Capital Structure and Investment Dynamics with Fire Sales

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1 Introduction

The financial crisis of 2007-2008 and the current sovereign debt crisis in Europe have focused attention on the macroeconomic consequences of debt financing. In this paper, we turn our attention to the use of debt finance in the corporate sector and study the general-equilibrium effects of debt finance on investment and growth. More precisely, we analyze underinvestment in equilibrium when markets are incomplete and firms use debt and equity to finance investment.

At the heart of our analysis is the determination of firms' capital structure. In the classical world of Modigliani and Miller (1958), capital structure is indeterminate. To obtain a determinate capital structure, subsequent authors appealed to frictions such as distortionary taxes, bankruptcy costs, and agency costs.[1] We follow this tradition and assume the optimal capital structure balances the tax advantages of debt against the risk of costly bankruptcy. Debt has a tax advantage because it is not subject to the corporate income tax. Bankruptcy is perceived as costly because it forces the firm to sell assets at firesale prices. In equilibrium, the firm will balance the perceived costs of debt and equity in choosing the equilibrium capital structure.

In our model, neither the corporate income tax nor the risk of bankruptcy represents a real burden on the representative consumer. The corporate tax revenue is returned to consumers in the form of lump sum transfers, so it has no direct effect on investors' wealth or income. Similarly, bankruptcy results in a fire sale of assets, but this is a transfer of value from creditors to the asset buyers that has no effect on the wealth or income of the representative investor. Nonetheless, a rational, value-maximizing manager of a competitive firm will perceive the tax as a cost of using equity finance and the risk of a fire sale in

[1] See, for example, Barnea, Haugen and Senbet (1981), Bradley, Jarrell and Kim (1984), Brennan and Schwartz (1978), Dammon and Green (1987), Fischer, Heinkel and Zechner (1989), Kim (1982), Leland and Toft (1996), Miller (1977), and Titman (1984) and Titman and Wessels (1988).

bankruptcy as a cost of using debt. These perceived costs act like a tax on capital and distort the investment decision.

The economy We assume that time is discrete and the horizon is infinite. There are two commodities at each date, a perishable consumption good and a durable capital good. The economy consists of two productive sectors, one for each commodity. In the capital sector, the consumption good is the sole input in the production of capital goods. Capital is produced subject to decreasing returns to scale. The consumption goods sector uses capital as the sole input. Consumption goods are produced subject to constant returns to scale.

Production of the capital good is instantaneous, so firms in the capital-producing sector choose inputs and outputs to maximize profits at each date. The profits are distributed to consumers. The consumption producing sector, by contrast, requires long-lived capital as an input. To finance the purchase of capital, firms issue debt and equity. Constant returns to scale ensure that interest, dividends and retained earnings exhaust the firm's revenue in each period. Competition for funding requires the manager to maximize the firm's market value, which is the sum of the market value of the debt and equity outstanding.

The representative consumer maximizes the discounted sum of lifetime utilities. He decides how much of his income to consume or save at each date, using savings to purchase debt and equity issued by firms, and receives the dividends and interest payments on the securities purchased in the past.

Bankruptcy In order to allow for the possibility of bankruptcy, we assume that the production of consumption goods is subject to productivity shocks in the form of stochastic depreciation of capital. Of course, default and bankruptcy are only possible if the firm issues a positive amount of debt. We follow Gale and Gottardi (2010) in modeling the bankruptcy process as an extensive-form game consisting of three stages: renegotiation, liquidation and settlement. A firm in distress first attempts to restructure its debt by making an offer to exchange new debt and equity claims for the old debt. If the attempt to renegotiate the debt fails, i.e., the creditors reject the firm's offer, then and only then will the firm be forced to liquidate its assets. The firm's assets are sold on a competitive capital market and the liquidated value is paid to the creditors in the settlement stage.

There always exists a sub-game perfect equilibrium of the game in which all creditors reject the firm's offer and renegotiation fails. To eliminate such trivial failures of the bankruptcy process, we assume that renegotiation succeeds if and only if there exists a sub-game perfect equilibrium in which a feasible offer is accepted. With this qualification, the bankruptcy process has a unique sub-game perfect equilibrium in which the firm fails to renegotiate its debt if and only if the present value of the liquidated assets is less than the face value of the debt. In other words, the debt can be rolled over unless the firm is insolvent in this sense.

Bankruptcy procedures have numerous flaws (see Bebchuk, 1988; Aghion, Hart and Moore, 1992; Shleifer and Vishny, 1992). In the present model, we focus on one potential source of market failure, the so-called *finance constraint*. The finance constraint refers to the fact that the potential buyer who values the assets most highly may not be able to raise enough finance to purchase the assets at their full economic value. In our highly simplified environment, all potential buyers value the assets symmetrically, so the only friction is the finance constraint, which takes the form of a requirement to settle in cash rather than issuing IOUs. In equilibrium, cash turns out to be "scarce" in the sense that there will never be enough cash to purchase all the capital goods at their full economic value. Despite this friction, bankruptcy is *ex post efficient*. Assets sold at fire sale prices represent a transfer of value from creditors to buyers, rather than a deadweight loss.

Capital structure As a baseline, we use the "frictionless" case in which the corporate tax rate is zero. In that case,

- the competitive equilibrium achieves the First Best.
- The equilibrium allocation is Pareto efficient and maximizes the utility of the representative agent subject to the usual feasibility constraints.
- Firms are financed by an (indeterminate) mixture of debt and equity.
- The equilibrium capital structure is indeterminate, although it is subject to the (macro) constraint that the amount of debt must be small enough that there is no risk of costly bankruptcy.

Bankruptcy is "costly" only in the sense that assets sold off in an illiquid market may fetch less than fair economic value. If the finance constraint, which requires liquidated assets to be purchased by cash rather than IOUs, is binding, the market price of the assets is determined by the amount of cash in the market, rather than by economic fundamentals. The illiquidity of the asset market is endogenous however. If there is enough liquidity, there will be no loss from fire sales.

With a zero corporate tax rate, the finance constraint never "binds" and bankruptcy is not "costly." When the corporate tax rate is positive, we get quite different results.

- Equilibrium is constrained inefficient.
- Firms are financed by positive amounts of risky debt and equity.
- The optimal capital structure is uniquely determined in equilibrium.
- Each firm faces a positive probability of bankruptcy and bankruptcy is costly in the sense that the liquidated value of the firm is less than its fundamental or economic value as a going concern.

It is interesting that the introduction of single friction (the positive corporate tax rate) changes so many features of equilibrium. It implies (i) that the capital structure is uniquely determined; (ii) that both debt and equity are used in equilibrium; and (iii) that bankruptcy is costly. The intuition for point (iii) is simple. If debt were not risky (the probability of bankruptcy equalled zero) or bankruptcy were not costly (bankrupt firms could be liquidated with no loss of value), then firms would use 100% debt finance to avoid the corporate tax. But, in equilibrium, 100% debt finance is inconsistent with both a zero probability of bankruptcy and no fire sales for bankrupt assets. A similar argument establishes point (ii). If firms used 100% equity finance, there would be no bankruptcy and hence no fire sales. But this means that a single firm could issue a small amount of debt at no cost in terms of bankruptcy and benefit from the tax hedge. The uniqueness of the capital structure, point (i), follows from the fact that a rational manager will equate the marginal costs of debt and equity financing in equilibrium and, under reasonable conditions, the marginal costs are increasing.

Constrained inefficiency The main contribution of the paper is the analysis of welfare in the presence of distortions. Of course, equilibrium is not Pareto efficient, but more interestingly, it is not *constrained* efficient. We conduct two experiments to give a sense of the scope for welfare-improving interventions. First, we consider a policy of controlling the level of investment. An increase in investment increases welfare by bringing the capital stock closer to the first best. Second, we consider a policy of controlling the probability of bankruptcy, for example, by manipulating the capital structure. Increasing the probability of bankruptcy above its equilibrium level increases welfare by increasing investment and bringing the capital stock closer to the first best level. Thus, contrary to what one might expect, there is not too much instability, but too little, in equilibrium. This seems to contradict the common intuition that firms have an incentive to use too much debt financing because of the tax deductibility of interest.

The fact that there is too little bankruptcy risk and, presumably, too little debt is surprising. There are two distortions in the model, one working to increase debt finance (the tax advantage) and the other working to reduce it (the risk of costly bankruptcy). It seems that the distortion could go either way, too much or too little debt. Nonetheless, given a fixed distortion in the form of the corporation tax rate, the optimal intervention is to increase the risk of bankruptcy. At the very least, this should give us pause when evaluating claims that less debt finance is a "good thing."

Inefficient hedging One limitation of the basic model is that the only source of "cash" in the market for liquidated assets is the output of the consumption at the beginning of the period. Thus, although the supply of liquidated assets is determined by the firms' capital structure, the demand for assets (supply of "cash") is a function of the capital stock only. It might be thought that allowing firms to accumulate liquid assets would relax the finance constraint and reduce the inefficiency associated with cost of default. In fact, the costliness of default is an equilibrium condition and providing more financial capital in the asset market will simply call forth more liquidated capital goods to ensure that in equilibrium the cost of default balances the tax advantage of debt.

To see how this works in the present setup, we introduce a safe technology that produces

B units of the consumption good per unit of capital and has a deterministic depreciation rate. We assume that the depreciation rate of the safe technology is equal to the average or expected depreciation rate of the risky technology. Then the risky technology will be used only if $B < A$. In fact, there exists a critical value, $B = \bar{B}$, such that the safe technology is not used if $B \leq \bar{B}$. If $B > \bar{B}$, then a positive amount of capital is invested in the safe technology and if $B \geq A$ then the safe technology dominates the risky technology. It turns out that it is never optimal for a firm to combine the two technologies: a firm either invests all its capital in the safe technology or invests all its capital in the risky technology. In equilibrium, with $A > B > \bar{B}$, both types of firms are present and earn the same return on capital. Firms operating the safe technology make capital gains that balance the lower productivity of their capital.

The main result for this extension is that, far from improving matters, the introduction of the safe technology makes matters worse. Compared to an economy in which $B \leq \bar{B}$ and no capital is invested in the safe technology, welfare is lower in an economy with $\bar{B} < B < \bar{B}+\varepsilon$, in which a positive amount of capital is invested in the safe technology. The safe firms have the liquidity to buy up assets and this raises the price of capital, but it cannot raise it too much because the risky firms respond by issuing more debt with the result that a larger fraction of them default. So the fire sale remains an equilibrium phenomenon in spite of the safe firms' efforts.

The presence of the safe firms does not solve the problem of the illiquid asset market. More importantly, it does not improve welfare. This is not really surprising. After all, the safe firms are only speculators: their gains are losses for the creditors of the bankrupt firms. Since there is no aggregate uncertainty, they do not provide any risk sharing. So the fact that their capital is less productive means that everyone is worse off. The bottom line is that there is too much liquidity rather than too little.

Of course, if the safe asset is sufficiently productive, it must improve welfare. This follows from the fact that we approach the first best as $B \nearrow A$.

Dynamics Using the reduced-form relationships, we characterize the steady-state equilibrium of the model, demonstrate its existence and uniqueness, and establish some comparative static properties. As the corporation tax rate increases, the price of capital decreases, the probability of bankruptcy increases, while changes in the fundamental economic value of assets, investment, and the capital stock are ambiguous. As the discount factor increases, the price of capital increases, the probability of bankruptcy decreases, the fundamental economic value of assets, investment, and the capital stock all increase, and the ratio of the fundamental value to the price of capital increases.

To get a sense of what happens outside the steady state, we consider an example in which the representative consumer is risk neutral. In this special case, we can show that the equilibrium probability of bankruptcy, the price of capital, the fundamental value of capital and the level of investment are all equal to their steady-state values. The only variable that moves outside of the steady state is the capital stock, which converges to its steady-state value. Thus, at least in this special case, the globally stable steady state uniquely characterizes the equilibrium variables other than the capital stock.

