Asset Pricing with Heterogeneous Preferences

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Motivation

- Consumption-based capital asset pricing model (cCAPM) of Lucas (1978) and Breeden (1979) does not perform well empirically (*e.g.*, Hansen & Singleton 1982, 1983).
- Equity premium puzzle (Mehra & Prescott, 1985) and risk-free rate puzzle (Weil, 1989) are widely viewed as unresolved.
- Some researchers tested heterogeneous-agent models (Brav, Constantinides, & Geczy 2002, Cogley 2002, Balduzzi & Yao 2007, Kocherlakota & Pistaferri 2009), but results are mixed. Toda & Walsh (2012) reject all specifications.

Questions

1. Almost all studies assume common (typically additive CRRA) preferences across agents, which may well be misspecified. Can we find a stochastic discount factor (SDF) that is robust to model misspecification?
2. If so, how does the SDF perform empirically?

Contribution

1. Characterize the general equilibrium with many agents with heterogeneous (time-, state-, and individual-dependent) preferences with common relative risk aversion γ.

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2. Show that $-\gamma$th power of market return is a valid SDF and hence consumption is irrelevant for asset pricing.

Contribution

1. Characterize the general equilibrium with many agents with heterogeneous (time-, state-, and individual-dependent) preferences with common relative risk aversion γ.
2. Show that $-\gamma$th power of market return is a valid SDF and hence consumption is irrelevant for asset pricing.
3. Estimate and test the model: $\gamma \approx 2$ explains the historical equity premium and risk free rate, and model not rejected.
4. Argue that asset pricing puzzles are puzzles in macroeconomics, not finance.

Literature

cAPM and tests Lucas (1978), Breeden (1979), Hansen & Singleton (1982, 1983).

Tests of heterogeneous-agent cCAPM Brav, Constantinides, & Geczy (2002), Cogley (2002), Balduzzi & Yao (2007), Kocherlakota & Pistaferri (2009), Toda & Walsh (2012).

Asset pricing puzzles and reviews Mehra & Prescott (1985), Weil (1989), Kocherlakota (1996), Campbell (2003).

Asset pricing without consumption Rubinstein (1976), Brown & Gibbons (1985), Campbell (1993).

Example

- Two period model. Agents have identical additive CRRA preferences
$$\frac{1}{1-\gamma}(c_0^{1-\gamma} + \beta \, \mathsf{E}[c_1^{1-\gamma}]).$$
- We all know that MRS
$$\beta \left(\frac{c_1}{c_0}\right)^{-\gamma}$$
is a valid stochastic discount factor (SDF).

Optimal consumption-portfolio problem

- Initial wealth w, no future endowment.
- Assets $k = 1, \dots, K$. ϕ^k: portfolio share (fraction invested in each asset, so $\sum_k \phi^k = 1$).
- R^k: gross return, $R(\phi) := \sum_k R^k \phi^k$ total return on portfolio.
- Budget constraint: $c_1 = R(\phi)(w - c_0)$.
- Optimal consumption-portfolio problem:

$$\max_{c_0,\phi} \frac{1}{1-\gamma}(c_0^{1-\gamma} + \beta \operatorname{E}[c_1^{1-\gamma}])$$
$$\text{s.t. } c_1 = R(\phi)(w - c_0).$$

Solution

- Problem breaks into two parts:

$$\text{Optimal portfolio:} \quad \frac{F}{1-\gamma} := \max_{\phi} \frac{1}{1-\gamma} \mathrm{E}[R(\phi)^{1-\gamma}],$$

$$\text{Optimal consumption:} \quad \max_{c} \frac{1}{1-\gamma} (c^{1-\gamma} + \beta F (w-c)^{1-\gamma}).$$

- By FOC of optimal portfolio problem,

$$\mathrm{E}[R(\phi^*)^{-\gamma}(R^k - R(\phi^*))] = 0$$

for any asset k, where ϕ^*: optimal portfolio.

- Hence $R(\phi^*)^{-\gamma}$ (times constant) is also a valid SDF.

