# A Foundation for Renegotiation-Proof Contracts*

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**Abstract**

This paper provides a foundation for renegotiation-proof contracts when the principal has the bargaining power and the agent has private information. In doing so, it extends the analysis of the Coase conjecture to contractual environments in which the seller and the buyer must also determine the quantity or the quality of the good being sold.

## 1 Introduction

In the standard analysis of the durable-good monopolist, any sale is efficient and definitive: buyer and seller cannot both benefit from modifying the price of the sale. In richer contractual environments, however, a signed contract may be inefficient. For example, the parties may benefit from increasing the quantity of the good initially sold, or by agreeing on a different quality of that good.

This issue is particularly important when the buyer holds private information, because his willingness to sign some contract is informative of his type, and may thus reveal some inefficiency of the contract that was just signed. This, in turn, prompts the seller to propose a new contract, and may distort the buyer's ex ante incentives to accept any given contract.

Contract renegotiation with a privately informed agent has traditionally been studied from two different angles. The first approach is axiomatic, and focuses on "renegotiation-proof" contracts.[1]

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[1]See Dewatripont (1989), Maskin and Tirole (1992), Battaglini (2007), Maestri (2012), and Strulovici (2011, 2013).

It essentially *assumes* that renegotiation leads to an efficient contract, even when one party holds private information. The second approach focuses on simple renegotiation protocols, in which the principal gets a single shot at renegotiating the contract, by making a take-it-or-leave-it offer. This approach typically results in *inefficient* contracts.[2]

The second approach seems incomplete: what, in reality, should prevent the principal from proposing a new contract after learning the inefficiency of the current contract? This paper studies a more flexible negotiation protocol, in which the principal is allowed to propose a new contract following any incoming information about the agent's type. Put differently, the principal cannot commit *not* to renegotiate a contract.

While such flexibility seems necessary to guarantee ex post efficiency, proving that it is sufficient raises complex issues. To appreciate the difficulty, consider again the standard durable-good monopolist. Efficiency, in that context, means that the good is sold without delay, and was established by Gul, Sonnenschein, and Wilson (1986) as the discount rate, or breakdown probability, goes to zero.[3] The proof is sophisticated even in this simple contractual environment, where each contract amounts to a single posted price. The key question is to determine whether the seller can benefit from distorting the allocation of the low-valuation buyer (i.e., by inefficiently delaying the sale) to extract some rent from the high-valuation buyer. In richer environments, the question is more complex because i) the signature of any contract may be followed by further negotiations (e.g., contractual covenants, changes in quantities or qualities may be added), ii) the principal may benefit from proposing multiple new contracts at each rounds instead of single one,[4] iii) each type of the agent can randomize over all such contracts, and iv) in many contracting problems, the utility of the agent need not be linear or even separable in the contract components.

The negotiation protocol considered here is as follows: at each round, the principal can propose a menu of new contracts. The agent then chooses a contract in that menu or hold on to the last accepted contract. At the end of each round, negotiation breaks down exogenously with a fixed

A similar approach has been used to study renegotiation in repeated games with complete information by Bernheim and Ray (1989) and Farrell and Maskin (1989).

[2]See Hart and Tirole (1988) and Fudenberg and Tirole (1990).

[3]The result is shown for the "gap" case and the "no gap" case under some Lipschitz condition on the distribution of types, for weak Markov equilibria (see also Sobel and Takahashi (1983) and Fudenberg, Levine, and Tirole (1985)). Ausubel and Deneckere (1989) show that the conjecture can fail when more general equilibria are allowed. The analysis Coase conjecture has been extended to various environments: interdependent values (Deneckere and Liang (2006)), flow of new buyers (Fuchs and Srzypacz (2010)), and outside options for the buyer (Board and Pycia (2013)). All these models focus on the case in which the buyer can only buy one unit of the good, and a single quality of the good is available.

[4]For example, the principal may propose one contract for each type of the agent.

probability $\eta$, in which case the last accepted contract is implemented. The breakdown probability captures negotiation frictions: when it is equal to 1, the protocol reduces to full commitment, and the principal typically distorts the allocation of one type of the agent, which leads to ex post inefficiency. The main finding of the paper is that, as $\eta$ goes to zero, all PBE outcomes of the negotiation game converge to efficient, renegotiation-proof contracts.

The model focuses on a binary type structure, which satisfies a single crossing condition. As a result, there is common knowledge of gains from renegotiation: as long as the types of the agent have not been fully separated, there is a strictly positive surplus to be extracted. In equilibrium, the principal always extracts *some* surplus from the relationship. Moreover, the closer to efficiency the contract is for one type of the agent, the more surplus the principal can extract from the other type. Intuitively, when the contract is close to being efficient for one type, say $L$, there is little room for renegotiation with that type, and hence the other type, $H$, cannot gain much from pretending to be $L$. This result stands in sharp contrast with the Coase conjecture for binary types. In that environment, the seller extracts surplus $v_L - c$ from the sale, where $v_L$ is the valuation of the low-type and $c$ is the principal's marginal cost; the high type gains $v_H - v_L$ from the sale. As the difference $v_L - c$ gets smaller, the initial contract (no sale) gets closer to efficiency for the low type. However, this does not improve the principal's ability to extract surplus from $H$ (in fact, the principal gets an even smaller share of the surplus if $v_L$ decreases, keeping $c$ fixed). In that sense, the present results suggest that the seller's inability to extract any surplus in the gap case of the Coase conjecture may rely more than was previously thought on the very specific contractual environment in which it has been analyzed.