1.1 Related literature

In a representative agent economy without distortions, competitive equilibrium is efficient because the agent's decision problem is identical to the planner's problem. In the presence of distortionary taxes, the situation is very different: there may exist multiple, Pareto-inefficient equilibria (Foster and Sonnenschein, 1970). Here we find a unique Pareto-inefficient equilibrium in spite of the existence of a representative consumer. Although consumers collectively own all the assets, individual managers' decisions are distorted by the presence of taxes and bankruptcy costs. Thus, even though tax revenues are returned to consumers and consumers end up holding the same assets after liquidation, the distortion of investment decisions imposes a welfare cost on the economy. Gale and Gottardi (2011) found similar results in a static model in which all investment was 100% debt financed.

The classical literature on the firm's investment decision excludes external finance constraints and bankruptcy costs and uses adjustment costs to explain the reliance of investment on Tobin's Q (see Eberly, Rebelo and Vincent, 2008, for a contemporary example). The new wave literature on investment, exemplified by Sundaresan and Wang (2006) and Bolton, Chen and Wang (2009), incorporates frictions of various types, such as agency costs and distress costs of debt. Hackbarth and Mauer (2012) investigate the interaction of financing and investment in a model where there are multiple debt issues with possibly different seniority. These papers study an individual firm in partial equilibrium, rather than a large number of firms in general equilibrium.

Gomes and Schmid (2010) study a "tractable general equilibrium model with heterogeneous firms making optimal investment and financing decisions." Kuehn and Schmid (2011) allow for endogenous assets in a structural model of default to account for credit risk. Miao and Wang (2010) develop a DSGE model of default and credit risk and calibrate it to match the persistence and volatility of output growth as well as credit spreads. All of these models assume a representative consumer and a continuum of heterogenous firms and use computational methods to derive the equilibrium properties of the model. We endogenize the cost of bankruptcy through the finance constraint in the market for liquidated assets, whereas these papers take the cost of bankruptcy as exogenous.

The interaction between illiquidity and incompleteness of asset markets is also studied in the literature on banking and financial crises. For models of firesales and their impact on bank portfolios, see Allen and Gale (2004a, 2004b).

The rest of the paper is organized as follows. In Section 2 we describe the primitives of the model and characterize the first-best allocation that would be implemented by a planner seeking to maximize the welfare of the representative agent. In Section 3 we describe the firms, markets and other institutions of the economy. Section 4 contains a reduced-form description of equilibrium. Section 4.1 contains the characterization of steady-state equilibrium, shows that it existence and uniqueness and provides some comparative static results. Section 5 contains an analysis of non-steady-state paths. Section 6 shows that the first best can be achieved when there is no tax on equity and then investigates the constrained inefficiency of equilibrium. This section also contains an extension of the model to allow for a safe technology and shows that its introduction may be welfare decreasing. A brief conclusion follows. All proofs are collected in the appendix.

2 The Economy

We consider an infinite horizon production economy. Time is described by a countable sequence of dates, $t = 0, 1, \ldots$. At each date there are two goods, a perishable consumption good and a durable capital good.

2.1 Consumers

There is a unit mass of identical, infinitely-lived consumers. The consumption stream of the representative consumer is denoted by $\mathbf{c} = (c_0, c_1, ...) \geq \mathbf{0}$, where c_t is the amount of the consumption good consumed at date t. For any $\mathbf{c} \geq \mathbf{0}$, the representative consumer's utility is denoted by $U(\mathbf{c})$ and given by

$$U(\mathbf{c}) = \sum_{t=0}^{\infty} \delta^t u(c_t), \tag{1}$$

where $0 < \delta < 1$ and $u : \mathbf{R}_+ \to \mathbf{R}$ has the usual properties: it is C^2 and such that $u'(c) > 0$ and $u''(c) < 0$ for any $c \geq 0$.

2.2 Production

There are two production sectors in the economy. In one, capital is produced using the consumption good as an input. In the other, the consumption good is produced using the capital good as an input.

Capital goods sector The technology for producing capital is given by a decreasing-returns-to-scale production function. If $I_t \geq 0$ is the amount of the consumption good used as an input at date t, the output is $\varphi(I_t) \geq 0$ units of capital at the end of the period, where $\varphi(.)$ is a C^2 function that satisfies $\varphi'(I_t) > 0$ and $\varphi''(I_t) < 0$, for any $I_t \geq 0$, as well as the following Inada conditions: $\lim_{I\to 0} \varphi'(I) = \infty$ and $\lim_{I\to\infty} \varphi'(I) = 0$.

Consumption goods sector The technology for producing the consumption good exhibits constant returns to scale. Each unit of capital used as an input at the produces $A > 0$ units of output beginning of date t. The capital good is assumed to depreciate at an average rate $1 - \bar{\theta}$, so for every unit of capital used in production at the beginning of date t, $\bar{\theta}$ units remain after production is completed.

In the decentralized model introduced later, we assume that production in the consumption goods sector is undertaken by a large number of firms with stochastic depreciation rates. The depreciation rates are assumed to be i.i.d. across firms with mean $1-\bar{\theta}$. For the purpose of characterizing the efficient allocation, we can ignore the heterogeneity and assume the average depreciation is deterministic.

2.3 Feasible allocations

At date 0, there is an initial stock of capital goods $\bar{k}_0 > 0$. A (symmetric) *allocation* is given by a sequence $\{c_t, k_t, I_t\}_{t=0}^{\infty}$ that specifies the consumption c_t, capital k_t, and investment I_t at each date t. The allocation $\{c_t, k_t, I_t\}_{t=0}^{\infty}$ is *feasible* if, for every date $t = 0, 1, ...,$ it satisfies non-negativity,

$$(c_t, k_t, I_t) \geq \mathbf{0}, \tag{2}$$

attainability for the consumption good,

$$c_t + I_t \leq Ak_t, \tag{3}$$

and the law of motion for capital,

$$k_{t+1} = \bar{\theta}k_t + \varphi\left(I_t\right), \tag{4}$$

together with the initial condition $k_0 = \bar{k}_0$.

It follows from the assumptions regarding the technology for producing the capital good that there exists a unique level of the capital stock, $0 < \hat{k} < \infty$, satisfying the condition

$$\varphi\left(A\hat{k}\right) = \left(1-\bar{\theta}\right)\hat{k}.$$

That is, the capital stock $\hat{k}$ remains constant when all the output of the consumption good is used for investment. It is then straightforward to show that $\hat{k}$ constitutes an upper bound on the permanently feasible levels of the stock of capital.

Proposition 1 *At any feasible allocation* $\{c_t, k_t, I_t\}_{t=0}^{\infty}$*, we have* $\limsup_{t\to\infty} k_t \leq \hat{k}$*.*

As a corollary, $A\hat{k}$ is an upper bound on the levels of consumption and investment that can be maintained indefinitely:

$$\limsup_{t\to\infty} c_t \leq A\hat{k}, \ \ \limsup_{t\to\infty} I_t \leq A\hat{k}.$$

2.4 Efficient allocations

A first-best, *socially optimal allocation* maximizes the utility of the representative consumer within the set of feasible allocations. More precisely, it is a sequence $\{c_t, k_t, I_t\}_{t=0}^{\infty}$ that solves the problem of maximizing the representative consumer's utility (1) subject to the feasibility constraints (2), (3), (4).

To characterize the properties of the first best, consider the necessary and sufficient conditions for an interior solution $\left(c_t^{FB}, k_t^{FB}, I_t^{FB}\right) \gg \mathbf{0}$, $t = 0, 1, ...$ of this problem, for every t,

$$\delta^t u'\left(c_t^{FB}\right) = \lambda_t,$$

$$\lambda_{t+1} A + \mu_{t+1}\bar{\theta} = \mu_t,$$

and

$$\mu_t \varphi'\left(I_t^{FB}\right) = \lambda_t.$$

for some non-negative multipliers $\{(\lambda_t, \mu_t)\}_{t=0}^{\infty}$, together with the feasibility conditions (2-4) and the initial condition $k_0 = \bar{k}_0$. The boundedness property established above implies that the *transversality condition*

$$\lim_{t\to\infty} \sum_{s=t}^{\infty} \delta^s u\left(c_s\right) = 0$$

is automatically satisfied.

Much of our analysis focuses on *steady states*, that is on allocations such that

$$(c_t, k_t, I_t) = (c, k, I)$$

for all t. It is interesting to see what the above first-order conditions imply for an optimal steady state:

Proposition 2 *At an optimal steady state, the capital stock is given by*

$$k^{FB} = \frac{\varphi\left(I^{FB}\right)}{1-\bar{\theta}}, \tag{5}$$

where I^* is determined by

$$\frac{\delta A}{1-\delta\bar{\theta}} = \frac{1}{\varphi'\left(I^{FB}\right)}. \tag{6}$$

Equation (6) has a natural interpretation in terms of marginal costs and benefits. The marginal revenue of a unit of capital at the end of period 0 is

$$\frac{\delta A}{1-\delta\bar{\theta}} = \delta A + \delta^2\bar{\theta}A + \ldots + \ \delta^t\bar{\theta}^{t-1}A,$$

because it produces $\bar{\theta}^{t-1}A$ units of the consumption good at each date $t > 0$ and the present value of that consumption is $\delta^t\bar{\theta}^{t-1}A$. The marginal cost of a unit of capital is $\frac{1}{\varphi'(I^{FB})}$ units of consumption at date 0. So optimality requires the equality of marginal cost and marginal revenue.

3 An incomplete markets economy

In characterizing the efficient allocation for the economy in Section 2, we ignored the decisions of the various economic agents in the economy and assumed the planner maximized the welfare of the representative agent. In order to complete the description of a decentralized economy, we have to reintroduce firms that make production decisions and explain the savings behavior portfolio choices of consumers.

3.1 Firms

In the capital goods sector, there is a unit mass of identical firms operating the technology. Since production is instantaneous and there is no capital, firms simply maximize maximize current profits in each period.

In the consumption sector, there is a continuum of infinitely-lived firms. The capital of each firm is subject to a distinct depreciation shock θ_t, which is assumed to be i.i.d. across firms as well as over time. Hence firms, while ex ante identical, are different ex post. The random variable θ_t has support $[0, 1]$ and a continuous p.d.f. $f(\theta)$. We denote the c.d.f. by $F(\theta)$. By the law of large numbers convention, there is no aggregate uncertainty and the aggregate depreciation rate is constant. The fraction of the capital stock that remains after depreciation is therefore equal to $\bar{\theta}$, the expected value of θ_t. If the aggregate capital stock in the economy is $k_t \geq 0$ at the beginning of date t, the total output of consumption good at t is Ak_t and the total amount of capital remaining after production has taken place is $\bar{\theta}k_t$. The only additional condition we impose on the distribution $F(\theta)$ is that the hazard rate $\frac{f(z)}{1-F(z)}$ is increasing.

Given the CRTS nature of the technology the mass of firms active in this sector is indeterminate. Further, since we allow for bankruptcy and the entry of new firms, the mass of active firms may change over time. To simplify the description of equilibrium, we will assume that a combination of entry and exit maintains the mass of firms equal to unity and that firms adjust their size so that each has the same amount of capital. This allows us to describe the evolution of the economy in terms of a representative firm with capital stock k_t.

At the initial date $t = 0$, we assume that all capital is owned by firms in the consumption good sector and that each of these firms has been previously financed entirely by equity. Each consumer has an equal shareholding in each firm in the two sectors.

3.2 Renegotiation and default

In a frictionless environment, where firms have access to a complete set of contingent markets to borrow against their future income stream and hedge the idiosyncratic depreciation shocks, the first-best allocation can be decentralized, in the usual way, as a perfectly competitive equilibrium.

In what follows, we consider instead an environment with frictions, where the first best is typically not attainable. More specifically, in this environment there are no markets for contingent claims, the firms' output is sold in spot markets for goods and firms are financed only with (short-term) debt and equity.

In the presence of uncertainty regarding the depreciation rate of the firm's capital, debt financing gives rise to the risk of bankruptcy, which may be costly. In the event of default, in fact, firms are required to liquidate their assets by selling them to the solvent firms. These firms may be finance-constrained in equilibrium and whenever this happens there will be a fire sale, in which assets are sold for less than their full economic value.

Equity financing, in contrast, entails no bankruptcy risk. The cost of equity is that firms must pay a linear (distortionary) tax on equity's returns. We assume for simplicity that the revenue of the tax on equity is used to make an equal lump sum transfer to all consumers.

A firm producing the consumption good must then choose each period the optimal composition between debt and equity financing of its purchases of capital, by trading off the costs and benefits of these two financial instruments. To analyze this decision formally we must first describe more in detail the structure of markets and the timing of the decisions taken within each period by firms and consumers.

