Capital Structure and Investment Dynamics with Fire Sales

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Introduction

- Corporate finance and the Modigliani Miller theorem
- Indeterminacy of capital structure
- Tradeoff between taxes and bankruptcy cost
- Our objective: understand GE effects of debt finance on investment and growth
- Competitive equilibrium when markets are incomplete
- Distortionary taxes and transfers
- Second best policies
- Related concern: capital adequacy regulation
Results

- The optimal capital structure is determinate, although individuals are indifferent between debt and equity.
- Introducing a corporate income tax implies bankruptcy is (a) costly and (b) occurs with positive probability.
- Equilibrium with a corporate income tax is constrained inefficient: welfare would be higher with:
  - a higher level of investment, other things being equal
  - or a higher probability of default (in the limit, higher debt-equity ratio)
- In fact, we can approach the first best by imposing near 100% debt finance.
- The introduction of a safe technology to increase the liquidity of the asset market can make everyone worse off.
Outline

- The planner’s problem
- An economy with frictions, debt, and equity
- Equilibrium
- Steady states
- Transition dynamics
- Constrained inefficiency
- Inefficient hedging
The Planner’s Problem
Production of capital

- Time is discrete and divided into a sequence of dates $t = 0, 1, \ldots$.
- There is a single perishable good which can be used for consumption or production.
- Capital is produced using the good as the sole input.
- An input of $I \geq 0$ units of the good produces $\varphi(I) \geq 0$ units of capital goods at the same date, where $\varphi$ is $C^2$ and satisfies
  \[ \varphi'(I) > 0 \text{ and } \varphi''(I) < 0, \text{ for any } I \geq 0 \]
- Investment is irreversible (capital cannot be consumed).
Production of consumption goods

- The good is produced using capital as the sole input, subject to constant returns to scale
- *Time to build*: one unit of capital produced at date $t$ is available to produce $A > 0$ units of goods at date $t + 1$
- *Depreciation*: one unit of capital used for production at date $t$ is transformed into $\bar{\theta}$ units of capital available for production at date $t + 1$
- *Boundedness*: There exists a constant $0 < \hat{k} < \infty$ such that
  \[
  \varphi(Ak) < (1 - \bar{\theta})k, \text{ for every } k > \hat{k}
  \]
- There is an initial endowment of $\bar{k}_0 > 0$ units of capital at date 0
Consumption

- There is a unit mass of identical, infinitely-lived consumers
- A consumption stream \( c = (c_0, c_1, \ldots) \geq 0 \) generates utility

\[
U(c) = \sum_{t=0}^{\infty} \delta^t u(c_t),
\]

where \( 0 < \delta < 1 \)

- The function \( u(\cdot) \) has the usual properties: \( u(c) \) is \( C^2 \) and satisfies

\[
u'(c) > 0 \text{ and } u''(c) < 0 \text{ for any } c \geq 0.
\]
The planner’s problem

Choose \( \{(c_t, k_t, l_t)\}_{t=0}^{\infty} \) to maximize

\[
\sum_{t=0}^{\infty} \delta^t u(c_t)
\]

subject to the constraints

\[
c_t + l_t = Ak_t, \quad t = 0, 1, \ldots,
\]

\[
k_{t+1} = \bar{\theta}k_t + \varphi(l_t), \quad t = 0, 1, \ldots,
\]

\[
(c_t, k_t, l_t) \geq 0, \quad t = 0, 1, \ldots,
\]

and

\[
k_0 = \bar{k}_0
\]
Suppose that \[ \{(c_t^*, k_t^*, l_t^*)\}_{t=0}^\infty \] is a solution to the planner’s problem and suppose that \[ (c_t^*, k_t^*, l_t^*) \gg 0, \quad t = 0, 1, \ldots \] Then there exist non-negative multipliers \[ \{(\lambda_t^*, \mu_t^*)\}_{t=0}^\infty \] such that \[ \delta^t u' (c_t^*) = \lambda_t^*, \quad t = 0, 1, \ldots, \]
\[ \lambda_{t+1} A + \mu_{t+1} \bar{\theta} = \mu_t^*, \quad t = 0, 1, \ldots, \]
\[ \mu_t^* \phi' (l_t^*) = \lambda_t^*, \quad t = 0, 1, \ldots. \]
Tranversality

