Complete and Incomplete Financial Markets in Multi-Good Economies *

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Abstract

In several multi-good models in the literature we see in equilibrium that the span of the stocks drop relative to the span of the dividends, which is not a desirable feature. Therefore, in this paper, we investigate conditions for endogenous completeness and incompleteness in continuous-time financial markets driven by diffusion processes with multiple consumption goods and heterogeneous agents. We show that for a class of utility functions, including unit elasticity of substitution, the span of the risky assets is strictly smaller than the span of dividends. We find that, unlike in one-good economies, preferences matter for completeness through relative commodity prices. A sufficient condition for market completeness is that a matrix depending on preferences and dividends is invertible. We show that market completeness can depend on the choice of numeraire good, as changing the numeraire good simultaneously implies a change of the risk-free asset and consequently the asset structure.

Keywords: Multi-Good Economies; Financial Market Incompleteness; Financial Market Completeness; Non-Separable Utility Functions; Unit Elasticity of Substitution

JEL Classification: G10; G11

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1 Introduction

In the Lucas (1978) asset pricing model agents are endowed with claims to exogenously specified fruit or Lucas trees, where the fruits represent dividends. In such a consumption based economy, agents trade in a risk free bond in zero net supply in addition to shares in stocks or claims to dividends. In the theoretical finance literature on asset pricing the Lucas tree economy represents the main workhorse model.[1]

The financial market in the Lucas economy is complete if agents can implement, by trading in the stocks and the bond, any consumption plan from an equivalent Arrow-Debreu equilibrium. Verifying if the market is complete appears challenging, however, as in general one needs to calculate the equilibrium stock prices and then check whether the optimal consumption plans from the Arrow-Debreu market can be implemented by trading in the available assets. Magill and Shafer (1990) show that in the real asset model of financial equilibrium theory the market is generically complete as long as the aggregate endowment satisfies a regularity condition, i.e., it spans all the uncertainty in the economy.[2] However, Cass and Pavlova (2004) show that the Lucas tree economy, although a special case of the real asset model, has some embedded structures that make it significantly different from the real asset model. Importantly, the generic existence result of Magill and Shafer (1990) does not apply, as one cannot perturb endowments independently of asset cash flows.

Recently, Anderson and Raimondo (2008) derived conditions for market completeness in a continuous time economy with a single consumption good. In general, implementing optimal consumption plans from an equivalent Arrow-Debreu equilibrium requires invertibility of the stock return covariance matrix. Anderson and Raimondo (2008) prove that imposing certain smoothness conditions on the primitives of an economy in addition to invertibility

[1] See Lucas (1978), Mehra and Prescott (1985), Epstein and Zin (1989, 1990, 1991), Duffie and Epstein (1992), Campbell and Cochrane (1999), Bansal and Yaron (2004), and Gabaix (2012). The Lucas model has been extended to economies populated by agents with heterogeneous risk aversion by Wang (1996) and Chan and Kogan (2002) and with heterogeneous beliefs by Detemple and Murthy (1994). For multiple Lucas trees the reader is referred to Cochrane et al. (2008). We mention just a few references, but there are many more Lucas type models by numerous authors.

[2] See also Duffie and Shafer (1985) and Duffie and Shafer (1986).

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of the payoff matrix imply invertibility of the stock return covariance matrix. Prior to Anderson and Raimondo (2008), every single model without a closed form solution for the stock return covariance matrix assumed in some form completeness (Duffie and Huang (1985); Duffie and Zame (1989)). More recently, Hugonnier et al. (2010), Kramkov and Predoiu (2011) and Riedel and Herzberg (2012) work out generalizations of the conditions for market completeness in a continuous time economy with a single consumption good.

However, we know less about incompleteness and completeness in multi-good economies than in one-good economies. What we do know is that in several multi-good models in the literature (e.g., Serrat (2001)) the span of the stocks drops relative to the span of the dividends. To our knowledge all the models where the span of the stocks drops relative to the span of the dividends use a variant of the Lucas model. To us it seems important to obtain imperfectly correlated stock market returns in multi-good economies when the span of the dividends allows for completeness.

Specifically, Serrat (2001), solves a continuous time international Lucas tree economy with multiple consumption goods and derives an explicit formula for stock price diffusion coefficients. Yet, it appears that even with an explicit formula for stock price diffusion coefficients it can be difficult to detect an inherently incomplete market. In the end, Kollmann (2006) shows that the economy studied by Serrat (2001) has incomplete markets and that portfolios are indeterminate. Importantly, Serrat (2001) claims that the presence of non-traded goods leads to portfolio home bias in stocks that are claims to traded goods, and hence apparently proposes a solution to the portfolio home bias puzzle. Unfortunately, given that portfolios are indeterminate, we cannot learn anything about portfolio home bias from Serrat (2001).

The above example illustrates that simple conditions for verification of market completeness or incompleteness might be useful when studying economies with multiple consumption goods and heterogeneous agents. This is even more so the case as the asset pricing literature with multiple consumption goods continues to grow.[3]

Therefore, in this paper we consider a continuous time Lucas tree economy with multiple consumption goods and agents with heterogeneous preferences. We derive sufficient conditions for market completeness and incompleteness. In particular, our conditions for market completeness and incompleteness do not require the computation of the equilibrium stock price diffusion matrix, but rely solely on the utility function of the representative agent and an invertibility condition on dividends.

First, we define a class of utility functions of which the span of the risky assets is strictly smaller than the span of the dividends. This class of utility functions covers the preferences employed, among others, in the following papers: Cole and Obstfeld (1991), Zapatero (1995), Serrat (2001), Cass and Pavlova (2004) and Berrada et al. (2007). Within this class is, for instance, the widely used Cobb-Douglas utility function. As Cole and Obstfeld (1991) illustrate, when the representative agent has Cobb-Douglas utility, then the relative price is proportional to the relative dividends, and consequently, dividends measured in units of numeraire perfectly correlate with one another. Hence, in equilibrium stock prices are linearly dependent.

Second, we extend Anderson and Raimondo (2008) and Hugonnier et al. (2010) to the multi-good case. Our completeness condition depends on preferences, not only on the structure of dividends as in the one good case. Preferences matter for completeness in a multi-good economy through relative prices that are used to transform quantities of different goods into units of the numeraire.

Third, market completeness depends on the choice of numeraire good, since changing the numeraire good implies that the original risk-free asset is non-tradable under the new numeraire good. Therefore, changing the numeraire good may move an economy from com-

[3]Recent general Lucas (1978) type asset pricing models with multiple consumption goods include Ait-Sahalia et al. (2004), Yogo (2006), Piazzesi et al. (2007), and Lochstoer (2009) among others. International asset pricing models with multiple consumption goods that are Lucas (1978) trees include Lucas (1982), Cole and Obstfeld (1991), Zapatero (1995), Baxter et al. (1998), Serrat (2001), Kollmann (2006), Pavlova and Rigobon (2007, 2008, 2010), and Li and Muzere (2011) among many others.

plete to incomplete and vice versa. Numeraire good invariance holds[4] when our sufficient condition for incompleteness is satisfied, thus for this class of utility functions the market remains incomplete under any numeraire good.

Fourth, even if the market is endogenously incomplete, the numeraire good might be important. We show that the choice of numeraire good might determine if the Arrow-Debreu equilibrium can be implemented by trading in the available assets. In an example, we show that for a certain choice of the numeraire good the equilibrium is of the peculiar type as in Cass and Pavlova (2004), even though agents do not have log-linear utility functions. However, the Arrow-Debreu equilibrium can only be implemented under one particular choice of the numeraire good, and for any other choice of the numeraire good, the endogenous incompleteness will have real effects.

Our work also relates to Berrada et al. (2007); they focus on autarky equilibrium and show that unit elasticity of substitution leads to fund separation. We note that for an autarky equilibrium with unit elasticity of substitution it is required that agents exhibit homogeneous preferences (except for the log utility case). We extend Berrada et al. (2007) to unit elasticity of substitution with heterogeneous preferences, risk aversion and taste, and show that portfolio autarky and fund separation still holds (i.e., there is no trade in financial markets). Yet, a heterogeneity in taste restores financial market completeness through trade in good markets.

We believe that the sufficient conditions for completeness and incompleteness might turn out to be especially useful for models solved numerically such as multi-country dynamic stochastic general equilibrium models that employ higher-order approximations.[5] More specifically, the advantage of our condition is that it only requires verification of an invertibility condition for one realization of the state variables at the terminal time. In contrast, to verify market completeness without such a condition requires the calculation of the stock

[4]We use the term *numeraire good invariance* to mean that the financial market does not switch from complete to incomplete when changing the numeraire good.

[5]See the literature review section in Pavlova and Rigobon (2012) for a brief overview about this literature.

price diffusion matrix and then to check if the matrix is invertible for every possible realization of the sate variables at every point in time. The latter is clearly infeasible without explicit closed form solutions.

2 The Economy

In this section we outline the model. The model setup is close to Hugonnier et al. (2010), but our results are easily modified to the setups in Anderson and Raimondo (2008), Kramkov and Predoiu (2011) or Riedel and Herzberg (2012). Our setup is close to the workhorse macro-finance model typically employed in the literature.[6]

We consider a frictionless continuous time pure exchange economy over a finite time span $[0, T]$. Uncertainty is represented by a filtered probability space, $\left(\Omega, \mathcal{F}, \mathcal{P}, \{\mathcal{F}_t\}_{t\geq 0}\right)$,[7] on which is defined an N-dimensional Brownian motion $Z = (Z_1, ..., Z_N)^\top$. In the following, all stochastic processes are assumed to be progressively measurable and all equalities are assumed to hold in the almost surely sense.