Each date t is divided into three sub-periods, labeled A, B, and C.

A. At the beginning of each period (sub-period A), the production of the consumption good occurs and the realization of the depreciation shock of each firm θ_t is learnt. Also, the debt liabilities of each firm are due. The firm has three options: it can repay the debt, renegotiate ("roll over") the debt, or default and declare bankruptcy. Renegotiation takes place via the game described in the next section, where the firm makes a take it or leave it offer to its bond holders and they simultaneously choose whether to accept or reject. Non defaulting firms may then distribute their earnings to equity holders or retain them to finance new purchases of capital.

B. In the intermediate sub-period (B), the market opens where bankrupt firms can sell their assets (their capital). A liquidity constraint applies, so that only agents with resources readily available, either solvent firms who retained earnings in sub-period A or consumers who received dividends in sub–period A, can purchase the assets on sale. Let q_t denote the market price of the liquidated capital.

C. In the final sub-period (C), the production of capital goods occurs. The profits of the firms who operate in this sector are then distributed to the consumers who own them. In addition, debt holders of defaulting firms receive the proceeds of the liquidation sales in sub-period B. The taxes on equity's returns are due and the lump sum transfers to consumers are also made in this sub-period. All other markets open; spot markets, where the consumption and the capital goods are traded, at a price respectively 1 and v_t, as well as asset markets, where debt and equity issued by firms (both surviving and newly formed) to acquire capital are traded. The consumers buy and sell these securities in order to fund future consumption and rebalance their portfolios.

3.2.1 Sub-period A: The renegotiation game

Consider a firm with k_t units of capital at the beginning of period t. The firm produces Ak_t units of the consumption good, has outstanding debt with face value $d_t k_t$,[2] and learns the realization of its depreciation shock θ_t. The renegotiation process that occurs in sub-period A between the firm and the creditors who purchased the firm's bonds at $t-1$ is represented by a two-stage game. Without loss of generality, we analyze the renegotiation game for the case where the firm has one unit of capital, i.e., $k_t = 1$.

S1 The firm makes a "take it or leave it" offer to the bond holders to rollover the debt, replacing each unit of the maturing debt with face value d_t with a combination of equity and debt maturing the following period. The new face value of the debt, d_{t+1}, determines the firm's capital structure since equity is just a claim to the residual value.

S2 The creditors simultaneously accept or reject the firm's offer.

Two conditions must be satisfied in order for the renegotiation to succeed. First, a majority of the creditors must accept the offer. Second, the rest of the creditors must be paid off in full. If either condition is not satisfied, the renegotiation fails and the firm is declared bankrupt. In that event, all the assets of the firm are frozen, nothing is distributed until the capital stock has been liquidated (sold in the market). After liquidation, the sale price of the liquidated assets is distributed to the bond holders in sub-period C. Obviously, there is nothing left for the shareholders in this case. Hence default is always involuntary: a firm acting so as to maximize its market value will always repay or roll over the debt unless it is unable to do so.

We show next that there is an equilibrium where renegotiation succeeds if and only if

$$d_t k_t \leq (A + q_t \theta_t) k_t, \tag{7}$$

that is, if the value of the firm's equity is negative when its capital is evaluated at its liquidation price q_t. Note that the condition is independent of k_t. Consider, with no loss of generality, the case of an individual creditor holding debt with face value d_t. If he rejects the offer and demands to be repaid immediately, he receives d_t in sub-period A. With this payment he can purchase $\frac{d_t}{q_t}$ units of capital in sub-period B. If the firm manages to roll over the debt, it can retain A and purchase $\frac{A}{q_t}$ units of capital in sub-period B. Then it will have $\frac{A}{q_t} + \theta_t$ units of capital at the end of the period. Therefore the most that the firm can offer the creditor is a claim to an amount of capital $\frac{A}{q_t} + \theta_t$ at the end of the period, with market value $v_t \left(\frac{A}{q_t} + \theta_t \right)$. So the firm's offer will be accepted only if the creditor rejecting the offer ends up with no more capital than by accepting, that is,

$$\frac{d_t}{q_t} \leq \frac{A}{q_t} + \theta_t,$$

[2]Here and in what follows, it is convenient to denote by d_t the face value of the debt issued per unit of capital acquired.

which is equivalent to (7). If (7) is satisfied, there exists a sub-game perfect equilibrium of the renegotiation game in which the entrepreneur makes an acceptable offer worth d_t to the creditors and all of them accept. To see this, note first that the shareholders receive a non-negative payoff from rolling over the debt, whereas they get nothing in the event of default, and the creditors will not accept a lower offer. Second, the creditors will accept the offer of d_t because they cannot get a higher payoff by deviating and rejecting it. Thus, we have the following simple result.

Proposition 3 *There exists a sub-game perfect equilibrium of the renegotiation game in which the debt is renegotiated if and only if (7) is satisfied.*

Proposition 3 leaves open the possibility that renegotiation may fail even if (7) is satisfied. Indeed it is the case that if every creditor rejects the offer, it is optimal for every creditor to reject the offer because a single vote has no effect. In the sequel, we ignore this trivial coordination failure among lenders and assume that renegotiation succeeds whenever (7) is satisfied. We do this because we want to focus on non-trivial coordination failures.

3.2.2 Sub-period B: Liquidation

Let z_t denote the break even value of θ_t, implicitly defined by the following equation

$$d_t \equiv A + q_t z_t. \tag{8}$$

Thus a firm is bankrupt if and only if $\theta_t < z_t$. When all firms active at the beginning of date t have the same size (k_t), the supply of capital to be liquidated by bankrupt firms in sub-period B is

$$\int_0^{z_t} \theta_t k_t f(\theta_t)\, d\theta_t.$$

It is a matter of indifference to shareholders whether solvent firms retain earnings or pay them out as dividends, since shareholders can sell shares to finance consumption and the manager operates the firm in the shareholders' interests. There is no loss of generality, therefore, in assuming that solvent firms ($\theta_t \geq z_t$) retain all of their earnings and have them available to purchase capital in sub-period B. The amount of resources available to purchase capital in sub-period B is so

$$A \int_{z_t}^{1} k_t f(\theta_t)\, d\theta_t = A\left(1 - F(z_t)\right) k_t.$$

If q_t, the price of capital in sub-period B, is greater than v_t, the price of capital in sub-period C, no firm will buy capital at the price q_t and the market cannot clear. This means that market clearing requires $q_t \leq v_t$ and, if the inequality is strict, all the available resources must be offered in exchange for liquidated capital. Thus, market-clearing situations is equivalent to $q_t \leq v_t$ and

$$q_t \int_0^{z_t} \theta_t k_t f(\theta_t)\, d\theta_t \leq A\left(1 - F(z_t)\right) k_t, \tag{9}$$

with equality if $q_t < v_t$.

3.2.3 Sub-period C: Settlement, investment and trades

Capital sector decisions The decision of the firms operating in the capital goods sector, in sub-period C, is simple. At any date t the representative firm chooses $I_t \geq 0$ to maximize current profits, $v_t \varphi(I_t) - I_t$. Because of the concavity of the production function, a necessary and sufficient condition for the input I_t to be optimal is

$$v_t \varphi'(I_t) \leq 1, \tag{10}$$

with strict equality if $I_t > 0$.

The profits from the capital sector, $\pi_t = \sup_{I_t \geq 0} \{v_t \varphi(I_t) - I_t\}$, are paid to consumers in the same sub-period.

Consumption sector decisions In the consumption goods sector, the firm's decision is more complicated because the production of consumption goods requires durable capital, which generates returns that repay the investment over time. So the firm has to issue securities to finance the purchase of capital. As we explained above, the number and size of firms in this sector are indeterminate because of constant returns to scale. We consider a symmetric equilibrium in which, at any date, a unit mass of firms are active and all of then have the same size, given by k_t units of capital[3] at the end of date t.

The representative firm chooses its capital structure to maximize its market value, that is the value of the outstanding debt and the equity claims on the firm. This capital structure is summarized by the break even point z_{t+1}. Whenever the firm's depreciation shock next period $\theta_{t+1} < z_{t+1}$, the firm defaults next period and its value is equal to the value of the firm's liquidated assets, $A + q_{t+1}\theta_{t+1}$. If $\theta_{t+1} > z_{t+1}$, the firm is solvent and can use its earnings A to purchase capital at the price q_{t+1}. Then the final value of the firm is $v_{t+1}\left(\frac{A}{q_{t+1}} + \theta_{t+1}\right)$, from which the amount due for the tax on equity's returns must be subtracted. The corporate income tax rate is denoted by $\tau > 0$ and the tax base is assumed to be the value of the firm's equity at the beginning of sub-period C.

To calculate the value of equity, we need two components. The first is the value of capital owned by the firm, $v_t\left(\frac{A}{q_t} + \theta_t\right)$. The second is the value of the (renegotiated) debt, $v_t\left(\frac{d_t}{q_t}\right)$. The tax base is the difference between these two values,

$$v_t\left(\frac{A}{q_t} + \theta_t\right) - v_t\left(\frac{d_t}{q_t}\right).$$

Hence, the tax payment due at date $t+1$ is

$$\tau \max\left\{v_{t+1}\left(\frac{A}{q_{t+1}} + \theta_{t+1}\right) - v_{t+1}\left(\frac{d_{t+1}}{q_{t+1}}\right), 0\right\} = \tau \max\left\{\frac{v_{t+1}}{q_{t+1}}\left(A + q_{t+1}\theta_{t+1} - d_{t+1}\right), 0\right\},$$

[3] Because of the default of a fraction of the firms, the surviving firms who acquire their capital may grow in size in sub-period B, but are then indifferent between buying or selling capital at v_t in sub-period C. Hence we can always consider a situation where the mass of active firms remains unchanged over time, while their size varies with k_t.

and the expected value of the firm at date $t+1$ is

$$\int_0^{z_{t+1}} (A + q_{t+1}\theta_{t+1})\, dF + \int_{z_{t+1}}^1 \left[v_{t+1}\left(\frac{A}{q_{t+1}} + \theta_{t+1}\right) - \tau \frac{v_{t+1}}{q_{t+1}} (A + q_{t+1}\theta_t - d_{t+1}) \right] dF. \quad (11)$$

Because there is no aggregate uncertainty and there is a continuum of firms offering debt and equity subject to idiosyncratic shocks, diversified debt and equity are risk-free and must bear the same rate of return. Denoting by r_t the risk-free interest rate between date t and $t+1$, the present value of the firm at t is given by the expression in (11) divided by $1+r_t$. Hence the firm's problem consists in the choice of its capital structure, as summarized by z_{t+1}, so as to maximize the following objective function

$$\int_0^{z_{t+1}} (A + q_{t+1}\theta_{t+1})\, dF + \int_{z_{t+1}}^1 \left[v_{t+1}\left(\frac{A}{q_{t+1}} + \theta_{t+1}\right) - \tau v_{t+1} (\theta_{t+1} - z_{t+1}) \right] dF \quad (12)$$

where we use (8) to substitute for d_{t+1}. The value of the firm at an optimum is then equal to the market value of capital, v_t. The solution of the firm's problem in (12) has a fairly simple characterization:

Proposition 4 *There is a unique solution z_t for the firm's optimal capital structure, given by $z_{t+1} = 0$ when $\left(1 - \frac{q_{t+1}}{v_{t+1}}\right) Af(0) \geq \tau$ and by $0 < z_t < 1$ satisfying*

$$\left(1 - \frac{q_{t+1}}{v_{t+1}}\right) (A + q_{t+1} z_{t+1}) \frac{f(z_{t+1})}{1 - F(z_{t+1})} = \tau$$

when $\left(1 - \frac{q_{t+1}}{v_{t+1}}\right) Af(0) < \tau$.