An interpretation of the efficiency result is that renegotiation provides a dynamic implementation, without commitment, of efficient allocations. The fact that almost efficient contracts are proposed immediately implies that *in equilibrium*, renegotiation has little impact on the contracts that are proposed initially, even though the *possibility* of renegotiation plays a major role on the contracts that are initially proposed. This suggests that one should not infer that renegotiation plays a minor role on equilibrium outcomes, even though the observed renegotiation activity seems negligible.

At a broader level, this paper analyzes a dynamic screening problem and bears some resemblance to the literature on reputation, in which some players are trying to determine whether other players have a "commitment" type.[5] Compared to this literature, the present analysis differs in several ways: i) the "actions" of the players (the types) are endogenous, because the principal chooses which contracts the agent chooses from in each round, ii) the state space is larged, because it

[5] See Fudenberg and Levine (1989), Schmidt (1993), Abreu and Gul (2000), Cripps et al. (2005), and Atakan and Ekmekci (2012).

includes the last accepted contract, in addition to the principal's belief, and iii) all types of the agent are strategic. The richer state space, in particular, requires specific tools, and the use of a number of inequalities which combine the nonlinear geometry of the problem (as captured by the agent's and the principal's utility over the contract space) with the incentives of the players.

Finally, a separate contribution of the paper is to establish the existence of a PBE for a negotiation game with a (relatively) rich contract space. In the present setting, backward induction techniques cannot be applied. Instead, the proof takes a two-step approach: first, prove the existence of an equilibrium in an auxiliary game of perfect information between the principal and the high type of the agent, based on Harris (1985). Second, use that equilibrium to construct an equilibrium of the negotiation game with private information.

# 2 Setting and Overview of the Results

There are two players, a principal (P) and an agent (A) who negotiate a contract lying in some compact and convex subset $\mathcal{C}$ of $\mathbb{R}^2$.

The agent has a utility function $u_\theta : \mathcal{C} \to \mathbb{R}$ where $\theta \in \{L, H\}$ denotes his type, and P has a cost function $Q : \mathcal{C} \to \mathbb{R}$. It is assumed throughout that i) $u_L$, $u_H$ and $Q$ are strictly increasing and twice continuously differentiable, ii) for each $\theta$, $u_\theta$ is concave and strictly concave in its second argument, and iii) $Q$ is convex.

A contract $C = (x_1, x_2) \in \mathcal{C}$ is $\theta$-*efficient* if it is the cheapest contract providing $\theta$ with some given utility level. For each $\theta$, let $\mathcal{E}_\theta$ denote the set of interior $\theta$-efficient contracts, i.e., at which $\theta$'s iso-utility curve and P's iso-cost curve are tangent.

The functions $u_L$ and $u_H$ satisfy a standard single-crossing property: iso-utility curves of $L$ are steeper than those of $H$ at their intersection point. This implies that the efficiency curve $\mathcal{E}_L$ lies to the lower right of $\mathcal{E}_H$. $\mathcal{C}$ can therefore be partitioned into three regions separated by the efficiency curves. Contracts in the inner region are said to be in the *No Rent* configuration, while contracts below $\mathcal{E}_L$ (above $\mathcal{E}_H$) are in the $H$-Rent ($L$-Rent) configuration. $\mathcal{H}$ will denote the set of contracts in the $H$-Rent configuration. To rule out pathological cases, it is assumed that the efficiency curves $\mathcal{E}_L, \mathcal{E}_H$ are upward sloping. To guarantee the applicability of differential techniques, it is further assumed that for any contract $C$ in the $H$-Rent or $L$-Rent configuration, the $\theta$-efficient contract that gives $\theta$ the same utility as $C$ lies on $\mathcal{E}_\theta$.[6]

[6] If the utility and cost functions are defined on some upper orthant $\mathcal{O} = [\ell_1, +\infty) \times [\ell_2, +\infty)$ containing $\mathcal{C}$, one can always extend $\mathcal{C}$ to a larger subset $\mathcal{C}'$ of $\mathcal{O}$ so as to satisfy this assumption. This is for example achieved by