There are $N+1$ securities, of which N are dividend paying stocks and one is a locally risk free asset. All dividend paying stocks are in unit supply, while the risk-free asset is in zero net supply. Dividends are paid as consumption goods. There are N different consumption goods, where stock $i = 1, ..., N$ pays out dividends in consumption good i.[8] Dividends of stock i are paid at a rate $\delta_i\left(X(t)\right)$, where δ_i denotes a nonnegative function and where $X(t)$ is a N-dimensional vector of state variables with dynamics

$$X(t) = X(0) + \int_0^t \mu_X(X(\tau), \tau)d\tau + \int_0^t \sigma_X(X(\tau), \tau)dZ(\tau). \tag{1}$$

[6]See the workhorse model in Pavlova and Rigobon (2011).

[7]The filtered probability space is defined over the finite horizon $[0, T]$, where Ω defines the state space, $\mathcal{F}$ denotes the σ-algebra, $\mathcal{P}$ is the probability measure, and the information structure or filtration $\mathcal{F}_{(.)}$ is generated by the Brownian motion of the state variable processes with $\mathcal{F}_T = \mathcal{F}$.

[8]It is easy to extended the economy to a setting where stocks pay out in more than one good without altering our results.

Assumption 1. *The unique solution of Equation (1) takes values in* $\mathcal{X} \subseteq R^N$ *and for all* $(x,t) \in R^N \times [0,T]$ *the diffusion of the state variable process is invertible, i.e.,* $rank(\sigma_X(x,t)) = N$.

Assumption 2. *The dividends,* $\delta_i(x)$, *are functions of class* $\mathcal{C}^2$.

Definition 1. *We define* I *to be the* $N \times N$ *identity matrix and for a vector* y, *with* $y \in R^N$, *we define* I_y *to represent a* $N \times N$*-dimensional matrix with* y_i *as element* (i,i) *and zero elsewhere.*

Given the process in Equation (1), an application of Ito's lemma to $\delta(X(t)) = (\delta_1(t), .., \delta_N(t))$ yields the diffusion of dividends: $\lambda(t) \equiv \lambda(X(t),t) = \frac{\partial \delta(X(t))}{\partial x}\sigma_X(X(t),t)$.[9]

Assumption 3. *The dividend diffusion matrix is invertible, i.e.,* $rank(\lambda(t)) = N$.

Assumption 3 ensures that the market is potentially complete, i.e., that the dividends span all the uncertainty in the economy.

Let $P = \{p_1, p_2, ..., p_N\}$ denote the vector of N commodity prices. Consumption good $l \in \{1, ..., N\}$ serves as numeraire. Thus, the price of consumption good l is normalized to one, $p_l(t) = 1$, for all $t \in [0,T]$. Commodity prices are determined in equilibrium. The N-dimensional commodity price evolves according to[10]

$$P(t) = P(0) + \int_0^t I_P(\tau)\mu_P(\tau)d\tau + \int_0^t I_P(\tau)\sigma_P(\tau)dZ(\tau) \tag{2}$$

where μ_P and σ_P denote expected growth rates and diffusion coefficients in R^N and $R^{N\times N}$, respectively.

Definition 2. *We define the* N*-dimensional dividend rate process in units of the numeraire as*

$$\widetilde{\delta}(t) = I_P(t)\delta(t). \tag{3}$$

[9] We will occasionally drop the explicit reference to the state variable X and write $F(t)$ rather than $F(X(t),t)$.

[10] To simplify notation we do not explicitly indicate the numeraire good, i.e., $P(t) = P^l(t)$ denotes the relative commodity prices when good l serves as numeraire. Yet, it is important to recognize that all equilibrium quantities depend on the choice of numeraire good.

There are N stocks, each representing a claim to its respective dividend rate process. In equilibrium, the N gains processes identify stock prices and are given by

$$G(t) = S(t) + \int_0^t \widetilde{\delta}(\tau)d\tau = S(0) + \int_0^t I_S(\tau)\mu(\tau)d\tau + \int_0^t I_S(\tau)\sigma(\tau)dZ(\tau). \tag{4}$$

The diffusion term $\sigma(t)$ denotes a $N \times N$ matrix with the i'th row given by $\sigma_i(t)^\top$. Both the N-dimensional drift rates and the $N \times N$-dimensional diffusion terms in Equation (3) represent endogenous quantities.

A locally risk-free asset in zero net supply pays out in the numeraire good. Its price process is given by

$$B(t) = B(0) + \int_0^t r(\tau)B(\tau)d\tau \tag{5}$$

with $B(0) = 1$. The risk-free rate, r, is to be determined endogenously in equilibrium.

The economy is populated by $J \geq 1$ agents indexed by j. The utility function is

$$U_j(C) = E\left[\int_0^T e^{-\rho\tau}u_j(C(\tau))d\tau\right] \tag{6}$$

where $\rho > 0$ and where u is a classical time-additive von Neumann-Morgenstern (VNM) utility function. In the above, $C = \{c_1, c_2, ..., c_N\}$ denotes the vector containing the N consumption goods.

Assumption 4. *The utility function $u_j : (0,\infty)^N \to R$ is assumed to be increasing and strictly concave function of class $\mathcal{C}^3$ that satisfies the multidimensional Inada conditions.*

Agent j maximizes $U_j(C)$ subject to the dynamic budget constraint

$$\begin{aligned} W^j(t) &= W^j(0) + \int_0^t W^j(\tau)r(\tau)d\tau - \int_0^t P(\tau)^\top C^j(\tau)d\tau \\ &\quad + \int_0^t \pi^j(\tau)^\top(\mu(\tau) - r(\tau)1_N)\,d\tau + \int_0^t \pi^j(\tau)^\top\sigma(\tau)dZ(\tau) \end{aligned} \tag{7}$$

where $\pi^j(t) = \left(\pi_1^j(t), \pi_2^j(t), ..., \pi_N^j(t)\right)$ is a vector process of amounts held in the stocks by agent j, and where $W^j(0) = \pi^j(0)^\top 1_N$, i.e., agents are endowed with initial shares in the stocks. We impose the condition that $\pi_i^j(0) \geq 0$ for all $i = 1, .., N$ and $j = 1, .., J$ and $\pi_k^j(0) > 0$ for at least one $k = 1, .., N$. This condition implies $W^j(0) > 0$ and, hence, if an agent never trades away from his initial portfolio, wealth remains positive for all times and states. This condition is similar to the condition in Equation (4) in Anderson and Raimondo (2008). However, in their setting, agents receive an endowment stream in addition to initial shares of equity, hence, they impose a less strict condition in that agents may be endowed with short positions.

Definition 3. *Equilibrium is a collection of allocations (C^j, π^j) for $j = 1, 2, ..., J$, and a price system (B, S, P) or price coefficients $(r, \mu, \sigma, \mu_P, \sigma_P)$, such that $\left(C^j, \pi^j, \pi_B^j\right)$ denote optimal solutions to agent j's optimization problem and good and financial markets clear*

$$\sum_j C^j(t) = \delta(t), \ \sum_j \pi^j(t) = S(t), \ \sum_j \pi_B^j(t) = 0$$

for $t \in [0, T]$ where π_B^j is the amount held in the bond market.

To derive sufficient conditions for completeness and incompleteness we first solve for the Arrow-Debreu equilibrium. Next we calculate the stock prices in the Arrow-Debreu equilibrium and then illustrate whether or not the span of the equilibriums stock prices are sufficient to complete the market. We impose the following assumption for existence of an Arrow-Debreu equilibrium.

Definition 4. *An Arrow-Debreu equilibrium is defined as a state price density, ξ, commodity prices, P, and consumption allocations, $(C^j)_{j=1}^J$, such that C^j maximizes U_j given the static budget constraint $E\left[\int_0^T \xi(\tau)P(\tau)^\top C^j(\tau)d\tau\right] \leq W^j(0)$ and all good markets clear.*

Assumption 5.

$$\sum_{j=1}^J E\left[\int_0^T e^{-\rho\tau} \frac{\partial u_j\left(\delta(\tau)/J\right)^\top}{\partial \delta} \delta(\tau)d\tau\right] < \infty. \tag{8}$$

Assumption 5 is similar to assumption C.1 in Hugonnier et al. (2010) adapted to a multi-good setting.

Proposition 1. *There exists an Arrow-Debreu equilibrium in which the state price density is*

$$\frac{\xi(t)}{\xi(0)} \equiv \frac{\xi(a,t)}{\xi(a,0)} = e^{-\rho t}\frac{\frac{\partial u(a,\delta(t))}{\partial \delta_l}}{\frac{\partial u(a,\delta(0))}{\partial \delta_l}}. \tag{9}$$

Moreover, the N-dimensional equilibrium commodity price vector is

$$P(t) \equiv P(a,t) = \frac{\frac{\partial u(a,\delta(t))}{\partial \delta}}{\frac{\partial u(a,\delta(t))}{\partial \delta_l}} \tag{10}$$

where the utility function of a representative agent —state by state and time by time— is given by

$$u(a,\delta) = \max_{\sum_j C^j = \delta} \sum_j a_j u_j(a, C^j) \tag{11}$$

with strictly positive weights $a = (a_1, .., a_J)$. In Equation (11), the utility weights, a, correspond to solutions to

$$E\left[\int_0^T \xi(a,\tau)P(a,\tau)^\top C^j(a,\tau)d\tau\right] = W^j(a,0) \tag{12}$$

where the above is evaluated at the optimal solution for $j = 1, ..., J$.

Given the Arrow-Debreu equilibrium, the natural candidate for a gains process or stock price is the discounted future value of dividends

$$G_i(t) = S_i(t) + \int_0^t \frac{\xi(\tau)}{\xi(0)} p_i(\tau)\delta_i\left(X(\tau)\right)d\tau = E_t\left[\int_0^T \frac{\xi(\tau)}{\xi(0)} p_i(\tau)\delta\left(X(\tau)\right)d\tau\right] \tag{13}$$

for $i = 1, .., N$.