- Feasibility and Boundedness imply that
  \[ \phi(I_t) \leq \phi(Ak_t) < (1 - \bar{\theta}) k_t, \quad \forall k_t > \hat{k} \]

- Hence, the law of motion
  \[ k_{t+1} = \bar{\theta} k_t + \phi(I_t) \leq k_t, \quad \forall k_t > \hat{k} \]

implies that \( \{k_t\} \) is bounded and

\[ \lim_{t \to \infty} \sum_{s=t}^{\infty} \delta^s u(c_s) = 0 \]
Steady states

- A steady-state solution satisfies

\[(\lambda_t^*, \mu_t^*, k_{t+1}^*, c_t^*, l_t^*) = (\delta^t \lambda^*, \delta^t \mu^*, k^*, c^*, l^*)], \quad \forall t = 0, 1, \ldots\]

and

\[u'(c^*) = \lambda^*, \]
\[\delta \lambda^* A + \delta \mu^* \bar{\theta} = \mu^*, \]
\[\mu^* \varphi'(l^*) = \lambda^*\]

- The second and third conditions can be rewritten as

\[\frac{\delta A}{1 - \delta \bar{\theta}} = \frac{\mu^*}{\lambda^*} = \frac{1}{\varphi'(l^*)}\]
Efficient steady states

- The feasibility conditions become
  \[ c^* + l^* = Ak^* \]
  \[ k^* = \bar{\theta} k^* + \varphi (l^*) \]

- Thus, the efficient steady state capital stock is
  \[ k^* = \frac{\varphi (l^*)}{1 - \bar{\theta}} \]

  where \( l^* \) is determined by
  \[ \varphi' (l^*) \frac{\delta A}{1 - \delta \bar{\theta}} = 1 \]
An Economy with Frictions, Debt, and Equity
An economy with frictions

In a frictionless world, the planner’s solution could be decentralized in the usual way; to make things more interesting, we impose a number of frictions

- Markets are *incomplete*: there are only spot markets for goods and assets
- Firms are financed using (short-term) *debt* and *equity*
- Firms pay a *distortionary* corporation tax but interest income is exempt
- In the event of default, firms are forced into bankruptcy and their assets are *liquidated*
- Liquidation is subject to a *finance constraint*
Firms

- The capital goods sector consists of a continuum of producers with identical technologies $\varphi(I)$.
- Producers maximize profits and pay dividends to consumers at the end of each date.
- Since production is instantaneous, finance is not required.
- The consumption goods sector consists of a continuum of producers with identical technologies $Ak$.
- Capital is subject to random depreciation, where $\theta_t \sim F(\theta)$ is i.i.d. across producers and $\int \theta_t dF = \bar{\theta}$ (no aggregate uncertainty).
- Capital is long lived so goods producers finance capital purchase with (short-term) debt and equity.
Bankruptcy, liquidation and settlement

Each date $t$ is divided into three sub-periods, labeled $A$, $B$, and $C$.

A. In sub-period $A$, a firm either renegotiates the debt (rolls it over) or defaults and declares bankruptcy

B. In sub-period $B$, bankrupt firms sell their capital at the market-clearing price

C. In sub-period $C$, firms issue debt and equity to purchase newly produced capital goods; they choose the capital structure so as to maximize the market value of the firm
Sub-period A: The renegotiation game

The renegotiation game at date $t$ consists of two stages:

1. The entrepreneur makes a “take it or leave it” offer to the bondholders to rollover the debt, replacing the maturing debt with face value $d_t$ with new assets (a combination of equity and debt maturing at the following date with face value $d_{t+1}$).

2. The creditors simultaneously accept or reject the firm’s offer.

The renegotiation succeeds if a majority of creditors accept and those who do not are paid $d_t$.

There are multiple equilibria; we focus on equilibria in which renegotiation succeeds whenever possible.
Without loss of generality, we consider the behavior of a representative firm with one unit of capital.

If renegotiation succeeds, a creditor can obtain at least $d_t/q_t$ units of capital.

The firm’s manager can ensure the firms ends the period with $A/q_t + \theta_t$ units of capital.