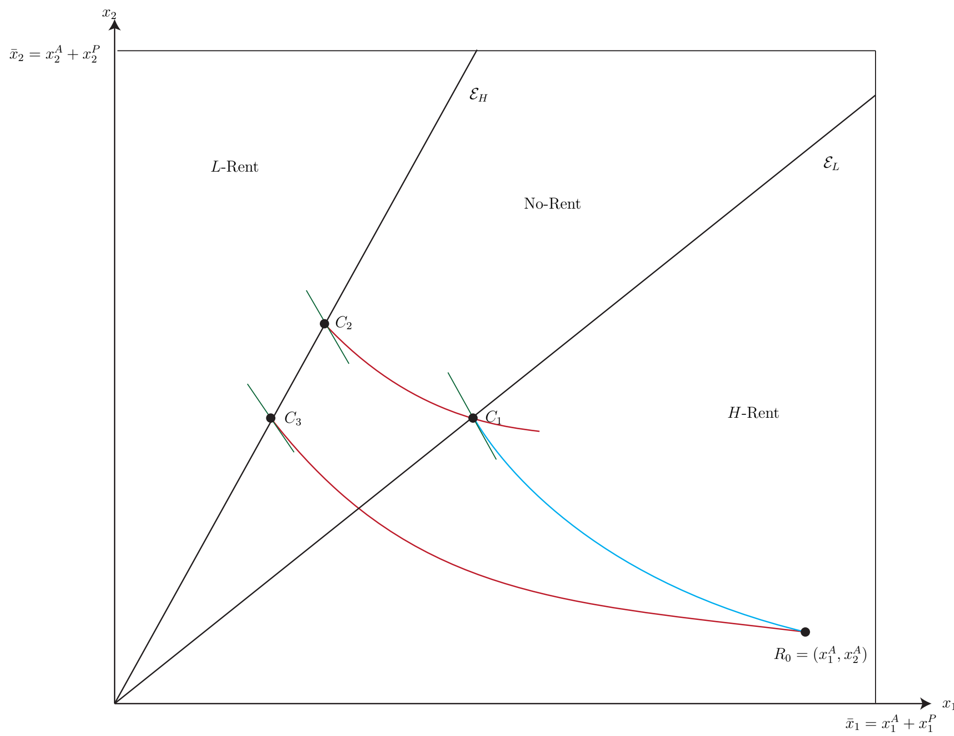


Figure 1: Setting (trade interpretation)

The situation is represented on Figure 1 for the case of a trade application. $\mathcal{C}$ represents and Edgeworth box, delimited by the sum of endowments of the agent and the principal. A contract $C$ represents the agent's final allocation, and the status quo $R_0$ represents the endowment of the agent, before any trade.

Setting (trade interpretation)

## The Negotiation Game

The game unfolds as follows: the agent's type is privately known, initially, and $\beta_0 = Pr(\theta = H)$ represents P's prior about $\theta$. The game starts with an initial contract $R_0 \in \mathcal{C}$ representing some status quo or the result of some earlier play. There are countably many potential rounds, indexed by $n \in \mathbb{N}$. At each round $n$, P can propose a menu $M_n$ of contracts in $\mathcal{C}$. We assume that the

taking $\mathcal{C}'$ to be the smallest rectangle including $\mathcal{C}$ and such that $\mathcal{E}_L$ (resp. $\mathcal{E}_H$) hits the boundary of $\mathcal{C}'$ on its right (resp. upper) edge.

number of contracts in $M_n$ is bounded above by some constant $G \geq 2$ that is arbitrary but fixed throughout the game. The agent chooses an item of $M_n$ or holds on to the last accepted contract, $R_n$. Any mixed strategy over the choice set $M_n \cup \{R_n\}$ is allowed. The contract $R_{n+1}$ that is selected by the agent becomes the new reference. At the end of each round, renegotiation breaks down with probability $\eta \in (0,1]$ and the last accepted contract, $R_{n+1}$, is implemented. Otherwise, negotiation moves on to the next round.

Letting $\{R_n\}$ denote the stochastic process of contracts entering each round $n$, the agent's expected utility is equal to $\mathcal{V}_\theta = E[\sum_{n\geq 0}(1-\eta)^n \eta u_\theta(R_{n+1})]$, while P's expected cost is $\mathcal{Q} = E[\sum_{n\geq 0}(1-\eta)^n \eta Q(R_{n+1})]$.

The parameter $\eta$ represents the *negotiation friction.* The objective of this paper is to characterize the PBEs of the game as the friction $\eta$ goes to zero, under the following assumption. Let $\beta_n$ denote the probability, at the beginning of round $n$, that P assigns to type $H$.