Lemma 1. *The commodity price diffusion coefficients, $\sigma_P(t)$, are given by*

$$\sigma_P(t) = \varepsilon(t)\lambda(t) \tag{14}$$

where $\varepsilon(t)$ is a $N \times N$ matrix with element (i, j) given by

$$\varepsilon_{i,j}(t) = \delta_j(t) \frac{\partial \ln MRS_{i,l}(t)}{\partial \delta_j} \tag{15}$$

where $MRS_{i,l}(t) = \frac{\frac{\partial u(a,\delta(t))}{\partial \delta_i}}{\frac{\partial u(a,\delta(t))}{\partial \delta_l}}$ stands for the marginal rate of substitution.

Proposition 2. *The diffusion coefficients of the dividend rate processes in units of the numeraire are given by*

$$\sigma_{\widetilde{\delta}}(t) = (I + \varepsilon(t))\,\lambda(t). \tag{16}$$

3 Incomplete Markets

In this section, we derive sufficient conditions for market incompleteness.

Definition 5. *A utility function $u : R_+^N \to R$ will be defined to be in U^{IC}, where IC stands for incompleteness, if it has a representation*

$$u(c_1, ..., c_N) = \varphi\left(c_2 c_1^{\beta_2}, ..., c_N c_1^{\beta_N}\right) \tag{17}$$

where $\varphi : R^{N-1} \to R$.

The next theorem establishes a sufficient condition for the market to be incomplete.

Theorem 1. *If the utility function of the representative agent, u, is such that $u \in U^{IC}$, then $\sigma(t)$ is non-invertible, and the market is incomplete.*

Theorem 1 illustrates that any economy with a representative agent exhibiting a utility function of the form in Equation (17) has incomplete financial markets. A key feature of this class of utility functions is that they can be reduced from being functions with N arguments (one for each good) to functions of less than N arguments. This reduction is the source of the incompleteness as the utility function is not "rich" enough to accommodate enough preference variation.

Remark 1. *We stress that Theorem 1 does not depend on the dynamics of dividends or on the continuous time assumption. Hence, any Lucas type economy with preferences as in Equation (17) leads to incomplete financial markets.*

For the two-goods case, Theorem 1 corresponds to a constraint on the elasticity of substitution. If the elasticity of substitution equals one, then the market is incomplete. This is the argument put forward in Proposition 2 in Berrada et al. (2007). Moving from the $two - good$ case to the $N - good$ case, however, implies that unit elasticity of substitution between any two goods is not necessary for the utility function to satisfy the condition in Theorem 1.

Theorem 1 is a statement about the representative agent's utility function. The next proposition illustrates that if agents have utility functions as in Definition 5 and have homogeneous tastes, i.e., $\beta = (\beta_2, ..., \beta_N)$ are the same for every agent, then the market is incomplete.

Proposition 3. *Let $u_j(C) = \varphi_j\left(c_2 c_1^{\beta_2}, ..., c_N c_1^{\beta_N}\right)$ for $j = 1, ..., J$, i.e., every agent's utility function is in U^{IC} and the preference parameters $\beta = (\beta_2, ..., \beta_N)$ are the same for every agent, then the market is incomplete.*

Although incomplete markets are certainly realistic and important to better understand the real economy, Section 3.1 shows that the incompleteness induced by preferences as in Theorem 1 has undesirable features. We close this discussion by pointing to Section 3.2 which illustrates an application of Theorem 1 to the economy in Serrat (2001) to emphasize its usefulness.

3.1 Log-linear utility

Let $J = 1$ and assume that there are two consumption goods. The first good (our numeraire good) pays out a constant dividend stream of $\delta_1(t) = \delta_1$ and the second dividend stream follows

$$d\delta_2(t) = \delta_2(t)\left(\mu_2 dt + \lambda_2 dZ\right). \tag{18}$$

Assume that the representative agent has a log-linear utility function

$$u(c_1, c_2) = log\left(c_1\right) + log\left(c_2\right). \tag{19}$$

Moreover, assume that the time discount factor, ρ, is zero. Note that the above utility function satisfies the condition in Theorem 1. In equilibrium, the stock prices are

$$S_1(t) = \delta_1\left(T - t\right),\ S_2(t) = \delta_1\left(T - t\right), \tag{20}$$

and, consequently, there is effectively only one asset in this economy and the market is incomplete. More importantly, the example illustrates that the above form of incompleteness is unsatisfactory from an economic point of view. The quantity of good two is stochastic, yet the stock price being a claim to the dividend stream is insensitive to it. Thus, in equilibrium unexpected growth or news about the output of the second good is not reflected in the stock price, as it is perfectly offset by its relative price.

3.2 Separation between Non-Traded and Traded Goods as in Serrat (2001)

Serrat (2001) studies an economy with two countries. Each country has access to three output processes where two goods are traded while the third good is non-traded. Kollmann (2006), however, proves that the diffusion matrix in the Serrat (2001) economy is non-invertible. We illustrate here that this result can be verified by applying Theorem 1. The utility function of the representative agent in Serrat (2001) can be expressed, with a small simplification, as follows

$$u(c_1, c_2, c_3, c_4) = \frac{1}{q}\left(c_1^q + c_2^q\right)\left(a c_3^\alpha + b c_4^\beta\right). \tag{21}$$

Equation (21) can be rewritten as

$$
\begin{aligned}
u(c_1, c_2, c_3, c_4) &= \frac{1}{q}\left(1+\left(\frac{c_2}{c_1}\right)^q\right)\left(a\left(c_3 c_1^{\frac{q}{\alpha}}\right)^\alpha + b\left(c_4 c_1^{\frac{q}{\beta}}\right)^\beta\right) \\
&= \varphi\left(c_2 c_1^{\beta_2}, c_3 c_1^{\beta_3}, c_4 c_1^{\beta_4}\right) \qquad (22)
\end{aligned}
$$

where $\varphi(v, w, z) = \frac{1}{q}(1+v^q)\left(aw^\alpha + bz^\beta\right)$ and where $\beta_2 = -1$, $\beta_3 = \frac{q}{\alpha}$ and $\beta_4 = \frac{q}{\beta}$. From Equation (22) we see that the utility function in Serrat (2001) implies incomplete markets since it satisfies the sufficient condition for incompleteness in Theorem 1.

4 Complete Markets

In this section we derive sufficient conditions for market completeness. To derive these conditions we need to introduce additional assumptions on the primitives of the economy. The first assumption imposes conditions on the state vector X. It corresponds to assumption $A(c)$ and $A(d)$ in Hugonnier et al. (2010)

Assumption 6. *The solution to Equation (1) admits a transition density $p(t, x, \tau, y)$ that is smooth for $t \neq \tau$. Moreover, there are locally bounded functions (K, L), a metric d that is locally equivalent to the Euclidean metric, and constants ϵ, α, $\phi > 0$ such that $p(t, x, \tau, y)$ is analytic with respect to $t \neq \tau$ in the set*

$$
\mathcal{P}_\epsilon^2 \equiv \{(t,\tau) \in \mathbb{C}^2 : \mathcal{R}t \geq 0, 0 \leq \mathcal{R}\tau \leq T \text{ and } |\mathcal{M}(\tau - t)| \leq \epsilon\mathcal{R}(\tau - t)\} \qquad (23)
$$

and satisfies

$$
|p(t, x, \tau, y)| \leq K(x)L(y)|\tau - t|^{-\alpha} e^{\phi|\tau - t| - d(x,y)^2/|\tau - t|} \equiv \bar{p}(t, x, \tau, y) \qquad (24)
$$

for all $(t, \tau, x, y) \in \mathcal{P}_\epsilon^2 \times \mathcal{X}^2$.

Assumption 7. *Dividends are real analytic functions.*

Assumption 8. *The utility function of agent $j = 1, ..., J$, u_j, is analytic and there are constants $R \leq \rho$ and $\nu > 1$ such that*

$$\int_0^T \sum_{j=1}^J \left(\int_{\mathcal{X}} e^{-R\tau} \frac{\partial u_j \left(\delta(y)/J\right)}{\partial \delta}^\top \delta(y)\bar{p}(0, x, \nu\tau, y) dy \right) d\tau < \infty. \tag{25}$$

Definition 6. *A utility function will be defined to be in $U^{C,l}$, where C stands for completeness and l denotes the numeraire good, if*

$$I + \varepsilon(T, x) \tag{26}$$

is invertible for at least one $x \in \mathcal{X}$.

Theorem 2. *If the utility function of the representative agent, u, is such that $u \in U^{C,l}$ then $\sigma(t)$ is invertible, and the market is complete when good l serves as the numeraire good.*

Our condition for completeness, unlike the condition in Anderson and Raimondo (2008), Hugonnier et al. (2010), Kramkov and Predoiu (2011) and Riedel and Herzberg (2012) involves marginal rates of substitution, ε, emanating from endogenous commodity prices. Nevertheless, $I+\varepsilon(T, x)$, is easy to compute. The fact that market completeness is guaranteed if $I + \varepsilon(T, x)$ is invertible at one point indeed makes the verification of completeness easy, even in a situation where the utility of the representative agent is not known in closed form. Using standard aggregation techniques, one can check if the resulting market is complete by numerically solving for the utility function of the representative agent and applying the theorem. In contrast, without the condition for market completeness in Theorem 2, one would have to calculate the stock price diffusion coefficients for every possible realization $(t, x) \in [0, T] \times \mathcal{X}$. Clearly, this seems, in general, computationally infeasible.

The condition in Anderson and Raimondo (2008) represents, a long anticipated, missing building block in the theory of continuous time asset pricing with heterogeneous agents. However, at least from Duffie and Huang (1985), Duffie and Zame (1989), Huang (1987) and Karatzas et al. (1990), it is expected that such equilibrium exists. Further, although essentially all models before Anderson and Raimondo (2008) assume complete markets in one way or another, many papers in this literature, especially the applied ones, contain examples or numerical work that shows that equilibrium holds at least for some parameter values.

In contrast, the literature on multi-goods, often (but not always) in an international context, contains a series of models that imply incomplete financial markets (e.g. Cole and Obstfeld (1991), Zapatero (1995) and Serrat (2001)). For instance, as shown by Cole and Obstfeld (1991), the much used Cobb-Douglas utility function implies that markets are incomplete. Hence, it appears that our condition for completeness might prove useful for future research.