The consumption savings decision The representative consumer has an income flow given by the initial endowment of capital k_0 and the payment of the profits π_t of the firms in the capital good sector and of the lump sum transfers T_t by the government at every date. Since he faces no income risk and can fully diversify, as we said, the idiosyncratic income risk of equity and corporate debt, the consumer effectively only trades each period a riskless asset. His choice problem reduces then to the maximization of the discounted stream of utility subject to the lifetime budget constraint:

$$\begin{array}{ll} \max & \sum_{t=0}^{\infty} \delta^t u(c_t) \\ \text{s.t.} & c_0 + \sum_{t=1}^{\infty} p_t c_t = Ak_0 + v_0 \bar{\theta} k_0 + \pi_0 + \sum_{t=1}^{\infty} p_t (T_t + \pi_t), \end{array} \quad (13)$$

where $p_t = \prod_{s=0}^{t-1} \frac{1}{1+r_s}$ is the discount rate between date 0 and date t, given the access to risk free borrowing and lending each period at the rate r_t.[4]

[4]The value of the initial endowment of capital k_0 equalss the value of the output Ak_0 produced with this capital in sub-period A plus the value of the capital left after depreciation in sub-period C, $\bar{\theta} k_0 v_0$. Also, while producers of capital good operate and hence distribute profits in every period $t \geq 0$, the first equity issue is at the end of date 0 and hence the first tax revenue on equity earnings is at date $t = 1$.

Market clearing The market-clearing condition for the consumption good is

$$c_t + I_t = Ak_t, \qquad \text{for all } t \geq 0 \tag{14}$$

The markets for debt and equity clear at any t if the amount of wealth the households want to carry forward into the next period is equal to the value of debt and equity issued by firms in that period. We show in the appendix that the market-clearing condition for the securities markets is automatically satisfied if the market-clearing condition for the goods market (14) is satisfied. This is just an application of Walras' law.

Finally, the market for capital clears if

$$k_{t+1} = \bar{\theta} k_t + \varphi\left(I_t\right) \tag{15}$$

4 Equilibrium

We are now ready to state the equations defining a competitive equilibrium in the environment described.

Definition 5 *A competitive equilibrium is a sequence of values* $\left\{\left(c_t^*, k_t^*, z_{t+1}^*, I_t^*, q_{t+1}^*, v_t^*, r_t^*\right)\right\}_{t=0}^{\infty}$ *satisfying the following conditions:*

1. Profit maximization in the capital good sector. *For every date* $t \geq 0$*,* I_t^* *solves:*

$$v_t^* \varphi'\left(I_t^*\right) \leq 1 \text{ and } \left(v_t^* \varphi'\left(I_t^*\right) - 1\right) I_t^* = 0.$$

2. Optimal capital structure. *For every date* $t \geq 0$*, the capital structure* z_{t+1}^* *of the firms in the consumption good sector satisfies:*

$$\left(1 - \frac{q_{t+1}^*}{v_{t+1}^*}\right)\left(A + q_{t+1}^* z_{t+1}^*\right) \frac{f\left(z_{t+1}^*\right)}{1 - F\left(z_{t+1}^*\right)} = \tau.$$

and the present value of the firms in this sector satisfies the law of motion

$$\left(1 + r_t^*\right) v_t^* = \left\{\int_0^{z_{t+1}^*} \left(A + q_{t+1}^* \theta_{t+1}\right) dF + \int_{z_{t+1}^*}^1 \left(v_{t+1}^* \left(\frac{A}{q_{t+1}^*} + \theta_{t+1}\right) - \tau v_{t+1}^* \left(\theta_{t+1} - z_{t+1}\right)\right) dF\right\}$$

3. Optimal consumption. *The sequence* $\left\{c_t^*\right\}_{t=0}^{\infty}$ *satisfies the following first-order conditions*

$$\frac{\delta u'\left(c_{t+1}^*\right)}{u'(c_t^*)} = \frac{1}{1 + r_t^*},$$

for every date $t \geq 0$, together with the budget constraint

$$c_0^* + \sum_{t=1}^{\infty} \left(\prod_{s=0}^{t-1} \frac{1}{1+r_s^*} \right) c_t^* = Ak_0 + v_0^* \bar{\theta} k_0 + v_0^* \varphi(I_0^*) - I_0^* +$$
$$\sum_{t=1}^{\infty} \left(\prod_{s=0}^{t-1} \frac{1}{1+r_s^*} \right) \left(\tau k_t^* v_t^* \int_{z_t^*}^{1} (\theta_t - z_t^*) f(\theta_t) \, d\theta_t + v_t^* \varphi(I_t^*) - I_t^* \right)$$

4. Liquidation market clearing. *For every date $t > 0$, the asset market clears in sub-period B:*

$$q_t^* < v_t^* \text{ and } q_t^* \int_0^{z_t^*} \theta_t dF = A \int_{z_t^*}^{1} dF.$$

5. Goods market clearing. *For every date $t \geq 0$, the goods market clears in sub-period C:*

$$Ak_t^* = c_t^* + I_t^*.$$

6. Capital market clearing. *For every date $t \geq 0$, the sequence $\{k_t^*\}$ satisfies the law of motion*

$$k_{t+1}^* = \bar{\theta} k_t^* + \varphi(I_t^*)$$

and $k_0^* = \bar{k}_0$.

Condition 1 requires firms in the capital goods sector to maximize profits at every date, taking the price of capital goods v_t^* as given. Condition 2 requires firms in the consumption goods sector to choose their capital structures optimally. Here we assume that the optimal capital structure occurs at an interior solution $0 < z_t^* < 1$. In fact, this is implied by Proposition 4 and the market-clearing condition for sub-period B (equation (9)). The law of motion for the value of the firm is simply the Bellman equation associated with the maximization problem in equation (12). Condition 3 requires that the consumption path solves the consumers' maximization problem (13) at every date. Conditions 4 – 6 are the market-clearing conditions for the liquidated capital goods in sub-period B and for capital goods and consumption goods in sub-period C. These conditions follow from equations (9), (15), and (14), respectively.

The equilibrium market prices of equity v_t^{e*} and debt v_t^{b*} at any date t are readily obtained from the other equilibrium variables. The returns on diversified equity and debt are deterministic, because there is no aggregate uncertainty. The rate of return on diversified debt must be equal to the rate of return on diversified equity. Thus, v_t^{e*} and v_t^{b*} must be such that the one-period expected returns on debt and equity are equal to the risk free rate.

Putting together the market-clearing condition for liquidated capital in sub-period B (9) with the optimality conditions for the firms in the consumption good sector (Proposition 4) we see that in equilibrium we must have an interior optimum for the firms' capital structure,

$z_t \in (0,1)$, and $q_t < v_t$. Thus, default is costly and occurs with probability strictly between zero and one:

$$0 < F(z_t) < 1.$$

Intuitively, if default were costless firms would choose 100% debt financing, but this implies default with probability one, which is inconsistent with market clearing. Similarly, 100% equity financing implies that there is no default and hence default is costless, so firms should use 100% debt financing instead. The only remaining alternative is a mixture of debt and equity and costly default.

We also see from the previous analysis that uncertainty only affects the returns and default decisions of individual firms. All other equilibrium variables, aggregate consumption, investment and market prices are deterministic.

4.1 Steady-state equilibria

Definition 6 *A steady state is a competitive equilibrium* $\{(c_t^*, k_t^*, z_t^*, I_t^*, q_t^*, v_t^*, r_t^*)\}_{t=0}^{\infty}$ *in which for all* $t \geq 0$,

$$(c_t^*, k_t^*, z_t^*, I_t^*, q_t^*, v_t^*, r_t^*) = (c^*, k^*, z^*, I^*, q^*, v^*, r^*).$$

The conditions defining a steady state are readily obtained by substituting the stationarity restrictions into Conditions 1 – 6 of Definition 5 of a competitive equilibrium.

Our first result shows that a steady state exists and is unique. In addition, the system of conditions defining a steady state can be reduced to a system of two equations.

Proposition 7 *Under the maintained assumptions, there exists a unique steady-state equilibrium, obtained as a solution of the following system of equations:*

$$q^* = \frac{A(1-F(z^*))}{\int_0^{z^*} \theta f(\theta) d\theta}, \tag{16}$$

$$v^* = \frac{\delta A}{1 - \delta\bar{\theta} + \tau \int_{z^*}^1 (\theta - z^*) dF}, \tag{17}$$

$$\left(1 - \frac{q^*}{v^*}\right)(A + q^* z^*) \frac{f(z^*)}{1 - F(z^*)} = \tau. \tag{18}$$

In a steady state, the risk free rate r^* is determined by the condition that the interest rate equals the subjective rate of time preference:

$$\frac{1}{1+r^*} = \delta,$$

while v^* is obtained by substituting q^*, r^* and z^* into Condition 2 of the definition of competitive equilibrium.

Having simplified the system of equations defining a steady state, we can also identify some of its comparative static properties.

Proposition 8 *(i) An increase in the tax rate τ increases the steady values of q^* and z^* (and hence the debt-equity ratio) but the effect on v^* (and hence I^* and k^*) is ambiguous. (ii) An increase in the discount factor δ decreases the steady-state values of q^* and z^* (and hence the debt-equity ratio), but the effect on v^* (and hence I^* and k^*) is ambiguous.*

5 Transition dynamics

The main focus of the rest of the paper will be on the welfare properties of equilibria, in particular on the efficiency of the investment and capital structure decisions of firms. To facilitate this analysis, we first complete the equilibrium analysis by studying the properties of the dynamics outside of the steady state. To make the analysis of the transitional dynamics tractable we will impose the additional assumption that consumers are *risk neutral*,

$$u(c_t) = c_t, \text{ for all } c_t \geq 0. \tag{19}$$

As a consequence, the stochastic discount factor is constant and equal to δ and, hence,

$$\frac{1}{1+r_t^*} = \delta, \text{ for all } t,$$

in any equilibrium. On the basis of assumption (19), we show in this section that the equilibrium dynamics converges monotonically to the steady state.

Under assumption (19), the equilibrium conditions outside the steady state can be reduced to a system of two equations. From the market-clearing condition in sub-period B (Condition 4), we have

$$q_t = \frac{A(1-F(z_t))}{\int_0^{z_t} \theta dF}, \tag{20}$$

which implies that $q_t = q(z_t)$ is a continuously decreasing function of z_t on the interval $[0,1]$, for all $t \geq 1$. The first-order condition for the optimal capital structure (Condition 2) can then be rewritten as

$$\left(1 - \frac{q(z_{t+1})}{v_{t+1}}\right)\left(\frac{A}{q(z_{t+1})} + z_{t+1}\right)\frac{f(z_{t+1})}{1-F(z_{t+1})} = \tau. \tag{21}$$

Holding v_{t+1} constant, an increase in z_{t+1} increases the left hand side of (21), so the change in v_{t+1} must decrease $\left(1 - \frac{q(z_{t+1})}{v_{t+1}}\right)$. In other words, an increase in z_{t+1} must decrease v_{t+1}. Thus, $v_{t+1} = v(z_{t+1})$ is a continuously decreasing function of z_t on the interval $[0,1]$, for all $t \geq 1$. The profit-maximization condition 1. of the capital good producers,

$$v_t \varphi'(I_t) = 1, \tag{22}$$

implies that the investment level $I_t = I(v_t)$ is a well defined and strictly increasing function of v_t; hence $I(v(z_t))$ is a well defined and decreasing function of z_t on the interval $[0,1]$, for all $t \geq 0$.

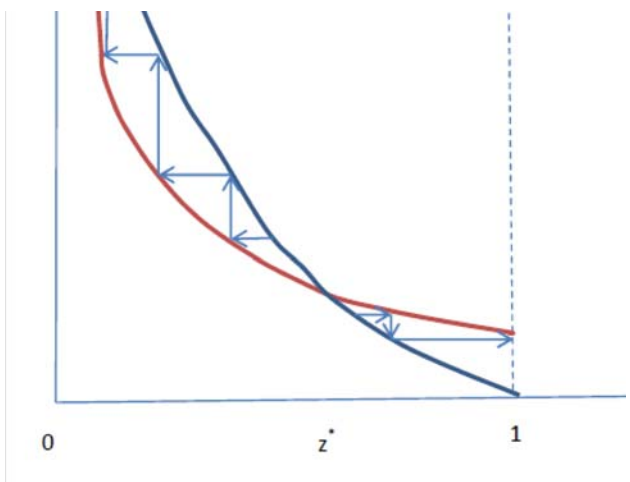
Substituting these functions for I, v, q into the expressions specifying the law of motion of the market value of the firms in the consumption good sector (in Condition 2)[5] and the capital market-clearing (Condition 6), we obtain the system of two difference equations below in z and k:

$$v(z_t) = \delta \left[A + v(z_{t+1})\bar{\theta} - \tau v(z_{t+1}) \int_{z_{t+1}}^{1} (\theta - z_{t+1}) dF \right] \tag{23}$$

$$k_{t+1} = \bar{\theta} k_t + \varphi(I(z_t)). \tag{24}$$

This dynamic system can be solved for the values $\{(k_t, z_t)\}_{t=1}^{\infty}$, subject to the initial conditions determining[6] k_1. This sequence defines an equilibrium trajectory if it belongs to an equilibrium as defined in Definition 5.