Thus, the manager can make an acceptable offer if and only if

$$d_t \leq A + q_t \theta_t$$

Renegotiation succeeds if and only if $\theta_t \geq z_t$, where the breakeven value $z_t$ is defined by

$$d_t = A + q_t z_t$$
Sub-period B: Liquidation

- In sub-period $B$, the supply of capital from bankrupt firms is
  \[ \int_0^{z_t} \theta_t k_t dF \]

- The amount of “cash” available to purchase this capital is
  \[ A \int_{z_t}^{1} k_t dF = A (1 - F (z_t)) k_t \]

- The market clears at a price $q_t$, where
  \[ q_t \int_0^{z_t} \theta_t k_t f (\theta_t) d\theta_t \leq A (1 - F (z_t)) k_t, \]
  and equality holds if $q_t < v_t$
Sub-period C: The capital market

- A firm in the capital goods sector chooses $l_t \geq 0$ to maximize $v_t \varphi(l_t) - l_t$, where $v_t$ is the price of capital.
- A firm in the consumption goods sector chooses $z_t$ to maximize the value of the firm.
- The goods market clears if $c_t + l_t = Ak_t$.
- The capital market clears if household wealth $w_t$ is sufficient to purchase all of the securities issued by firms $v_t (\bar{\theta} k_t + \varphi(l_t))$ plus the consumption $c_t$.

$$c_t + v_t (\bar{\theta} k_t + \varphi(l_t)) = w_t$$
Taxes

- Tax base = value of the firm - renegotiated value of the debt:
  \[ v_t \left( \frac{A}{q_t} + \theta_t \right) - v_t \left( \frac{d_t}{q_t} \right) \]

- The tax rate \( \tau > 0 \) and the tax on equity at date \( t \) is then:
  \[ \tau \max \left\{ \frac{v_t}{q_t} (A + q_t \theta_t - d_t) , 0 \right\} \]

- Let \( z \) denote the breakeven point defined by \( A + q_t z_t = d_t \); then tax on equity is
  \[ \tau \max \left\{ v_t (\theta_t - z_t) , 0 \right\} \]

- Taxes are returned to consumers as a lump sum transfer \( T_t \) in sub-period C
Asset pricing

- Let $v_t^e$ and $v_t^b$ denote the market price of this equity and debt at $t$
- The return on diversified equity and debt is known with certainty at date $t$ because there is no aggregate uncertainty.
- The one-period returns on debt and equity are equal:

$$1 + r_t = \frac{\int_0^1 \min \left\{ A + q_{t+1} \theta_{t+1}, \frac{v_t^{t+1} d_{t+1}}{q_{t+1}} \right\} f(\theta_{t+1}) d\theta_{t+1}}{v_t^b}$$

$$= \frac{\int_0^{z_{t+1}} \left( \frac{v_t^{t+1}}{q_{t+1}} - \tau \right) (A + q_{t+1} \theta_{t+1} - d_{t+1}) f(\theta_{t+1}) d\theta_{t+1}}{v_t^e}.$$
In a symmetric equilibrium,

- 100% debt finance implies firms default with probability one and $q_t = 0$
- 100% equity finance implies default with probability zero and $q_t = v_t$

Thus, if $\tau > 0$, equilibrium requires costly default

$$q_t < v_t$$

and equilibrium default occurs with probability strictly between zero and one:

$$0 < F(z_t) < 1.$$
Equilibrium
Profit maximization in the capital goods sector

In what follows, we consider only symmetric equilibria

- In the capital goods sector, the firm’s decision is trivial: firms simply choose \( I_t \geq 0 \) to maximize profits

\[
v_t \varphi (I_t) - I_t
\]

- The optimal output is determined uniquely by the first-order condition

\[
v_t \varphi' (I_t) \leq 1,
\]

with strict equality if \( I_t > 0 \)

- The profits \( \pi_t (v_t) = \max \{ v_t \varphi (I_t) - I_t \} \) are paid to consumers in sub-period C
Value maximization in the consumption goods sector