The next proposition characterizes two cases in which completeness can be directly induced from the individual agents utility functions. The first condition is related to separable utility functions (excluding log-linear utility) while the second condition is based on homogeneous tastes over the various consumption goods.

Proposition 4. *If either*

1. $u_j(C) = \sum_{i=1}^{N} u_{ji}(c_i)$ *for* $j = 1, .., J$ *and* $i = 1, .., N$ *and* $u_{ij}(c_i) \neq A_{ji} log(c_i) + B_{ji}$ *for some constants* A_{ij}, B_{ij}*, or*

2. $u_j(C) = \varphi_j\left(\hat{C}(C)\right)$ *is in* $U^{C,l}$ *for* $j = 1, .., J$ *and* $\hat{C} : R^N \to R$ *is homogeneous of degree* k*,*

then the market is complete.

5 Numeraire Good Invariance

In this section, we show that the choice of the risk-free asset, and hence the numeraire good, might be important for completeness and for whether the Arrow-Debreu equilibrium is implementable. Numeraire good invariance might fail since choosing a different good as numeraire also changes the assets available to the agents. The risk-free asset under one numeraire good is a non-tradable asset under another numeraire good.

The next proposition shows that if a utility function satisfies the condition in Theorem 1, then the market is incomplete under any numeraire good.

Proposition 5. *If $u \in U^{IC}$, then the market is incomplete under all numeraire goods.*

The next proposition establishes an equivalence between satisfying Theorem 1 and not satisfying the sufficient condition for completeness in Theorem 2 under any choice of numeraire good for the case of two goods.

Proposition 6. *The following conditions are equivalent*

1. $u \in U^{IC}$.

2. $u \notin U^{C,l}$ *for* $l = 1, 2$.

The next example illustrates how numeraire good invariance may fail to hold. In the example, the utility function includes a subsistence point. This is similar to the setup in Ait-Sahalia et al. (2004) with a negative subsistence point for luxury goods.[11]

5.1 Subsistence Point

Consider the following utility function defined over the two goods c_1 and c_2

$$u(c_1, c_2) = log(c_1 + b) + log(c_2) \tag{27}$$

where $b > 0$.[12]

[11]Pavlova and Rigobon (2007) consider a setting with preference shocks and log-linear utility. They show that preference shocks can change the market from incomplete to complete. In a previous version of the paper, we showed that depending on the correlation structure between preference shocks and output the equilibrium might be sensitive to the choice of numeraire good.

[12]Note that the utility function in Equation (5.1) does not satisfy the Inada conditions. This is, however, not important for the point of the example. An alternative example is to choose $u(c_1, c_2) = \frac{c_1^{1-\gamma}}{1-\gamma} + log(c_2)$ with $\gamma \neq 1$. In this case, one can show that if dividends are uncorrelated then the market will be incomplete if good one is the numeraire and complete if good two is the numeraire.

First, consider the case in which the first good serves as numeraire. The relative price of the second good in terms of the first good is given by

$$p_2(t) = \frac{\delta_1(t) + b}{\delta_2(t)}. \tag{28}$$

Applying Theorem 2, we have

$$det\left(I + \epsilon(t)\right) = \delta_2(t) \frac{\partial \ln MRS_{2,1}(t)}{\partial \delta_2} + 1 = -\frac{\delta_2(t)}{\delta_2(t)} + 1 = 0. \tag{29}$$

Hence, the market is not guaranteed to be complete when the first good is the numeraire. In fact, if δ_1 and δ_2 evolve as uncorrelated geometric Brownian motions, we then obtain the following expressions for stock prices

$$S_1(t) = (\delta_1(t) + b)\, E_t \left[\int_t^T \frac{\delta_1(\tau)}{\delta_1(\tau) + b} d\tau \right], \; S_2(t) = (\delta_1(t) + b)\,(T - t)\,. \tag{30}$$

Since δ_1 is only driven by one of the two Brownian motions, the market must be incomplete. Next consider the case in which the second good is the numeraire. The relative price of the first good in terms of the second good is given by

$$p_1(t) = \frac{\delta_2(t)}{\delta_1(t) + b}. \tag{31}$$

Again applying Theorem 2, we have

$$det\left(I + \epsilon(t)\right) = \delta_1(t) \frac{\partial \ln MRS_{1,2}(t)}{\partial \delta_1} + 1 = -\frac{\delta_1(t)}{\delta_1(t) + b} + 1 \neq 0. \tag{32}$$

Here the utility function is in $U^{C,l}$. Thus the market is complete.

The above example also illustrates that unit elasticity of substitution is only a sufficient condition for market incompleteness. When the first good is used as the numeraire, the market might be incomplete even though the elasticity of substitution differs from one.

5.2 Peculiar Financial Equilibrium and the Choice of Numeraire Good

In this subsection we illustrate that, in general, endogenous incompleteness will have real effects as the agents cannot implement the Arrow-Debreu equilibrium by trading in the available assets. However, as illustrated by Cass and Pavlova (2004), if agents have log-linear utility, then the market is incomplete but the Arrow-Debreu equilibrium can be implemented by trading in the available assets. They label such an equilibrium as Peculiar Financial Equilibrium (PFE). We show that agents might be able to implement the Arrow-Debreu equilibrium even though the market is incomplete and the agents have utility that is not log-linear. However, the next proposition shows that whether the agents can implement the Arrow-Debreu equilibrium depends on the choice of the numeraire good. In particular, assume that there are two agents and two goods. The utility function of agent $j = 1, 2$ is

$$u_j(c_1^j, c_2^j) = \varphi_j \left(c_2^j \left(c_1^j \right)^{\beta} \right). \tag{33}$$

Above we assume that tastes are the same for the two agents.[13] By Proposition 3, we know that the market will be incomplete. However, the agents might still be able to implement the Arrow-Debreu equilibrium through trading in the assets as in Cass and Pavlova (2004). Below we show that in general this will not be the case. Thus, endogenous completeness will matter for more than just portfolio indeterminacy.

Proposition 7. *Let preferences be as in Equation (33), and output of good $i = 1, 2$ evolve according to*

$$d\delta_i(t) = \delta_i(t) \left(\mu_i dt + \sigma_{i,1} dZ_1(t) + \sigma_{i,2} dZ_2(t) \right). \tag{34}$$

Then we have the following

- *If either good one or two is used as a numeraire good, the Arrow-Debreu equilibrium*

[13] If β differs across the two agents, then the market will be complete as illustrated by Proposition 9 unless the utility function is log-linear.

cannot be implemented by trading in the available assets, unless φ_j is such that the optimal consumption is $c_i^j(t) = f\delta_i(t)$, for all t for some $f \in (0,1)$.

- *If the consumption basket $\delta_2(t)\delta_1(t)^\beta$ is used as numeraire good, then the Arrow-Debreu equilibrium can be implemented by trading in the available assets and the market is effectively complete.*

5.3 Completing the Financial Market

Following up on the arguments of the previous subsection, it is expected that introducing bonds denominated in each consumption good always completes the financial market.

Proposition 8. *Assume that there are N pure discount bonds with maturity T where bond i pays out one unit of good i. Then, the financial market is complete.*

Importantly, the introduction of N pure discount bonds does not resolve the counter factual equilibrium property that stock market returns are perfectly correlated. However, it allows for solving models in which the market is incomplete without the bond contracts.[14]

6 Unit Elasticity of Substitution

In this section, we restrict our attention to the case with only two types of agents. To further simplify, we consider the case of two goods. Agents have utility functions with heterogeneity in taste given by

$$u_j(c_1^j, c_2^j) = \varphi_j\left(c_1^j\left(c_2^j\right)^{\alpha_j}\right). \tag{35}$$

Note that the utility function in Equation (35) exhibits unit elasticity of substitution. Assume that $\alpha_1 \neq \alpha_2$ and that $\varphi_j(x) \neq A_j ln(x) + B_j$ for some constants A_j, B_j. The next

[14] Note that the complete market equilibrium with N pure discount bonds differs, in general, from the incomplete market.

proposition states that the market is complete and characterizes equilibrium portfolio policies.

Proposition 9. *The optimal portfolio of agent* 1 *is given by*

$$\pi_1^1(t) = \frac{1 - B_2}{B_1 - B_2} S_1(t), \ \pi_2^1(t) = -\frac{B_2}{B_1 - B_2} S_2(t) \tag{36}$$

with $B_i = \frac{1}{(1+\alpha_i)}$. *Moreover, the market is complete.*

Several insights emerge from the above proposition. First, even though the market is incomplete if only one of the two agents is present in the economy, the market is complete if both are present. Thus, a small degree of heterogeneity in taste, moves the economy from incomplete to complete. Second, the portfolio policies are constant over time. That is, even though the agents have heterogeneous tastes and may or may not have heterogeneous risk aversion, the optimal portfolio shares are never re-balanced except maybe at $t = 0$. Third, optimal portfolios are such that the agent takes a long position in the stock that is a claim to the good that he prefers and a short position in the other stock. Thus, there is a strong degree of portfolio "home" or "familiarity" bias in the sense that the agent will take long positions in the good that he prefers and a short position in the other good. The reason for this is that as the preferred good becomes relative scarce, the price of the good increases and so does the stock price being a claim to the good. Thus, having a long position in the stock that is a claim to the preferred good is a hedge. Finally, less heterogeneity in taste implies larger differences in the portfolios of the two agents. This last insight is a consequence of the fact that the stocks become more and more correlated as heterogeneity in taste becomes smaller.