Robustness to misspecification

- SDF $\beta(c_1/c_0)^{-\gamma}$ is not robust to model misspecification because it is valid only for additive CRRA preference.
- SDF $R(\phi^*)^{-\gamma}$ is robust because valid for any homothetic CRRA preference.
- Hence "$R(\phi^*)^{-\gamma}$ is valid" is a weaker null hypothesis than "$\beta(c_1/c_0)^{-\gamma}$ is valid".
- Furthermore, tests of $R(\phi^*)^{-\gamma}$ do not require consumption data (which is subject to measurement errors, especially in micro data): asset returns data are sufficient.

Agents

- $i \in I = \{1, \dots, I\}$: agent. $t = 0, 1, \dots, T$: time.
- $(\Omega, \mathcal{F}, P)$: probability space. $\{\mathcal{F}_{it}\}_{t=0}^{T} \subset \mathcal{F}$: filtration representing information of agent i at each period. $\mathcal{F}_t = \bigcap_i \mathcal{F}_{it}$: public information.
- Time t state variables other than wealth: $\mathbf{X}_{it} = (X_{it}^1, X_{it}^2, \dots)$.
- (Individual-, time-, and state-dependent) recursive preferences:

$$U_{it} = f_{it}\left(c_{it}, \mathcal{M}_{it}(U_{i,t+1}), \mathbf{X}_{it}\right), \quad t = 0, \dots, T-1,$$

  where $\mathcal{M}_{it}(U_{i,t+1})$ is certainty equivalent of next period's utility.

Example 1

Example (Additive CRRA utility)

If $a_T = 1$ and the aggregator and certainty equivalent are given by

$$f_t(c, v, \mathbf{X}) = (c^{1-\gamma} + \beta v^{1-\gamma})^{\frac{1}{1-\gamma}},$$

$$\mathcal{M}_t(U_{t+1}) = \mathrm{E}\left[U_{t+1}^{1-\gamma} \,\middle|\, \mathcal{F}_t \right]^{\frac{1}{1-\gamma}},$$

then

$$U_t = \mathrm{E}\left[\sum_{s=t}^{T} \beta^{t-s} c_s^{1-\gamma} \,\middle|\, \mathcal{F}_t \right]^{\frac{1}{1-\gamma}},$$

the standard additive CRRA utility with discount factor β and relative risk aversion γ.

Example 2

Example (Recursive CRRA/CEIS utility)

If $a_T = 1$ and the aggregator and certainty equivalent are given by

$$f_t(c, v, \mathbf{X}) = (c^{1-\sigma} + \beta v^{1-\sigma})^{\frac{1}{1-\sigma}},$$

$$\mathcal{M}_t(U_{t+1}) = \mathrm{E}\left[U_{t+1}^{1-\gamma} \,\middle|\, \mathcal{F}_t\right]^{\frac{1}{1-\gamma}},$$

then U_t is the Epstein-Zin (1989) constant relative risk aversion (CRRA), constant elasticity of intertemporal substitution (CEIS) recursive utility with discount factor β, relative risk aversion γ, and elasticity of intertemporal substitution $1/\sigma$.

Example 3

Example (Habit formation)

If

$$f_t(c, v, x) = \left[(c/x)^{1-\sigma} + \beta v^{1-\sigma}\right]^{\frac{1}{1-\sigma}},$$

where $x_t = c_{t-1}$ is the consumption of the previous period, then we get habit formation (Abel, 1990). In this case the agent gets utility not from consumption but from consumption *growth*.

Assets

- Assets: $k \in K = \{1, \ldots, K\}$, with price P_t^k, dividend D_{t+1}^k, market capitalization W_t^k, and return $R_{t+1}^k = (P_{t+1}^k + D_{t+1}^k)/P_t^k$.
- $\phi \in \Pi_t \subset \mathbb{R}^K$, $\sum \phi_k = 1$: portfolio with return

$$R_{t+1}(\phi) = \sum_{k=1}^{K} R_{t+1}^k \phi^k.$$

- Budget constraint is $w_{i,t+1} = R_{t+1}(\phi_{it})(w_{it} - c_{it}) \geq 0$.