Assumption 1 (No Expanding Support) *If $\beta_n \in \{0,1\}$, then $\beta_m = \beta_n$ for all $m \geq n$.*

Theorem 1 *For each $\eta \in (0,1]$, there exists a PBE satisfying Assumption 1.*

For any contract $R \in \mathcal{C}$, let $(E_H(R), E_L(R))$ denote the cheapest pair of $H$- and $L$-efficient contracts such that each type $\theta$ weakly prefers $E_\theta(C)$ to $E_{\theta'}(C)$ and to $C$. That pair is uniquely defined: if $R$ is in the No-Rent configuration, $E_\theta(R)$ is simply the $\theta$-efficient contract that gives $\theta$ the same utility as $R$. If $R$ is in the $H$-Rent configuration, then $E_L(R)$ is similarly defined, while $E_H(R)$ is the $H$-efficient contract that gives $H$ the same utility as $E_L(R)$. Because that contract gives a strictly higher utility to $H$ than the initial contract $R$, $H$ must be get a positive rent in any equilibrium, hence the name of that configuration. A symmetric construction obtains if $R$ is instead in the $L$-Rent configuration. Figure 2 represents these concepts for the case of CRRA utility functions and a linear cost function, and where $\mathcal{C}$ is the Cartesian product $[0,\bar{x}_1] \times [0,\bar{x}_2]$.

Theorem 2 *Consider any initial contract $R_0$ and belief $\beta_0$, and fix any $\varepsilon > 0$. There exists $\bar{\eta}(\varepsilon) > 0$ such that the following statements hold for any $\eta \leq \bar{\eta}(\varepsilon)$ and corresponding PBE.*
*A: the expected utility of each type $\theta$ is bounded below by $u_\theta(E_\theta(R_0)) - \varepsilon$.*
*B: the probability that each type $\theta$ gets a contract within a distance[7] $\varepsilon$ of $E_\theta(R_0)$ when renegotiation breaks down is greater than $1-\varepsilon$.*

Statement B implies that the outcomes of renegotiation must get arbitrarily close to ex-post efficiency as the renegotiation friction $\eta$ goes to zero, since each contract $E_\theta(R_0)$ is $\theta$-efficient. The

[7]The statement holds for any norm on $\mathbb{R}^2$.

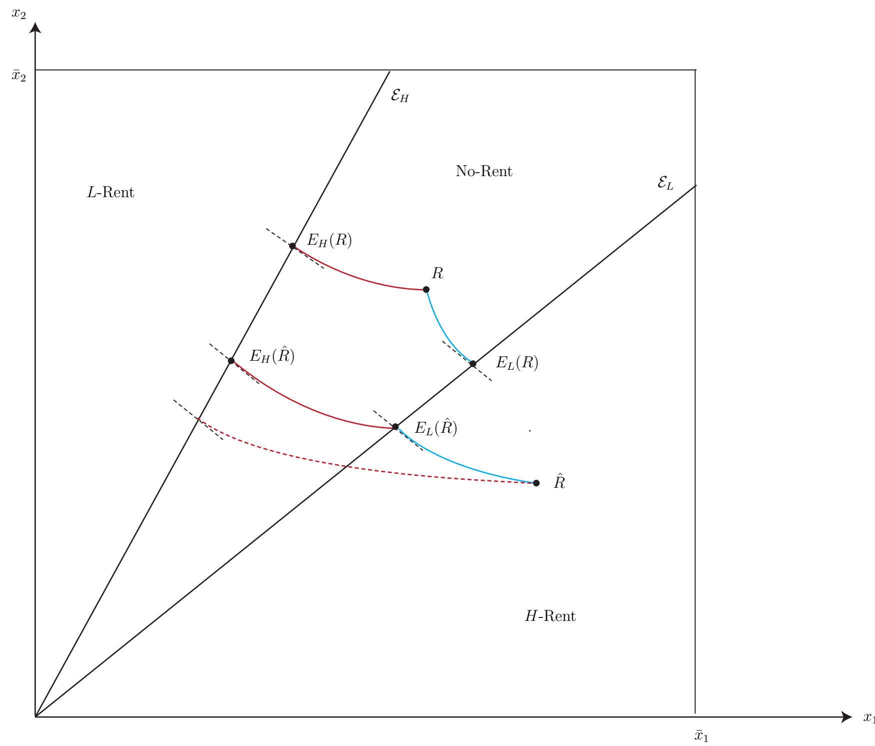


Figure 2: Renegotiation outcomes

statement is a relatively straightforward consequence of Statement A, to which the quasi-totality of the proof is devoted.

Theorem 2 implies that P always gets some of the surplus from negotiation. When the contract is in the No-Rent configuration, P extracts all the surplus. Even when $R_0$ is in the $H$-Rent configuration, P gets the surplus from negotiation that he would obtain only facing $L$, but also extracts some additional surplus in case he is facing $H$. Strikingly, the surplus extracted from $H$ is increasing as $L$'s utility change so as to make the initial contract closer to efficiency: at the limit, if $R_0$ is $L$-efficient, P extracts all the surplus from renegotiating $R_0$ with $H$.