7 Conclusion

In this paper, we investigated the determinants of financial market completeness in a continuous time Lucas (1978) model with multiple consumption goods and agents with heterogeneous preferences. We show that for a class of utility functions, including the Cobb-Douglas utility function, the market becomes endogenously incomplete. The endogenous incompleteness will in general prevent the agents from implementing the Pareto optimal allocations, and consequently the incompleteness has real effects. We derive a sufficient condition for market completeness that only depends on the properties of aggregate output and the utility function of the representative agent. The condition is easy to verify, even in cases in which the utility function of the representative agent is not known in closed form. The major advantage of our condition is that it only requires verification of an invertibility condition for one realization of the state variables at the terminal time. In contrast, to verify market completeness without such a condition would require the calculation of the stock price diffusion matrix and checking if the matrix is invertible for every possible realization of the sate variables at every point in time. Clearly, this is infeasible in cases without closed form solutions.

In an economy with multiple consumption goods, one has to make a stand on the numeraire good. The numeraire good will in turn determine how the risk free asset is defined. We illustrate that whether a market is complete or not, might depend on the specification of the risk free asset, and hence the numeraire. Put differently, the market might be complete under one specification of the numeraire good and incomplete under another. In addition, we illustrate that even in the case when the market is guaranteed to be endogenously incomplete, the choice of numeraire good might be crucial in terms of the real effect of incompleteness. In particular, depending on the choice of numeraire, the agents might or might not be able to implement the Pareto optimal allocations by trading in the available assets.

We believe our condition for market completeness might be useful in applied work in asset pricing theory. A large body of the literature in asset pricing theory is based on the

Lucas tree economy. Recently, more papers consider economies with multiple consumption goods, even in settings outside international finance. In general, the properties of the stock price diffusion matrix might be hard to explore due to complexities of the models, and a simple condition for verifying market completeness will in such cases be useful. Serrat (2001) is an example for which our condition for market completeness would have been particularly useful.

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A Auxiliary Results

In this appendix, we introduce several auxiliary results used to prove the portfolio policies in Proposition 9. The individual preferences used have unit elasticity of substitution.

Assume that the utility of the representative agent is non-separable in δ_1 and δ_2. This can be expressed as

$$\frac{\partial^2 u(\delta(t);a)}{\partial\delta_1\partial\delta_2} \neq 0. \tag{A.1}$$

Let the optimal consumption profiles of agent 1 and agent 2 be given by

$$c_1^1(t) = f_1^1(\delta_1(t),\delta_2(t)) \tag{A.2}$$
$$c_1^2(t) = \delta_1(t) - f_1^1(\delta_1(t),\delta_2(t)) \tag{A.3}$$
$$c_2^1(t) = f_2^1(\delta_1(t),\delta_2(t)) \tag{A.4}$$
$$c_2^2(t) = \delta_2(t) - f_2^1(\delta_1(t),\delta_2(t)) \tag{A.5}$$

where f is a functional mapping of aggregate dividends onto optimal consumption.

The sharing rule above is separable if

$$c_1^1(t) = f_1^1(\delta_1(t)),\ c_2^1(t) = f_2^1(\delta_2(t)), \tag{A.6}$$

i.e., optimal consumption of the first agent of the first good (second good) is only a function of aggregate output of the first good (second good). From the FOCs, we see that

$$\frac{\partial u(\delta(t);a)}{\partial\delta_1} = a_1\frac{\partial u_1(c^1(t))}{\partial c_1^1}. \tag{A.7}$$

Assume that the sharing rule is separable. We then obtain the following result (see Hara (2006))

Proposition 10. *If the representative agent's utility function is non-separable and the sharing rules are separable, then*

$$f_1^1(\delta_1) = A\delta_1,\ f_2^1(\delta_2) = A\delta_2 \tag{A.8}$$

for $A \in R_{++}$.

The next proposition shows that the utility function of the representative agent is non-separable.

Proposition 11. *Consider the utility function in Equation (35) with $\alpha_1 \neq \alpha_2$ and with $\varphi_j(x) \neq ln(x)$; then the utility function of the representative agent is non-separable in δ_1 and δ_2.*

The next proposition deals with linear sharing rules.

Proposition 12. *If $\alpha_1 \neq \alpha_2$ and if $\varphi_j(x) \neq ln(x)$, then equilibrium cannot be implemented by linear sharing rules.*

B Proofs

Proof. Proposition 1: The result follows from standard aggregation in complete markets extended to a multiple good setting (see Huang (1987), Duffie and Zame (1989), Karatzas et al. (1990), Dana and Pontier (1992)) and Hugonnier et al. (2010). For completeness we will sketch parts of the proof. The details can be found in Anderson and Raimondo (2008), Huang (1987), Karatzas et al. (1990), Dana and Pontier (1992), and Hugonnier et al. (2010), with slight modification to accommodate for multiple consumption goods, and Lakner (1989) for the case of multiple commodities. First, consider agent j's optimization problem in Equations (6) and (7) when prices are given and markets are complete. It can be shown that the dynamic optimization problem can be reduced to the static optimization problem (Cox and Huang (1989), Karatzas et al. (1990))

$$\max_{C^j} E\left[\int_0^T e^{-\rho s} u_j(C^j(\tau))d\tau\right] \quad \text{s.t.} \quad E\left[\int_0^T \xi(\tau)P(\tau)^\top C^j(\tau)d\tau\right] \leq W^j(0). \tag{B.1}$$

The utility gradient, $\nabla u_j : (0,\infty)^N \to (0,\infty)^N$, has an inverse $I_j : (0,\infty)^N \to (0,\infty)^N$ that inherits the basic properties of ∇u_j. We have that

$$e^{-\rho t} u_j\left(I_j(e^{\rho t}\eta)\right) - \eta^\top I_j(e^{\rho t}\eta) = max\left\{e^{-\rho t} u_j(C(t)) - \eta^\top C(t)\right\} \tag{B.2}$$

holds for any non-negative η. It can then be shown that $C^j(t) = I_j\left(y_j e^{\rho t}\Psi(t)\right)$ solves the maximization problem in Equation (B.1), where $\Psi(t) = \xi(t)P(t)$ and y_j is the solution to

$$E\left[\int_0^T \xi(\tau)P(\tau)^\top I_j\left(y_j e^{\rho s}\Psi(\tau)\right)d\tau\right] = W^j(0). \tag{B.3}$$

Since equilibrium is Pareto optimal, we can consider the following social planner problem

$$u(\delta; a) = \max_{\sum_j C^j = \delta} \sum_j a_j u_j(C^j) \tag{B.4}$$

where $a \in (0, \infty)^J$ is the vector of utility weights. Next, it can be shown that the maximization in Equation (B.4) is achieved by

$$C^j(t) = I_j \left(\frac{e^{\rho t} \Psi(t)}{a_j} \right) \tag{B.5}$$

where $\Psi(t) = e^{-\rho t} \nabla u(\delta(t); a)$ and where the weights, a, are solutions to

$$E \left[\int_0^T e^{-\rho s} \nabla u(\delta(\tau); a)^\top I_j \left(\frac{e^{\rho s} \nabla u(\delta(\tau); a)}{a_j} \right) d\tau \right] = W^j(a, 0). \tag{B.6}$$

Comparing Equation (B.6) with Equation (B.2) we can identify $a_j = \frac{1}{y_j}$. Defining $e(a) = E\left[\int_0^T e^{-\rho s} \nabla u(\delta(\tau); a)^\top I_j \left(\frac{e^{\rho s} \nabla u(\delta(\tau); a)}{a_j} \right) d\tau\right] - W^j(a, 0)$ as the excess utility map, one can show that $e(a)$ has all the properties of a finite dimensional demand function (see Lemma C.1 in Hugonnier et al. (2010) with slight modification to accommodate for multiple consumption goods), and consequently there exists some strictly positive a^* such that $e(a^*) = 0$. □

Proof. Lemma 1: In equilibrium, the commodity price vector $P(t)$ is given by

$$P(t) = \frac{\nabla u(\delta(t))}{\frac{\partial u(\delta(t))}{\partial \delta_l(t)}} = \left[MRS_{1,l}(t), ..., MRS_{N,l}(t) \right]^\top. \tag{B.7}$$

Applying Ito's lemma to $P(t)$ yields the lemma. □

Proof. Proposition 2: The lemma follows directly from applying Ito's lemma to the consumption process in units of the numeraire. □

Proof. Theorem 1: Assume that there exists a solution to the following equation

$$a_1 \frac{\partial u(\delta(t))}{\partial \delta_1} \delta_1(t) + a_2 \frac{\partial u(\delta(t))}{\partial \delta_2} \delta_2(t) + ... + a_N \frac{\partial u(\delta(t))}{\partial \delta_N} \delta_N(t) = 0. \tag{B.8}$$

Integrate from 0 to T to get

$$\int_0^T \left(a_1 \frac{\partial u(\delta(\tau))}{\partial \delta_1} \delta_1(\tau) + a_2 \frac{\partial u(\delta(\tau))}{\partial \delta_2} \delta_2(\tau) + ... + a_N \frac{\partial u(\delta(\tau))}{\partial \delta_N} \delta_N(\tau) \right) d\tau = 0. \tag{B.9}$$

Take conditional expectation on both sides

$$E_t \int_0^T \left(a_1 \frac{\partial u(\delta(\tau))}{\partial \delta_1} \delta_1(\tau) + a_2 \frac{\partial u(\delta(\tau))}{\partial \delta_2} \delta_2(\tau) + ... + a_N \frac{\partial u(\delta(\tau))}{\partial \delta_N} \delta_N(\tau) \right) d\tau = 0. \tag{B.10}$$

From dividing the above equation by $\xi(t)$ and comparing it with the pricing formula in

Equation (13), we can infer that the following equation is satisfied

$$a_1G_1(t) + a_2G_2(t) + ... + a_NG_N(t) = 0 \tag{B.11}$$

for all t. Hence, the gain processes are linearly dependent and the financial market is incomplete. Finally, note that any utility function $u \in U^{IC}$ satisfies the partial differential in Equation (B.8). □