The first of the two equations, (23), only depends on z_t. Hence the dynamics for z_t is determined by that equation, and does not depend on k. We show in the Appendix that the dynamics for z_t is as described in the following figure:



where the red line is the graph of the term on the right hand side of (23), and the blue line is the graph of the term on the left hand side, both regarded as functions of z. The two curves intersect at the unique steady-state value $z = z^*$. At that point the slope of the red curve is flatter than the slope of the blue curve. Also, both curves are negatively sloped. This implies that, starting at any initial point $z_1 \neq z^*$, the trajectory $\{z_t\}$ satisfying the difference equation must diverge away from z^*. In fact, if $z_1 > z^*$, z_t is monotonically increasing until it reaches a value, strictly smaller than one, beyond which a solution to (23) no longer exists.[7] On the other hand, if $z_1 < z^*$ both curves diverge to infinity and the dynamics is monotonically decreasing approaching zero. This is also unfeasible, since we see

[5] We also use (20) to simplify the expression in (23), as we did in the proof of Proposition 7.

[6] The initial conditions are given by $k_1 = \bar{\theta}\bar{k}_0 + \varphi(I_0)$, with I_0 determined by $\delta \left[A + v(z_1)\bar{\theta} - \tau q(z_1) \int_{z_1}^{1} (\theta - z_1) dF \right] \varphi'(I_0) = 1$ as a function of z_1.

[7] If $z \to 1$, the term on the right hand side converges to A and the one on the left hand side converges to 0. Thus for some finite value of t there is no value of z_{t+1} that satisfies the (23).

from (21), (20), (22) that when $z \to 0$, v, q, I and hence also k tend to infinity, which violate the boundedness property established in Proposition 1.

This shows that in any competitive equilibrium we must have $z_t = z^*$ for all t. From this it follows that prices and the investment level are constant along the equilibrium path, at the levels $q_t = q(z^*) = q^*$, $v_t = v(z^*) = v^*$ and $I_t = I(z^*) = I^*$, for all t, while the dynamics of the capital stock is determined by the law of motion

$$k_{t+1} = \bar{\theta} k_t + \varphi(I^*),$$

with k_1 determined by the initial conditions. Then

$$\begin{aligned} k_{t+\tau+1} &= \left(\bar{\theta} + \bar{\theta}^2 + \cdots + \bar{\theta}^\tau\right) \varphi(I^*) + \bar{\theta}^{\tau+1} k_t \\ &\to \frac{\varphi(I^*)}{1-\bar{\theta}} = k^* \text{ as } \tau \to \infty. \end{aligned}$$

So the capital stock converges to its steady-state value. We have thus established the following:

Proposition 9 *Let $\{(k_t, z_t)\}_{t=1}^{\infty}$ be a solution of (23), (24), satisfying $k_0 = \bar{k}_0$. Then $\{(k_t, z_t)\}_{t=1}^{\infty}$ is an equilibrium trajectory only if, for all $t \geq 1$, $z_t = z^*$, where z^* is the uniquely determined steady-state capital structure. Furthermore, k_t converges monotonically to its steady-state value, k^*.*

6 Welfare analysis

6.1 The inefficiency of equilibrium

If we compare the conditions for a Pareto efficient steady state derived in Proposition 2 with the conditions for a steady-state equilibrium derived in Section 4.1, we see that steady-state equilibria are Pareto efficient if and only if $I^* = I^{FB}$, which happens when the equilibrium market value of capital is given by

$$v^* = \frac{\delta}{1-\delta\bar{\theta}} A.$$

From the equilibrium conditions, in particular, Condition 2, it can be seen immediately that the condition above can hold only if $\tau = 0$. In that case, there is no cost of issuing equity and the firms in the capital good sector will choose 100% equity finance. On the other hand, when $\tau > 0$, as we have been assuming, the equilibrium market value of capital v^* is strictly lower than $\frac{\delta}{1-\delta\bar{\theta}} A$ and I^* and k^* are strictly less than the corresponding first-best values. The financial frictions of incomplete markets and the perceived costs of default and equity financing, imply that firms invest too little nd the equilibrium stock of capital is inefficiently low. Even with a representative consumer, competitive equilibria are Pareto inefficient.

6.2 Constrained inefficiency

It is not surprising that equilibrium is Pareto-inefficient in the presence of distortionary taxation. A more surprising result is that, even in the presence of frictions, regulation of a single variable, while allowing other variables to reach their equilibrium levels, can lead to a welfare improvement. That is, competitive equilibria are also constrained inefficient. We consider two possible types of interventions. In the first, we control the level of aggregate investment. In the second we control the breakeven level of debt.

Controlling investment I Starting in a steady-state equilibrium, we consider an increase in investment at some date t. Thus, at date t, I is no longer determined by Condition 1 of Definition 5, but set equal to $I^* + \Delta I$. The rest of the equilibrium variables are determined by the agents' optimizing decisions and market-clearing conditions. In particular, the output of capital goods at all subsequent dates responds to the exogenous change in investment. The law of motion of the capital stock is now

$$\begin{aligned} k_{t+1} &= \bar{\theta} k^* + \varphi\left(I^* + \Delta I\right), \\ k_{t+i} &= \bar{\theta} k_{t+i-1} + \varphi(I_{t+i-1}), \text{ for all } i > 1, \end{aligned}$$

while the market-clearing condition in the liquidation market, the optimality condition for the firms' capital structure and the law of motion for v_t are still given by (20), (21) and (23). Similarly, at each subsequent date $t + i > t$, the level of I is determined by the profit-maximization condition of the capital good producers, (22). Since equations (20), (21) and (23) are unchanged, the analysis of the transition dynamics in Section 5 still applies and implies that their solution is given by $z_{t+i} = z^*$, $v_{t+i-1} = v^*$, $q_{t+i} = q^*$, for all $\iota > 0$. It also follows that $I_{t+i} = I^*$ after date t.

The dynamics for consumption are given by the following equations:

$$\begin{gathered} c_t = Ak^* - \left(I^* + dI\right), \\ c_{t+1} = A(\bar{\theta} k^* + \varphi\left(I^* + dI\right)) - I^*, \\ c_{t+i} = A(k^* + \bar{\theta}^{i-1}\left(\varphi\left(I^* + \Delta I\right) - \varphi\left(I^*\right)\right)) - I^*, \text{ for all } i > 1. \end{gathered}$$

Hence, the effect on welfare of this intervention is given by the sign of

$$\left(-1 + \varphi'(I^*) A \delta \sum_{t=0}^{\infty} \left(\delta \bar{\theta}\right)^t\right) \Delta I.$$

The term in brackets in this expression is strictly positive because, as we showed in the previous section, in a steady-state equilibrium we always have

$$A \frac{\delta}{1 - \delta \bar{\theta}} > \frac{1}{\varphi'(I^*)} = v^*.$$

Hence, a temporary increase of investment above its equilibrium value increases welfare by bringing the stock of capital closer to its first-best level.

Notice that if we allow for repeated interventions of the kind described, setting the level of the investment $I = I^{FB}$ at all dates $t+i$ for $i > 0$, it is possible to attain the Pareto-efficient allocation after a transition of one period. Consider, for instance, the first-best steady state. It is easy to verify, by a similar argument analogous, that the following intervention

$$\begin{aligned} \text{at date } t, \text{ set } I_t \text{ such that } k^{FB} &= \bar{\theta}k^* + \varphi(I_t) \\ \text{at all } t+\iota,\ \iota &> 0,\ \text{set } I_{t+\iota} = I^{FB} \end{aligned}$$

induces the following equilibrium consumption sequence:

$$\begin{aligned} c_t &= Ak^* - I_t \\ c_{t+\iota} &= Ak^{FB} - I^{FB} = c^{FB} \text{ for all } \iota > 1. \end{aligned}$$

Controlling the breakeven point z Now consider an alternative intervention, consisting of a change in the capital structure of the firms producing in the consumption good sector, with all other variables determined as in equilibrium. In particular, we consider a permanent[8] change Δz starting at some fixed but arbitrary date $t+1$. The induced changes in the equilibrium variables q and v are obtained from the market-clearing condition in sub-period B, (20), and the law of motion of v, (23). After substituting the new value of z, the new values of q and v are determined by

$$A\left(1 - F(z^* + \Delta z)\right) = q_{t+i}\int_0^{z^*+\Delta z}\theta dF = q_{t+i}\left[z^* f(z^*) + \int_0^{z^*}\theta dF\right], \tag{25}$$

and [9]

$$v_{t+i} = \delta\left\{A + v_{t+i+1}\bar{\theta} - \tau v_{t+i+1}\int_{z^*+\Delta z}^1 (\theta - z^* - \Delta z)dF\right\}, \tag{26}$$

for all $i > 0$. We see from (25) that the new equilibrium value for q_{t+i} is the same for all i and from (26) we obtain a first-order difference equation in v. The solution of this equation diverges monotonically since the coefficient on v_{t+1+1} has absolute value

$$\begin{aligned} \left|\bar{\theta} - \tau\int_{z^*+\Delta z}^1 (\theta - z^* - \Delta z)dF\right| &< \max\left\{\bar{\theta}, \tau\int_{z^*+\Delta z}^1 (\theta - z^* - \Delta z)dF\right\} \\ &\leq \max\left\{\bar{\theta}, \tau\bar{\theta}\right\} = \bar{\theta} < 1. \end{aligned}$$

[8] We focus attention on a permanent, rather than a temporary, intervention to make the analysis simpler, but it is fairly easy to verify that the same welfare result holds in the case of a temporary intervention.

[9] Note that expressions (25) and (26) give us the new equilibrium levels of q and v also for any discrete change Δz, as long as we have $v \geq q$, that is as long as $z + \Delta z$ is not too close to 0.

Hence, the only admissible solution is obtained by setting v_{t+i} equal to its steady-state value:

$$v_{t+i} = v_{t+i+1} = v^* + \Delta v = \frac{\delta A}{1 - \delta\bar{\theta} + \delta\tau \int_{z^*+\Delta z}^{1} (\theta - z^* - \Delta z) dF}. \tag{27}$$

The remaining equilibrium variables are determined by the optimality condition for the capital goods producers, (22), and the capital market clearing condition, (24), which are both unchanged. Since, by the previous argument, v_{t+1} is equal to ts new steady-state equilibrium value, $v^* + \Delta v$, we have $I_{t+i} = I^* + \Delta I$ for all $i > 0$, where the sign of ΔI equals the sign of Δv.

The effect on welfare is then determined as in the case of the first intervention considered by the change in I and hence in k, and consumers' welfare increases if and only if $dI > 0$. From (27) it is then easy to verify that sign $\Delta v =$ sign Δz, since

$$\frac{d}{dz^*} \int_{z^*}^{1} (\theta - z^*) dF = - \int_{z^*}^{1} dF < 0$$

and so, in the limit,

$$\frac{dI}{dz} = \frac{dv}{dz}\frac{dI}{dv} = \frac{dv}{dz}\left(-\frac{\varphi''}{v\varphi} \right) > 0$$

Hence, welfare is increased by a permanent increase in z above its steady-state equilibrium value.

When z is increased above z^*, the equilibrium value of v increases, as we see from (26), but the tax liability $\tau v \int_{z}^{1} (\theta - z) dF$ decreases, as we can see from (27). In fact, it is because the tax liability falls that the value of capital increases. The reason firms do not choose a higher value of z in equilibrium is that they are price takers. They choose z to maximize (12), without taking into account the effect of z on q and v. This is why the higher value of z, $z^* + \Delta z$, cannot be sustained as a competitive equilibrium. If left free to choose its capital structure, each firm would respond to the lower value of q, $q^* + \Delta q$, by choosing a lower value of z rather than the higher one, so as to profit from the larger gains it can make when solvent. These gains disappear, however, when we take into account the effect of lowering z on the equilibrium value of q. Even though debt financing entails a cost of default, $v - q$, and this difference is going to increase as a result of the intervention considered, we see from (27) that, once we consider the equilibrium relationship between z and q, this cost has no effect on v, only the tax cost remains.