- Suppose a firm has one unit of capital and the breakeven level $z_{t+1}$

- If $\theta_{t+1} < z_{t+1}$ the firm is bankrupt at $t + 1$ and the liquidated value is
  
  $$A + q_{t+1}\theta_{t+1}$$

- If $\theta_{t+1} > z_{t+1}$ the firm is solvent and, w.l.o.g., the firm retains all its earnings, uses them to purchase liquidated capital, and the firm’s value in sub-period $C$ is
  
  $$v_{t+1} \left( \frac{A}{q_{t+1}} + \theta_{t+1} \right)$$

- The firm pays the corporation tax $\tau v_{t+1} (\theta_{t+1} - z_{t+1})$ in sub-period $C$ if it is solvent
Optimal capital structure

- The expected future value of the firm is

\[
\int_0^{z_{t+1}} (A + q_{t+1}\theta_{t+1}) \, dF + \\
\int_{z_{t+1}}^1 \left( \frac{v_{t+1}}{q_{t+1}} (A + q_{t+1}\theta_{t+1}) - \tau v_{t+1} (\theta_{t+1} - z_{t+1}) \right) \, dF.
\]

- The breakeven level \( z_{t+1} \) is chosen to solve

\[
v_t = \max_{z_{t+1}} \frac{\delta u'(c_{t+1})}{u'(c_t)} \left\{ \int_0^{z_{t+1}} (A + q_{t+1}\theta_{t+1}) \, dF + \\
\int_{z_{t+1}}^1 \left( \frac{v_{t+1}}{q_{t+1}} (A + q_{t+1}\theta_{t+1}) - \tau v_{t+1} (\theta_{t+1} - z_{t+1}) \right) \, dF \right\}.
\]
Optimal capital structure

- The derivative of the firm’s value with respect to $z_{t+1}$ is

$$\frac{dv_t}{dz_{t+1}} = \frac{\delta u'(c_{t+1})}{u'(c_t)} \left\{ (A + q_{t+1}z_{t+1}) f(z_{t+1}) - \frac{v_{t+1}}{q_{t+1}} (A + q_{t+1}z_{t+1}) \right\}$$

$$\tau v_{t+1} (\theta_{t+1} - z_{t+1}) f(z_{t+1}) + \tau v_{t+1} (1 - F(z_{t+1}))$$

- The first-order condition can be written as

$$\left(1 - \frac{q_{t+1}}{v_{t+1}}\right) (A + q_{t+1}z_{t+1}) \frac{f(z_{t+1})}{1 - F(z_{t+1})} = \tau.$$ 

- The solution to this equation will be unique and satisfy $0 < z < 1$ if the hazard rate is increasing in $z_{t+1}$ and $\left(1 - \frac{q_{t+1}}{v_{t+1}}\right) Af(0) < \tau$. 
The consumption decision

- Consumers maximize
  \[ \sum_{t=0}^{\infty} \delta^t u(c_t) \]
  subject to
  \[ \sum_{t=0}^{\infty} p_t c_t = v_0 k_0 + \sum_{t=0}^{\infty} p_t (T_t + \pi_t), \]
  where
  \[ p_t = \prod_{s=1}^{t} \left( \frac{1}{1+r_s} \right), \]
  the initial capital stock is \( k_0 \), \( \pi_t \) are the profits of firms in the capital sector and \( T_t \) are lump sum transfers.
Market clearing

- In sub-period $B$, market clearing requires
  \[ q_t \int_0^{z_t} \theta_t k_t f(\theta_t) d\theta_t = A(1 - F(z_t)) k_t, \]
  and equality holds if $q_t < v_t$.
- In sub-period $C$, the goods market clears if
  \[ c_t + l_t = A k_t \]
- By Walras’ Law, the securities market clears if the goods market clears
Equilibrium conditions

1. \( \{c_t^*\}_{t=0}^\infty \) maximizes the consumers discounted utility subject to the budget constraint.

