Applying Negishi's method to stochastic models with overlapping generations*

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Abstract

In this paper we develop a Negishi approach to characterize recursive equilibria in stochastic models with overlapping generations. When competitive equilibria are Pareto-optimal, using Negishi-weights as a co-state variable has three major computational advantages over the standard approach of using the natural state: First, the endogenous state space is a unit simplex and thus easy to handle. Second, the number of unknown functions characterizing equilibrium dynamics is orders of magnitude smaller. Third, approximation errors have a compelling economic interpretation.

Our main contribution is to show that the Negishi approach extends naturally to models with borrowing-constraints and incomplete financial markets where the welfare theorems fail. Many of the computational advantages carry over to this setting. We derive sufficient conditions for the existence of Markov equilibria in the complete markets model as well as for models with incomplete markets and borrowing constraints.

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1 Introduction

In stochastic infinite horizon exchange economies with overlapping generations and complete financial markets the welfare theorems hold if one assumes that there is a Lucas tree in unit net supply. Thus, Negishi's (1960) approach to characterize equilibrium allocations as the solution to a social planner's problem can be employed (see Kehoe et al. (1992)). Unfortunately, Negishi's approach is typically thought of as useful only when the number of commodities is larger than the number of agents and in the OLG model both the number of agents and the number of commodities are countably infinite. Moreover, in the presence of borrowing constraints or when financial markets are incomplete, competitive equilibria fail to be Pareto-efficient and Negishi's method appears no longer to be applicable. In this paper we show that using Negishi's method to compute equilibria in OLG models can result in large efficiency gains compared to conventional methods that approximate recursive equilibria on a natural state space, e.g. agents' portfolios. We also show that Negishi's method can still be applied in the presence of borrowing constraints and incomplete markets. We consider an OLG exchange economy with L perishable commodities and Markovian fundamentals. Each period H agents enter the economy, they live for A periods and maximize time-separable expected utility. We start with a model with complete financial markets where equilibrium allocations are Pareto-efficient and can be obtained as a solution to a planner's problem that maximizes the sum of all agents' utility weighted by Pareto-weights that ensure budget-balance for each agent. At each node of the tree we define *instantaneous* Negishi weights as the discounted weights associated with agents currently alive. Since utility is time- and state-separable, individuals' consumption at each node is a simple function of the instantaneous Negishi-weights. As a second step we show how the approach extends naturally to debt-constrained models with possibly incomplete financial markets where the welfare theorems fail. We argue that it is still advantageous to use instantaneous Negishi-weights as the endogenous state variable. Obviously, they can no longer be interpreted as welfare weights for a social planner's problem, but they are still a sufficient statistic for current consumption.

Models with overlapping generations have widespread applications in public finance, macroeconomics, and finance (see e.g. Kotlikoff and Auerbach (1983) or Storesletten et al. (2007)). In the presence of aggregate risk the computation of equilibria in these models becomes very difficult (see e.g. Krueger and Kubler (2004)). One of the difficulties stems from the fact that it is common to use individuals' *cash-at-hand*, i.e. the value of their beginning of period portfolio-holdings, as an endogenous state variable and that in these models the domain of policy functions is endogenous and possibly quite large. For models with infinitely lived agents, there have been various papers that use instead individuals' consumption or instantaneous Negishi-weights as a co-state to facilitate the computation of equilibria (e.g. Cuoco and He (1994), Chien and Lustig (2011), Chien et al. (2011) and Dumas and Lyasoff (2012)). We show that for models with overlapping generations, instantaneous Negishi-weights do not only have the advantage that one can take a unit-simplex as the state space. One also needs to approximate fewer functions to characterize the equilibrium dynamics of the economy. For the case of complete markets, one needs only the map from the current state to Negishi weights of the current young. These are SH functions, where S is the number of exogenous shocks and H is the number of agents per generation. In contrast, one needs $H(A-1)S^2$ functions if one wants to approximate the map from current cash-at-hand across agents to their cash-at-hand next period for each combination of shocks in the current and the subsequent period. Furthermore, the use of Negishi weights as states allows for a straightforward error analysis. Approximation errors can be interpreted as transfers that are necessary to obtain the computed allocation as an equilibrium allocation. Last but not least, using Negishi's method the computational burden barely increases with the number of physical commodities, while it substantially increases if the natural state space is used.

It is well known that recursive equilibrium might not exist in stochastic models with overlapping generations if one uses beginning of period asset holdings as the endogenous state variable (see Kubler and Polemarchakis (2004)). Sufficient conditions for existence are generally not applicable to models used in applications (see e.g. Citanna and Siconolfi (2010)). We examine existence of equilibria for which the Negishi-weights follow a Markov-process. While we do not know of counterexamples to existence in our framework, it seems likely that one can construct them. However, we show for the case of 2 period lived agents that recursive equilibria always exist, even if there is intra-generational heterogeneity and there are several goods (i.e. in the set-up of Kubler and Polemarchakis' counterexample). It is known that in our unconstrained model with complete markets equilibria are unique if all agents utility functions satisfy the gross-substitute property (see Kehoe et al. (1991)). We show that even if agents' are borrowing constrained, as long as they can trade in a full set of Arrow-securities, the assumption still guarantees uniqueness and hence the existence of a recursive equilibrium. In this paper we focus on pure exchange economies. The introduction of a neo-classical production sector is straightforward – however our existence results that rely on a gross substitute property do not carry over. Existence of recursive equilibria in production economies is an open issue that is subject to further research.

The rest of the paper is organized as follows. In Section 2 we introduce the basic model with complete markets, characterize recursive equilibria and explain the computational advantages of our approach. Although this is only a special case of our general model, it is useful to examine it first as it makes the computational advantages most transparent. In Section 3 we introduce borrowing constraints and incomplete markets and derive conditions for the existence of Markov equilibria in this general setup. In Section 4 we discuss two special cases. First we assume that markets are incomplete but that agents are not borrowing constraint. Then we examine the case where agents can trade in a full set of Arrow securities but are borrowing constrained. For this latter case, we proof existence of recursive equilibria under a gross substitutes assumption. In Section 5 we give an interpretation of the instantaneous Negishi-weights as welfare weights in a modified planner's problem.

2 Complete markets

To fix ideas we start with an economy with complete financial markets and no borrowing constraints. We assume that there is a Lucas-tree in unit net supply – this guarantees Pareto-efficiency of equilibrium allocations (see e.g. Demange (2002)). In the next section we generalize the model by allowing for arbitrary financial securities and borrowing constraints.

2.1 The physical economy

Time is indexed by $t = 0, 1, 2, \ldots$. Exogenous shocks $s_t \in \mathcal{S} = \{1, \ldots, S\}$ follow a Markov chain. A finite history of shocks $\sigma = s^t = (s_0, s_1, \ldots, s_t)$ is also called date-event or node of the event-tree. We use the symbols σ and s^t interchangeably. To indicate that $s^{t'}$ is a successor of s^t (or s^t itself) we write $s^{t'} \succeq s^t$. The set of all possible date-events s^t is denoted by Σ. We consider an exchange economy with L perishable commodities available for consumption at each date-event.

At each date-event H agents enter the economy; they live for A periods. An individual is identified by the date-event of his birth, $\sigma = s^t$, and his type, $h \in \mathcal{H} = \{1, \ldots, H\}$. The age of an individual is denoted by $a = 1, \ldots, A$; he consumes and has endowments at all nodes $s^{t+a-1} \succeq s^t$, $a = 1, \ldots, A$. At a given date-event s^t we can uniquely identify agents who consume at that node by their age and type, (a, h). We denote the set of all these agents by $\mathcal{A} = \{(a, h) : 1 \leq a \leq A, h \in \mathcal{H}\}$ and the set of all agents except for generation i by $\mathcal{A}_{-i} = \mathcal{A} \setminus \{(a, h) : a = i, h \in \mathcal{H}\}$. We will use (a, h) and (s^t, h) interchangeably to refer to a specific agent.

We denote individual endowments by $\omega_{a,h}(s^t) \in \mathbb{R}^L_+$ and assume that they are positive time-invariant functions of the current shock alone. Each agent has an intertemporal time-separable utility function,

$$U_{s^t,h}(x) = u_{1,h}\left(x(s^t)\right) + \sum_{a=1}^{A-1} \sum_{s^{t+a} \succeq s^t} \delta_{a,h}(s^{t+a}) u_{a+1,h}\left(x(s^{t+a})\right),$$

where $x(\sigma) \in \mathbb{R}^L_+$ denotes consumption of agent (s^t, h) at date-event σ, and x denotes consumption over the lifecycle

$$\{x(s^{t+a})\}_{0 \leq a \leq A-1, s^{t+a} \succeq s^t}.$$

The probability-discount factors $\delta_{a,h}(s^t) > 0$ are assumed to be stationary in that $\delta_{1,h}(s^t) = 1$ and $\delta_{a,h}(s^{t+a}) = \delta_{a-1,h}(s^{t+a-1})\delta_{a,h}(s_{t+a-1}, s_{t+a})$ for some $\delta_{a,h}(s, s') > 0$. This formulation encompasses heterogeneous beliefs as well as heterogeneous and age-varying discounting. The Bernoulli-functions

$u_{a,h} : \mathbb{R}^L_+ \to \mathbb{R}$ are assumed to be C^2 on $\mathbb{R}^L_{++}$, strictly increasing, strictly concave and satisfy an Inada condition: for all $y \gg 0$ the closure of the set $\{x \in \mathbb{R}^L_+ : u(x) > u(y)\}$ lies in $\mathbb{R}^L_{++}$.

There is a Lucas tree in unit net supply paying dividends $d(s^t) \in \mathbb{R}^L_+$, $d(s^t) > 0$. Dividends are a function of the shock alone, so $d(s^t) = d(s_t)$ for some function $d : \mathcal{S} \to \mathbb{R}^L_+$. At time $t = 0$, in addition to the H new agents (s^0, h), $h \in \mathcal{H}$, there are individuals of each age $a = 2, \ldots, A$ and each type $h = 1, \ldots, H$ present in the economy. We denote these individuals by (s^{1-a}, h) for $h = 1, \ldots, H$ and $a = 2, \ldots, A$. They have initial tree holdings, $\theta^{s^{1-a},h}(s^{-1})$, summing up to one:

$$\sum_{a=2}^{A} \theta_{s^{1-a},h}(s^{-1}) = 1.$$

These holdings determine the 'initial condition' of the economy. The aggregate endowment in the economy is $\bar{\omega}(s^t) = \bar{\omega}(s_t) = d(s_t) + \sum_{(a,h)\in\mathcal{A}} \omega_{a,h}(s_t)$.

2.1.1 Complete financial markets

At each node there are complete spot markets for the L commodities. Prices are $\pi(s^t) \in \mathbb{R}^L_{++}$ with the normalisation that $\pi_1(s^t) = 1$ for all s^t. Agents can trade the Lucas tree after dividends are payed out. Let $\theta_{s^t,h}(s^t) \geq 0$ denote the Lucas tree holding of individual (s^t, h) at date-event s^t and let $q(s^t)$ denote the price of the tree at that node. In addition there are S Arrow securities available for trade at each node. Arrow security $s \in \mathcal{S}$ pays one unit of the numéraire commodity $(l = 1)$ exactly if the subsequent shock is s and it is traded at a price p_s. An agent's portfolio of Arrow securities is denoted by $\phi(s^t) \in \mathbb{R}^S$, and the vector of Arrow security prices is denote by $p \in \mathbb{R}^S_{++}$. While we rule out short-sales in the Lucas tree, we impose no restrictions on trades in the financial assets. Since in this setting the payoffs of the tree are spanned by the other assets, the constraints on short-sales are irrelevant. However, they become important in the next section.

A sequential competitive equilibrium is a collection of prices and choices of individuals

$$\Big(\pi(s^t), p(s^t), q(s^t), \big(\phi_{a,h}(s^t), \theta_{a,h}(s^t), x_{a,h}(s^t)\big)_{(a,h)\in\mathcal{A}}\Big)_{s^t\in\Sigma}$$

such that markets clear and agents optimize, i.e. (1) and (2) hold.

(1) Market clearing equations:

$$\sum_{(a,h)\in\mathcal{A}_{-A}} \theta_{a,h}(s^t) = 1, \quad \sum_{(a,h)\in\mathcal{A}_{-A}} \phi_{a,h}(s^t) = 0, \quad \sum_{(a,h)\in\mathcal{A}} x_{a,h}(s^t) = \bar{\omega}(s_t), \qquad \text{for all } s^t \in \Sigma.$$

(2) For each s^t and $h = 1, \ldots, H$, individual (s^t, h) maximizes utility:

$$(x_{s^t,h}, \theta_{s^t,h}, \phi_{s^t,h}) \in \arg \max_{x\geq 0, \theta\geq 0, \phi} U_{s^t,h}(x) \quad \text{s.t. the constraints (i)-(ii).}$$

(i) Budget constraint for $a = 1$:

$$\pi(s^t) \cdot (x(s^t) - \omega_{1,h}(s_t)) + p(s^t) \cdot \phi(s^t) + q(s^t)\theta(s^t) \leq 0.$$

(ii) Budget constraints for all $a = 2, \ldots, A$ and $s^{t+a-1} \succ s^t$:

$$\begin{aligned}
&\pi(s^{t+a-1}) \cdot (x(s^{t+a-1}) - \omega_{a,h}(s_{t+a-1})) - \\
&\big(\phi_{s_{t+a-1}}(s^{t+a-2}) + \theta(s^{t+a-2})(q(s^{t+a-1}) + \pi(s^{t+a-1}) \cdot d(s_{t+a-1}))\big) + \\
&\big(p(s^{t+a-1}) \cdot \phi(s^{t+a-1}) + q(s^{t+a-1})\theta(s^{t+a-1})\big) \leq 0, \\
&\phi(s^{t+A-1}) = \theta(s^{t+A-1}) = 0.
\end{aligned}$$

The utility maximization problems for the agents (s^{1-a}, h), $a = 2, \ldots, A$, $h = 1, \ldots, H$, who are initially alive at $t = 0$ are analogous to the optimization problems for agents (s^t, h).