## Applications

1. **Durable Good Monopolist** Agent A is a buyer with quasi-linear utility $u_\theta(C) = \theta u(x_2) + x_1$, where $x_2$ is the quantity of the good sold by P, $x_1$ is A's cash holding, and $u$ is his concave utility function. The initial contract $R_0$ is equal to $(0, \bar{x}_2)$ where $\bar{x}_1$ is A's initial wealth. P's cost is

$Q(x_1, x_2) = cx_2 + x_1$, where $c > 0$ is the marginal cost for producing the good and $x_2$ captures, formally, how much wealth "P leaves to A".[8]

2. **Labor Contract** P is a potential employer and A is a worker. $-x_2$ represents A's effort and $x_1$ is his wage. A gets utility $u_\theta(C) = \theta\psi(-x_2) + x_1$, where the cost $\psi$ of effort is increasing in its argument, and $\theta$ is a worker-specific cost of effort. The status quo $R_0 = (0, 0)$ represents unemployment, while P's profit is $\Pi(x_1, x_2) = -Q(x_1, x_2) = p(-x_2) - x_1$, where $p > 0$ is the unit price of the good.

3. **Consumption Smoothing and Insurance** There are two periods and a single good. The dimensions of $\mathcal{C}$ represent A's consumption in each period. P is a social planner or a bank who can help the agent smooth his consumption. The type $\theta$ may be a privately known patience/discount factor, or a distribution parameter that describes how likely the agent is to value the good in the second period. For example $u(x_1, x_2) = v(x_1) + \theta v(x_2)$ or $u(x_1, x_2) = v(x_1) + E[w(x_2, \tilde{\rho})|\theta]$ where $\tilde{\rho}$ is a taste shock whose distribution FOSD increases in $\theta$ and $w$ is supermodular. $R_0$ is A's autarkic income stream. $Q(x_1, x_2) = p_1 x_1 + p_2 x_2$, where $p_t$ is the market price for the good in period $t$.

4. **Trade** More generally, the model describes a trade environment in which the dimensions of $\mathcal{C}$ represent distinct goods, with $x_i$ denoting the quantity of good $i$ consumed by A. Type $L$ cares more about the first good than the second, relative to $H$. P has convex preferences, and $Q$ is the negative of a utility function representing those preferences. $R_0$ denotes the agent's initial holdings of the goods.

## Overview of the main results

The argument for Theorem 2 starts with three cases of increasing complexity. First, if the type of the agent is known (and given Assumption 1), the problem reduces to a bargaining problem with perfect information in which P has all the bargaining power. The equilibrium outcome is essentially unique, with P extracting the entire surplus from renegotiation.

The second case is when the initial contract $R_0$ is efficient for one type, say type $L$. In that case, P cannot make any gain on that type, and that situation gives him leverage with respect to the other type: $H$ knows that there is nothing to be gained, compared to his current situation, by mimicking $L$. The situation is therefore somewhat similar to a setting with perfect information. One difference is that the argument must consider $H$'s maximum utility not only over all PBEs but also over all beliefs that P may hold about A.

[8] In effect, P's profit is $\Pi(t, x_2) = t - cx_2$, where $t$ is how much the agent pays P. Letting $t = \bar{x}_1 - x_1$, we obtain the formulation in terms of the cost function $Q$.

The third case is when $R_0$ is in the No-Rent configuration. The intuition is similar to the previous case, because the efficient contract that P wants to propose to each type in unattractive to the other type. The agent cannot leverage his private information to build any bargaining power. The argument works somewhat differently, however, by building upper bounds on P's expected cost over all PBEs, and working its implications for the continuation utility of each type.[9] When $R_0$ is in the $L$-Rent ($H$-Rent) configuration, a similar argument yields a simple upper bound on the rent achieved by $L$ ($H$).

All these results hold for any friction level.

The harder part is to show that efficiency obtains as the breakdown probability $\eta$ goes to zero, when $R_0$ is in the $L$- or $H$-Rent configuration. This result is clearly false for arbitrary $\eta$: if $\eta = 1$, in particular, the situation amounts to full commitment, and the optimum proposal involves distorting one of the two types. If, for example, $R_0$ is the $H$-rent configuration, P will propose an inefficient contract to $L$ in order to reduce $H$'s rent. $L$ gets no rent and an inefficient allocation, while $H$ gets a positive rent and an efficient allocation. These observations are standard in the contracting literature with commitment, and their proofs are omitted.

**Coasian dynamics and its limits**

The first part of the argument is similar to the proof of the Coase conjecture (Gul et al., 1986). Suppose that $R_0$ is in the $H$-Rent configuration. To avoid giving too much rent to $H$, P tries to exploit $H$'s concern about getting an early inefficient breakdown and resulting willingness to forego some of his rents. However, P is also harmed by such breakdowns. For the scheme to be attractive, $H$ must therefore accept the low-rent contracts with sufficiently high probability. This, however, quickly leads to a very low posterior probability of facing $H$, which makes P want to jump to the efficient contract for $L$ that entails leaving all the rent to $H$. As the breakdown probability goes to zero, this mechanism forces P to leave all the rent to $H$, as in the Coase conjecture.