2. For every \( t \), \( v_t^* \varphi'(l_t^*) \leq 1 \) and equality holds if \( l_t^* > 0 \).

3. For every \( t \), \( z_t^* \) maximizes the value of the consumption-producing firm.

4. For every \( t \), the asset market clears in sub-period \( B \):

\[
q_t^* \int_0^{z_t^*} \theta_t f(\theta_t) d\theta_t = A \int_{z_t^*}^1 d\theta_t,
\]

5. For every \( t \), the goods market clears in sub-period \( C \)

\[
Ak_t^* = c_t^* + l_t^*
\]

6. For every \( t \), \( \{k_t^*\} \) satisfies the law of motion

\[
k_{t+1}^* = \bar{\theta} k_t^* + \varphi(l_t^*) \quad \text{and} \quad k_0^* = \bar{k}_0
\]
Steady States
Steady-state equilibrium conditions

- Optimal consumption requires the first-order condition

\[
\frac{1}{1 + r^*} = \delta \frac{u'(c^*)}{u'(c^*)} = \delta,
\]

and the budget constraint

\[
\frac{1}{1 - \delta} c^* = v^* k^* + \frac{1}{1 - \delta} \left( \tau k^* q^* \int_{z^*}^{1} (\theta - z^*) dF + v^* \varphi (l^*) - l^* \right)
\]

- Profit maximization requires the first-order condition

\[
v^* \varphi' (l^*) = 1
\]
Steady-state equilibrium conditions

- Value maximization requires

\[ v^* = \delta \left\{ \int_0^{z^*} (A + q^* \theta) \, dF + \int_{z^*}^{1} \left( \frac{v^*}{q^*} (A + q^* \theta) - \tau v^* (\theta - z^*) \right) \, dF \right\} \]

and the first-order condition

\[ \left( 1 - \frac{q_{t+1}}{v_{t+1}} \right) (A + q_{t+1} z_{t+1}) \frac{f(z_{t+1})}{1 - F(z_{t+1})} = \tau. \]
Steady-state equilibrium conditions

- The asset-market clears
  \[ q^* \int_0^{z^*} \theta dF = A \int_{z^*}^1 dF \]

- The goods-market clears
  \[ Ak^* = c^* + l^*. \]

- The law of motion
  \[ k^* = \bar{\theta} k^* + \varphi (l^*) \]

is satisfied, where \( k^* = \bar{k}_0 \)
Existence and uniqueness

**Theorem**

There exists a unique steady-state equilibrium, obtained as a solution to the system of equations:

\[
q^* = \frac{A(1 - F(z^*))}{\int_0^{z^*} \theta dF},
\]

\[
v^* = \frac{\delta A}{1 - \delta \bar{\theta} + \tau \int_{z^*}^1 (\theta - z^*) dF}
\]

\[
\left(1 - \frac{q_{t+1}}{v_{t+1}}\right) (A + q_{t+1}z_{t+1}) \frac{f(z_{t+1})}{1 - F(z_{t+1})} = \tau
\]
Transition Dynamics
Non-steady-state paths

- Suppose that consumers are risk neutral, \( u(c) \equiv c \); then the equilibrium conditions reduce to

\[
\nu_t \varphi'(I_t) = 1
\]

\[
q_t = \frac{A(1 - F(z_t))}{\int_0^{z_t} \theta dF}
\]

\[
\nu_t = q_t + \tau (q_t)^2 \frac{1 - F(z_t)}{f(z_t)(A + q_t z_t)}
\]

- These equations yield solutions for \( I_t, q_t \) and \( \nu_t \) as a function of \( z_t \)

- We are left with a of two-equation system

\[
\nu(z_t) = \delta \left[ A + \nu(z_{t+1}) \bar{\theta} - \tau \nu(z_{t+1}) \int_{z_{t+1}}^{1} (\theta - z_{t+1}) dF \right] \quad (1)
\]

\[
k_{t+1} = \bar{\theta} k_t + \varphi(I(z_t)) \quad (2)
\]

where \( \nu_t = \nu(z_t) \) is the solution for \( \nu_t \) as a function of \( z_t \) and equation (1) can be solved for \( z_{t+1} \) in terms of \( z_t \).
Instability and uniqueness

- If we let $w(z)$ denote the right hand side of (1), then we are interested in the behavior of the dynamical system

$$v(z_t) = w(z_{t+1})$$

- We note that $v(z_t)$ and $w(z_{t+1})$ have the properties

$$\lim_{z \to 1} v(z_t) = 0 \text{ and } \lim_{z \to 1} w(z_{t+1}) = \delta A$$

and

$$\lim_{z \to 0} v(z_t) = \lim_{z \to 0} w(z_{t+1}) = \infty$$

- We can show that, for any $z$,

$$v'(z) < w'(z) < 0$$

and, as before, there is a unique solution $0 < z^* < 1$ to the equation

$$v(z^*) = w(z^*)$$
Instability and uniqueness

These properties are illustrated below, where the blue line is $v(z)$ and the red line is $w(z)$.
Instability and uniqueness

- From the picture, it is clear that $z_t > z^*$ implies that for some $T > t$,
  
  \[ \nu(z_T) < w(1) \]

  so the equilibrium condition (1) cannot be satisfied at $T$.