2.2 Negishi's approach to analyze equilibrium

As Kehoe et al. (1992) point out, the presence of a Lucas tree ensures that competitive equilibria in this economy are Pareto efficient and that there must exist summable Pareto weights $\{\eta_{s^t,h}\}_{s^t \in \Sigma, h \in \mathcal{H}}$ such that competitive equilibrium allocations satisfy

$$(x_{s^t,h})_{s^t \in \Sigma, h \in \mathcal{H}} = \arg\max_{x \geq 0} \sum_{s^t \in \Sigma, h \in \mathcal{H}} \eta_{s^t,h} U_{s^t,h}(x_{s^t,h}) \text{ s.t. } \sum_{s^t \in \Sigma, h \in \mathcal{H}} x_{s^t,h}(\sigma) \leq \bar{\omega}(\sigma) \text{ for all } \sigma \in \Sigma. \quad (1)$$

Since we assume time-separable expected utility, we can characterize equilibrium also by using instantaneous Negishi weights, $\lambda(s^t) = (\lambda_{a,h}(s^t))_{(a,h) \in \mathcal{A}}$, defined by

$$\lambda_{a,h}(s^t) = \eta_{s^{t-a+1},h} \delta_{a,h}(s^t).$$

Individuals' consumption is then given as a function $X : \mathcal{S} \times \mathbb{R}^{AH}_{+} \to \mathbb{R}^{AHL}_{+}$ of the shock and the instantaneous weights with

$$X(s, \lambda) = \arg\max_{x \in \mathbb{R}^{AHL}_{+}} \sum_{(a,h) \in \mathcal{A}} \lambda_{a,h} u_{a,h}(x_{a,h}) \text{ s.t. } \sum_{(a,h) \in \mathcal{A}} x_{a,h} \leq \bar{\omega}(s). \quad (2)$$

For $\lambda_{a,h} = 0$ we take $X_{a,h} = 0$ to be the optimal solution (although utility might be minus infinity at that consumption bundle). Given a process of instantaneous Negishi weights $(\lambda(\sigma))_{\sigma \in \Sigma}$, $\lambda(s^t) \in \mathbb{R}^{AH}_{++}$ for all s^t, we define for each node s^t, $x_{a,h}(s^t) := X_{a,h}(s_t, \lambda(s^t))$.

Then a sequence of Negishi weights

$$\big((\lambda_{a,h}(s^t))_{(a,h) \in \mathcal{A}}\big)_{s^t \in \Sigma}$$

characterizes a financial markets equilibrium if the following two conditions (E1)-(E2) hold.

(E1) Intertemporal Euler equations.

For all $h \in \mathcal{H}$ and all $a = 2, ..., A$ it holds that $\lambda_{a,h}(s^t) = \delta_{a,h}(s_t, s_{t-1})\lambda_{a-1,h}(s^{t-1})$.

(E2) Budget constraints.

Defining the budget of agent (a,h) at node s^t for all $h \in \mathcal{H}$ recursively by

$$w_{A,h}(s^t) := D_x u_{A,h}(x_{A,h}(s^t)) \cdot \left(x_{A,h}(s^t) - \omega_{A,h}(s^t)\right), \text{ and for } a = 1, ..., A-1 \text{ by}$$

$$w_{a,h}(s^t) := D_x u_{a,h}(x_{a,h}(s^t)) \cdot \left(x_{a,h}(s^t) - \omega_{a,h}(s^t)\right) + \sum_{s^{t+1} \succeq s^t} \delta_{a+1,h}(s^t, s_{t+1}) w_{a+1,h}(s^{t+1}),$$

it holds for all $h \in \mathcal{H}$ that: $w_{1,h}(s^t) = 0$.

It is easy to verify that for a sequence of Negishi weights that satisfies conditions (E1)-(E2), there exist initial conditions and a sequential equilibrium,

$$\left(\bar{\pi}(s^t), \bar{p}(s^t), \bar{q}(s^t), \left(\bar{\phi}_{a,h}(s^t), \bar{\theta}_{a,h}(s^t), \bar{x}_{a,h}(s^t)\right)_{(a,h)\in\mathcal{A}}\right)_{s^t \in \Sigma},$$

with $\bar{x}_{a,h}(s^t) = X_{a,h}(s_t, \lambda(s^t))$, $\bar{\pi}(s^t) = \frac{1}{\partial_{1,1}(s^t)} D_x u_{1,1}(x_{1,1}(s^t))$ and $\bar{p}_{s_{t+1}}(s^t) = \frac{\delta_{2,1}(s_t, s_{t+1}) \partial_{2,1}(s^{t+1})}{\partial_{1,1}(s^t)}$ for all $s_{t+1} \succeq s_t$, where $\partial_{a,h}(s^t) := \partial_{a,h}(s_t, \lambda(s^t)) := \frac{\partial u_{a,h}(X_{a,h}(s,\lambda))}{\partial x_1}$. The budget constraints imply that $\bar{q}(s^t) = -d(s_t) + \sum_{(a,h)\in\mathcal{A}} \frac{w_{a,h}(s^t)}{\partial_{a,h}(s^t)}$.

It is somewhat misleading to refer to (E1) as an 'intertemporal Euler equation'. The evolution of the instantaneous weight $\lambda_{a,h}(\sigma)$ simply follows from the definition of the planner's problem. However, once we introduce incomplete markets we can no longer work with a social planner and the evolution of instantaneous Negishi weights is then determined by an actual Euler equation.

2.3 Recursive equilibria

Using Negishi's approach to compute equilibria is useful only if the Negishi weights follow a Markov process. We refer to such equilibria as 'recursive equilibria', although it should be clear that they might or might not be recursive when one uses beginning-of-period portfolios as the state, which we call the natural state in what follows. A recursive equilibrium is described by a policy function mapping the state into all agents' consumption, $X(s,\lambda)$, and a state transition

$$\Lambda : \mathcal{S} \times \mathcal{S} \times \mathbb{R}^{AH}_{++} \to \mathbb{R}^{AH}_{++}.$$

We take as given a sequential equilibrium described by $(\lambda(s^t))_{s^t \in \sigma}$. Clearly this sequential equilibrium might fail to be a recursive equilibrium which is the case if the state transition can only be described by a correspondence and not a function. However, if there is a transition function, we can describe $w(s^t)$ recursively using the functions $X(s,\lambda)$ and $\Lambda(s,s,\lambda)$ as follows. Define for all types $h \in \mathcal{H}$ the value of their excess consumption (i.e. their budget) to be

$$W_{A,h}(s,\lambda) := D_x u_{a,h}(X_{A,h}(s,\lambda)) \cdot (X_{A,h}(s,\lambda) - \omega_{A,h}(s)), \quad (3)$$

$$W_{a,h}(s,\lambda) := D_x u_{a,h}(X_{a,h}(s,\lambda)) \cdot (X_{a,h}(s,\lambda) - \omega_{a,h}(s)) + \sum_{s'} \delta_{a+1,h}(s,s') W_{a+1,h}(s', \Lambda(s,s',\lambda)) \quad (4)$$

for all $a = 1, ..., A-1$.

Definition 1 *A recursive equilibrium is a function* $\Lambda : \mathcal{S} \times \mathcal{S} \times \mathbb{R}^{AH}_{++} \to \mathbb{R}^{AH}_{++}$ *that satisfies for all* $s, s' \in \mathcal{S}$ *and all* $\lambda \in \mathbb{R}^{AH}_{++}$*,*

$$W_{1,h}(s', \Lambda(s, s', \lambda)) = 0 \text{ for all } h \in \mathcal{H} \text{ and}$$

$$\Lambda_{a,h}(s, s', \lambda) = \delta_{a,h}(s, s')\lambda_{a-1,h} \text{ for all } (a, h) \in \mathcal{A}_{-1}.$$

It is clear that if such a transition function Λ exists, then it does indeed characterize an equilibrium. However, there is in general no guarantee that equilibria exist for which Negishi weights are Markov. As Kubler and Polemarchakis (2004) point out, in economies with overlapping generations recursive equilibria for the natural state space might fail to exist. Although we have no counter-example to existence of equilibria that are recursive using Negishi weights, there is no good reason to believe that these equilibria always exist. However, as we will show below they always exist for the case of two-period lived agents, $A = 2$. Moreover, we also show that they always exist if utility satisfies a gross substitute property. Citanna and Siconolfi (2010) have the clever insight that with sufficiently many agents per generation recursive equilibrium must exist generically. They prove their result using the natural state space, but it is clear that a similar approach can be used to prove generic existence of equilibria with Markovian Negishi weights. However, we want to use the fact that Negishi weights are Markov to approximate equilibria using Negishi's method in models where agents live for many periods. In this case, the number of agents needed for Citanna and Siconolfi's result to apply becomes astronomical very fast. It is subject to further research to use their approach to tackle models with a continuum of ex ante identical agents within each generation, for all other models it seems of little practical relevance.

When it comes to computing recursive equilibrium below, we consider only *minimal recursive equilibrium*, which we define as a recursive equilibrium for which the current $(\lambda_{1,h})_{h\in\mathcal{H}}$ is a function of $(\lambda_{a,h})_{(a,h)\in\mathcal{A}_{-1}}$. Thus, a recursive equilibrium is said to be minimal if there exists a function $\ell(s, (\lambda_{a,h})_{(a,h)\in\mathcal{A}_{-1}})$ so that $\Lambda_{1,h}(s, s', \lambda) = \ell_h\left(s', (\Lambda_{a,h}(s, s', \lambda))_{(a,h)\in\mathcal{A}_{-1}}\right)$ for all $h \in \mathcal{H}$. While our concept of recursive equilibrium is consistent with the idea of Markov-equilibria where the state is required to follow a Markov chain, the concept of minimal recursive equilibrium is actually more easily comparable to recursive equilibria using the natural state space. In that concept, the endogenous state at a node s^t consists only of variables that were determined at s^{t-1}. Clearly for each h, $\lambda_{1,h}(s^t)$ is typically not 'predetermined' and hence it is not obvious whether it should be included in the state space. In the computational approach below we search for a minimal recursive equilibrium. Obviously conditions that ensure existence of a minimal recursive equilibrium must be stronger than conditions for the existence of a recursive equilibrium. We revisit the existence of recursive equilibrium and of minimal recursive equilibrium in the next two sections where we prove general results that apply to models with and without borrowing constraints.

2.4 Computation

We describe and discuss a simple time iteration collocation method to numerically approximate minimal recursive equilibria. Time iteration is one of several standard approaches to solve dynamic non-optimal models (see e.g. Section 7.2. of Judd et al. (2003) or Krueger and Kubler (2004)). Obviously there are several other approaches which have advantages and disadvantages compared to time iteration. However, we chose to discuss this algorithm because it makes it easy to compare our approach to the conventional approach of doing time-iteration using the natural state-space. It also serves as a basis for computing large-scale models in practice.

We take as given that the functions $X(s, \lambda)$ can be approximated with high accuracy and negligible computational cost. For standard calibrations that assume identical homothetic utility this function is linear after a change of variable. For the case of one commodity there are several other classes of preferences for which closed-form solutions are known.

As we solve for a minimal recursive equilibrium, the endogenous state consist of $(\lambda_{a,h})_{(a,h)\in\mathcal{A}_{-1}}$ while $(\lambda_{1,h})$ is described by the function ℓ. Note that we could normalize $(\lambda_{a,h})_{(a,h)\in\mathcal{A}_{-1}}$ to lie on the unit simplex, yet it more straightforward to describe the algorithm without this normalization. To simplify notation, we define a vector $\tilde{\lambda} := (\lambda_{a,h})_{(a,h)\in\mathcal{A}_{-1}}$. Given $\ell(s, \lambda)$, we are only interested in $W(s, \lambda)$ for values of λ that satisfy $(\lambda_{1h})_{h\in\mathcal{H}} = \ell(s, \tilde{\lambda})$. We therefore define $\tilde{W}(s, \tilde{\lambda}) := W(s, (\ell(s, \tilde{\lambda}), \tilde{\lambda})))$.