Proof. Proof of Proposition 3. Let φ_{ji} be the partial derivative of φ_j with respect to argument i. Consider the FOC from the central planner problem

$$a_j \sum_{i=1}^{N-1} \varphi_{ji} c_{i+1}^j \beta_i \left(c_1^j\right)^{\beta_i - 1} = y_1 \tag{B.12}$$

$$a_j \varphi_{ji} \left(c_1^j\right)^{\beta_{i-1}} = y_i \quad i = 2, .., N. \tag{B.13}$$

We can rewrite the above as

$$c_1^j = \sum_{i=1}^{N-1} \frac{y_{i+1}}{y_1} \beta_i c_{i+1}^j. \tag{B.14}$$

Summing over Equation (B.14) for $j = 1, ..., J$ and applying market clearing yields

$$\delta_1 = \sum_{i=1}^{N-1} \frac{y_{i+1}}{y_1} \beta_i \delta_{i+1}. \tag{B.15}$$

Finally, noting that $\frac{y_{i+1}}{y_1} = \frac{p_{i+1}}{p_1}$ we have that the following equation must hold

$$\sum_{i=1}^{N} \beta_i p_i \delta_i = 0, \tag{B.16}$$

which implies that the market is incomplete. □

Proof. Theorem 2: First we establish that the consumption prices $\Psi(t) = \xi(t)P(t)$ are jointly real analytic in $(t, x) \in (0, T) \times \mathcal{X}$. This follows from Theorem 2.3.5 (analytic implicit function theorem) in Krantz and Parks (2002) (Theorem B.2 in Anderson and Raimondo (2008)). According to the analytic implicit function theorem (Theorem 2.3.5 in Krantz and Parks (2002)), the utility function of the representative agent is real analytic in $(t, x) \in (0, T) \times \mathcal{X}$. The fact that the candidate price functions S are jointly real analytic in (t,x) follows from Proposition 2 in Hugonnier et al. (2010). Following Hugonnier et al. (2010) adapted to the multiple good setting, one can show that the diffusion matrix of the candidate stock prices is

$$\sigma(t,x) = (T-t)\sigma_{\widetilde{\delta}}(T,x) + o(T-t) = (T-t)\left(I+\varepsilon(T,x)\right)\lambda(T,x) + o(T-t). \qquad \text{(B.17)}$$

First note that as $S(t,x)$ is jointly real analytic it follows that $\sigma(t,x)$ is jointly real analytic. Equation (B.17) show that the stock price diffusion coefficients, $\sigma(t,x)$, are proportional to the diffusion coefficients of the dividends in unit of the numeraire in a neighorhood of the terminal time. Following Theorem 1 in Hugonnier et al. (2010) one can show that a sufficient condition for the market to be complete is that $det\left(\left(I+\varepsilon(T,x)\right)\lambda(T,x)\right) = det\left(I+\varepsilon(T,x)\right)det\left(\lambda(T,x)\right)$ is non-zero. As $det\left(\lambda(T,x)\right) \neq 0$ by Assumption 3, it follow that $det\left(I+\varepsilon(T,x)\right) \neq 0$ for at least one $x \in \mathcal{X}$ is sufficient for market completeness. □

Proof. Proposition 4: To prove 1., first note that when the utility function is separable, i.e., $u_j\left(C\right) = \sum_{i=1}^{N} u_{ji}\left(c_i\right)$, then we have from the FOCs of the central planner problem

$$a_j \frac{\partial u_{ji}\left(c_i^j\right)}{\partial c_i^j} = a_k \frac{\partial u_{ki}\left(c_i^k\right)}{\partial c_i^k}, \qquad \text{(B.18)}$$

for $j,k = 1,..,J$. From equation (B.18), we have that the optimal consumption of good i for agent j is only a function of the aggregate ouput of good i

$$c_i^j = f_i^j\left(\delta_i\right), \qquad \text{(B.19)}$$

and, consequently, the marginal utility of agent j with respect to good i in equilibrium is only a function of good i

$$\frac{\partial u_{ji}\left(c_i^j\right)}{\partial c_i^j} = g_i^j\left(\delta_i\right). \qquad \text{(B.20)}$$

From equation (B.20), we have the the relative price of good i in terms of the numeraire good l is

$$P_i = MRS_{i,l} = \frac{\frac{\partial u_{ji}\left(c_i^j\right)}{\partial c_i^j}}{\frac{\partial u_{jl}\left(c_l^j\right)}{\partial c_l^j}} = \frac{g_i^j\left(\delta_i\right)}{g_l^j\left(\delta_l\right)}. \qquad \text{(B.21)}$$

Next we have that

$$\frac{\partial log\left(MRS_{i,l}\right)}{\partial \delta_k} = \frac{\partial log\left(g_i^j\left(\delta_i\right)\right)}{\partial \delta_k} - \frac{\partial log\left(g_l^j\left(\delta_l\right)\right)}{\partial \delta_k}. \qquad \text{(B.22)}$$

Equation (B.22) is zero if either $i = l$ or $k \neq i, l$. If $k = i$, it is $\frac{\partial log\left(g_i^j(\delta_i)\right)}{\partial \delta_i}$ and if $k = l$ it is

$-\frac{\partial log\left(g_l^j(\delta_l)\right)}{\partial \delta_l}$. Next we apply Theorem 2. The market is complete if we can find one x such that $I+\epsilon\,(T,x)$ is invertible. Given the partial derivative in equation (B.22) we have that $I+\epsilon(t,x)$ is lower triangular for all (t,x), and thus the determinant is

$$det\,(I+\epsilon(t,x)) = \prod_{i=2}^{N}\left(1+\epsilon_{i,i}(t,x)\right). \tag{B.23}$$

From equation (B.23), we must have that for $I+\epsilon\,(t,x)$ to be non-invertible, $(1+\epsilon_{i,i}(t,x))$ must be zero for at least one $i=2,..,N$. As $(1+\epsilon_{i,i}(t,x)) = 1+\delta_i\frac{\partial log\left(MRS_{i,l}\right)}{\partial \delta_i}$, one can show that this is zero only if $MRS_{i,l}=\frac{v(\delta_l)}{A\delta_i}$. As the marginal utility of good i is only a function of δ_i, we must have the the marginal utility with respect to good i is proportional to $\frac{1}{A\delta_i}$ and, consequently, the utility function must be of the form $A_{ji}log\,(c_i)+B_{ji}$. As we have ruled out by assumption any such utility function, the market must be complete. This completes the proof of 1.

To prove 2., consider the FOCs of the central planner problem

$$a_j\varphi_j'\left(\hat{C}\left(C^j\right)\right)\frac{\partial\hat{C}\left(C^j\right)}{\partial c_i^j} = y_i, \tag{B.24}$$

for $j=1,..,J$ and $i=1,..,N$ and where y_i denotes the Lagrange multiplier. As $\hat{C}$ is homogeneous of degree k, $\frac{\partial\hat{C}\left(C^j\right)}{\partial c_i^j}$ is homogeneous of degree $k-1$. Due to the homogeneity assumption and that the function $\hat{C}$ is the same for every agent, we have that the fraction of aggregate output of good i consumed by agent j is the same for all $i=1,...,N$, thus the marginal rate of substition is

$$\begin{aligned} MRS_{i,l} &= \frac{\frac{\partial\hat{C}\left(C^j\right)}{\partial c_i^j}}{\frac{\partial\hat{C}\left(C^j\right)}{\partial c_l^j}} = \frac{\frac{\partial\hat{C}(f_j\delta)}{\partial c_i^j}}{\frac{\partial\hat{C}(f_j\delta)}{\partial c_l^j}} \\ &= \frac{f_j^{k-1}\frac{\partial\hat{C}(\delta)}{\partial c_i^j}}{f_j^{k-1}\frac{\partial\hat{C}(\delta)}{\partial c_l^j}} = \frac{\frac{\partial\hat{C}(\delta)}{\partial c_i^j}}{\frac{\partial\hat{C}(\delta)}{\partial c_l^j}}, \end{aligned} \tag{B.25}$$

where $f_j=\frac{C_i^j}{\delta_i}=\frac{C_h^j}{\delta_h}$ for $i,h=1,...,N$. As $u_j\,(C)$ is in $U^{C,l}$ the market must be complete. □

Proof. Proposition 5: This follows directly from noting that the PDE in Equation (B.8) does not depend on the choice of numeraire. □

Proof. Proposition 6: We first show that condition 1 implies condition 2. We then show that condition 2 implies condition 1. First, note that by definition $MRS_{2,1} = \frac{1}{MRS_{1,2}}$ holds. Now, assume that the utility function satisfies the PDE in Equation (B.8), which then implies

$$a\frac{\partial u(\delta(t))}{\partial \delta_1}\delta_1(t) + \frac{\partial u(\delta(t))}{\partial \delta_2}\delta_2(t) = 0. \tag{B.26}$$

Rearranging the above yields

$$MRS_{2,1}(t) = b\left(\frac{\delta_1(t)}{\delta_2(t)}\right). \tag{B.27}$$

After taking the log of the MRS above, we obtain

$$\ln MRS_{2,1}(t) = \ln b + \ln \delta_1(t) - \ln \delta_2(t). \tag{B.28}$$

Applying Theorem 2 leads to

$$\frac{\partial \ln MRS_{2,1}(t)}{\partial \delta_2}\delta_2(t) = -\frac{1}{\delta_2(t)}\delta_2(t) = -1. \tag{B.29}$$

Now we perform the above steps once more for the case when good two serves as numeraire

$$\ln MRS_{1,2}(t) = -\ln b + \ln \delta_2(t) - \ln \delta_1(t) \tag{B.30}$$

$$\frac{\partial \ln MRS_{1,2}(t)}{\partial \delta_1}\delta_1(t) = -\frac{1}{\delta_1(t)}\delta_1(t) = -1. \tag{B.31}$$

This proves that condition 1 implies condition 2. Now assume that the utility function satisfy Theorem 2 under both numeraires. When good one serves as numeraire, we have

$$\frac{\partial \ln MRS_{2,1}}{\partial \delta_2} = -\frac{1}{\delta_2}. \tag{B.32}$$