- For $z_t < z^*$, the equilibrium condition (1) can be satisfied for all $t$, however, $z_t \to 0$ which implies $\nu(z_t) \to \infty$ and $q(z_t) \to \infty$.

- Now $\nu(z_t) \to \infty$ implies $l_t \to \infty$ and $k_t \to \infty$ but this is clearly in violation of the boundedness of $\{k_t\}$.

- So any equilibrium must satisfy $z_t = z^*$ for all $t$. 
Constrained Inefficiency
The First Best

- From the planner’s problem we know that

\[ k^* = \frac{\phi(l^*)}{1 - \bar{\theta}} \]

\[ \phi'(l^*) \frac{\delta A}{1 - \delta \bar{\theta}} = 1 \]

- If \( \tau = 0 \) then \( z^* = 1 \) and

\[ \nu^* = \delta \{ A + \nu^* \bar{\theta} \} = \frac{\delta A}{1 - \delta \bar{\theta}} \]

- Then the equilibrium condition \( \nu^* \phi'(l^*) = 1 \) implies that

\[ \phi'(l^*) \frac{\delta A}{1 - \delta \bar{\theta}} = 1 \]
Temporary increase in investment

- Suppose $dl_t = \varepsilon > 0$ and $dl_\tau = 0$ for $\tau \neq t$
- The effect on welfare of the policy change is given by

$$\left( \varphi'(I^*) A\delta \sum_{t=0}^{\infty} (\delta \bar{\theta})^t - 1 \right) u'(c^*) \varepsilon + o(\varepsilon)$$

- The term in parenthesis is positive iff

$$A \frac{\delta}{1 - \delta \bar{\theta}} > \frac{1}{\varphi'(I^*)} = \nu^*$$

- But this follows from the equilibrium condition

$$\nu^* = \frac{\delta}{1 - \delta \bar{\theta}} \left( A - \int_{z^*}^{1} \tau q^*(\theta - z^*) f(\theta) d\theta \right)$$

if $\tau > 0$ and $z^* < 1$
Increase in debt-equity ratio

- Consider a permanent increase starting from some date $t + 1$
- The future value of $q_\tau$ is constant and future values of $v_\tau$ are given by a first-order difference equation that is divergent, so the only admissible solution is the new steady-state value
- The remaining variables are determined by
  \[
  k_{t+1} = \bar{\theta} k^* + \varphi (l_{t+1}),
  \]
  \[
  \nu_{t+1} \varphi'(l_{t+1}) = 1.
  \]
- Then $l_\tau$ is constant and greater than $l^*$ iff $v_\tau > v^*$ and we can show that
  \[
  \frac{dv_\tau}{dz} > 0
  \]
Increase in debt-equity ratio

- What happens as \( z \to 1 \) (equilibrium is not defined when \( z = 1 \))?  
- From the market-clearing condition, \( z \to 1 \) implies \( q \to 0 \)  
- As \( q \to 0 \), we see that
  \[
  v \to \frac{\delta A}{1 - \delta \bar{\theta}} = v^{FB}
  \]
  and hence \( I \to I^{FB} \)
- Also, as \( z \to 1 \) we can show that
  \[
  \frac{v^b}{v^e} \to \infty
  \]
  so the debt-equity ratio is increasing in the limit.
Intuition

- Equilibrium is constrained inefficient because agents take prices as given and ignore the pecuniary externality of changes in $v$ and $q$.
- At the margin, $z$ balances the two costs but the ‘tax’ on investment distorts $I$.
- An exogenous increase in $z$ increases the $v$ and decreases $q$ but the tax on equity
  \[
  \tau v \int_z^1 (\theta - z) \, dF
  \]
decreases.
- In fact, it is the lower tax that explains the increase in $v$; the fall in $q$ does not affect $v$.
- An increase in $v$ increases $k$ and hence increases welfare.
Inefficient Hedging
Introducing a safe technology