Also, both the default and tax costs 'wash out' in the welfare analysis, since they only entail a redistribution of wealth between debt holders, equity holders, and taxpayers. Given the homogeneity of consumers, such a redistribution has no effect on welfare. The only effect on welfare comes from the change in investment and capital. Any intervention that increases I and k is welfare improving.

The intervention acts directly on the threshold below which the firm has to default on its debt. To claim that an increase of this threshold corresponds to an increase in the debt-equity ratio, the change in the market value of debt and equity should also be taken into account, that is we should look at

$$\frac{v^b}{v^e} = \frac{\int_0^z (A + q_{t+\iota}\theta)\, dF + \int_z^1 \frac{v_{t+i}(A+q_{t+\iota}z)}{q_i} dF}{\int_z^1 \left(\frac{v_{t+i}}{q_{t+i}} - \tau\right) q_{t+i} (\theta - z)\, dF} \tag{28}$$

The effect of a marginal increment in z, starting from z^* on the value of the debt equity ratio $\frac{v^b}{v^e}$, is not straightforward to determine in general. We will show in what follows that, for a discrete, sufficiently large increment in z we have an unambiguous increase in the debt equity ratio $\frac{v^b}{v^e}$.

Consider a sequence of discrete changes Δz, such that $z + \Delta z$ approaches 1. Along such sequence q goes to zero and we also see from (27) that v approaches $\frac{\delta A}{1-\delta\theta}$ and hence, by the argument of the previous section, I approaches I_{FB}. That is, in the limit, the equilibrium corresponding to such an ntervention converges to the steady-state, first-best allocation.[10] Also, as $z \to 1$, we have

$$\frac{v^b}{v^e} = \frac{\int_0^z (A + q_{t+i}\theta)\, dF + \int_z^1 \frac{v_{t+i}(A+q_{t+\iota}z)}{q_{t+i}} dF}{\int_z^1 \frac{v_i}{q_{t+i}} (1-\tau)(\theta - z)\, dF} \to \frac{A + vA \lim_{z\to 1} \int_z^1 \frac{1}{q_{t+i}} dF}{0} = \infty$$

Hence, we can indeed say that the debt equity ratio is increasing, at least in the limit, as a result of such intervention.

6.3 Inefficient hedging

Fire sales are a necessary element of equilibrium, as we have shown. Equity is dominated by debt finance unless banruptcy is perceived to be costly and, in equilibrium, both debt and equity finance must be used. One might think that speculators would have an incentive to accumulate liquidity in order to buy assets at fire sale prices, but speculation does nothing to increase the efficiency of equilibrium. As long as liquid assets yield a low return, speculators will not hold them unless they can expect capital gains from buying assets in the fire sale. The supply of liquidity will never be sufficient to eliminate fire sales. In fact, the presence of liquid assets can make equilibrium less efficient. As we have pointed out, the costs of bankruptcy are an illusion, because they represent a transfer rather than a true economic cost. For the same reason, the capital gains from buying assets in fire sales are also an illusion. So although holding low-yielding liquid assets in order to buy up assets in a fire sale is always inefficient. In fact, it can make everyone worse off than in an economy without liquid assets.

To represent the possibility of speculative arbitrage to provide liquidity in the market, we extend the analysis by introducing an additional, "safe" technology to produce the consumption good using the capital good, also subject to constant returns to scale. We assume that one unit of capital applied to this technology produces B units of the good and that after

[10] Note that the equilibrium condition (25) has an admissible solution for all $z + \Delta z < 1$, but not in the limit for $z + \Delta z = 1$.

depreciation the amount of capital remaining is $\bar{\theta}$. The two technologies have then the same *average* depreciation rate but the depreciation rate of the safe technology is deterministic. We assume that $B < A$; otherwise, the safe technology would dominate the risky technology.

Each firm in the consumption good sector now faces a technology choice, in addition to the choice of its capital structure. Otherwise, the definition of a competitive equilibrium is unchanged.

To analyze the firms' technology choice, consider a firm which has one unit of capital at date t. If the capital is entirely invested in the safe technology, the optimal capital structure is full debt financing, since there is no default risk in this case. At date $t+1$ the firm produces B units of goods which it retains and uses to buy $\frac{B}{q_{t+1}}$ units of capital. Then, at the end of date $t+1$, the firm has $\frac{B}{q_{t+1}} + \bar{\theta}$ units of capital which is valued at $v_{t+1}\left(\frac{B}{q_{t+1}} + \bar{\theta}\right)$. In equilibrium, it is optimal for the firm to invest all its capital in the safe technology if and only if

$$v_t = \frac{1}{1+r_t} v_{t+1} \left(\frac{B}{q_{t+1}} + \bar{\theta}\right). \tag{29}$$

In addition, the zero profit condition requires that the nominal value of debt issued by the firm fully investing in the safe technology is equal to $d_{t+1} = B + q_{t+1}\bar{\theta}$.

We establish first some properties of the equilibrium technology choice.

Proposition 10 *At a competitive equilibrium, if $v_{t+1} > q_{t+1}$ it is never optimal for a consumption good producer to use both technologies at the same time.*

This proposition is the result of the non-convexity of the firm's objective function associated with costly bankruptcy. If the firm has a positive amount of debt and a positive probability of default, the firm can increase its value by shifting all its production to the risky technology, keeping the default probability and the default cost unchanged and enjoying the higher returns of the technology, or to the safe technology which allows to avoid all the default risk and cost.

We show next that, as in the previous specification, in equilibrium we always have $v_{t+1} > q_{t+1}$. Suppose not, that is we have $v_{t+1} = q_{t+1}$. In that case there is no default cost, hence firms by investing in the risky technology and fully financing with debt attain a higher value, since $A > B$ and there is no cost attached to debt financing. But if all firms only invest in the risky technology we have shown in the previous section there can be no equilibrium where $v_{t+1} = q_{t+1}$.

Having shown that $v_{t+1} > q_{t+1}$, the market clearing condition in the liquidation market implies that at least a positive fraction of firms invest in the risky technology. Hence at a competitive equilibrium two possible cases arise. The first one is a situation where all firms invest in the risky technology. The equilibrium is then the same as in the previous section. More precisely, a competitive equilibrium $\left\{\left(c_t^*, k_t^*, z_{t+1}^*, I_t^*, q_{t+1}^*, v_t^*, r_t^*\right)\right\}_{t=0}^{\infty}$ according to Definition 5 is also an equilibrium when consumption good producers also face a choice between a risky and a safe technology provided the equilibrium values satisfy the following condition, for all t,

$$v_t^* \geq \frac{1}{1+r_t^*} v_{t+1}^* \left(\frac{B}{q_{t+1}^*} + \bar{\theta} \right), \tag{30}$$

that is, no producer can gain at these prices by switching form the risky to the safe technology.

The second case arises when (30) is violated, in which case the competitive equilibrium is different and entails a positive fraction $(1-\ell_t^{**}) \in (0,1)$ of firms using the safe technology. In this case, the equilibrium conditions need to be partly modified, in particular the liquidation market clearing condition, which becomes

$$q_t^{**} \int_0^{z_t^*} \theta_t dF = \ell_t^{**} A \left(1 - F\left(z_t^{**}\right)\right) + \left(1 - \ell_t^{**}\right) B, \tag{31}$$

to reflect the fact that the buyers of capital goods now include the solvent firms investing in the risky technology and all the firms investing in the safe technology, as well as the good market clearing condition,

$$A\ell_t^{**} k_t^{**} + B\left(1 - \ell_t^{**}\right) k_t^{**} = c_t^{**} + I_t^{**}, \tag{32}$$

to reflect the differing productivities of the two technologies. In addition, condition (29), requiring that firms must be indifferent between the safe and risky technologies, must also hold.

We investigate in what follows the welfare properties of these equilibria. We show in particular that the availability of an alternative, safe technology, which allows firms to avoid the default risk, generates an additional source of inefficiency.

Proposition 11 *The exists a unique value of B, denoted by $\bar{B} > 0$, such that if $B \leq \bar{B}$ then $\ell^{**} = 1$ in any steady-state equilibrium $(c^{**}, k^{**}, z^{**}, I^{**}, q^{**}, v^{**}, r^{**}, \ell^{**})$. By contrast, for some $\varepsilon > 0$ and any $B \in \left(\bar{B}, \bar{B} + \varepsilon\right)$, $\ell^{**} < 1$ and the consumption level c^{**} is lower than in the equilibrium with $\ell^{**} = 1$.*

The critical value of B, denoted by $\bar{B}$, is defined by

$$\bar{B} = \frac{q^{**}\left(1 - \bar{\theta}\right)}{\delta},$$

where q^* is the price of liquidated capital at a steady-state equilibrium of the economy with no safe technology. At this steady-state equilibrium price, (30) holds with equality when $B = \bar{B}$, hence firms are indifferent between using the safe and risky technologies. At $\bar{B} + dB > \bar{B}$, (30) no longer holds, the steady-state equilibrium involves a positive fraction of firms $1 - \ell^{**} > 0$ adopting the safe technology and a higher steady-state equilibrium value of q,

$$q^{**} = \frac{\delta(\bar{B} + dB)}{1 - \delta\bar{\theta}}. \tag{33}$$

We show in what follows that it equilibrium welfare is lower at a new steady-state equilibrium than at the original one. Since the original allocation, with all firms investing in the risky technology, clearly remains feasible, this shows that the equilibrium indeed exhibits an inefficient technology choice, with excessive investment in the safe technology, as claimed.

From the market-clearing condition (32), we see that the change in the equilibrium consumption when B is increased from $\bar{B}$ to $\bar{B}+dB$ depends on the change in the equilibrium values of ℓ, k and I. Since conditions 1 and 6 of Definition 5 still hold, as shown in the previous sections both I and k increase if v increases. Consider the equation determining the value of the firm at a steady-state equilibrium (condition 2):

$$v^{**} = \delta \left\{ \int_0^{z^{**}} \left(A + q^{**}\theta \right) dF + \int_{z^{**}}^1 \left(\frac{v^{**}}{q^{**}} \left(A + q^{**}\theta \right) - \tau v^{**} \left(\theta - z^{**} \right) \right) dF \right\} \tag{34}$$

The change in the equilibrium value of v when B is increased to $\bar{B}+dB$ is obtained by differentiating this equation with respect to B and evaluating the derivative at $B=\bar{B}$:

$$\begin{aligned} \frac{dv}{dB} = \delta \Bigg\{ & \left(\int_0^{z^{**}} \theta dF \right) \frac{dq}{dB} + \left(\frac{1}{q^{**}} \int_{z^{**}}^1 \left(A + q^{**}\theta \right) dF \right) \frac{dv}{dB} \\ & - \left(\frac{v^{**}}{q^{*2}} \int_{z^{**}}^1 A dF \right) \frac{dq}{dB} - \left(\tau \int_{z^{**}}^1 \left(\theta - z \right) dF \right) \frac{dv}{dB} \Bigg\}. \end{aligned}$$

Rearranging, we get

$$\begin{aligned} & \left(1 - \frac{\delta}{q^{**}} \int_{z^{**}}^1 \left(A + q^{**}\theta \right) dF + \tau \int_{z^{**}}^1 \left(\theta - z \right) dF \right) \frac{dv}{dB} = \\ & \qquad \delta \left(\int_0^{z^{**}} \theta dF - \frac{v^{**}}{q^{**2}} \int_{z^{**}}^1 A dF \right) \frac{dq}{dB}. \end{aligned}$$

Now, using the market-clearing condition (32) and noting that $\ell^{**} = 1$, we see that

$$\begin{aligned} \frac{\delta}{q^*} \int_{z^*}^1 \left(A + q^*\theta \right) dF &= \frac{\delta}{q^* *} \left(A \left(1 - F \left(z^{**} \right) \right) + \int_{z^{**}}^1 q^{**}\theta dF \right) \\ &= \frac{\delta}{q^*} \left(q^* \int_0^{z^*} \theta dF + q^* \int_{z^*}^1 \theta dF \right) \\ &= \delta \bar{\theta} < 1. \end{aligned}$$

So,

$$\begin{aligned} 1 - \frac{\delta}{q^{**}} \int_{z^{**}}^1 \left(A + q^{**}\theta \right) dF + \tau \int_{z^{**}}^1 \left(\theta - z \right) dF &= 1 - \delta\bar{\theta} + \tau \int_{z^{**}}^1 \left(\theta - z \right) dF \\ &> 1 - \delta\bar{\theta} > 0. \end{aligned}$$

Similarly,

$$\int_0^{z^{**}} \theta dF - \frac{v^{**}}{q^{**2}} \int_{z^{**}}^1 AdF - \tau \int_{z^{**}}^1 (\theta - z)\, dF = \int_0^{z^{**}} \theta dF - \frac{v^{**}}{q^{**}} \frac{A\,(1 - F\,(z^{**}))}{q^{**}} - \tau \int_{z^{**}}^1 (\theta - z)\, dF$$
$$< \int_0^{z^{**}} \theta dF - \int_0^{z^{**}} \theta dF - \tau \int_{z^{**}}^1 (\theta - z)\, dF$$
$$= -\tau \int_{z^{**}}^1 (\theta - z)\, dF < 0.$$

From these two inequalities, it follows that $\frac{dv}{dB}$ and $\frac{dq}{dB}$ have opposite signs. Since $\frac{dq}{dB} > 0$ follows from (33), we have proved that $\frac{dv}{dB} < 0$.