Solving the above PDE yields

$$\ln MRS_{2,1} = \ln f_1(\delta_1) - \ln \delta_2 \quad \text{and} \quad MRS_{2,1} = \frac{f_1(\delta_1)}{\delta_2}. \tag{B.33}$$

Next, consider the case when we choose the second good as numeraire. We then have

$$\frac{\partial \ln MRS_{1,2}}{\partial \delta_1} = -\frac{1}{\delta_1}. \tag{B.34}$$

Solving the above PDE results in

$$\ln MRS_{1,2} = \ln f_2(\delta_2) - \ln \delta_1 \quad \text{and} \quad MRS_{1,2} = \frac{f_2(\delta_2)}{\delta_1}. \tag{B.35}$$

Using Equation (B.33) and Equation (B.35) implies that $f_1(x) = bx$ and that $f_2(x) = \frac{1}{b}x$. Thus we have

$$MRS_{2,1} = b\left(\frac{\delta_1(t)}{\delta_2(t)}\right). \tag{B.36}$$

Rearranging yields,

$$a\frac{\partial u(\delta(t))}{\partial \delta_1}\delta_1(t) + \frac{\partial u(\delta(t))}{\partial \delta_2}\delta_2(t) = 0 \tag{B.37}$$

which is the PDE in Equation (B.8). This concludes the proof. □

Proof. Proof of Proposition 7. First we derive the optimal consumption allocations in the Arrow-Debreu equilibrium. Next, we calculate the dynamics of the stock prices and check whether the Arrow-Debreu consumption allocations can be implemented by trading in the available assets. The FOC of the central planner problem is

$$a_j\varphi_j'\left(c_2^j\left(c_1^j\right)^\beta\right)c_2\beta\left(c_1^j\right)^{\beta-1} = y_1 \tag{B.38}$$

$$a_j\varphi_j'\left(c_2^j\left(c_1^j\right)^\beta\right)\left(c_1^j\right)^\beta = y_2 \tag{B.39}$$

$$\tag{B.40}$$

for $j = 1, 2$. This implies that

$$\frac{c_1^1}{c_2^1} = \frac{c_1^2}{c_2^2} = \frac{\delta_1}{\delta_2}, \tag{B.41}$$

where the last equality follow from the market clearing. Furthermore this means that optimal consumption allocations take the form

$$c_i^1 = f\delta_1 \tag{B.42}$$

$$c_i^2 = (1-f)\,\delta_1 \tag{B.43}$$

Define $e = \delta_2\delta_1^\beta$. Then we have by Ito's lemma

$$de(t) = e(t)\left(\mu_e dt + \sigma_e^\top dZ(t)\right), \tag{B.44}$$

where $\sigma_e = \sigma_2 + \beta\sigma_1$. Inserting e into the FOC we get

$$a_1\varphi_1'\left(f^{1+\beta}e\right)f^\beta = a_2\varphi_2'\left((1-f)^{1+\beta}e\right)(1-f)^\beta. \tag{B.45}$$

Thus, by the implicition function theorem, the consumption share f will be a function of the consumption basket e. Next, consider the stock prices

$$
\begin{aligned}
\xi(t)S_1(t) &= E_t\left[\int_t^T \xi(u)\delta_1(u)du\right] \\
&= E_t\left[\int_t^T \varphi_1'\left(f(e(u))^{1+\beta}e(u)\right)f(e(u))^{\beta}e(u)du\right],
\end{aligned} \tag{B.46}
$$

and $\xi(t)S_2(t) = \frac{\xi(t)}{\beta}S_1(t)$, and hence the market is incomplete. Now, assume that the first good is the numeraire good.[15] Then, then in equilibrium the state price density, ξ, is proportional to $\varphi_j'\left(c_2^j\left(c_1^j\right)^{\beta}\right)c_2\beta\left(c_1^j\right)^{\beta-1}$. Apply Ito's lemma on the left hand side of Equation (B.46) and Clark-Ocone Theorem of Malliavin Calculus on the right hand side of Equation (B.46) and then equate the diffusion coefficient on both side we get

$$
\sigma_S(t) = A_e(t)\sigma_e + (\beta-1)\sigma_1 + \sigma_2 + B_S(t)\sigma_e, \tag{B.47}
$$

where $A_e(t) = \frac{\partial}{\partial e}log\left(\varphi_1'\left(f(e(u))^{1+\beta}e(u)\right)f(e(u))^{\beta}\right)$ and $B_S(t) = \frac{E_t\left[\int_t^T \frac{\partial}{\partial e}\left(\varphi_1'\left(f(e(u))^{1+\beta}e(u)\right)f(e(u))^{\beta}e(u)\right)du\right]}{E_t\left[\int_t^T \varphi_1'\left(f(e(u))^{1+\beta}e(u)\right)f(e(u))^{\beta}e(u)du\right]}$. Next, consider the wealth of agent 1

$$
\begin{aligned}
\xi(t)W_1(t) &= E_t\left[\int_t^T \xi(u)\left(c_1^1(u)+p(u)c_2^1(u)\right)du\right] \\
&= (1+\beta)\,E_t\left[\int_t^T \varphi_1'\left(f(e(u))^{1+\beta}e(u)\right)f(e(u))^{1+\beta}e(u)du\right],
\end{aligned} \tag{B.48}
$$

Applying Ito's lemma on the left hand side and Clark Ocone Theorem of the right hand side we get

$$
\sigma_{W_1}(t) = A_e(t)\sigma_e + (\beta-1)\sigma_1 + \sigma_2 + B_{W_1}(t)\sigma_e, \tag{B.49}
$$

where $B_{W_1}(t) = \frac{E_t\left[\int_t^T \frac{\partial}{\partial e}\left(\varphi_1'\left(f(e(u))^{1+\beta}e(u)\right)f(e(u))^{1+\beta}e(u)du\right)\right]}{E_t\left[\int_t^T \varphi_1'\left(f(e(u))^{1+\beta}e(u)\right)f(e(u))^{1+\beta}e(u)du\right]}$. Note that as B_S and B_{W_1} are not identical, the optimal wealth will load onto a different linear combination of the two Brownian motions than the stock price, and hence it is not possible to implement the Arrow-Debreu equilibrium in the case if good one is the numeraire good.[16] Next, consider the case when e is the numeraire good. Then, in equilibrium the state price density, ξ, is proportional to $\varphi_j'\left(c_2^j\left(c_1^j\right)^{\beta}\right)$. Defining $\hat{A}_e(t) = \frac{\partial}{\partial e}log\left(\varphi_1'\left(f(e(u))^{1+\beta}e(u)\right)\right)$ and following a similar calcula-

[15] The proof when the second good is the numeraire good follow similarly.

[16] If $f(e(t)) = f$ for all $e(t)$, then the Arrow-Debreu equilibrium would be implementable. However, we have ruled out any utility function that would lead to such a no-trade equilibrium.

tion we have that the stock price diffusion coefficients are

$$\sigma_S(t) = \left(\hat{A}_e(t) + B_S(t)\right)\sigma_e, \tag{B.50}$$

and the diffusion of the wealth of the first agent is

$$\sigma_{W_1}(t) = \left(\hat{A}_e(t) + B_{W_1}(t)\right)\sigma_e, \tag{B.51}$$

and thus effectivly the market is complete as the optimal wealth and the stock prices depend on the same linear combination of the two Brownian motions. □

Proof. Proposition 8: To verify market completeness we need to consider the expanded diffusion matrix

$$\Sigma(T) = \begin{bmatrix} \sigma_{\widetilde{\delta}}(T) \\ \sigma_P(T) \end{bmatrix}. \tag{B.52}$$

The market is complete if there exists one $x \in \mathcal{X}$ such that $rank\,(\Sigma(T)) = N$. To prove that $rank\,(\Sigma(T)) = N$, subtract row $N+i$ from row i to get

$$\overline{\Sigma}(T) = \begin{bmatrix} \lambda \\ \sigma_P(T) \end{bmatrix}. \tag{B.53}$$

This expression has $rank = N$ because of $rank(\lambda) = N$, which proves that a market with N bonds paying out in the N goods always completes the market. □

Proof. Proposition 10: The FOCs imply the following

$$\frac{\partial u(\delta(t);a)}{\partial \delta_1} = a_1 \frac{\partial u_1(c^1(t))}{\partial c_1^1}, \;\; \frac{\partial u(\delta(t);a)}{\partial \delta_2} = a_1 \frac{\partial u_1(c^1(t))}{\partial c_2^1}. \tag{B.54}$$

Differentiate the above with respect to δ_2 (δ_1) under the assumption of separable sharing rules thus

$$\frac{\partial^2 u(\delta(t);a)}{\partial \delta_1 \partial \delta_2} = a_1 \frac{\partial u_1(c^1(t))}{\partial c_1^1 \partial c_2^1} \frac{d}{d\delta_1} f_1^1(\delta_1), \;\; \frac{\partial^2 u(\delta(t);a)}{\partial \delta_2 \partial \delta_1} = a_1 \frac{\partial u_1(c^1(t))}{\partial c_2^1 \partial c_1^1} \frac{d}{d\delta_2} f_2^1(\delta_2). \tag{B.55}$$

As $\frac{\partial^2 u(\delta(t);a)}{\partial \delta_1 \partial \delta_2} = \frac{\partial^2 u(\delta(t);a)}{\partial \delta_2 \partial \delta_1} \neq 0$ and $\frac{\partial u_1(c^1(t))}{\partial c_1^1 \partial c_2^1} = \frac{\partial u_1(c^1(t))}{\partial c_2^1 \partial c_1^1} \neq 0$, which together with non-separability of the representative agent's utility function implies that $\frac{d}{d\delta_1} f_1^1(\delta_1) = \frac{d}{d\delta_2} f_2^1(\delta_2)$. As this must hold for all δ_1 and δ_2, we obtain the following result

$$f_1^1(\delta_1) = A\delta_1, \;\; f_2^1(\delta_2) = A\delta_2. \tag{B.56}$$

This ends the proof. □

Proof. Proposition 11: A necessary condition for the utility function of the representative agent to be separable is

$$\frac{\partial u\left(\delta(t)\right)}{\partial \delta_1} = g(\delta_1(t)) \tag{B.57}$$

for some function g, i.e., the partial derivative with respect to the first good only depends on the first good. In equilibrium, the following relation holds

$$\frac{\partial u\left(\delta(t)\right)}{\partial \delta_1} = a_1 \frac{\partial u_1\left(c^1(t)\right)}{\partial c_1^1}. \tag{B.58}$$