We assume that the unknown functions $\tilde{W}(s, \tilde{\lambda})$ and $\ell(s, \tilde{\lambda})$ can be well approximated by some $\hat{W}(s, \tilde{\lambda})$ and $\hat{\ell}(s, \tilde{\lambda})$ that are uniquely determined by the requirement that $\hat{W}_{a,h}(s, \tilde{\lambda}^i) = W_{a,h}(s, \tilde{\lambda}^i)$ and $\hat{\ell}_h(s, \tilde{\lambda}^i) = \ell(s, \tilde{\lambda}^i)$ for some finite number G of so-called 'collocation points' $\tilde{\lambda}^i \in \mathcal{G} \subset \mathbb{R}^{(A-1)H}_{++}$, $i = 1, ..., G$. Examples of functions commonly used for collocation methods include Smolyak-polynomials as in Krueger and Kubler (2004) or splines as in Judd et al. (2003). Recall that for expositional reasons, we do not normalize the state variable to lie in the unit simplex. Thus, all functions are homogeneous of degree zero in $\tilde{\lambda}$, thus we can easily redefine them over compact domains. The main steps of the algorithm are as follows.

1. Set $n = 0$ and start with an initial guess $\hat{W}^0 : \mathcal{S} \times \mathbb{R}^{(A-1)H}_{+} \to \mathbb{R}^{(A-1)H}$.

2. Given $\hat{W}^n$, for each $s \in \mathcal{S}$ and each $\tilde{\lambda}^i \in \mathcal{G}$, compute $\hat{\ell}^{n+1}(s, \tilde{\lambda}^i)$ as a solution to the system of equations

$$D_x u_{1,h}(X_{1,h}(s, \lambda^i(s))) \cdot (X_{1,h}(s, \lambda^i(s)) - \omega_{1,h}) + \sum_{s'} \delta_{2,h}(s, s')\hat{W}^n_{2,h}(s', \tilde{\lambda}^i(s')) = 0, \ h \in \mathcal{H}, \quad (5)$$

   where $\lambda^i(s) = (\hat{\ell}^{n+1}(s, \tilde{\lambda}^i), \tilde{\lambda}^i)$ and $\tilde{\lambda}^i(s') = (\delta_{a,h}(s, s')\lambda^i_{a-1,h}(s))_{(a,h)\in\mathcal{A}_{-1}}$.

3. For each $s \in \mathcal{S}$ and each $\tilde{\lambda}^i \in \mathcal{G}$, let $\lambda^i(s)$ and $\tilde{\lambda}^i(s')$ be given as before and compute for all

$(a, h) \in \mathcal{A}_{-1}$

$$\hat{W}^{n+1}_{a,h}(s, \tilde{\lambda}^i) = D_x u_{a,h}(X_{a,h}(s, \lambda^i(s)))\cdot(X_{a,h}(s, \lambda^i(s)) - \omega_{a,h}) + \sum_{s'} \delta_{a+1,h}(s, s')\hat{W}^n_{a+1,h}(s', \tilde{\lambda}^i(s')),$$

where $\hat{W}^{n+1}_{A+1,h}(s, \tilde{\lambda}^i) := 0$.

4. For each $s \in \mathcal{S}$, interpolate $\{\hat{W}^{n+1}(s, \tilde{\lambda}^i), i = 1, ..., G\}$ to obtain approximating functions $\hat{W}^{n+1}(s, .)$.

5. Check some error criterion. If error criterion is not met, increase n by 1 and go to 2.

6. Set $\hat{W}^* = \hat{W}^{n+1}$ and interpolate $\hat{\ell}^n(s, \tilde{\lambda}^i)$ to obtain $\hat{\ell}^*(s, .)$.

In Section 4 below, we give conditions under which the algorithm converges, assuming an idealized situation without error in the function-approximation. In general, the system of equations (5) might have no solutions, or it might have several solutions. Furthermore, there is no guarantee that $\hat{W}^n$ converges as n tends to infinity. This is obvious as there is generally no guarantee that a (minimal recursive) equilibrium exists. Feng et al. (2013) develop a method which can be used to compute generalized Markov equilibria in this setting. However, for reasonable values of A the method is not applicable as it suffers from a severe curse of dimensionality.

In this simple framework with complete markets, it is clear that using Negishi weights as state variables has important advantages over the 'standard' approach that uses beginning-of-period cash-at-hand. Most importantly, the computational complexity barely increases with the number of goods, L. Only the computation of $X(s, \lambda)$ and of $D_x u_{a,h}(X(s, \lambda))$ depends on L. This is in stark contrast to the case of cash-at-hand as endogenous state variable where one needs to solve for spot-prices and allocations simultaneously with portfolios and asset-prices. We will return to this issue in Section 4 below.

Even for the case of a single commodity the Negishi-approach has three important advantages over conventional methods: First, as policies are homogeneous of degree zero in Negishi weights, policy functions may be defined over the $(A - 1)H - 1$ dimensional unit simplex (for given states today and tomorrow). Thus the admissible set is simple and can easily be worked with, while it can be arbitrarily complicated in the case of the natural state space. Since agents do not face borrowing constraints, young agents typically borrow substantial amounts and one thus has to determine the 'natural' borrowing limits as one step of the computations. For models with large H and/or A this can result in substantial difficulties. Second, along the time iteration the only costly computation consists of solving for $\hat{\ell}(s, \tilde{\lambda}^i)$ in Step 2 above. For each s and $\tilde{\lambda}^i$, this is a non-linear system of H equations in as many unknowns. In contrast, for the case of cash-at-hand, one needs to solve all agents' first order conditions plus market clearing conditions simultaneously to obtain optimal choices and prices. This results in $(A - 1)HS$ equations for each s and $\tilde{\lambda}^i$ (if market clearing conditions are used to express one agent's portfolio in terms of all others'). Even for moderate A and S this can be an order of magnitude larger, thus an enormous efficiency gain can be realized if Negishi-weights are used. Note also, that the dynamics of the economy is fully captured by the HS functions $(\ell_h(s,.))_{s\in S,h\in\mathcal{H}}$. If cash-at-hand is used instead, one needs to keep track of $(A-1)HS^2$ functions, for each current shock s mapping cash-at-hand of all generations but the oldest into their cash-at-hand at all successor nodes. Thus, the Negishi approach reduces both the number of equations that have to be solved simultaneously as well as the number of functions that are needed to characterize equilibrium dynamics by a factor of $(A-1)S$. Third, and perhaps most importantly, error analysis is trivial to conduct if we use Negishi-weights. As mentioned above, the error in computing $X(s,\lambda)$ can typically be taken to be negligible. Given a transition $\hat{\ell}$, the errors in the computation of $\hat{W}$ are pure function-approximation errors and there are reliable methods to bound them above. As explained for example in Kubler and Schmedders (2005) it is generally impossible to find bounds on how close a computed approximation is to an exact equilibrium. In the current context, it is impossible to determine how close the computed evolution of λ is to the exact equilibrium evolution. Given approximations $\hat{W}$ and $\hat{\ell}$ for the unknown policy functions, the only relevant error is $MAXERR = \sup_{h\in\mathcal{H},s\in\mathcal{S},\tilde{\lambda}\in\mathbb{R}^{(A-1)H}} \|\frac{\hat{W}_{1,h}(s',\hat{\ell}(s,\tilde{\lambda}))}{\partial_{1,h}(s,\tilde{\lambda})}\|$. This can be interpreted as the maximal transfer necessary to turn the computed allocation into an equilibrium allocation. That is, while we cannot guarantee in general that the computed allocation is close to an exact equilibrium allocation, it is always close to an equilibrium allocation of an economy with transfers. The size of the transfers is bounded by $MAXERR$. Kubler and Schmedders (2005) suggest a similar interpretation for the case of the natural state. However, in their method one needs to transform the error in the computation into an error that has an economic interpretation. Using Negishi-weights as state variable has the advantage that the error in the computation translates directly to an interpretable approximation error.

3 The general model

We now generalize the above model by introducing borrowing constraints and the possibility of incomplete financial markets. We assume that there are $J \leq S$ financial securities with stationary payoffs, i.e. security j pays $b_j(s)$ units of the numéraire commodity in shock s. Budget constraints and borrowing constraints for agent (s^t, h) read as follows

(i) Budget constraint for $a = 1$:

$$\pi(s^t)\cdot(x(s^t)-\omega_{1,h}(s_t)) + p(s^t)\cdot\phi(s^t) + q(s^t)\theta(s^t) \leq 0.$$

(ii) Budget constraints for all $a = 2,\ldots,A$ and $s^{t+a-1} \succeq s^t$:

$$\pi(s^{t+a-1}) \cdot (x(s^{t+a-1}) - \omega_{a,h}(s_{t+a-1})) -$$
$$\left(\phi(s^{t+a-2}) \cdot b(s^{t+a-1}) + \theta(s^{t+a-2})(q(s^{t+a-1}) + \pi(s^{t+a-1}) \cdot d(s_{t+a-1}))\right) +$$
$$p(s^{t+a-1}) \cdot \phi(s^{t+a-1}) + q(s^{t+a-1})\theta(s^{t+a-1}) \leq 0,$$
$$\phi(s^{t+A-1}) = \theta(s^{t+A-1}) = 0.$$

(iii) Borrowing constraint for all $a = 1, ..., A-1$ and $s^{t+a} \succ s^t$:

$$\phi(s^{t+a-1}) \cdot b(s^{t+a-1}) + \theta(s^{t+a-1})(q(s^{t+a}) + \pi(s^{t+1}) \cdot d(s_{t+a})) \geq -D(s^{t+a}).$$

This borrowing constraint demands that the value in s^{t+a} of the portfolio bought in s^{t+a-1} has to exceed $-D(s^{t+a})$, i.e. the net repayment obligation in s^{t+a} may not exceed $D(s^{t+a})$. On the left hand side of the borrowing constraint, short positions in financial assets might be offset by long positions in other financial assets or the Lucas tree. This means that these long positions may be used as collateral for borrowing. The additional amount that agents are able to borrow, which is represented by $D(s^{t+a})$, is determined by the assumption that agents may borrow against part of their future endowments. We denote this part of their endowments by $f_{a,h}(s^t) \geq 0$. These are tangible resources that can be pledged to finance consumption and asset purchases at a time before they are received (see Gottardi and Kubler (2012)). The remaining part of an agents endowments is non-pledgeable and assumed to be positive for all s^t. It is denoted by $e_{a,h}(s^t)$. Total endowments of agent (a, h) are thus given by

$$\omega_{a,h}(\sigma) = e_{a,h}(\sigma) + f_{a,h}(\sigma) \text{ for all } \sigma.$$

We assume that both components depend only on the current shock: $e_{a,h}(s^t) = e_{a,h}(s_t)$, $f_{a,h}(s^t) = f_{a,h}(s_t)$. To understand how the debt limit $D(s^{t+a})$ is determined it is helpful to first consider the case where agents can trade in a complete set of assets. In this case agents can borrow against the current value of their future f-endowments. Thus, for all $a = 1, ..., A$ and $s^{t+a} \succ s^t$:

$$D(s^{t+a}) = \sum_{s^{t+i} \succeq s^{t+a}} \frac{\rho(s^{t+i})}{\rho(s^{t+a})} \pi(s^{t+i}) \cdot f_{i,h}(s^{t+i}),$$

were $\rho(s^t)$ denotes the Arrow-Debreu price of consumption of the numeraire good at node s^t.

If the set of available assets is not complete, then the debt limit cannot be derived in closed-form. Instead, we provide a recursive definition of the debt limit starting with the final period of an agent's life. In that period, an agent has to repay all his debt. Thus, the repayment obligations that he enters that period with may not exceed the f-endowments that he earns in that period:

$$D(s^{t+A-1}) = \pi(s^{t+A-1}) \cdot f_{A,h}(s^{t+A-1}).$$

Given debt limits at all $s^{t+a} \succ s^{t+a-1}$, the debt limit for s^{t+a-1} is given by

$$D(s^{t+a-1}) = \pi(s^{t+a-1}) \cdot f_{a,h}(s^{t+a-1}) - \max_{\phi \in \mathbb{R}^S, \theta \geq 0} (p(s^{t+a-1}) \cdot \phi + q(s^{t+a-1})\theta)$$
$$\text{s.t. } \phi \cdot b(s_{t+a}) + \theta(q(s^{t+a+1}) + \pi(s^{t+a+1}) \cdot d(s_{t+1})) + D(s^{t+a}) \geq 0 \text{ for all } s^{t+a} \succ s^{t+a-1}.$$

This is saying that the repayment obligations that an agent faces at date-event s^{t+a-1} may not exceed the sum of his f-endowments plus the maximum amount that he may borrow subject to the constraint that he does not violate his debt limit in any of the subsequent periods. In other words, debt must be repayable by a trading strategy that finances itself entirely through f-endowments.

Note that despite the fact that we throughout impose the restrictions $\theta(s^t) \geq 0$, it is important to allow trading in the tree in the definition of $D(s^{t+a-1})$. If an agent has high f-endowments in one state, but the price of the tree is high in another state, the agent might want to buy the tree to be able to short a risk-free asset that allows him to borrow against both states.