A major complication arises, however, owing to the richer contract space and bargaining possibilities: P does not have to wait for $H$ to accept low-rent contracts in order to reduce the breakdown inefficiency: he can also propose new contracts to $L$ (and $H$) that reduce this inefficiency. Unlike the standard analysis of the Coase conjecture, in which the seller is reduced to posting a single price at each period, here the principal can propose arbitrarily many contracts. In particular, P could have $L$ accept gradually more efficient contracts for $L$, and $H$ randomize between those contracts and low-rent $H$-efficient contracts. To pursue the comparison with the Coase conjecture, the seller

[9] Compared to an approach based on rationalizability, the argument uses the full force of the equilibrium concept, which imposes common knowledge of P's belief following any move. If each type of the agent held a different belief about P's belief conditional on a particular move, the present argument would not apply.

could propose *probabilities* of getting the good: the high valuation buyer would buy the good at a high price with some probability. With the remaining probability he, like the low-valuation buyer, would buy a small probability of getting the good, which would gradually increase over time. In general, there is no guarantee that proposing only two contracts at each round is without loss of generality.[10]

### Stalling negotiation and relaxed problem

Proposition 3 states that, in any PBE, the posterior probability of facing $H$ must remain sufficiently high as long as renegotiation has not reached a critical level of efficiency related to the breakdown probability $\eta$. The argument exploits fine comparisons between the gain of extracting rent from $H$ versus the losses caused by inefficient breakdowns on both $L$ and $H$. The argument works by contradiction, and may be summarized as follows: suppose that negotiation has reached a stage at which $H$ is highly unlikely. Then, P's main concern is to reduce the inefficiency for $L$. At the same time, $H$ is only willing to reveal himself at a given round if the contracts that he will receive in the coming rounds do not offer a significant increase in utility. This means that negotiation must proceed very slowly. For that slow-negotiation PBE to be profitable for P (instead of leaving all the rent to $H$), $H$ must be accepting low-rent contracts with high probability, which increases P's focus on $L$'s inefficiency, creating a cycle that reduces the speed of negotiation even more. Eventually, the argument shows that such slow-negotiation PBE must stalls with positive probability before getting to efficiency, which leads to a contradiction.

The argument for proving Proposition 3 uses geometry to evaluate and approximate the speed of negotiation near efficiency. It also uses a relaxed problem for the principal, in which his incentives at each round are weakened to a couple of necessary conditions. To facilitate computations, the relaxed problem is again modified by changing the mixing probabilities of the agent so as to preserve each type's and the principal's weakened IC constraints but obtain tractable formulas.

### Equilibrium existence and the lack of backward induction

Because of the richness of the contract space, it is impossible to use backward induction to characterize equilibrium behavior near efficiency (unlike in the Coase conjecture with binary types, where setting the price at $L$'s valuation is optimal for low enough probabilities of facing $H$). This prevents the use of constructive proofs for the existence of an equilibrium.

[10] As Bester and Strausz (2001) have shown that the set of implementable outcomes can require more "messages" (or contracts) than the number of types of the agent. Since we consider all equilibria, their modified revelation principle for optimal contracts without commitment cannot be applied here. The extent to which that principle applies to models with an infinite horizon is also unclear.

Instead, equilibrium existence is shown in two steps. In the first step, the setting is simplified to a game between P and $H$, and $L$'s behavior is taken as given: in period $n$, P proposes two contracts, $R_{n+1}$ and $C_n$; $H$ randomizes between the two contracts, while $L$ accepts $R_{n+1}$ with probability 1. This reduced game has perfect information in the sense Harris (1985),[11] and satisfies the compactness and continuity assumptions used in that paper to guarantee the existence of an equilibrium. In the second step, the equilibrium of this simplified, perfect information game is used to construct a PBE of the original game, by appropriately specifying reactions and beliefs following any possible deviation.

## 3 Results holding for all friction levels

PROPOSITION 1 *The following holds for any PBE and $\eta$:*

*i) If the prior $\beta$ puts probability 1 on some type $\theta$, P immediately proposes the $\theta$-efficient contract that leaves $\theta$'s utility unchanged and $\theta$ accepts it.*

*ii) If $R_0$ is $\theta$-efficient, P immediately proposes $E_{\theta'}(R_0)$ ($\theta' \neq \theta$), and $\theta'$ accepts it.*

*iii) If $R_0$ is in the No-Rent configuration, P immediately proposes $E_L(R_0)$ and $E_H(R_0)$, and each type $\theta$ accepts $E_\theta(R_0)$.*

*iv) If $R_0$ is in the H-Rent (L-Rent) configuration, H's (L's) expected utility is bounded above by $u_H(E_H(R_0))$ ($u_L(E_L(R_0))$).*

The next result is crucial for the analysis: for any PBE and round $n$, P can always propose the contracts $E_H(R_n)$ and $E_L(R_n)$ and have them accepted by types $H$ and $L$, respectively. This deviation puts an upper bound on P's continuation cost as a function of the current contract $R_n$. The deviation will henceforth simply be referred to as the "jump."

