k = 2$, then it is easy to decentralize the model if we think of it as a two-sided matching problem
    - Standard results from assignment games (nonempty core, competitive equilibrium existence)
- For $k > 2$, we do not have a decentralization result:
    - Consider matching groups of experts with identical firms
    - Problem does not satisfy gross substitutes condition of Kelso and Crawford (1982) (or Gul and Stacchetti (1999))
- With fractional assignment, easy decentralization result

CIC: Decentralized Market Solution

Failure of Gross Substitutes Condition

- The firm solves:

$$\max_{A \subseteq \mathcal{I}} v\left(\sum_{i \in A} x_i\right) - \sum_{i \in A} w(x_i)$$

- Let $D(w)$ be the set of solutions
  - GS: If $A^* \in D(w)$ and $w' \geq w$, then there is a $B^* \in D(w')$ such that $T(A^*) \subseteq B^*$, $T(A^*) = \{i \in A^* | w(x_i) = w'(x_i)\}$.
- The following example shows that GS fails in our model.
  - Firm and experts 1, 2, and 3, with $x_1 = 1$, $x_2 = 2$, and $x_3 = 3$
  - Assume $\pi = \tau = 1$
  - If $w = ((1/12) - \varepsilon, 1/12, 1/6)$, then $A^* = \{1, 2\}$
  - Let $w = ((1/12) - \varepsilon, 1/6, 1/6)$, so that only the wage of expert 2 has increased. If $\varepsilon < 1/30$, then $B^* = \{3\}$
  - Hence, GS fails since $T(A^*) = \{1\} \not\subseteq B^*$
  - Same example shows that if the firm were constrained to hire at most two experts, GS would still fail

CIC: Decentralized Market Solution

- With fractional assignment, given prices w_j, firm n solves:

$$\begin{aligned} \max_{\mu_{jn}} \quad & v\left(\sum_j \mu_{jn} x_j\right) - \sum_j \mu_{jn} w_j \\ \text{s.t.} \quad & \sum_j \mu_{jn} = k \end{aligned}$$

- There are J first order conditions for all N firms:

$$\text{FOC}_{jn} : v'\left(\sum_j \mu_{jn} x_j\right) x_j - w_j + \phi_n = 0, \forall j, n$$

- Coincides with the planner's solution

$$\text{FOC}_{jn} : v'\left(\sum_j \mu_{jn} x_j\right) x_j + \lambda_j + \phi_n = 0, \forall j, n$$

Discussion

Modeling Assumptions

- How restrictive is the normal/quadratic-payoff group problem?
  - Affords analytical solutions
  - Widely used (teams a la Marschak-Radner, global games, etc.)
  - Beyond this set up:
    - Binary case
    - Other canonical model with submodular value function: probabilistic information arrival
    - Stumbling block: Non-concavity in value of information (Radner-Stiglitz (1984), Chade-Schlee (2002))
- How general are our matching results?
  - More general than problem of matching information
  - The CIC shows that it holds for any matching problem with V concave in the sum of types $\sum_i x_i^S$

Discussion

Stochastic Sorting

- Matching information is an example of stochastic sorting
  - Type is a family $\{f(\sigma_i|s, x_i)\}_s$ indexed by x_i
  - Blackwell informativeness orders the distributions
  - 'Matching distributions'
- Other problems with stochastic sorting
- Becker (1973) with uncertainty: $f(\sigma_i|x_i)$, $v(\vec{\sigma})$
  - $f(\sigma_i|\cdot)$ ordered by FOSD
  - $f(\sigma_i|\cdot)$ ordered by MPS
  - Results for this case
- Legros-Newman (2007) with NTU and uncertainty
  - Risk sharing problems with FOSD and MPS
  - Examples and some results

Concluding Remarks

- Matching where information is the sorting characteristic
- Competition for members $\Rightarrow$ diversification of expertise
- Maximally balanced teams
- Extensions:
  1. More general framework
  2. Algorithms
  3. Allowing for different group sizes
     - $\rho = 0$, six agents, 2, 2, 7, 7, 8, 10
     - If $k = 3$, then $\{2, 7, 10\}, \{2, 7, 8\}$, with $X_1 = 19$ and $X_2 = 17$
     - If different sizes allowed, then $\{2, 2, 7, 7\}, \{8, 10\}$, with $X_1 = X_2 = 18$, a strict improvement
     - Not an issue with fractional assignment
  4. Non transferable utility
  5. Decentralization without gross substitutes