Competitive equilibrium (with borrowing constraints and possibly incomplete markets) is defined as before - agents maximize utility subject to constraints (i) - (iii) and markets clear. Our general description of borrowing constraints has three well known special cases. First, full spanning in which case the borrowing constraints simplifies considerably as we have seen above. Second, if e-endowments are all zero, then our borrowing constraint is nothing but a natural borrowing limit. Third, if f-endowments are all zero, then agents can borrow only against the value of their long positions in the Lucas tree or financial securities, i.e. agents face a collateral constraint. With incomplete markets, the constraint needs to take into account that the future endowments can only be sold through the existing assets. In this case, the assumption that default is not possible is crucial as default might result in payoffs that are not possible through trade in the incomplete set of financial securities. If one wants to allow for default, one needs to model asset-specific margins (see Gottardi and Kubler (2012)) – many of the results below will also hold for such a model, however, in order to make the analysis consistent we choose to rule out default both in the case of complete markets and incomplete markets.

3.1 Equilibrium characterization via Negishi weights

As in the complete market case, we can characterize competitive equilibria by a sequence of instantaneous Negishi weights. Without the interpretation of Negishi weights as welfare weights in a social planner's problem, there are infinitely many sequences of weights that give rise to the same allocation. We can therefore choose a normalization each period and it is useful to choose one that simplifies to our earlier definition in the case of complete markets.

Somewhat similarly to the 'Cass-trick' (see Cass (2006)) we take the evolution of one agent's Negishi weight λ to be the same as in complete markets. Without loss of generality, we take this to be agent $(1,1)$.

It is easy to see that competitive equilibrium is characterized by budget equations and first order conditions (see e.g. Kubler and Polemarchakis (2004)). A sequence of Negishi weights

$$\left((\lambda_{a,h}(s^t))_{(a,h)\in\mathcal{A}}\right)_{s^t\in\Sigma}$$

characterizes a financial markets equilibrium if there exist portfolios and multipliers

$$\left((\phi_{a,h}(\sigma),\theta_{a,h}(\sigma),\zeta_{a,h}(\sigma),\chi_{a,h}(\sigma))_{(a,h)\in\mathcal{A}}\right)_{\sigma\in\Sigma},$$

such that conditions (E1)-(E4) below hold for all s^t given the following definitions. As before, we use $x_{a,h}(s^t) := X_{a,h}(s^t,\lambda(s^t))$, $\partial_{a,h}(s^t) := \frac{\partial u_{a,h}(X_{a,h}(s,\lambda))}{\partial x_1}$, and we define the budget of agents recursively by

$$w_{A,h}(s^t) := D_x u_{A,h}(x_{A,h}(s^t)) \cdot (x_{A,h}(s^t) - \omega_{A,h}(s^t)) \text{ for all } h = 1,\ldots,H$$

and for all $(a,h) \in \mathcal{A}_{-A}$,

$$\begin{aligned} w_{a,h}(s^t) &:= D_x u_{a,h}(x_{a,h}(s^t)) \cdot (x_{a,h}(s^t) - \omega_{a,h}(s^t)) + \\ &\sum_{s^{t+1}\succ s^t} \delta_{a+1,h}(s_t,s_{t+1}) \left(1 + \frac{\zeta_{a+1,h}(s^{t+1})}{\partial_{a+1,h}(s^t)}\right) w_{a+1,h}(s_{t+1}). \end{aligned}$$

Prices are defined by:

$$\begin{aligned} \pi(s^t) &:= \frac{1}{\partial_{1,1}(s^t)} D_x u_{1,1}(x(s^t)) \\ q(s^t) &:= \sum_{(a,h)\in\mathcal{A}} \frac{w_{a,h}(s^t)}{\partial_{a,h}(s^t)} - d(s^t) \\ p(s^t) &:= \sum_{s^{t+1}\succ s^t} \delta_{2,1}(s_t,s_{t+1}) \frac{\partial_{2,1}(s^{t+1}) + \zeta_{2,1}(s^{t+1})}{\partial_{1,1}(s^t)} b(s_{t+1}) \end{aligned}$$

The value of f-endowments is defined recursively by

$$v_{A,h}(s^t) := D_x u_{a,h}(x_{a,h}(s^t)) \cdot (x_{A,h}(s^t) - e_{A,h}(s^t)) \text{ for all } h = 1,\ldots,H$$

and for all $(a,h) \in \mathcal{A}_{-A}$,

$$\begin{aligned} v_{a,h}(s^t) &:= \max_{\phi\in\mathbb{R}^S,\theta\geq 0} D_x u_{a,h}(x_{a,h}(s^t)) \cdot (x_{a,h}(s^t) - e_{a,h}(s^t)) - \partial_{a,h}(s^t)(p(s^t)\cdot\phi + q(s^t)\theta) \text{ s.t.} \\ &\phi \cdot b(s_{t+1}) + \theta \cdot (p(s^{t+1}) + \pi(s^{t+1}) \cdot d(s_{t+1})) + \frac{v_{a+1,h}(s^{t+1})}{\partial_{a+1,h}(s^{t+1})} \geq 0 \text{ for all } s^{t+1} \succ s^t. \end{aligned}$$

With these definitions the equilibrium conditions read as follows.

(E1) Intertemporal Euler equations.

For agent (1,1) we have

$$\lambda_{2,1}(s^{t+1}) = \delta_{2,1}(s_t,s_{t+1})\lambda_{1,1}(s^t) + \eta_{2,1}(s^{t+1}) \text{ for all } s^{t+1} \succ s^t,$$

where $\eta_{2,1}(s^{t+1}) := \zeta_{2,1}(s^{t+1})\frac{\lambda_{1,1}(s^t)\delta_{2,1}(s_t,s_{t+1})}{\partial_{2,1}(s^{t+1})}$.

For all $(a,h) \in \mathcal{A}_{-A}$ with $(a,h) \neq (1,1)$ we have

$$\begin{aligned} q(s^t) &= \chi_{a,h}(s^t) + \sum_{s^{t+1}\succ s^t} \frac{\delta_{a+1,h}(s_t,s_{t+1})\lambda_{a,h}(s^t) + \eta_{a+1,h}(s^{t+1})}{\lambda_{a+1,h}(s^{t+1})} \\ &\quad \delta_{2,1}(s_t,s_{t+1}) \frac{\partial_{2,1}(s^{t+1}) + \zeta_{2,1}(s^{t+1})}{\partial_{1,1}(s^t)} (q(s^{t+1}) + \pi(s^{t+1}) \cdot d(s_{t+1})) \\ p(s^t) &= \sum_{s^{t+1}\succ s^t} \frac{\delta_{a+1,h}(s_t,s_{t+1})\lambda_{a,h}(s^t) + \eta_{a+1,h}(s^{t+1})}{\lambda_{a+1,h}(s^{t+1})} \delta_{2,1}(s_t,s_{t+1}) \frac{\partial_{2,1}(s^{t+1}) + \zeta_{2,1}(s^{t+1})}{\partial_{1,1}(s^t)} b(s_{t+1}), \end{aligned}$$

where $\eta_{a+1,h}(s^{t+1}) := \zeta_{a+1,h}(s^{t+1})\frac{\partial_{1,1}(s^t)}{\partial_{2,1}(s^{t+1})+\zeta_{2,1}(s^{t+1})}\frac{\lambda_{a+1,h}(s^{t+1})\delta_{a+1,h}(s^{t+1})}{\partial_{a,h}(s^t)\delta_{2,1}(s_t,s_{t+1})}$.

(E2) Budget constraints.

For all $h \in \mathcal{H}$ it holds that:

$$w_{1,h}(s^t) = 0 \text{ for all } h \in \mathcal{H} \text{ and all } s^t.$$

(E3) Short sale constraints and spanning conditions.

For all s^t and all $(a,h) \in \mathcal{A}_{-A}$:

$$\theta_{a,h}(s^t) \geq 0, \theta_{a,h}(s^t)\chi_{a,h}(s^t) = 0,$$

and for all $s^{t+1} \succ s^t$,

$$\frac{w_{a+1,h}(s^{t+1})}{\partial_{a+1,h}} = \theta_{a,h}(s^t)\left(q(s^{t+1}) + \pi(s^{t+1}) \cdot d(s_{t+1})\right) + \phi_{a,h}(s^t) \cdot b(s_{t+1}).$$

(E4) Debt constraints.

For all $(a,h) \in \mathcal{A}$ and all s^t it holds that:

$$v_{a,h}(s^t) \geq 0, \quad \zeta_{a,h}(s^t)v_{a,h}(s^t) = 0.$$

Clearly, the equilibrium definition via Negishi weights is more complicated in the general setup than in the complete markets case. Two parts of the definition require an explanation.

First, the evolution of the Negishi weights is the same as in complete markets only for agent $(1,1)$. For all other agents, the evolution is implicitly given by the intertemporal Euler equations

$$p(s^t) = \sum_{s^{t+1} \succ s^t} \frac{\delta_{a+1,h}(s_t,s_{t+1})\lambda_{a,h}(s^t) + \eta_{a+1,h}(s^{t+1})}{\lambda_{a+1,h}(s^{t+1})}\delta_{2,1}(s_t,s_{t+1})\frac{\partial_{2,1}(s^{t+1}) + \zeta_{2,1}(s^{t+1})}{\partial_{1,1}(s^t)}b(s_{t+1}).$$

To rewrite this Euler equation, we use that

$$\begin{aligned}
&\frac{\delta_{a+1,h}(s_t,s_{t+1})\lambda_{a,h}(s^t)}{\lambda_{a+1,h}(s^{t+1})}\delta_{2,1}(s_t,s_{t+1})\frac{\partial_{2,1}(s^{t+1}) + \zeta_{2,1}(s^{t+1})}{\partial_{1,1}(s^t)} \\
&= \delta_{a+1,h}(s_t,s_{t+1})\frac{\lambda_{a,h}(s^t)}{\lambda_{a+1,h}(s^{t+1})}\frac{\left(\delta_{2,1}(s_t,s_{t+1})\lambda_{1,1}(s^t) + \eta_{2,1}(s^{t+1})\right)\partial_{2,1}(s^{t+1})}{\lambda_{1,1}(s^t)\partial_{1,1}(s^t)} \\
&= \delta_{a+1,h}(s_t,s_{t+1})\frac{\lambda_{a,h}(s^t)}{\lambda_{a+1,h}(s^{t+1})}\frac{\lambda_{2,1}(s^{t+1})\partial_{2,1}(s^{t+1})}{\lambda_{1,1}(s^t)\partial_{1,1}(s^t)} = \delta_{a+1,h}(s_t,s_{t+1})\frac{\partial_{a+1,h}(s^{t+1})}{\partial_{a,h}(s^t)}
\end{aligned}$$

and

$$\frac{\eta_{a+1,h}(s^{t+1})}{\lambda_{a+1,h}(s^{t+1})}\delta_{2,1}(s_t,s_{t+1})\frac{\partial_{2,1}(s^{t+1}) + \zeta_{2,1}(s^{t+1})}{\partial_{1,1}(s^t)} = \delta_{a+1,h}(s_t,s_{t+1})\frac{\zeta_{a+1,h}(s^{t+1})}{\partial_{a,h}(s^t)},$$

which results in the standard intertemporal Euler equation

$$p(s^t) = \sum_{s^{t+1} \succ s^t} \delta_{a+1,h}(s_t,s_{t+1})\frac{\partial_{a+1,h}(s^{t+1}) + \zeta_{a+1,h}(s^{t+1})}{\partial_{a,h}(s^t)}b(s_{t+1}).$$

Second, debt constraints are now more complicated as well. To understand why $v_{a,h}(s^t) \geq 0$ captures the borrowing constraint (iii) above, we show by induction that

$$v_{a,h}(s^t) = D_{a,h}(s^t) + \phi(s^{t-1}) \cdot b(s^t) + \theta(s^{t-1})(q(s^t) + \pi(s^t) \cdot d(s_t)).$$

We start the induction with $v_{A,h}(s^{t+a-1})$ where we just need the budget constraint at that age to show that the above equation holds. For $a < A$ we get from the definition of $v_{a,h}$ and the budget constraint that

$$\begin{aligned}
&v_{a,h}(s^t) = \pi(s^t) \cdot f(s^t) + \phi(s^{t-1}) \cdot b(s_t) + \theta(s^{t-1}) \cdot (q(s^t) + \pi(s^t) \cdot d(s_{t+1})) \\
&-(p(s^t) \cdot \phi(s^t) + q(s^t)\theta(s^t)) - \max_{\tilde{\phi} \in \mathbb{R}^S, \tilde{\theta} \geq 0} (p(s^t) \cdot \tilde{\phi} + q(s^t)\tilde{\theta}) \\
&\text{ s.t. } \tilde{\phi} \cdot b(s_{t+1}) + \tilde{\theta}(q(s^{t+1}) + \pi(s^{t+1}) \cdot d(s_{t+1})) + v_{a+1,h}(s^{t+1}) \geq 0 \text{ for all } s^{t+1} \succ s^t.
\end{aligned}$$

Using the induction hypothesis we can rewrite the side condition for the maximization as

$$\text{s.t. } (\phi(s^t) + \tilde{\phi}) \cdot b(s_{t+1}) + (\theta(s^t) + \tilde{\theta})(q(s^{t+1}) + \pi(s^{t+1}) \cdot d(s_{t+1})) + D_{a+1,h}(s^{t+1}) \geq 0 \text{ for all } s^{t+1} \succ s^t.$$

Defining $\hat{\phi} = (\phi + \tilde{\phi})$ and $\hat{\theta} = (\phi + \tilde{\theta})$, we get

$$\begin{aligned}
&v_{a,h}(s^t) = \pi(s^t) \cdot f(s^t) + \phi(s^{t-1}) \cdot b(s_t) + \theta(s^{t-1}) \cdot (q(s^t) + \pi(s^t) \cdot d(s_{t+1})) \\
&- \max_{\hat{\phi} \in \mathbb{R}^S, \hat{\theta} \geq 0} (p(s^t) \cdot \hat{\phi} + q(s^t)\hat{\theta}) \\
&\text{ s.t. } \hat{\phi} \cdot b(s_{t+1}) + \hat{\theta}(q(s^{t+1}) + \pi(s^{t+1}) \cdot d(s_{t+1})) + D_{a+1,h}(s^{t+1}) \geq 0 \text{ for all } s^{t+1} \succ s^t.
\end{aligned}$$

Comparing this representation of $v_{a,h}(s^t)$ with the definition of $D_{a,h}(s^t)$ shows that

$$v_{a,h}(s^t) = D_{a,h}(s^t) + \phi(s^{t-1}) \cdot b(s^t) + \theta(s^{t-1})(q(s^t) + \pi(s^t) \cdot d(s_t)),$$

which implies that $v_{a,h}(s^t) \geq 0$ is equivalent to the borrowing constraint (iii).