The change in the steady-state equilibrium consumption level is then obtained by differentiating (32) with respect to B and evaluating the derivative at $B = \bar{B}$:

$$dc^{**} = Adk^{**} - dI^{**} + (1 - B)k^{**}d\ell^{**}, \tag{35}$$

since $\ell^* = 1$. The profit maximization condition $v^*\varphi'\,(I^{**}) = 1$ implies that a reduction in v^{**} reduces I^{**}. The law of motion $k^{**} = \varphi\,(I^{**}) + \bar{\theta}k^{**}$implies that a reduction in I^{**} reduces k^{**}:

$$dk^{**} = \frac{\varphi'\,(I^{**})}{1 - \bar{\theta}} dI^{**}$$

Also, $d\ell^{**} \leq 0$ since $\ell^{**} = 1$ at the equilibrium associated with $B = \bar{B}$. Then inspection of (35) yields

$$\begin{aligned} dc^{**} &< Adk^{**} - dI^{**} \\ &= \left(\frac{A\varphi'\,(I^{**})}{1 - \bar{\theta}} - 1 \right) dI^{**} \\ &= \left(\frac{A}{v^{**}\,\left(1 - \bar{\theta}\right)} - 1 \right) dI^{**}, \end{aligned}$$

so a sufficient condition for $dc^{**} < 0$ is $v^{**}\left(1 - \bar{\theta}\right) < A$. But it is clear from the definition of v^{**} in (34) that

$$\begin{aligned} v^{**} &\leq \delta A \left\{ 1 + \delta\bar{\theta} + \left(\delta\bar{\theta}\right)^2 + \cdots \left(\delta\bar{\theta}\right)^k + \cdots \right\} \\ &= \frac{\delta A}{1 - \delta\bar{\theta}} < \frac{A}{1 - \bar{\theta}}. \end{aligned}$$

This completes the proof of the proposition.

7 Conclusion

[To be completed]

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Proofs

Proof of Proposition 1 From the strict concavity of φ and the gradient inequality, it follows that, for any $k < \hat{k}$,

$$\begin{aligned} \varphi(Ak) &\leq \varphi\left(A\hat{k}\right) + \varphi'\left(A\hat{k}\right) A\left(k - \hat{k}\right) \\ &= \left(1 - \bar{\theta}\right)\hat{k} + \varphi'\left(A\hat{k}\right) A\left(k - \hat{k}\right) \\ &< \left(1 - \bar{\theta}\right)\hat{k}. \end{aligned}$$

Hence

$$\varphi(Ak) + \bar{\theta}k < \left(1 - \bar{\theta}\right)\hat{k} + \bar{\theta}k < \hat{k}.$$

For any $k > \hat{k}$,

$$\begin{aligned} \varphi(Ak) &\leq \varphi\left(A\hat{k}\right) + \varphi'\left(A\hat{k}\right) A\left(k - \hat{k}\right) \\ &< \left(1 - \bar{\theta}\right)\hat{k} + \left(1 - \bar{\theta}\right)\left(k - \hat{k}\right) \\ &= \left(1 - \bar{\theta}\right)k, \end{aligned}$$

where the second inequality follows from the assumptions made on $\varphi(\cdot)$, implying the existence of a unique solution for $\hat{k}$. Thus,

$$k_t > \hat{k} \Longrightarrow k_{t+1} < k_t$$

and

$$k_t < \hat{k} \Longrightarrow k_{t+1} < \hat{k}.$$

Proof of Proposition 2 At an optimal steady state the multipliers $\{(\lambda_t^*, \mu_t^*)\}_{t=0}^{\infty}$ satisfy

$$\lambda_t^* = \delta^t u'(c^*) = \delta^t \lambda_0^*,$$

and hence

$$\mu_t^* = \frac{\lambda_t^*}{\varphi'(I^*)} = \frac{\delta^t \lambda_0^*}{\lambda_0^*/\mu_0^*} = \delta^t \mu_0^*,$$

for every t. The first-order conditions for the steady-state optimum can then be written as

$$u'(c^*) = \lambda_0^*, \tag{36}$$

$$\delta\lambda_0^* A + \delta\mu_0^*\bar{\theta} = \mu_0^*, \tag{37}$$

$$\mu_0^*\varphi'(I^*) = \lambda_0^*. \tag{38}$$

Conditions (37) and (38) can be rewritten as

$$\frac{\delta A}{1 - \delta\bar{\theta}} = \frac{\mu_0^*}{\lambda_0^*} = \frac{1}{\varphi'(I^*)} \tag{39}$$

The feasibility conditions become

$$c^* + I^* = Ak^*$$

and

$$k^* = \bar{\theta} k^* + \varphi(I^*).$$

Thus,

$$k^* = \frac{\varphi(I^*)}{1 - \bar{\theta}},$$

where I^* is determined by (39).

Proof of Proposition 4 The firm's choice problem (12) can be rewritten as follows:

$$v_t = \max_{z_{t+1}} \frac{1}{1+r_t} \left\{ \int_0^{z_{t+1}} (A + q_{t+1}\theta_{t+1}) f(\theta_{t+1}) d\theta_{t+1} + \int_{z_{t+1}}^1 \left(v_{t+1} \left(\frac{A}{q_{t+1}} + \theta_{t+1} \right) - \tau v_{t+1} (\theta_{t+1} - z_{t+1}) \right) f(\theta_{t+1}) d\theta_{t+1} \right\}.$$

The derivative of the expression on the right hand side with respect to z_{t+1} is easily calculated to be

$$\begin{aligned} & \frac{1}{1+r_t} \left\{ (A + q_{t+1}z_{t+1}) f(z_{t+1}) - v_{t+1} \left(\frac{A}{q_{t+1}} + z_{t+1} \right) f(z_{t+1}) + (z_{t+1} - z_{t+1}) f(z_{t+1}) + \tau v_{t+1} (1 - F(z_{t+1})) \right\} \\ &= \frac{1}{1+r_t} \left\{ \left(1 - \frac{v_{t+1}}{q_{t+1}} \right) Af(z_{t+1}) + \left(1 - \frac{v_{t+1}}{q_{t+1}} \right) q_{t+1}z_{t+1}f(z_{t+1}) + \tau v_{t+1} (1 - F(z_{t+1})) \right\}. \end{aligned}$$

The first-order condition for an interior solution of the firm's problem requires this expression to equal zero, a condition which can be written as

$$\left(\frac{v_{t+1}}{q_{t+1}} - 1 \right) (A + q_{t+1}z_{t+1}) f(z_{t+1}) = \tau v_{t+1} (1 - F(z_{t+1})),$$

or

$$\left(1 - \frac{q_{t+1}}{v_{t+1}} \right) (A + q_{t+1}z_{t+1}) \frac{f(z_{t+1})}{1 - F(z_{t+1})} = \tau.$$

Since all terms on the left hand side are positive, the solution to this equation, if it exists, is unique if all these terms are increasing in z_{t+1}. Obviously, $A + q_{t+1}z_{t+1}$ is increasing in z_{t+1}, and so is $\frac{f(z_{t+1})}{1-F(z_{t+1})}$ under the assumption of an increasing hazard rate.

From the above expression of the derivative we also see that a corner solution with $z_{t+1} = 0$ obtains when $\left(1 - \frac{q_{t+1}}{v_{t+1}}\right) Af(0) > \tau$. In contrast, it is easy to verify that a corner solution with $z_{t+1} = 1$ never exists. By the continuity of the objective function in z_{t+1}, a solution always exists, so it follows that an interior solution exists when $\left(1 - \frac{q_{t+1}}{v_{t+1}}\right) Af(0) < \tau$.

Market clearing in the securities market The markets for debt and equity clear at any t if the amount of wealth the households want to carry forward into the next period is equal to the value of debt and equity issued by firms in that period. The latter is equal to the market value of depreciated capital, $v_t\bar{\theta}k_t$, plus the value of newly produced capital goods, $v_t\varphi(I_t)$. To find the consumer's savings, we need first to find the value of the consumer's wealth in sub-period C of date t. This is equal to the *sum* of the profits from the capital good sector, the proceeds from the liquidation of firms which defaulted in this period, and the value of the firms that did not default in the period *minus* the corporation tax and the lump sum transfer from the government. The corporation tax and the transfer cancel in equilibrium and can be ignored. Hence, the consumer's wealth, w_t, is given by

$$\begin{aligned} w_t = v_t\varphi(I_t) - I_t + A\int_0^{z_t} k_t f(\theta_t)\,d\theta_t + q_t\int_0^{z_t}\theta_t k_t f(\theta_t)\,d\theta_t + v_t\bar{\theta}k_t + A\int_{z_t}^1 k_t f(\theta_t)\,d\theta_t \\ - q_t\int_0^{z_t}\theta_t k_t f(\theta_t)\,d\theta_t \\ = v_t\varphi(I_t) - I_t + Ak_t + v_t\bar{\theta}k_t. \end{aligned}$$

Therefore, the securities market clears at date t if

$$\begin{aligned} w_t - c_t &= v_t\varphi(I_t) - I_t + Ak_t + v_t\bar{\theta}k_t \\ &= v_t\left(\bar{\theta}k_t + \varphi(I_t)\right) \end{aligned}$$

or

$$c_t + I_t = Ak_t.$$

So market clearing in the goods market implies market clearing in the securities markets.

Proof of Proposition 7 Equation (16) comes directly from Condition 4 of the definition of competitive equilibrium, applied to a steady state. Equation (18) is simply the first-order condition from Condition 2 of the definition of competitive equilibrium.

The law of motion of the value of the firm can be written as

$$\begin{aligned} v^* &= \delta\left\{\int_0^{z^*}(A+q^*\theta)dF + \int_{z^*}^1\left(\frac{v^*}{q^*}(A+q^*\theta) - \tau v^*(\theta - z^*)\right)dF\right\} \\ &= \delta\left\{\int_0^{z^*}(A+v^*\theta-(v^*-q^*)\theta)dF + \int_z^1\left(A+\left(\frac{v^*}{q^*}-1\right)A + v^*\theta - \tau v^*(\theta-z^*)\right)dF\right\} \\ &= \delta\left\{\int_0^{z^*}\left(A+v^*\theta-\left(\frac{v^*}{q^*}-1\right)q^*\theta\right)dF + \int_{z^*}^1\left(A+v^*\theta+\left(\frac{v^*}{q^*}-1\right)A - \tau v^*(\theta-z^*)\right)dF\right\} \\ &= \delta\left\{A + v^*\bar{\theta} - \int_z^1 \tau v^*(\theta - z^*)dF\right\} \end{aligned}$$

where in the last step we used (16) to simplify the expression. Solving the last equation we obtained for v, we get:

$$v^*\left(1-\delta\bar{\theta}+\tau\int_{z^*}^1(\theta-z^*)\,dF\right) = \delta A$$

or

$$v^* = \frac{\delta A}{1 - \delta\bar{\theta} + \tau \int_{z^*}^{1} (\theta - z^*)\, dF},$$

which is equation (17).