- What is the effect of introducing a safe technology?
- Suppose there is an alternative technology for producing the consumption good: one unit of capital produces $B$ units of the good and leaves $\bar{\theta}$ units of capital.
- The risky technology is dominated unless $B < A$.
- It is never optimal for firms to use both technologies.
- A fraction $\ell$ use the risky technology and $1 - \ell$ use the safe technology.
Equilibrium with a choice of risk

- Risky firms choose $z^*$ to maximize the value of the firm $v^*$
- Safe firms use 100% debt financing
- Firms are indifferent between the two technologies if

$$v^* = \delta \frac{v^*}{q^*} (B + q^* \bar{\theta}) \iff q^* = \frac{\delta B}{1 - \delta \bar{\theta}}$$

- If $\ell^*$ is the fraction of the capital stock devoted to the risky technology, the market-clearing condition at sub-period B is

$$q^* \ell k^* \int_0^{z^*} \theta dF = A \ell^* k^* \int_{z^*}^1 \theta dF + B (1 - \ell^*) k^*$$

and the market-clearing condition at sub-period C is

$$c^* = A \ell^* k^* + B (1 - \ell^*) k^* - l^*$$
Welfare

Let \((k^*, c^*, z^*, q^*, v^*)\) be the steady-state equilibrium with only the risky technology and let \(\bar{B}\) satisfy

\[ q^* = \frac{\delta B}{1 - \delta \theta} \]

Then \(\ell^* = 1\) is an equilibrium with two technologies for \(B \leq \bar{B}\) and \(0 < \ell^* < 1\) for \(\bar{B} < B < A\).

A small increase in \(B\) at \(\bar{B}\) will lead to the allocation of a small amount of capital to the safe technology.

Under the usual assumptions of risk neutrality and uniform distribution of \(\theta\),

\[ \frac{dv^*}{dB} \bigg|_{B=\bar{B}} < 0 \]

The change in \(c^*\) evaluated at \(B = \bar{B}\) and \(\ell^* = 1\) is

\[ \frac{dc^*}{dB} \bigg|_{B=\bar{B}} = 4A \frac{dv^*}{dB} - 2v^* \frac{dv^*}{dB} + (A - B) 4v^* \frac{d\ell^*}{dB} \bigg|_{B=\bar{B}, \ell^* = 1} \]
Welfare

- Assume that
  \[ \varphi(I) = 2I^{\frac{1}{2}} \]

- Then \( v^* \varphi'(I) = 1 \) implies that
  \[ I^* = (v^*)^2 \]

  and
  \[ k^* = \frac{\varphi(I^*)}{1 - \bar{\theta}} = 4v^* \]

- Then
  \[
  \left. \frac{dc^*}{dB} \right|_{B=\bar{B}} = 4A \frac{dv^*}{dB} - 2v^* \frac{dv^*}{dB} + (A - B)4v^* \frac{d\ell^*}{dB} \bigg|_{B=\bar{B}, \ell^*=1}
  \]

  \[ = (4A - 2v^*) \frac{dv^*}{dB} + (A - B)2v \frac{d\ell^*}{dB} \bigg|_{B=\bar{B}, \ell^*=1} \]

- Then \( \frac{dc^*}{dB} < 0 \) since \( \frac{dv^*}{dB} < 0 \) and \( \frac{d\ell^*}{dB} \leq 0 \) and

  \[ v^* < \frac{\delta}{2 - \delta} 2A \]
Numerical example

Parameters: $A = 2$, $\delta = 0.9$, $\tau = 0.35$, $\theta \sim U [0, 1]$
Conclusion

- Introducing a corporate income tax *implies* that bankruptcy is costly and occurs with positive probability.
- The optimal capital structure is determinate, although individuals are indifferent between diversified debt and equity.
- Equilibrium with a corporate income tax is constrained inefficient: welfare would be higher with:
  - a higher level of investment, other things being equal
  - or a higher probability of default (in the limit, higher debt-equity ratio)
- In fact, we can approach the first best by imposing near 100% debt finance.
- If firms have a choice of a safe and a risky technology, the use of the safe technology makes them worse off.