3.2 Recursive equilibrium

As before we aim to find equilibria for which $(s_t, \lambda(s^t))$ follow a Markov process. We say that the function $\Lambda : \mathcal{S} \times \mathcal{S} \times \mathbb{R}^{AH}_{+} \to \mathbb{R}^{AH}_{+}$ characterizes a recursive equilibrium if for each initial condition s^0 and $\lambda(s^0)$ there exists a financial markets equilibrium with $\lambda(s^{t+1}) = \Lambda(s_t, s_{t+1}, \lambda(s^t))$ for all s^t and $s^{t+1} \succ s^t$. In this case we can define $W_{a,h}(s_t, \lambda(s^t)) := w_{a,h}(s^t)$ and $V_{a,h}(s_t, \lambda(s^t)) := v_{a,h}(s^t)$.

In the case of incomplete markets, it is not obvious that the use of instantaneous Negishi weights as endogenous state variables provides large computational advantages. We discuss the issue in Section 4.1 below.

As in the complete markets case general existence of a recursive equilibrium is an open issue. However, it is easy to show that for $A = 2$ recursive equilibria always exist (this is in contrast to the case of the natural state space, where Kubler and Polemarchakis (2004) show that recursive equilibria might fail to exist even if $A = 2$. This follows directly from the existence of a so-called 'generalized Markov equilibrium' where additional endogenous variables enter the state.

3.2.1 Generalized Markov equilibria

Following the proof in Kubler and Polemarchakis (2004) or in Citanna and Siconolfi (2011), one can show that there exist correspondences $\mathbf{\Lambda} : \mathcal{S} \times \mathcal{S} \times \mathbb{R}_+^{AH} \rightrightarrows \mathbb{R}_+^{AH}$, $\mathbf{V} : \mathcal{S} \times \mathbb{R}_+^{AH} \rightrightarrows \mathbb{R}^{AH}$, and $\mathbf{W} : \mathcal{S} \times \mathbb{R}_+^{AH} \rightrightarrows \mathbb{R}^{AH}$, such that there exists a financial markets equilibrium with $\lambda(s^{t+1}) \in \mathbf{\Lambda}(s_t, s_{t+1}, \lambda(s^t))$ for all s^t and all $s^{t+1} \succ s^t$, where $w_{a,h}(s^t) \in \mathbf{W}_{a,h}(s_t, \lambda(s^t))$, $v_{a,h}(s^t) \in \mathbf{V}_{a,h}(s_t, \lambda(s^t))$. The key step of the proof[1] is to apply Proposition 1 in Kubler and Polemarchakis (2004) in our framework. To do this, one has to redefine the expectations correspondence using (E1)-(E4). The only technical difficulty then lies in showing existence of a T-horizon equilibrium for arbitrary first period Negishi-weights. It is standard to show existence for an open set of initial conditions (i.e. tree-holdings among the initially alive). The proof in Kubler and Polemarchakis (2004) can be directly applied. In order to show existence for arbitrary initial Negishi weights, one simply alters the best-response correspondence of the initially young. For each agent initially alive, by the intermediate value theorem, one can find a continuous map from prices and first period consumption to life-time consumption after the initial period and first period initial cash-at-hand so that with that cash-at-hand, given prices, life-time consumption maximizes utility given the budget constraints. This map is continuous in prices and a standard fixed point argument shows existence of an equilibrium for arbitrary first period consumption.

To show that there exists a 'generalized' Markov equilibrium where the state space is enlarged to contain not only instantaneous weights but also current values of v and w, we need to show that there exists a function $T(s, s', \lambda, v, w)$ such that

$$(\lambda(s^{t+1}), v(s^{t+1}), w(s^{t+1})) = T(s_t, s_{t+1}, \lambda(s^t), v(s^t), w(s^t))$$

for all s^t and all $s^{t+1} \succ s^t$. To do so, first note that we must have that for all s^t: $q(s^t) + d(s_t) = \sum_{(a,h)\in\mathcal{A}} \frac{w_{a,h}(s^t)}{\partial_{a,h}(s^t)}$. Second, note that the requirements (E1)-(E4) all only involve values of v, w, q and λ in the current period and at all nodes of the subsequent period. Therefore if there is an equilibrium that is not Markov in the enlarged state space (v, w, λ) we can construct a new equilibrium that is. I.e. if at two nodes σ and $\tilde{\sigma}$ with the same current shock s we have that $(v(\sigma), w(\sigma), \lambda(\sigma)) = (v(\tilde{\sigma}), w(\tilde{\sigma}), \lambda(\tilde{\sigma}))$ but there are direct successor σ_+ of σ and $\tilde{\sigma}_+$ of $\tilde{\sigma}$ with the same shock but different values for v, w, λ, i.e. $(v(\sigma_+), w(\sigma_+), \lambda(\sigma_+)) \neq (v(\tilde{\sigma}_+), w(\tilde{\sigma}_+), \lambda(\tilde{\sigma}_+))$, we can just replace all endogenous variables at the direct successor nodes of $\tilde{\sigma}$ by the endogenous variables of the direct successor nodes of σ. The conditions (E1)-(E4) must then be satisfied and since they are necessary and sufficient for equilibrium, this must be a new equilibrium that is Markov in the extended state space. Citanna and Siconolfi (2011) and Duffie et al (1994, Section 2.5)) develop a similar argument for the case of cash-at-hand as a state variable.

Finally note that in equilibrium we must always have that $\mathbf{W}_{1,h}(s, \lambda) = \{0\}$ and that there is

[1] A full proof is available from the authors upon request.

a function from $(s, \lambda, (v_{a,h}, w_{a,h})_{(a,h) \in \mathcal{A}_{-1}})$ to $(v_{1,h})_{h \in \mathcal{H}}$. As in the above argument, if in a given equilibrium there occur different values of $v_{1,h}$ for given $(s, \lambda, (v_{a,h}, w_{a,h})_{(a,h) \in \mathcal{A}_{-1}})$ we can construct a new equilibrium by picking one of them.

These observation gives rise to the following simple lemma.

LEMMA 1 *Given a generalised Markov equilibrium with* $\mathbf{V}(s, \lambda)$ *and* $\mathbf{W}(s, \lambda)$, *there exists a recursive equilibrium if* $\mathbf{V}_{a,h}(s, \lambda)$ *and* $\mathbf{W}_{a,h}(s, \lambda)$ *are single valued for all* $(a, h) \in \mathcal{A}_{-1}$.

It is noteworthy that in the case of no constraints, the correspondence $\mathbf{V}$ is irrelevant. In this case, $w_{a,h}(s^t)$ simply denotes agent (a, h)'s cash-at-hand at node s^t. If a given Negishi-weight can only be supported by a single value for cash-at-hand across agents there must be a recursive equilibrium.

3.2.2 Two period lived agents

It is easy to see that for the case $A = 2$ the existence of a generalized Markov equilibrium directly implies existence of a recursive equilibrium. This is because we obtain

$$\mathbf{V}_{2,h}(s, \lambda) = \{\frac{1}{\partial_{2,h}(s^t)} D_x u_{2,h}(X_{2,h}(s, \lambda)) \cdot (X_{2,h}(s, \lambda) - e_{2,h})\}, \text{ and}$$

$$\mathbf{W}_{2,h}(s, \lambda) = \{\frac{1}{\partial_{2,h}(s^t)} D_x u_{2,h}(X_{2,h}(s, \lambda)) \cdot (X_{2,h}(s, \lambda) - \omega_{2,h})\},$$

which shows that both correspondences are single valued. By Lemma 1 there must therefore exist a function $\Lambda(s, s, \lambda)$ that describes the equilibrium evolution of λ - a recursive equilibrium always exists.

This simple general existence result is in contrast to the case of the natural state space. In this case, as Kubler and Polemarchakis (2004) show, recursive equilibrium might fail to exist even if $A = 2$. However, this seems to be an artefact of the assumption of two-period lived agents. In particular, the result does not imply that there always exist minimal recursive equilibria. In fact the strategy from Kubler and Polemarchakis (2004) can be used to find simple counter-examples.

As it is well known, in the case without constraints and with complete financial markets, a model with A-period lived agents can always be reduced to a two period model (see e.g. Balasko et al. (1980)). For this two-period model, recursive equilibria always exist. However, if one recalls how the construction works it becomes clear that existence of a recursive equilibrium in the two-period reduction does not say more than the existence of a generalised Markov equilibrium. In the construction all agents born within $A - 1$ periods are grouped together. The state then consists of all these agents instantaneous Negishi weights – even if $H = 1$, we then have a $A - 1 + \sum_{a=1}^{A-1} S^a$ dimensional endogenous state space. This is clearly not in the spirit of recursive equilibrium.

3.2.3 Log-utility

We now assume as in Huffman (1987) that for all $a > 2$ and all h, $u_{a,h}(x) = \sum_{l=1}^{L} \alpha_{a,h,l} \log x_l$ and $\omega_{a,h} = 0$. For this very restricted case recursive equilibria always exist, independently of A, the market structure and borrowing constraints. Since for this case $D_x u(x) \cdot x$ is a constant, independent of x, it is easy to see that for $a > 2$, $v_{a,h}$ and $w_{a,h}$ are constants, just depending on (a, h) and the current state s. Therefore $\mathbf{V}_{a,h}(s, \lambda)$ and $\mathbf{W}_{a,h}(s, \lambda)$ must be single-valued for all $a > 1$ and all h. By the previous argument, a recursive equilibrium must always exist.

4 Two important special cases

It is useful to discuss in detail two important special cases of our model. First we assume that financial markets are incomplete but that agents face no constraints on their trades (i.e. neither borrowing constraints nor short-sale constraints on the stock). This case plays an important role in finance and macro-economics and we want to compare the computational burden of the Negishi approach with that of the standard approach that uses cash-at-hand as a state-variable.

Secondly, we examine the case where agents can trade in a complete set of Arrow securities but face borrowing constraints. We refer to this case as 'full-spanning'. Gottardi and Kubler (2012) examine this case for a model with infinitely lived agents and it turns out that competitive equilibria are often constrained inefficient. However, they also show that the assumption of gross substitutes guarantees existence if agents are infinitely lived. We show that this assumption suffices for existence of recursive equilibria in OLG models with borrowing constraints.

4.1 Unconstrained incomplete markets

We first assume that markets are incomplete but that there are no e-endowments and discuss computational issues. We modify the model slightly in that we allow for short-sales in the stock – while this can lead to failures of existence of a competitive equilibrium, these cases are non-robust and the assumption of unlimited short-sales allows us to focus on the consequences of missing financial markets.