LEMMA 1 (JUMP) *If $R_n$ is in the H-Rent configuration and P proposes the contracts $E_H(R_n)$ and $E_L(R_n)$, with $E_H(R_n)$ augmented by an arbitrarily small amount $\varepsilon > 0$, then H accepts $E_H(R_n)$ with probability 1 and L accepts $E_L(R_n)$ with probability 1. Therefore, P's continuation cost is bounded above by $\bar{Q}_n = \beta_n Q(E_H(R_n)) + (1-\beta_n)Q(E_L(R_n))$*

*Proof.* The result follows from Part iv) of Proposition 1: $E_H(R_n)$ plus any small amount gives a strictly higher utility to $H$ than what he can get under any continuation utility, and also gives him

[11] Negotiation breakdown, which is a stochastic event, can also be encoded into the players' payoff and formally emptied of its stochastic content

strictly more utility than $E_L(R_n)$. Therefore, $H$ accepts the contract with probability 1. Because $L$ strictly prefers $E_L(R_n)$ to $E_H(R_n)$ in the $H$-Rent configuration, and because the agent's type is revealed in round $n$ unless $L$ takes the strictly suboptimal contract $E_H(R_n)$, it is optimal for $L$ to accept $E_L(R_n)$. ■

LEMMA 2 *If $R_0$ is in the $H$-Rent configuration, then in any PBE, L accepts only contracts that are in the $H$-Rent configuration.*

Given any PBE, any contract sequence $\{R_n\}$ that is accepted by $L$ with positive probability (until the exogenous negotiation breakdown) will be called a **choice sequence**. When $R_0$ is in the $H$-Rent configuration, choice sequences will play a particular role: we will see that, without loss of generality, any accepted contract sequence is a choice sequence, until $H$ accepts an $H$-efficient contract. Moreover, choice sequences have several important properties. First, as indicated by Lemma 2, any choice sequence consists of contracts that are in the $H$-Rent configuration. Other properties are described by the following lemma.

PROPOSITION 2 *Suppose that $R_0$ is in the $H$-Rent configuration. Along any choice sequence $\{R_n\}$ i) $\beta_n$ converges to zero, and ii) $R_n$ converges to an $L$-efficient contract, denoted $\bar{C}_L$.*

## 4 Proof of Theorem 2

Without loss of generality, it suffices to prove the theorem when $R_0$ is in the $H$-Rent configuration: Proposition 1 already addresses the case in which $R_0$ is in the No-Rent configuration, and the $L$-Rent configuration can be proved by symmetry. Let us thus assume that $R_0 \in \mathcal{H}$. From Lemma 2, $L$ accepts only contracts in $\mathcal{H}$. Moreover, any contract $C_n$ that is only accepted by $H$ with positive probability can be replaced by an $H$-efficient contract $\tilde{C}_n$ that gives $H$ the same utility, without affecting anyone's incentive (assuming that $H$ accepts $\tilde{C}_n$ with the same probability as he was accepting $C_n$) and reduces P's cost. Therefore, we can without loss of generality focus on PBEs in which P only proposes, at each round, a number of contracts in $\mathcal{H}$, and the $H$-efficient contract that gives $H$ his continuation utility. This assumption is maintained throughout the analysis.

### ORGANIZATION OF THE PROOF

The proof of Theorem 2 (Part A) proceeds by contradiction. We suppose that there exists $\varepsilon > 0$, a decreasing sequence $\{\eta_m\}_{m\in\mathbb{N}}$ of breakdown probabilities that converges to zero, and a PBE associated to each $\eta_m$ for which $H$'s expected utility $u_H(0)$ at round 0 is below $u_H(E_H(0)) - \varepsilon$. (Throughout, $u_\theta(n)$ will denote $\theta$'s continuation utility at the beginning of round $n$.)

In what follows, we focus entirely on that sequence of $\eta$'s and corresponding PBEs. The expression "as $\eta$ goes to zero" will refer to the elements of that sequence and corresponding PBEs.[12]

The difference $w_0 = u_H(E_H(0)) - u_H(0)$ can be thought of as a *rent extraction index* for type $H$. It defines how much rent P is extracting from $H$, relative to the immediate jump: $u_H(0)$ is $H$'s continuation utility while $u_H(E_H(0))$ is the maximal utility that P may concede to give to $H$, as shown by Proposition 1, part iv).

The proof consists in the following steps.

**Step 1:** For each PBE of the sequence, we construct a choice sequence which ends at some finite round $\tilde{N}$ for which the *augmented* rent extraction index $\hat{w}_{\tilde{N}} = \max_{m\leq\tilde{N}}\{u_H(E_H(\tilde{m}))\} - u_H(\tilde{N})$ is of order $\eta$, and there exists $d > 0$ such that either a) $\beta_{\tilde{N}} \geq \eta^d$ and $w_0 \leq \hat{w}\sqrt{\eta}$, where $\hat{w} > 0$ is exogenous, or b) $\beta_{\tilde{N}} < \eta^d$. Proving that step is the object of Part I below. Of course, Case a) implies that $w_0$ could not have been greater than $\varepsilon$, for $\eta$ small enough. Therefore, it suffices to rule out Case b).