Sequential partial equilibrium

Definition

Given asset prices, dividends, and market capitalization $\left\{(P_t^k, D_t^k, W_t^k)_{k\in K}\right\}_{t=0}^T$, the profile of individual consumption, wealth, and portfolio $\{(c_{it}, w_{it}, \phi_{it})_{i\in I}\}_{t=0}^T$ constitutes a sequential partial equilibrium if

1. given asset returns $R_{t+1}^k = (P_{t+1}^k + D_{t+1}^k)/P_t^k$, (c_{it}, ϕ_{it}) solves the optimal consumption-portfolio problem,
2. asset markets clear, *i.e.*, for each asset k and time t we have $\sum_{i=1}^I \phi_{it}^k (w_{it} - c_{it}) = W_t^k$, and
3. individual wealth evolves according to the budget constraint
$$w_{i,t+1} = R_{t+1}(\phi_{it})(w_{it} - c_{it}).$$

- Can also formulate GE (see paper).

Assumption 1

Assumption (Irrelevance of wealth)

For any $\mathcal{F}$-measurable function f, we have

$$\mathrm{E}\left[f(\mathbf{X}_{i,t+1}) \,|\, \mathcal{F}_{it}\right] = g(\mathbf{X}_{it})$$

for some g, that is, agent's wealth is irrelevant for predicting a function of next period's state variables other than wealth.

- Essentially, agents are price takers.

Assumption 2

Assumption (Homothetic CRRA preference)

The continuation utilities $\{U_{it}\}_{t=0}^{T}$ *satisfy the recursion* $U_{iT} = a_{iT}(\mathbf{X}_{iT})c_{iT}$ *and*

$$U_{it} = f_{it}\left(c_{it}, \mathrm{E}\left[U_{i,t+1}^{1-\gamma} \,\middle|\, \mathcal{F}_{it}\right]^{\frac{1}{1-\gamma}}, \mathbf{X}_{it}\right), \quad t = 0, \dots, T-1,$$

where $a_{iT} > 0$ *is some function of the state variables* $\mathbf{X}_{iT}$, $\gamma > 0$ *is the common relative risk aversion coefficient, and the aggregator*

$$f_{it} : \mathbb{R}_+ \times \mathbb{R}_+ \times \mathbb{R}^{\dim \mathbf{X}_{it}} \to \mathbb{R}_+$$

is increasing and homogeneous of degree 1 in the first two arguments.

- γ may even be stochastic (γ_t), but for now assume constant.

Assumption 3

Let $\mathbf{R}_{t+1} = (R^k_{t+1})_{k \in K}$ be asset returns.

Assumption (Conditional independence)

For each i, t, next period's state variables $\mathbf{X}_{i,t+1}$ and asset returns $\mathbf{R}_{t+1}$ are independent conditional on time t information $\mathcal{F}_{it}$.

- In particular, $\mathbf{R}_t \notin \mathbf{X}_{it}$, but $\mathbf{X}_{it}$ may contain volatility.
- Example: $\mathbf{X}_{it} = (\mu_t, \Sigma_t)$ consists of expected return and variance matrix, which evolves according to some stochastic process; $\log \mathbf{R}_{t+1} \sim N(\mu_t, \Sigma_t)$ independent of past information.)
- Thus GARCH returns are allowed.

Assumption 4

Assumption (Efficient market hypothesis)

For each i and t, the distribution of asset returns $\mathbf{R}_{t+1}$ conditional on private information $\mathcal{F}_{it}$ is the same as the distribution conditional on public information $\mathcal{F}_t$.

Definition taken from Bewley (1982).

Individual decision

Agents maximize recursive utility subject to budget constraint:

$$V_{it}(w, \mathbf{X}_{it}) = \sup_{\substack{c \geq 0 \\ \phi \in \Pi_t}} f_t \left(c, \mathrm{E}\left[V_{i,t+1}(w', \mathbf{X}_{i,t+1})^{1-\gamma} \,\middle|\, \mathcal{F}_{it} \right]^{\frac{1}{1-\gamma}}, \mathbf{X}_{it} \right)$$

subject to $w' = R_{t+1}(\phi)(w - c)$.

Theorem

Under Assumptions 1–4, the value function is linear in wealth w, $V_{it}(w, \mathbf{X}_{it}) = a_{it}(\mathbf{X}_{it})w$, and the optimal portfolio problem at time t reduces to

$$\sup_{\phi \in \Pi_t} \frac{1}{1-\gamma} \mathrm{E}\left[R_{t+1}(\phi)^{1-\gamma} \,\middle|\, \mathcal{F}_t \right],$$

which is common across all agents. Hence "market" portfolio is individually optimal.