Let $W_{a,h}(s, \lambda)$ be defined as in (3) and (4), and $\partial_{a,h}(s, \lambda) = \frac{\partial u_{a,h}(X_{a,h}(s,\lambda))}{\partial x_1}$.
Define prices by:

$$\pi(s, \lambda) := \frac{1}{\partial_{1,1}(s, \lambda)} D_x u_{1,1}(x_{a,h}(s, \lambda)), \quad q(s, \lambda) := \sum_{(a,h) \in \mathcal{A}} \frac{W_{a,h}(s, \lambda)}{\partial_{a,h}(s, \lambda)} - d(s),$$

$$p(s, \lambda) := \sum_{s' \succ s} \delta_{2,1}(s, s', \Lambda_{2,1}(s, s', \lambda)) \frac{\partial_{2,1}(s')}{\partial_{1,1}(s)} b(s').$$

A transition function $\Lambda(s, s', \lambda)$ with $\Lambda_{2,1}(s, s', \lambda) = \delta_{2,1}(s, s')\lambda_{1,1}$ defines a recursive equilibrium with incomplete markets if it satisfies, in addition to the requirement that $W_{1h}(s', \Lambda(s, s', \lambda)) = 0$

for all $h \in \mathcal{H}$, that for all $(a,h) \in \mathcal{A}_{-A} \setminus \{(1,1)\}$,

$$q(s,\lambda) = \sum_{s'} \frac{\delta_{a+1,h}(s,s')\lambda_{a,h}}{\Lambda_{a+1,h}(s,s',\lambda)} \cdot \delta_{2,1}(s,s')\partial_{2,1}(s',\Lambda(s,s',\lambda)) \left(\frac{q(s',\Lambda(s,s',\lambda))}{\partial_{1,1}(s,\lambda)} + \pi(s,\Lambda(s,s',\lambda))) \cdot d(s') \right) \tag{6}$$

$$p(s,\lambda) = \sum_{s'} \frac{\delta_{a+1,h}(s,s')\lambda_{a,h}}{\Lambda_{a+1,h}(s,s',\lambda)} \delta_{2,1}(s,s') \frac{\partial_{2,1}(s',\Lambda(s,s',\lambda))}{\partial_{1,1}(s,\lambda)} b(s'), \tag{7}$$

and if for each $(a,h) \in \mathcal{A}_{-A} \setminus \{(1,1)\}$ there exists $\phi \in \mathbb{R}^J$ and $\theta \in \mathbb{R}$ such that

$$\frac{W_{a+1,h}(s',\Lambda(s,s',\lambda))}{\partial_{a+1,h}(s',\Lambda(s,s',\lambda))} = \theta_{a,h} \left(q(s',\Lambda(s,s',\lambda)) + \pi(s,\Lambda(s,s',\lambda)) \cdot d(s') \right) + \phi_{a,h} \cdot b(s'). \tag{8}$$

Note that the definition simplifies considerably if there exist Arrow-securities for some states but not for others. For the shocks where agents can trade in an Arrow security the instantaneous Negishi-weight evolves according to the same rule as for complete markets. We want to argue that even outside of this special case, using Negishi-weights faciliates the computation of equilibria. As Chien et al. (2011) point out the use of Negishi-weights also has large advantages in models with limited market participation. If some agents can trade in a complete set of Arrow securities while others only have access to a limited set of assets our definition of recursive equilibrium simplifies for those who trade in Arrow-securities.

4.1.1 Computation

The computational algorithm is as in Section 2 above with the big difference that in Step 2 one needs to solve a more complicated system of non-linear equations. Instead of solving a system with H equations and unknowns we now need to solve $H + ((A-1)H-1)S + ((A-1)H-1)J$ equations. The additional equations arise because of missing financial markets. Given $\hat{W}^n$, for a given shock s and a given collocation-point $\tilde{\lambda}^i$, one needs to solve

$$D_x u_{1,h}(X_{1,h}(s,\lambda^i(s))) \cdot (X_{1,h}(s,\lambda^i(s)) - \omega_{1,h}) + \sum_{s'} \delta_{2,h}(s,s')\hat{W}^n(s',\hat{\Lambda}(s,s',\tilde{\lambda}^i(s'))) = 0, \;\; h \in \mathcal{H}$$

together with (6), (7) and (8), resulting in $H + ((A-1)H-1)(J+1) + ((A-1)H-1)S$ equations in the unknowns

$$(\lambda^i_{1,h})_{h\in\mathcal{H}}, \;\hat{\Lambda}(s,s',\lambda^i), \text{ and } (\theta_{a,h},\phi_{a,h})_{(a,h)\in\mathcal{A}_{-A}\setminus\{(1,1)\}},$$

where $\hat{\Lambda}(s,s',\lambda^i)$ just consists of $\lambda_{a,h}$ for $(a,h) \in \mathcal{A}_{-1} \setminus \{(2,1)\}$. A significant efficiency gain can be obtained by noting that for given $(\lambda_{1,h})_{h\in\mathcal{H}}$ and $(\theta_{ah},\phi_{a,h})$, Equations (8) can be solved separately to obtain $\hat{\Lambda}(s,s',\lambda^i)$. Along the iteration, this can typically be done efficiently using starting-points from previous iterations. Taken as given a map from $(\lambda_{1,h})_{h\in\mathcal{H}}$ and $(\theta_{ah},\phi_{a,h})$ to $\hat{\Lambda}(s,s',\lambda^i)$, the computational burden reduces to solving $H + ((A-1)H-1)J$ equations and unknowns.

This is comparable to the case of cash-at-hand where one needs to solve for agents' cash-at-hand in the subsequent period – since prices depend on cash-at-hand this is also a fixed-point problem that gives cash-at-hand at all S shocks next period given the portfolios today. However, for the case of one good, the number of equations one needs to solve in this case might actually be slightly smaller since one does not need to solve for $(\lambda_{1,h})_{h\in\mathcal{H}}$ resulting in H fewer equations and unknowns. Yet one does need to solve for $J+1$ prices, but if J is small and H is large the resulting system might be smaller.

However, for the case of several commodities, the number of equations to be solved in the Negishi-approach is independent of L while each additional good results in $H+1$ additional equations if one uses cash-at-hand. It is clear that if there are several goods, the Negishi-approach results in large efficiency gains.

It is an open question whether the Negishi-approach has significant advantages for the case of one commodity. Dumas and Lyasoff (2011) argue that for infinitely lived agents this is the case – it is certainly true that the problem of bounding the endogenous state space might turn out to be insurmountable if one uses cash-at-hand as the state. On the other hand, it is no longer true that the number of unknown functions is much smaller for the Negishi approach than it is for the 'natural state-space' approach.

4.2 Full spanning with constraints

In this subsection, we show that in the presence of a full set of Arrow-securities the model with borrowing constraints is tractable and in many respects quite similar to the complete markets model. It is useful to first spell out how our definition of recursive equilibrium simplifies in the case of full spanning. As before it is useful to evaluate all consumptions in terms of agents' marginal utility. In the case of complete markets, i.e. full spanning without constraints, this was straightforward. With constraints, we have to change the setup slightly and modify the definition of the functions $W(s,.)$.

Note that for all nodes s^t it follows from the first order conditions of the planner problem that $\lambda_{a,h}(s^t)D_x u_{a,h}(X_{a,h}(s_t,\lambda(s^t)))$ are identical across all agents alive at that node. Furthermore, we must have that for each subsequent node there is at least one agent that is unconstrained. For this agent we have that $\lambda_{a+1,h}(s^{t+1}) = \delta_{a+1,h}(s_t,s_{t+1})\lambda_{a,h}(s^t)$. Therefore we must have that for any agent, throughout his life-time, $\lambda_{a,h}D_x u_{a,h}$ are collinear to market-prices even if $D_x u_{a,h}$ are not because of constraints. We define for all types $h\in\mathcal{H}$

$$
\begin{aligned}
W_{A,h}(s,\lambda) &= \lambda_{a,h}D_x u_{a,h}(X_{A,h}(s,\lambda))\cdot(X_{A,h}(s,\lambda)-\omega_{A,h}(s)), \text{ and} \\
W_{a,h}(s,\lambda) &= \lambda_{a,h}D_x u_{a,h}(X_{a,h}(s,\lambda))\cdot(X_{a,h}(s,\lambda)-\omega_{a,h}(s))+\sum_{s'}W_{a+1,h}(s',\Lambda(s,s',\lambda))
\end{aligned}
$$

for all $a = 1, ..., A-1$. Similarly, we define the value of excess consumption to be

$$\begin{aligned} V_{A,h}(s,\lambda) &= \lambda_{A,h} D_x u_{a,h}(X_{A,h}(s,\lambda)) \cdot (X_{A,h}(s,\lambda) - e_{A,h}(s)), \text{ and} \\ V_{a,h}(s,\lambda) &= \lambda_{a,h} D_x u_{a,h}(X_{a,h}(s,\lambda)) \cdot (X_{a,h}(s,\lambda) - e_{a,h}(s)) + \sum_{s'} V_{a+1,h}(s', \Lambda(s,s',\lambda)) \end{aligned}$$

for all $a = 1, ..., A-1$. We then have the following definition of a recursive equilibrium.

Definition 2 *A recursive equilibrium is a function* $\Lambda : \mathcal{S} \times \mathcal{S} \times \mathbb{R}^{AH}_{++} \to \mathbb{R}^{AH}_{++}$ *that satisfies for all* $h \in \mathcal{H}$, *all* $\lambda \in \mathbb{R}^{AH}_{++}$, *all* $s, s' \in \mathcal{S}$, *and some* $\eta \in \mathbb{R}^{AH}_{+}$

$$W_{1,h}(s', \Lambda(s,s',\lambda)) = 0, \ \Lambda_{1,h}(s,s',\lambda) = \eta_{1,h}, \ \textit{and}$$

$$\Lambda_{a,h}(s,s',\lambda) = \delta_{a,h}\lambda_{a-1,h} + \eta_{a,h},$$

$$V_{a,h}(s', \Lambda(s,s',\lambda)) \geq 0, \ V_{a,h}(s', \Lambda(s,s',\lambda))\eta_{a,h} = 0 \ \textit{for all } a = 2, ..., A.$$

Note that this definition is substantially simpler than the requirements (E1)-(E4) above. The fact that there is a full set of Arrow-securities and that for each Arrow-security at least one of the agents is always unconstrained simplifies the characterization of equilibrium substantially. As in the complete markets case, we define a recursive equilibrium to be *minimal* if there is a function $\ell(s, (\lambda_{a,h})_{(a,h)\in\mathcal{A}_{-1}})$ so that $\Lambda_{1,h}(s,s',\lambda) = \ell_h\left(s', (\Lambda_{a,h}(s,s',\lambda))_{(a,h)\in\mathcal{A}_{-A}}\right)$ for all $h \in \mathcal{H}$.

4.2.1 Gross substitutes

For the case of full spanning with constraints we can show that the assumption of gross substitutes ensures that minimal recursive equilibria always exist (as in Gottardi and Kubler (2012) for the case of infinitely lived agents and a single commodity). The following definition is standard.

Definition 3 *A function* $F : \mathbb{R}^m_+ \to \mathbb{R}^m$ *satisfies the strict gross substitute property if for all* $y \in \mathbb{R}^m_+$ *and all* $x \in \mathbb{R}^m_+$ *with* $x_i = 0$ *for some* $i = 1, ..., m$ *it holds that* $F_i(y) > F_i(y+x)$. *It satisfies the weak gross substitute property if the inequality holds weakly.*

The gross substitutes assumption is easy to verify if there is a single commodity per state, but we formulate the result generally, for L commodities. In order to state a gross substitute assumption for the case of several commodities it is useful to define the set of fundamentals, $\mathcal{F} \subset \mathbb{R}^L_{++}$, to be the set of all possible realization of endowments and dividends, i.e. $\mathcal{F} = \{e_{a,h}(s), f_{a,h}(s), d(s); (a,h) \in \mathcal{A}, s \in \mathcal{S}\}$. We have the following assumption on Bernoulli functions across all agents.

Assumption 1 *For all agents* (a,h) *and all shocks* $s \in \mathcal{S}$, *the function* $D_x u_{a,h}(X(s,\lambda)) \cdot X(s,\lambda)$ *satisfies the weak gross substitute property in* λ. *Moreover, for each* $y \in \mathcal{F}$ *the function* $-D_x u_{a,h}(X(s,\lambda)) \cdot y$ *satisfies the strict gross substitute property in* λ.

For the case of a single good, Dana (1994) uses these assumptions to show uniqueness in a model with complete markets. It is easy to see that for this case, the assumption is satisfied whenever all agents' relative risk aversion is less than or equal to one. For the case of several commodities the assumption is difficult to verify if $u_{a,h}(.)$ is non-separable, but it is a well-defined assumption on fundamentals.

The assumption guarantees that minimal recursive equilibria always exist. To see this, note that existence of a minimal recursive equilibrium can only fail if given some initial shock, s_0 there exist two distinct competitive equilibria $(\lambda(\sigma), \lambda'(\sigma))_{\sigma\in\Sigma}$ with $\lambda_{a,h}(s_0) = \lambda'_{a,h}(s_0)$ for all $(a,h) \in \mathcal{A}_{-1}$. Putting it differently, we need to rule out that given an initial 'state' $(s_0, (\lambda_{a,h}(s_0))_{(a,h)\in\mathcal{A}_{-1}})$ there can be two different 'continuation equilibria'. Suppose to the contrary, that there exist two competitive equilibria $(\lambda(\sigma), \lambda'(\sigma))_{\sigma\in\Sigma}$ with $\lambda'(s^1) \neq \lambda(s^1)$ for at least one node at $t = 1$, while $\lambda(s_0) = \lambda'(s_0)$. Define $\underline{\lambda}_{a,h}(s^t) = \min[\lambda_{a,h}(s^t), \lambda'_{a,h}(s^t)]$ for all $(a,h) \in \mathcal{A}$ and all s^t. We will show below that the sequence of Negishi weights $(\underline{\lambda}(\sigma))_{\sigma\in\Sigma}$ does not lead to a feasible consumption allocation, which contradicts the assumption that there exist two equilibria $(\lambda(\sigma))_{\sigma\in\Sigma} \neq (\lambda'(\sigma))_{\sigma\in\Sigma}$ as characterized above.