Now the value of $v^* = v(z^*)$ given by equation (17) is an increasing function of z^* and the value of $q^* = q(z^*)$ given by (16) is a decreasing function of z^*. Writing the first-order condition (18) as

$$\left(1 - \frac{q(z^*)}{v(z^*)}\right)\left(\frac{A}{q(z^*)} + z^*\right)\frac{f(z^*)}{1 - F(z^*)} = \tau, \tag{40}$$

it is clear from inspection that the terms on the left hand side are increasing in z^*, so there exists at most one steady state.

To see that there exists a solution to (40), note that as $z^* \to 0$, $q(z^*) \to \infty$ and $v(z^*) \to \frac{\delta A}{1-\delta\bar{\theta}+\tau\bar{\theta}}$, so for some finite value $z^* > 0$, $q(z^*) = v(z^*)$ and the left hand side of (40) equals zero. Next, consider what happens as $z^* \to 1$ and note that $q(z^*) \to 0$ and $v(z^*) \to \frac{\delta A}{1-\delta\bar{\theta}} > 0$, so the left hand side of (40) is positive. Thus, by continuity, there exists a value of $0 < z^* < 1$ satisfying (40).

Proof of Proposition 8 Consider first the effect of a change in τ. From equation (16) it is clear that $q^* = q(z^*)$ is independent of τ whereas equation (17) shows that $v^* = v(z^*, \tau)$ is decreasing in τ. Then the first-order condition (40) can be rewritten as

$$\left(1 - \frac{q(z^*)}{v(z^*, \tau)}\right)\left(\frac{A}{q(z^*)} + z^*\right)\frac{f(z^*)}{1 - F(z^*)} = \tau.$$

An increase in τ increases the right hand side and, by decreasing $v(z^*, \tau)$, it decreases the left hand side. Thus, to satisfy the first-order condition, the left hand side must be increased and that requires an increase in z^*. Thus, an increase in τ increases z^* and, hence, reduces $q^* = q(z^*)$. Since $v(z^*, \tau)$ is increasing in z^* and decreasing in τ, the net effect on v^* (and hence the effect on I^* and k^*) is uncertain. What we can say, from equation (17), is that v^* (and hence I^* and k^*) will increase if the tax revenue $\tau \int_{z^*}^{1} (\theta - z^*)\, dF$ declines as a result of the increase in τ.

Now consider the impact of an increase in δ. Again, $q^* = q(z^*)$ is independent of δ, whereas $v^* = v(z^*, \delta)$ is increasing in δ according to (17). Rewriting the first-order condition (40) as follows

$$\left(1 - \frac{q(z^*)}{v(z^*, \delta)}\right)\left(\frac{A}{q(z^*)} + z^*\right)\frac{f(z^*)}{1 - F(z^*)} = \tau,$$

it is clear that an increase in δ will increase $v(z^*, \delta)$ and hence increase the left hand side. To satisfy the first-order condition, the left hand side must be decreased, which requires a decrease in z^*. From (16), q^* must fall as a result of the decrease in z^*; however, the effect of the change in δ on v^* (and hence on I^* and k^*) is ambiguous.

Proof of Proposition 9 At the unique steady-state value, $z = z^*$, the derivative of the term on the left hand side of (23), $v'(z^*)$, is negative and strictly smaller than the derivative of the term on the right hand side. Denoting the term on the right hand side by $\varphi(z_{t+1})$, where $\varphi(z)$ is defined by

$$\varphi(z) = \delta\left[A + v(z)\bar{\theta} - \tau v(z)\int_z^1 (\theta - z)dF\right],$$

we have

$$\begin{aligned}\varphi'(z^*) &= \delta v'(z^*)\,\bar{\theta} - \delta\tau v'(z^*)\int_z^1 (\theta - z)dF + \delta\tau v(z^*)\,(1 - F(z^*)) \\ &> \delta v'(z^*),\end{aligned}$$

since the second and third terms on the right hand side are positive. Then the fact that $v'(z^*)$ is negative and $0 < \delta < 1$ implies that $\varphi'(z^*) > v'(z^*)$ as claimed. Since the steady state is unique, this proves that

$$v(z) \gtrless \varphi(z) \text{ as } z \lesseqgtr z^*,$$

for all $0 < z < 1$.

We show next that $\varphi'(z) < 0$, for all $0 < z < 1$. This is equivalent to

$$-\frac{v'}{v} > \frac{\tau(1 - F(z))}{\bar{\theta} - \tau\int_z^1 (\theta - z)dF}. \tag{41}$$

Differentiating (21) we get

$$\frac{v'}{v} = \frac{d}{dz}\left(\frac{\tau}{\left(\frac{A}{q} + z\right)\frac{f}{1-F}}\right)\frac{v}{q} + \frac{q'}{q}.$$

Also, from (20) we get

$$q' = \frac{-Af\int_0^z \theta dF - A(1-F)zf}{\left(\int_0^z \theta dF\right)^2}.$$

Since $-\frac{q'}{q} > 0$, a sufficient condition for (41) to hold is that

$$-\frac{d}{dz}\left(\frac{\tau}{\left(\frac{A}{q} + z\right)\frac{f}{1-F}}\right)\frac{v}{q} = -\frac{v\tau}{q}\frac{\left[\left(\frac{A}{q^2}q' - 1\right)\left(\frac{f}{1-F}\right) - \left(\frac{A}{q} + z\right)\frac{d}{dz}\left(\frac{f}{1-F}\right)\right]}{\left(\frac{A}{q} + z\right)^2\left(\frac{f}{1-F}\right)^2} > \frac{\tau(1-F)}{\bar{\theta} - \tau\int_z^1 (\theta - z)dF}.$$

Recall that the hazard rate $f/(1-F)$ is assumed to be increasing. Hence the above inequality holds if

$$\frac{v}{q}\frac{\left(1 - \frac{A}{q^2}q'\right)}{\left(\frac{A}{q} + z\right)^2\left(\frac{f}{1-F}\right)} > \frac{(1-F)}{\bar{\theta} - \tau\int_z^1 (\theta - z)dF}.$$

Substituting the expression for q' derived above, this inequality can be rewritten as follows

$$\frac{v}{q}\frac{1+\frac{A}{q^2}\frac{Af\int_0^z\theta dF+A(1-F)zf}{\left(\int_0^z\theta dF\right)^2}}{\left(\frac{A}{q}+z\right)^2}>\frac{f}{\bar{\theta}-\tau\int_z^1(\theta-z)dF},$$

or, using (20) to substitute for q,

$$\frac{v}{q}\frac{\frac{1}{f}+\frac{\int_0^z\theta dF+(1-F)z}{(1-F)^2}}{\left(\frac{\int_0^z\theta dF}{1-F}+z\right)^2}>\frac{1}{\bar{\theta}-\tau\int_z^1(\theta-z)dF}.$$

Note that the term on the left hand side can be rewritten as

$$\frac{v}{q}\left[\frac{(1-F)^2/f}{\left(\int_0^z\theta dF+z(1-F)\right)^2}+\frac{1}{\int_0^z\theta dF+z(1-F)}\right]>\frac{1}{\int_0^z\theta dF+z(1-F)},$$

where the inequality sign follows from the fact that $v/q>1$. Hence (41) holds if

$$\bar{\theta}-\tau\int_z^1(\theta-z)dF>\int_0^z\theta dF+z(1-F),$$

or

$$\bar{\theta}-\int_0^z\theta dF-\tau\int_z^1\theta dF=\int_z^1\theta dF(1-\tau)>z(1-F)(1-\tau),$$

always satisfied. This completes the proof that $\varphi'(z)<0$, for all $0<z<1$.

Denoting, as usual, the solution of (20) by $q(z)$, we note that $q(z)\to\infty$ as $z\to 0$ and $q(z)\to 0$ as $z\to 1$. Since the first-order condition (**??**) implies that $v_{t+1}>q_{t+1}$, we have $v(z)\to\infty$ as $z\to 0$. And since $A/q(z)\to\infty$ as $z\to 1$, we must have $v(z)\to 0$ as $z\to 1$. Then $v(z)\to\infty$ (resp. 0) as $z\to 0$ (resp. 1), whereas $\varphi(z)$ behaves like $\delta\left[A+(1-\tau)\bar{\theta}v(z)\right]$ as $z\to 0$, that is, $\varphi(z)\to\infty$ as $z\to 0$, and $\varphi(z)\to\delta A$ as $z\to 1$.

Any sequence $\{z_t\}$ satisfying the difference equation (21) that does not begin at z^* will diverge either to 0 or 1. If $z_t\to 1$, then within a finite number of steps $v(z_t)<\delta A<\varphi(z)$ for any z, so there does not exist a continuation value z_{t+1} that satisfies $v(z_t)=\varphi(z_{t+1})$. If $z_t\to 0$, on the other hand, then $v_t(z_t)\to\infty$, which implies that $I_t\to\infty$ and $k_t\to\infty$, violating the boundedness property established in Proposition 1. Thus, no divergent sequence corresponds to an equilibrium and the only possible equilibrium sequence is $z_t=z^*$ for all t.

Proof of Proposition 10 To see this, suppose to the contrary that at some date t a firm with one unit of capital at its disposal devotes a fraction ℓ of it to the risky technology and the remaining fraction $1-\ell$ to the safe technology. As usual, we define the break even point z_{t+1} for a debt with nominal value d_{t+1} as:

$$\ell A+(1-\ell)B+q_{t+1}\left(\ell z_{t+1}+(1-\ell)\bar{\theta}\right)=d_{t+1}.$$

Then the expected value of the firm at date $t+1$ is given by

$$\int_0^{z_{t+1}} \left(\ell A + (1-\ell) B + q_{t+1} \left(\ell \theta_{t+1} + (1-\ell) \bar{\theta}\right)\right) dF +$$
$$\int_{z_{t+1}}^1 \left[\frac{v_{t+1}}{q_{t+1}} \left(\ell A + (1-\ell) B + q_{t+1} \left(\ell \theta_{t+1} + (1-\ell) \bar{\theta}\right)\right) - \tau v_{t+1} \ell \left(\theta_{t+1} - z_{t+1}\right)\right] dF,$$

since the tax base is

$$\ell A + (1-\ell) B + q_{t+1} \left(\ell \theta_{t+1} + (1-\ell) \bar{\theta}\right) - d_{t+1} =$$
$$\ell A + (1-\ell) B + q_{t+1} \left(\ell \theta_{t+1} + (1-\ell) \bar{\theta}\right) - \left[\ell A + (1-\ell) B + q_{t+1} \left(\ell z_{t+1} + (1-\ell) \bar{\theta}\right)\right] =$$
$$q_{t+1} \ell \left(\theta_{t+1} - z_{t+1}\right).$$

We show below that the firm can achieve a higher value by splitting into two separate entities, of size respectively ℓ and $1-\ell$, the first one investing fully in the risky technology and the second one investing fully in the safe one. The nominal value of the debt in the second entity is set at $(1-\ell) B + q_{t+1} (1-\ell) \bar{\theta} = d_{t+1}^{II}$ while the one in the first entity is $\ell A + q_{t+1} \ell z_{t+1} = d_{t+1}^{I}$, that is the break even point is kept at z_{t+1}. The sum of the value of these two entities is then

$$\ell \left[\int_0^{z_{t+1}} \left(A + q_{t+1} \theta_{t+1}\right) dF + \int_{z_{t+1}}^1 \left(\frac{v_{t+1}}{q_{t+1}} \left(A + q_{t+1} \theta_{t+1}\right) - \tau v_{t+1} \left(\theta_{t+1} - z_{t+1}\right)\right) dF\right] +$$
$$(1-\ell) \left[\frac{v_{t+1}}{q_{t+1}} (B + q_{t+1} \bar{\theta})\right]$$

which is clearly strictly greater than the value of the combined firm above, as long as $v_{t+1} > q_{t+1}$. Moreover, the firm can also achieve a higher value by investing all the capital at its disposal in the risky technology, if the first term in square brackets is larger than the second one, and otherwise in the safe technology.