Proof.

Trivial if $t = T$. If true for $s = t+1, \dots, T$, then

$$
\begin{aligned}
& V_{it}(w, \mathbf{X}_{it}) \\
&= \sup_{\substack{0 \le c \le w \\ \phi \in \overline{\Pi}_t}} f_{it}\left(c, (w-c)\,\mathrm{E}_{it}[a_{i,t+1}(\mathbf{X}_{i,t+1})^{1-\gamma} R_{t+1}(\phi)^{1-\gamma}]^{\frac{1}{1-\gamma}}, \mathbf{X}_{it}\right) \\
&= \sup_{\substack{0 \le c \le w \\ \phi \in \overline{\Pi}_t}} f_t\left(c, (w-c) b_{it}(\mathbf{X}_{it})\,\mathrm{E}_{it}[R_{t+1}(\phi)^{1-\gamma}]^{\frac{1}{1-\gamma}}, \mathbf{X}_{it}\right) \\
&= \sup_{0 \le \tilde{c} \le 1} w f_t\left(\tilde{c}, (1-\tilde{c}) b_{it}(\mathbf{X}_{it}) \sup_{\phi \in \Pi_t} \mathrm{E}_t[R_{t+1}(\phi)^{1-\gamma}]^{\frac{1}{1-\gamma}}, \mathbf{X}_{it}\right),
\end{aligned}
$$

where $b_{it}(\mathbf{X}_{it}) := \mathrm{E}_{it}[a_{i,t+1}(\mathbf{X}_{i,t+1})^{1-\gamma}]^{\frac{1}{1-\gamma}}$ and $\tilde{c} = c/w$. □

Asset pricing in partial equilibrium

Theorem

Let everything be as in Theorem. Let $\mathcal{F} = \mathcal{F}_t$ be time t information and $R_m = R_{t+1}(\phi_t^*)$ be the return on the "market" portfolio. If the portfolio constraint $\phi \in \Pi_t$ does not bind at the market portfolio ϕ_t^* for asset k, then

$$\mathrm{E}\left[R_m^{-\gamma}(R^k - R_m) \middle| \mathcal{F}\right] = 0.$$

In particular, the one period risk-free rate is

$$R_f = \frac{\mathrm{E}\left[R_m^{1-\gamma} \middle| \mathcal{F}\right]}{\mathrm{E}\left[R_m^{-\gamma} \middle| \mathcal{F}\right]}.$$

Implications

- Consumption is irrelevant for asset pricing.
- Relative risk aversion γ can be estimated using only asset returns data by GMM, using moment condition $\mathrm{E}\left[R_m^{-\gamma}(R^k - R_m)\,\middle|\,\mathcal{F}\right] = 0.$
- Hence rejection of a particular model using Euler equation is a rejection of the particular structure of the model, not necessarily the asset pricing implications, and
- "consumption volatility puzzle" (volatility of consumption growth is low relative to stock market volatility) is not an asset pricing puzzle.

GMM estimation

- Data: monthly returns on stock portfolio (10 size portfolios or Fama-French 25 portfolios) and risk-free rate for 1926–2011 from CRSP. Assume "market return" = value weighted average stock market return.
- $u_t(\gamma) = R_{m,t}^{-\gamma}(\mathbf{R}_t - R_{m,t}\mathbf{1}_K) \otimes \mathbf{z}_t$: pricing error at t, where $\mathbf{R}_t$: vector of real returns, $\mathbf{z}_t$: vector of instruments (constant and/or previous year's dividend yield).
- $g_T(\gamma) = \frac{1}{T}\sum_{t=1}^{T} u_t(\gamma)$: average pricing error.
- GMM estimation minimizes $g_T(\gamma)' W g_T(\gamma)$, where W is weighting matrix.

Results

Table: GMM estimation results of $\mathrm{E}[R_m^{-\gamma}(R^k - R_m)] = 0$.