Again, define $v(s^t; (\lambda(\sigma)))$ and $w(s^t; (\lambda(\sigma)))$ as the (possibly non-Markovian) values of e- and f-endowments, where the definition follows the definition of V and W, i.e. for example for $a < A$,

$$w_{a,h}(s^t; (\lambda(\sigma))) = \lambda_{a,h}(s^t) D_x u_{a,h}(X_{a,h}(s_t, \lambda(s^t)))\cdot(X_{a,h}(s_t, \lambda(s^t)) - \omega_{a,h}(s_t)) + \sum_{s^{t+1}\succ s^t} w_{a+1,h}(s^{t+1}; (\lambda(\sigma)))$$

We have the following two lemmas.

LEMMA 2 *For all s^t and for any $(a,h) \in \mathcal{A}$ and any $y \in \mathcal{F}$, it must be true that*

$$\underline{\lambda}_{a,h}(s^t) D_x u_{a,h}(X_{a,h}(s_t, \underline{\lambda}(s^t))) \cdot y \leq$$
$$\min\left[\lambda_{a,h}(s^t) D_x u_{a,h}(X_{a,h}(s_t, \lambda(s^t))) \cdot y, \lambda'_{a,h}(s^t) D_x u_{a,h}(X_{a,h}(s_t, \lambda'(s^t))) \cdot y\right].$$

Proof. W.l.o.g. take $\lambda_{a,h}(s^t) \leq \lambda'_{a,h}(s^t)$, thus $\underline{\lambda}_{a,h}(s^t) = \lambda'_{a,h}(s^t)$. Since $-D_x u_{a,h}(X_{a,h}(s,.)) \cdot y$ satisfies the gross substitute property, we have

$$\underline{\lambda}_{a,h}(s^t) D_x u_{a,h}(X_{a,h}(s_t, \underline{\lambda}(s^t))) \cdot y \leq \lambda_{a,h}(s^t) D_x u_{a,h}(X_{a,h}(s_t, \lambda(s^t))) \cdot y.$$

Moreover, define $\hat{\lambda}$ by $\hat{\lambda}_{a',h'} = \underline{\lambda}_{a',h'}(s^t)$ for $(a',h') \neq (a,h)$ and $\hat{\lambda}_{a,h} = \lambda'_{a,h}(s^t)$. By the gross substitute property we must have

$$\begin{aligned} \lambda'_{a,h}(s^t) D_x u_{a,h}(X_{a,h}(s_t, \lambda'(s^t))) \cdot y &\geq \hat{\lambda}_{a,h}(s^t) D_x u_{a,h}(X_{a,h}(s_t, \hat{\lambda}(s^t)) \cdot y \\ &\geq \underline{\lambda}_{a,h}(s^t) D_x u_{a,h}(X_{a,h}(s_t, \underline{\lambda}(s^t))) \cdot y, \end{aligned}$$

where the second inequality follows from the fact that $\hat{\lambda}_{a,h}(s^t) D_x u_{a,h}(X_{a,h}(s_t, \hat{\lambda}(s^t)))$ is identical across all agents, (a,h), and similarly for $\underline{\lambda}$. □

Lemma 3 *For all s^t and all (a,h),*

$$w_{a,h}(s^t;(\underline{\lambda}(\sigma))) \geq \min\left[w_{a,h}(s^t;(\lambda(\sigma))), w_{a,h}(s^t;(\lambda'(\sigma)))\right], \tag{9}$$

with the inequality holding strict for some s^t and some h.

Proof. Applying Assumption 1 and Lemma 2 to

$$v_{A,h}(s^t;\underline{\lambda}(\sigma)) = \underline{\lambda}_{A,h}(s^t) D_x u_{A,h}(X_{A,h}(s_t,\underline{\lambda}(s^t))) \cdot (X_{A,h}(s_t,\underline{\lambda}(s^t)) - e_{a,h}(s_t)),$$

we find that the following is satisfied for $a = A$:

$$v_{a,h}(s^t;\underline{\lambda}(\sigma)) \geq \begin{cases} v_{a,h}(s^t;\lambda(\sigma)), & \text{if } \underline{\lambda}_{A,h} = \lambda_{A,h}, \\ v_{a,h}(s^t;\lambda'(\sigma)), & \text{if } \underline{\lambda}_{A,h} = \lambda'_{A,h}, \end{cases} \quad \text{for all } \sigma \text{ and } h. \tag{10}$$

We now show that if (10) holds for $a+1$, then it also does for a. Suppose w.l.o.g. that $\underline{\lambda}_{a,h}(s^t) = \lambda_{a,h}(s^t)$. For each $s^{t+1} \succ s^t$ there are two cases possible. In the first case, $\lambda_{a+1,h}(s^{t+1}) = \delta_{a+1,h}(s_{t+1},s_t)\lambda_{a,h}(s^t)$, then $\underline{\lambda}_{a+1,h}(s^{t+1}) = \lambda_{a+1,h}(s^t)$, and thus $v_{a+1,h}(s^{t+1};\underline{\lambda}(\sigma)) \geq v_{a+1,h}(s^{t+1};\lambda(\sigma))$ by the induction hypothesis. In the second case, $\lambda_{a+1,h}(s^{t+1}) > \delta_{a+1,h}(s_{t+1},s_t)\lambda_{a,h}(s^t)$, then $v_{a+1,h}(s^{t+1};\underline{\lambda}(\sigma)) \geq v_{a+1,h}(s^{t+1};\lambda(\sigma)) = 0$. Summing up, we find that (10) holds for a.
Again suppose w.l.o.g. that $\underline{\lambda}_{a,h}(s^t) = \lambda_{a,h}(s^t)$. By (10) and Lemma 2, we have for all σ and h:

$$\begin{aligned} w_{a,h}(s^t;\underline{\lambda}(\sigma)) &= v_{a,h}(s^t;\underline{\lambda}(\sigma)) - \sum_{i=0}^{A-a-1} \sum_{s^{t+i} \succeq s^t} \underline{\lambda}_{i,h}(s^{t+i}) D_x u_{a+i,h}(X_{a+i,h}(s_{t+i},\underline{\lambda}(s^{t+i})) \cdot f_{a+i,h}(s_{t+i}) \\ &\geq v_{a,h}(s^t;\lambda(\sigma)) - \sum_{i=0}^{A-a-1} \sum_{s^{t+i} \succeq s^t} \lambda_{i,h}(s^{t+i}) D_x u_{a+i,h}(X_{a+i,h}(s_{t+i},\lambda(s^{t+i})) \cdot f_{a+i,h}(s_{t+i}) \\ &= w_{a,h}(s^t;\lambda(\sigma)). \end{aligned}$$

This finishes the proof. □

The equilibrium conditions require that

$$w_{1,h}(s_0;(\lambda(\sigma)) = 0, \sum_{(a,h)\in\mathcal{A}_{-1}} w_{a,h}(s_0,\lambda(\sigma)) - \lambda_{a,h}(s_0) D_x u_{a,h}(c_{a,h}(s_0))\left(q(s_0;(\underline{\lambda}(\sigma))) + d(s_0)\right) = 0,$$

and similarly for λ'. Using Lemmas 2 and 3, this implies that

$$\sum_{(a,h)\in\mathcal{A}} w_{a,h}(s_0,\underline{\lambda}(\sigma)) - \underline{\lambda}_{a,h}(s_0) D_x u_{a,h}(c_{a,h}(s_0))\left(q(s_0;(\underline{\lambda}(\sigma))) + d(s_0)\right) > 0, \tag{11}$$

where we use that

$$q(s_0;(\lambda(\sigma))) = \sum_{t'>0} \sum_{s^{t'} \succ s_0} \frac{D_x u_{1,1}(X_{1,1}(s'_t,\lambda(s^{t'}))}{D_x u_{1,1}(X_{1,1}(s_0,\lambda(s_0)))} \lambda_{1,1}(s^{t'}) d(s_{t'}).$$

As (11) is a contradiction to feasibility of $(\underline{\lambda}(\sigma))_{\sigma\in\Sigma}$ we have proved that there cannot be two continuation equilibria and thus there exists a minimal recursive equilibrium.

4.2.2 Computation

As before we consider a simple time-iteration collocation method. Since there are important differences to the unconstrained case, it is useful to describe the algorithm in some detail. The main steps are as follows.

1. Set $n = 0$ and start with initial guesses $\hat{W}^0 : \mathcal{S} \times \mathbb{R}_+^{(A-1)H} \to \mathbb{R}^{(A-1)H}$ and $\hat{V}^0 : \mathcal{S} \times \mathbb{R}_+^{(A-1)H} \to \mathbb{R}^{(A-1)H}$.

2. Given $\hat{V}^n$, for each $s \in \mathcal{S}$ and each $\tilde{\lambda}^i \in \mathcal{G}$, compute $\hat{\eta}^n(s, \tilde{\lambda}^i)$ as a solution to the non-linear complementarity problem

$$\hat{V}^n(s, \tilde{\lambda}^i + \hat{\eta}^n(s, \tilde{\lambda}^i)) \geq 0, \quad \hat{\eta}^n(s, \tilde{\lambda}^i)) \geq 0$$
$$\hat{V}^n_{a,h}(s, \tilde{\lambda}^i + \hat{\eta}^n(s, \tilde{\lambda}^i))\hat{\eta}^n_{a,h}(s, \tilde{\lambda}^i)) = 0 \text{ for all } (a,h) \in \mathcal{A}_{-1}$$

   Interpolate $\{\hat{\eta}^n(s, \tilde{\lambda}^i), i = 1, ..., G\}$ to obtain approximating functions $\hat{\eta}^n(s, .)$.

3. Given $\hat{\eta}^n$, $\hat{V}^n$ and $\hat{W}^n$, for each $s \in \mathcal{S}$ and each $\tilde{\lambda}^i \in \mathcal{G}$, compute $\hat{\ell}^{n+1}(s, \tilde{\lambda}^i)$ as the solution to

$$\hat{\ell}^{n+1}(s, \tilde{\lambda}^i) D_x u_{1,h}(X_{1,h}(s, \lambda^i)) \cdot (X_{1,h}(s, \lambda^i) - \omega_{1,h}) + \sum_{s'} \delta_{2,h}(s, s') \hat{W}^n_{2,h}(s', \tilde{\lambda}(s')) = 0, \ h \in \mathcal{H},$$

   where $\lambda^i = (\hat{\ell}^{n+1}(s, \tilde{\lambda}^i), \tilde{\lambda}^i)$ and $\tilde{\lambda}^i(s') = \left(\delta_{a,h}(s, s')\lambda^i_{a-1,h} + \hat{\eta}^n(s', \delta_{a,h}(s, s')\lambda^i_{a,h})\right)_{(a,h)\in\mathcal{A}_{-1}}$. With these values for λ^i and $\tilde{\lambda}^i(s')$ compute for all $(a,h) \in \mathcal{A}_{-1}$

$$\hat{W}^{n+1}_{a,h}(s, \tilde{\lambda}^i) = \lambda_{a,h} D_x u_{a,h}(X_{a,h}(s, \lambda^i)) \cdot (X_{a,h}(s, \lambda^i) - \omega_{a,h}) + \sum_{s'} \delta_{a+1,h}(s, s') \hat{W}^n_{a+1,h}(s', \tilde{\lambda}^i(s')),$$

$$\hat{V}^{n+1}_{a,h}(s, \tilde{\lambda}^i) = \lambda_{a,h} D_x u_{a,h}(X_{a,h}(s, \lambda^i)) \cdot (X_{a,h}(s, \lambda^i) - e_{a,h}) + \sum_{s'} \delta_{a+1,h}(s, s') \hat{V}^n_{a+1,h}(s', \tilde{\lambda}^i(s')),$$

   where $\hat{W}^{n+1}_{A+1,h}(s, \tilde{\lambda}^i) := 0$, and $\hat{V}^{n+1}_{A+1,h}(s, \tilde{\lambda}^i) := 0$.

4. For each $s \in \mathcal{S}$, interpolate $\{\hat{W}^{n+1}(s, \tilde{\lambda}^i), i = 1, ..., G\}$ to obtain approximating functions $\hat{W}^{n+1}(s, .)$ and interpolate $\{\hat{V}^{n+1}(s, \tilde{\lambda}^i), i = 1, ..., G\}$ to obtain approximating functions $\hat{V}^{n+1}(s, .)$.

5. Check some error criterion. If error criterion not met, increase n by 1 and go to 2.

6. Set $\hat{W}^* = \hat{W}^{n+1}$, $\hat{V}^* = \hat{V}^{n+1}$ and interpolate $\hat{\ell}^n(s, \tilde{\lambda}^i)$ to obtain $\hat{\ell}^*(s, .)$.

While the algorithm appears more complicated than for the case of complete markets, it is actually only Step 2 that is different and that is computationally expensive. For many realistic calibrations one can expect the borrowing constraint to bind rarely for older agents which can simplify the computations in Step 2 considerably.