Using the sharing rules and the utility function in Equation (35), we obtain

$$\frac{\partial u\left(\delta(t)\right)}{\partial \delta_1} = a_1 \varphi_1^{'}\left(f_1^1(\delta_1(t),\delta_2(t)) f_2^1(\delta_1(t),\delta_2(t))^{\alpha_1}\right) f_2^1(\delta_1(t),\delta_2(t))^{\alpha_1}. \tag{B.59}$$

This above function depends only on δ_1 if φ_1 is given by $\varphi_1(x) = ln(x)$. □

Proof. Proposition 12: Consider that sharing rules are linear; then the following holds

$$a_1 \varphi_1^{'}\left(A\delta_1(t)(B\delta_2(t))^{\alpha_1}\right)\right)\left(B\delta_2(t)\right)^{\alpha_1} = a_2 \varphi_2^{'}\left((1-A)\delta_1(t)((1-B)\delta_2(t))^{\alpha_2})\right)\left((1-B)\delta_2(t)\right)^{\alpha_2} \tag{B.60}$$

for some positive constants A and B. Since dividends are less than perfectly correlated, the above cannot hold unless agents have log-linear preferences or homogeneous taste. □

Proof. Proposition 9: The strategy of the proof is as follows

1. Prove that $|corr_t\left(W^1(t), W^2(t)\right)| < 1$.
2. Prove that portfolio policies takes the form as in Proposition 9.
3. Prove that this implies that $|corr_t\left(S_1(t), S_2(t)\right)| < 1$.

To prove (1): From the FOC it follows that

$$p_2(t) = \alpha_1 \left(\frac{c_1^1}{c_2^1}\right) \quad \text{and that} \quad \alpha_1 \left(\frac{c_1^1}{c_2^1}\right) = \alpha_2 \left(\frac{c_1^2}{c_2^2}\right). \tag{B.61}$$

The wealth of agent 1 is given by

$$W^1(t) = \frac{E_t\left[\int_0^T \left(\xi(\tau)c_1^1(\tau) + \xi(\tau)p_2(\tau)c_2^1(\tau)\right)d\tau\right]}{\xi(t)}. \tag{B.62}$$

Using Equation (B.61), we can rewrite wealth as

$$W^1(t) = \frac{A_1 E_t \left[\int_0^T \xi(\tau) c_1^1(\tau) d\tau\right]}{\xi(t)} \tag{B.63}$$

with $A_1 = (1 + \alpha_1)$. Similarly, we can write the wealth of agent 2 as

$$W^2(t) = \frac{A_2 E_t \left[\int_0^T \xi(\tau) c_1^2(\tau) d\tau\right]}{\xi(t)} \tag{B.64}$$

with $A_2 = (1 + \alpha_2)$. Next, we show that W^1 and W^2 are linearly independent. To this end, we use the Anderson and Raimondo (2008) technique. Let

$$dc_1^1(t) = \phi_1(t)dt + \Sigma_1(t)^T dZ(t), \;\; dc_1^2(t) = \phi_2(t)dt + \Sigma_2(t)^T dZ(t) \tag{B.65}$$

and

$$\Sigma(t) = \begin{bmatrix} \Sigma_1(t)^T \\ \Sigma_1(t)^T \end{bmatrix} = \begin{bmatrix} \Sigma_{11}(t) & \Sigma_{12}(t) \\ \Sigma_{21}(t) & \Sigma_{22}(t) \end{bmatrix}. \tag{B.66}$$

If there exists an $x \in \mathcal{X}$ such that $\det(\Sigma(T, x)) \neq 0$, then $W^1(t)$ and $W^2(t)$ are linearly independent a.a. $(t, x) \in (0, T) \times \mathcal{X}$. Calculating $\Sigma(t)$ by Ito's lemma, we have

$$\Sigma(t) = J(c_1(t)) I_{\delta(t)} \lambda \tag{B.67}$$

where $J(c_1(t))$ denotes the Jacobian of $c_1(t) = (c_1^1(t), c_1^2(t))$ and is given by

$$J(c_1(t)) = \begin{bmatrix} \frac{\partial c_1^1(t)}{\partial \delta_1} & \frac{\partial c_1^1(t)}{\partial \delta_2} \\ \frac{\partial c_1^2(t)}{\partial \delta_1} & \frac{\partial c_1^2(t)}{\partial \delta_2} \end{bmatrix}. \tag{B.68}$$

Note that $\det(\Sigma(T)) = \det(J(c_1(T))) \det\left(I_{\delta(T)}\right) \det(\lambda)$. By definition, $\det\left(I_{\delta(T)}\right) \neq 0$ and $\det(\lambda) \neq 0$, implying that $\det(\Sigma(T)) \neq 0$ if and only if $\det(J(c_1(T))) \neq 0$. From the clearing of the commodity market, we obtain

$$c_1^1(t) + c_1^2(t) = \delta_1(t). \tag{B.69}$$

Taking the derivative with respect to δ_2, we get

$$\frac{\partial c_1^1(t)}{\partial \delta_2} = -\frac{\partial c_1^2(t)}{\partial \delta_2}. \tag{B.70}$$

We need to show that Equation (B.70) is not zero. To this end, note that by utilizing Proposition (12), we have established that the sharing rule is non-linear. Moreover, according to Proposition (11), we know that the utility function of the representative agent is non-separable in δ_1 and δ_2. By referencing Proposition (10), we know that separable sharing rules and non-separable utility functions are only consistent with linear sharing rules. As we do not have linear sharing rules, this implies that the sharing rule must be non-separable. Non-separable sharing rules guarantee that Equation (B.70) is non-zero and that

$$\det\left(J(c_1(T))\right) = J_{11}J_{22} - J_{12}J_{21} \neq 0 \tag{B.71}$$

since $J_{11}J_{22} < 0$ and $J_{12}J_{21} > 0$ or $J_{11}J_{22} > 0$ and $J_{12}J_{21} < 0$ where J_{ij} denotes element (ij) of $J(c_1(T))$. This proves 1.

To prove (2): Combining Equation (B.63) with Equation(B.64) yields

$$B_1 W^1(t) + B_2 W^2(t) = S_1(t) \tag{B.72}$$

with $B_j = 1/A_j$. In equilibrium, total wealth must be equal to the value of the stock market, i.e.,

$$W^1(t) + W^2(t) = S_1(t) + S_2(t). \tag{B.73}$$

Using Equation (B.72) and Equation (B.73), we obtain

$$\begin{bmatrix} B_1 & B_2 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} W^1(t) \\ W^2(t) \end{bmatrix} = \begin{bmatrix} S_1(t) \\ S_1(t) + S_2(t) \end{bmatrix}. \tag{B.74}$$

This has a unique solution if $B_1 \neq B_2$, which holds whenever $\alpha_1 \neq \alpha_2$. This proves 2.

To prove (3): Subtracting Equation (B.63) from Equation (B.64) leads to

$$(1 - B_1)\, W^1(t) + (1 - B_2)\, W^2(t) = S_2(t). \tag{B.75}$$

We have now expressed stock prices as linear combinations of the wealth of agent 1 and agent 2 (see Equation (B.72) and Equation (B.75)). Let the wealth of the agents evolve accordingly to

$$dW^1(t) = \mu_{W^1}(t)dt + \sigma_{W^1}(t)^T dZ(t),\ dW^2(t) = \mu_{W^2}(t)dt + \sigma_{W^2}(t)^T dZ(t) \tag{B.76}$$

which implies

$$\begin{bmatrix} B_1 & B_2 \\ 1-B_1 & 1-B_2 \end{bmatrix} \begin{bmatrix} \sigma_{W_1^1}(t) & \sigma_{W_2^1}(t) \\ \sigma_{W_1^2}(t) & \sigma_{W_2^2}(t) \end{bmatrix} = \begin{bmatrix} S_1(t)\sigma_{11}(t) & S_1(t)\sigma_{12}(t) \\ S_2(t)\sigma_{21}(t) & S_2(t)\sigma_{22}(t) \end{bmatrix}. \tag{B.77}$$

As $\begin{bmatrix} \sigma_{W_1^1}(t) & \sigma_{W_2^1}(t) \\ \sigma_{W_1^2}(t) & \sigma_{W_2^2}(t) \end{bmatrix}$ is invertible, $\begin{bmatrix} S_1(t)\sigma_{11}(t) & S_1(t)\sigma_{12}(t) \\ S_2(t)\sigma_{21}(t) & S_2(t)\sigma_{22}(t) \end{bmatrix}$ is invertible if $\begin{bmatrix} B_1 & B_2 \\ 1-B_1 & 1-B_2 \end{bmatrix}$ is invertible. Finally, $\begin{bmatrix} B_1 & B_2 \\ 1-B_1 & 1-B_2 \end{bmatrix}$ is invertible if $\alpha_1 \neq \alpha_2$. This ends the proof. □

Remark 2. *Note that the only requirement for the proof above to work is less than perfect correlation between the wealth of agent one and agent two. In fact, we could solve the optimal portfolio even if the market was intrinsically incomplete, i.e., in a setting with more than two Brownian motions but only two risky securities.*

Remark 3. *The above proof also relies on the number of goods and agents. We have two goods and two agents. This allows us to write down a system with two equations in two unknowns. If there where more goods than agents, we would not be able to identity all of the stock price processes from the above method.*