Test assets	10 size portfolios (S10)		Fama-French 25 (FF25)	
Period	1926–2011		1931–2011	
Conditional?	no	yes	no	yes
RRA, $\hat{\gamma}$	2.0	2.15	2.05	1.55
S.E.	0.65	0.64	0.58	0.52
T	1032	1032	966	966
# moments	11	22	26	52
P (J test)	0.123	0.125	0.0015	0.106
P (Subsampling)	0.072	0.20	0.21	0.12

Asset pricing puzzles

- Equity premium and risk-free rate explained by $\gamma \approx 2$, hence no equity premium puzzle (Mehra & Prescott, 1985) or risk-free rate puzzle (Weil, 1988).
- How about consumption volatility puzzle (volatility of consumption growth is low relative to stock market volatility)?

Model

- Representative agent model with two technologies.
- Preference additive CRRA in luxury, $\mathrm{E}_0 \sum_{t=0}^{\infty} \beta^t \frac{l_t^{1-\gamma}}{1-\gamma}$. (See also Aït-Sahalia, Parker, & Yogo (2004).)
- Necessity $n_t = (1+g)^t n_0$ grows at constant rate. $c_t = n_t + l_t$: total consumption.
- One technology is risky with productivity $\log A \sim N(\mu, \sigma^2)$, the other is risk-free with return $1 + r$. $R(\theta) = A\theta + (1+r)(1-\theta)$: total return.
- $w_0 = k_0 - \sum_{t=0}^{\infty} \frac{n_t}{(1+r)^t}$: initial effective wealth, where k_0: initial capital.

Calibration

- Calibrate economy at quarterly frequency for 15 years.
- Annual values are $\beta = 0.96$, $\gamma = 3$, $g = 0$, $\mu = 0.07$, $\sigma = 0.17$, $r = 0.01$.
- Choose n_0 such that $w_0/k_0 = 0.1$. (Effective wealth is 10% of initial capital.)
- Then optimal portfolio $\theta^* = 0.8593$, propensity to consume luxury out of effective wealth $= 0.0103$.

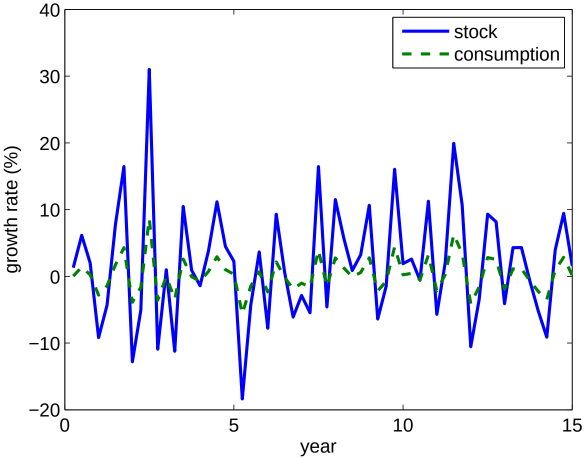

Figure: Typical sample paths of stock market return (blue solid) and consumption growth (green dashed).

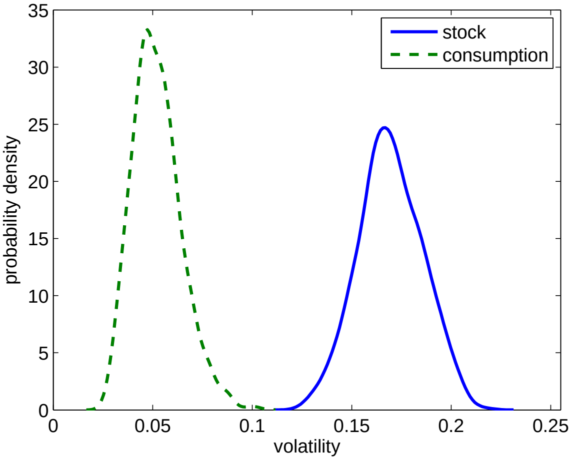

Figure: Kernel density estimate of the distribution of sample volatility of stock market return (blue solid) and consumption growth (green dashed) for 1,000 simulations.

Conclusion

1. Showed that $-\gamma$th power of market return is a valid stochastic discount factor under very general conditions and hence robust to model misspecification.
2. Estimated $\gamma \approx 2$ and model not rejected. Hence no equity premium puzzle or risk-free rate puzzle.
3. Since consumption is irrelevant for asset pricing, "consumption volatility puzzle" belongs to macroeconomics (consumption/saving puzzle), not finance.
4. Introduction of luxury resolves consumption volatility puzzle.