Under the gross substitute assumption from above, we can prove that the algorithm converges under the idealized scenario where the function approximation is exact. To formalize this, let

$W^0_{a,h}(\lambda, s) = \lambda_{a,h} D_x u_{a,h}(X_{a,h}(s,\lambda)) \cdot (X_{a,h}(s,\lambda) - \omega_{a,h})$ and $V^0_{a,h}(\lambda, s) = \lambda_{a,h} D_x u_{a,h}(X_{a,h}(s,\lambda)) \cdot (X_{a,h}(s,\lambda) - e_{a,h})$ and define $\eta^n(s,\lambda)$ as well as $V^n(s,\lambda)$ and $W^n(s,\lambda)$ to solve Steps 2 and 3 of the algorithm exactly (i.e. for all λ). We prove the following theorem in the Appendix.

THEOREM 1 *Under Assumption 1, for each $n = 1, \ldots$ the functions V^n and W^n are well defined and as $n \to \infty$, $V^n \to V^*$ and $W^n \to W^*$ for some function V^*, W^* that describe a recursive equilibrium.*

5 Interpretation

The planner's maximization problem (1) obviously only has a solution if Pareto-weights are summable. While this is guaranteed in the presence of a Lucas tree, Pareto-efficiency of equilibrium allocations itself is not enough to ensure this. However, in the absence of a tree as long as markets are complete, one can always, even if the equilibrium allocation is dynamically inefficient work with the concept of Malinvaud efficiency (see e.g. Aliprantis et al. (1990)). An allocation is Malinvaud efficient if there exist no Pareto-dominating allocation that differs at only finitely many nodes. For these allocations one can construct Negishi-weights through a limit argument. The analysis in Section 2 goes through without any changes.

More interestingly, in models with incomplete markets and in models with borrowing constraints (see Gottardi and Kubler (2012)) competitive equilibria are typically constrained suboptimal – there is no social planner's problem that determines the equilibrium allocation. Nevertheless, we show in this paper that it can be useful to employ (instantaneous) Negishi weights as a state variable. The question then arises what these weights 'represent' and how errors in the computations can be interpreted economically (recall that for the case of complete markets and efficient equilibria a straightforward interpretation was possible). Although sequential equilibria are not (constrained) Pareto-efficient for the model where agents live for A periods, they are efficient if we reformulate the model and assume that agents only live for one period while *dynasties* live for A periods. Within a dynasty, an agent derives utility from his consumption and the consumption of his successors in the dynasty. But of course, Pareto-optimality means that it is impossible to improve all agents, not all dynasties. More formally, an agent is now identified by the date-event of the birth of his dynasty, s^t, by the age of his dynasty, a and by the type, h. He derives utility from his own consumption and the consumption of successors within the dynasty:

$$U_{a,h,s^t}(x) = \delta_{a,h}(s^t) u_{a,h}(x(s^t)) + \sum_{i=a+1}^{A} \delta_{i,h}(s^{t+i}) u_{i,h}(x(s^{t+i})).$$

We say that an allocation $(x_{a,h}(s^t))$ is D-Pareto-efficient if there is no allocation where all agents (a, h, s^t) are weakly better off and some strictly.

Given a summable sequence of instantaneous Negishi-weights $(\lambda(\sigma))_{\sigma\in\Sigma}$ the allocation $(x(\sigma))$ with $x_{a,h}(s^t) = X_{a,h}(s_t, \lambda(s^t))$ for all (a, h) and all s^t must be the solution to the maximization problem

$$\max_x \sum_{(a,h)\in\mathcal{A},s^t} \lambda_{a,h}(s^t)u_{a,h}(x_{a,h}(s^t)) \text{ s.t. } \sum_{(a,h)\in\mathcal{A}} x_{a,h}(s^t) = \bar{\omega}(s_t) \text{ for all } s^t,$$

which can be rewritten as

$$\max_x \sum_{(a,h)\in\mathcal{A},s^t} \eta_{a,h}(s^t)U_{a,h,s^t}(x) \text{ s.t. } \sum_{(a,h)\in\mathcal{A}} x_{a,h}(s^t) = \bar{\omega}(s_t) \text{ for all } s^t,$$

where

$$\eta_{1,h}(s^t) = \lambda_{1,h}(s^t), \quad \eta_{a,h} = \lambda_{a,h}(s^t) - \delta_{a,h}(s_{t-1}, s_t)\lambda_{a-1,h}(s^{t-1}), a = 2, \ldots, A.$$

An allocation is D-Pareto-efficient if for summable $(\lambda_{a,h}(s^t))$ the resulting weights satisfy $\eta_{a,h}(s^t) \geq 0$ for all $(a,h) \in \mathcal{A}$ and all s^t. Since we can normalize instantaneous Negishi weights node by node, the requirement that $(\lambda_{a,h}(\sigma))$ are summable in itself is vacuous – given a sequence that is not summable, we can define a new sequence that characterizes the same allocation and is summable. However, the crucial requirement is that the resulting $\eta_{a,h}(\sigma)$ are non-negative, i.e. that $\lambda_{a,h}(s^t) \geq \delta_{a,h}(s_t, s_{t-1})\lambda_{a-1,h}(s^{t-1})$ for all $(a,h) \in \mathcal{A}_{-1}$ and all s^t. This alone imposes no restrictions, as any allocation characterized by a sequence of Negishi-weights is D-Malinvaud efficient, in the sense that there is no other allocation that D-Pareto-dominates it and differs only at finitely many nodes.

When there is a complete set of Arrow-securities our construction of λ in the definition of recursive equilibrium ensures that the welfare weights for the dynasty model, $(\eta_{a,h}(s^t))$, are always positive. Moreover, the weights must be summable because the price of the tree is finite and we have

$$\lambda_{1,1}(s^t)\partial_{1,1}(s^t)q(s^t) \geq \sum_{\sigma \succ s^t} \lambda_{1,1}(\sigma)\partial_{1,1}(\sigma)d(\sigma).$$

When markets are incomplete the allocation might fail to be D-Pareto optimal and welfare weights need to be interpreted as limits as in the Malinvaud case. Different assumptions on financial markets and borrowing constraints then simply translate into different restrictions on bequest. To see that it is useful to define an agent's optimization problem recursively. Given prices $(q(\sigma), p(\sigma), \pi(\sigma))$, define $\mathbf{U}_{A+1,h}(s^t, \kappa) = 0$, and for $a = 1, ..., A$, define

$$\begin{aligned}
\mathbf{U}_{a,h}(s^t, \kappa) \quad = \quad & \max_{x\in\mathbb{R}^L_+,\theta\geq 0,\phi} u_{a,h}(x) + \sum_{s^{t+1}\succeq s^t} \delta_{a+1,h}(s_t, s_{t+1})\mathbf{U}_{a+1,h}(s^{t+1}, \kappa(s^{t+1})) \text{ subject to} \\
& \kappa + \pi(s^t)\cdot\omega_{a,h}(s_t) = \pi(s^t)\cdot x + q(s^t)\theta + p(s^t)\cdot\phi \\
& \kappa(s^{t+1}) = \theta(q(s^{t+1}) + \pi(s^t)\cdot d(s_{t+1})) + \phi\cdot b(s_{t+1}) \text{ for all } s^{t+1} \succeq s^t \\
& \kappa(s^{t+1}) \in \mathcal{K}_{a+1,h}(s^{t+1}),
\end{aligned}$$

where $\mathcal{K}(s^t)$ is some set that can depend on current and future prices as well as on agents' endowments. A competitive equilibrium for the dynasty economy consists of asset prices and agents' choices, i.e. consumption choices and bequest-portfolios so that markets clear and all agents maximize their utility. It is easy to see that depending on the market structure and on the specification of the sets $\mathcal{K}(s^t)$ we can construct economies for which equilibrium allocations and prices will be identical to the ones in the various OLG economies considered in this paper.

As in the case discussed in Section 2, approximation errors in computations can now be interpreted as transfers necessary to obtain the computed D-efficient allocation as an equilibrium allocation.

6 Appendix A: Proof of Theorem 1

To prove the theorem we require two lemmas. Yet first of all, we introduce the following notation: For $\alpha, x, y \in \mathbb{R}^{AH}$ we write $x = (x_{a,h})_{(a,h)\in\mathcal{A}}$ and define

$$z = x \oplus_\alpha y \Leftrightarrow \text{ for all } h = 1, \ldots, H : z_{1,h} = y_{1,h} \text{ and } z_{a,h} = \alpha_{a,h} x_{a-1,h} + y_{a,h},\ a = 2, ..., A.$$

Lemma 4 *Suppose $F : \mathbb{R}^{AH}_+ \to \mathbb{R}^{AH}$ satisfies the strict gross substitute property and is homogeneous of degree zero. Given any $x, \alpha \in \mathbb{R}^{AH}_{++}$ suppose there exist $\eta, \eta' \in \mathbb{R}^{AH}_+ \setminus \mathbb{R}^{AH}_{++}$ with $\eta_{a,h} > 0, \eta'_{a,h} > 0$ for some $(a,h) \in \mathcal{A}$. If $F(x \oplus_\alpha \eta) \geq 0$, $F(x \oplus_\alpha \eta') \geq 0$ and $\eta_{a,h} F_{a,h}(x \oplus_\alpha \eta) = 0$, $\eta'_{a,h} F_{a,h}(x \oplus_\alpha \eta') = 0$, for all $(a,h) \in \mathcal{A}$, then $\eta = \eta'$.*

Proof. Suppose to the contrary that $\eta, \eta' \notin \mathbb{R}^{AH}_{++}$ and $\eta \neq \eta'$. Then there is an (a,h) and a $\xi > 0$ such that $\eta_{a,h} > 0$, $(x \oplus_\alpha \eta)_{a,h} = \xi (x \oplus_\alpha \eta')_{a,h}$, and $\xi(x \oplus_\alpha \eta') > x \oplus_\alpha \eta$. The latter inequality holds strict because $\eta \neq \eta'$ and because both are not strictly positive. But since $F(.)$ is homogeneous of degree zero we must have that

$$F_{a,h}\left(\xi(x \oplus_\alpha \eta')\right) = F_{a,h}(x \oplus_\alpha \eta') \geq 0.$$

On the other hand, by the strict gross substitute property and since $\eta_{a,h} > 0$ we must have

$$F_{a,h}\left(\xi(x \oplus_\alpha \eta')\right) < F_{a,h}(x \oplus_\alpha \eta) = 0,$$

which is a contradiction. □

Lemma 5 *Suppose $F : \mathbb{R}^{AH}_+ \to \mathbb{R}^{AH}$ satisfies the strict gross substitute property and is homogeneous of degree zero. For any $\alpha, x, y \in \mathbb{R}^{AH}_{++}$ with $y > x$ suppose there exist $\eta^x, \eta^y \in \mathbb{R}^{AH}_+$ such that $F(x \oplus_\alpha \eta^x) \geq 0$, $F(y \oplus_\alpha \eta^y) \geq 0$ and $\eta^x_{a,h} F_{a,h}(x \oplus_\alpha \eta^x) = 0$ and $\eta^y_{a,h} F_{a,h}(y \oplus_\alpha \eta^y) = 0$ for all $(a,h) \in \mathcal{A}$. If $x_{a,h} = y_{a,h}$ for some a, h then it must hold that*

$$F_{a,h}(x \oplus_\alpha \eta^x) \geq F_{a,h}(y \oplus_\alpha \eta^y).$$

Proof. If $\eta^y_{a,h} > 0$ or if $\eta^y_{a,h} = 0$ and $\eta^x = 0$, then the result holds by construction.

If $\eta^y_{a,h} = 0$ and $\eta^x > 0$, then we must have $x \oplus_\alpha \eta^x = y \oplus_\alpha \eta^y$. If this were not the case, there must exist an (a,h) with $\eta^x_{a,h} > 0$ and a $\xi > 0$ such that $(x \oplus_\alpha \eta^x)_{a,h} = \xi (y \oplus_\alpha \eta^y)_{a,h}$ and

$\xi(y \oplus_\alpha \eta^y) > (x \oplus_\alpha \eta^x)$. As in the previous proof this leads to a contradiction since $F_{a,h}\left(\xi(y \oplus_\alpha \eta^y)\right) = F_{a,h}(y \oplus_\alpha \eta^y) \geq 0$ while $F_{a,h}\left(\xi(y \oplus_\alpha \eta^y)\right) < F_{a,h}(x \oplus_\alpha \eta^x) = 0$. $\square$

To prove the theorem first note that under Assumption 1, W^0 and V^0 satisfy the gross substitute property. It is a standard argument to show that equilibrium exists for each finitely truncated economy and that therefore the complementarity problem that determines $\eta^0(s, \lambda)$ has a solution. By Lemma 4 this solution must be unique. Given W^n, V^n and η^n with V^n, W^n satisfying the gross substitute property, by existence there must exist $\ell^{n+1}(s, \lambda)$. W^{n+1} and V^{n+1} are well defined and Lemma 5 implies that they satisfy the gross substitute property. Normalizing λ to lie in a compact set, it is easy to see that $\ell^n(s, \lambda)$ is uniformly bounded across n. Therefore there must exist some finite liminf and some finite limsup as $n \to \infty$. It is easy to see that both must describe a competitive equilibrium. Our argument in Section 4.2.1 implies that under Assumption 1 there must be a unique equilibrium and hence the liminf must equal to the limsup. With ℓ^n converging one can then verify that also W^n and V^n must converge.

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