**Step 2:** Show that in Case b) there must exist a round $N \geq \tilde{N}$ such that $\hat{w}_N \leq \frac{\eta D}{2a}$ but $\hat{w}_N \geq \underline{w}\eta$ and $\beta_N \leq \eta^d$, for some positive constants $a, D, \underline{w}$. This is done in Part II.

**Step 3:** Show that at round $N$, one must have $\hat{w}_N \leq \bar{w}\eta^{1+d}$ for some $\bar{w} > 0$. This contradicts, for $\eta$ small enough, the inequality of Step 2 that involves $\underline{w}$. (Part III).

Once Part A) of Theorem 2 has been proven, showing Part B) is straightforward. The argument is at the end of the Appendix.

## Part I: Macro Level

The strategy of the proof is to build a sequence of *stages* (each consisting of finitely many rounds), and choice sequence going through these stages, with the following properties: i) at each stage, for the PBE to be profitable to P compared to an immediate jump, $H$ must accept $H$-efficient contracts with a high enough probability, which drives the posterior $\beta$ closer to zero, by a controlled amount,

[12] Without loss of generality we focus on $\varepsilon$ small enough so that the constant $D(2\varepsilon)$ defined in the Appendix (see (39)), is strictly positive.

and ii) P's potential gain, conditional on facing type $H$, shrinks geometrically across stages. This construction ends at some terminal stage, $K$, such that P's maximal potential gain on $H$ is of order $\eta$, and the posterior $\beta(K)$ is bounded above by $g^K\beta_0$ for some factor $g < 1$. When $\beta(K) > \eta^d$ for some power $d > 0$ that is judiciously chosen, this yields an upper bound on the number $K$ of stages which, by using the geometric series backwards, implies that the initial gain on $H$ must have been small as well, for $\eta$ small enough, contradicting the existence of a sequence $\{\eta_m\}$ and corresponding PBEs for which the initial rent index $w_0$ always exceeds $\varepsilon$. The ulterior parts (Parts II and III) of the proof establish that $\beta(K) > \eta^d$ is the only viable case when $\eta$ is small enough.

We begin the proof by the following observation. For any round $n$ and choice sequence up to round $n$, let $\bar{e}_n = \max\{u_H(E_H(R_m)) : m \leq n\}$ and $\bar{w}_n = \bar{e}_n - u_H(n)$.

LEMMA 3 *If $u_H(E_H(R_0)) > u_H(0) + \varepsilon$, there exists a choice sequence and a round $n_0$ such that i) $\beta_{n_0} \leq \beta_0$ and ii) $\bar{w}_{n_0} \in [\varepsilon/2, \varepsilon]$.*

Stage 1 starts at the round $n_0$ guaranteed by Lemma 3, so that $\bar{w}_{n_0} = \bar{e}_{n_0} - u_H(n_0) \in (\varepsilon/2, \varepsilon)$. Let $\hat{u}_0 = u_H(n_0)$, $\hat{e}_0 = \bar{e}_{n_0}$, and $\hat{\beta}_0 = \beta_{n_0} \leq \beta_0$. We construct the end of Stage 1 as follows. First, define $\hat{u}_1$ by

$$\frac{\hat{e}_0 - \hat{u}_0}{\hat{u}_1 - \hat{u}_0} = t > 1$$

where $t$ is a fixed threshold, greater than 1, to be determined shortly. Also let $n_1 = \inf\{n : u_H(n) \geq \hat{u}_1\}$ denote the first round at which $H$'s continuation utility exceeds the threshold $\hat{u}_1$. Because $\hat{u}_1 < \hat{e}_0$, Lemma 14, in the Appendix, guarantees that $n_1$ is finite with probability 1, as it implies that $u_H(R_n)$ must eventually exceed any utility level $\hat{u}_1$ such that $\hat{u}_1 < \max\{u_H(E_H(R_m)) : m \leq n_0\}$, along any choice sequence, as $n$ gets large enough.

Stage 1 finishes at round $n_1$. To get a control on how much the posterior must have dropped across that stage, let $\mu_0$ denote the probability, evaluated at round $n_0$, that $H$ accepts only contracts in $\mathcal{H}$ until round $n_1$ (i.e., the probability that $H$ does not fully reveal himself). Lemma 15, in the Appendix, shows that there must exist a *pushdown* choice sequence such that, upon observing that sequence up to $\hat{u}_1$, the posterior probability $\hat{\beta}_1$ of facing $H$ satisfies

$$\hat{\beta}_1 \leq \frac{\hat{\beta}_0\mu_0}{\hat{\beta}_0\mu_0 + (1 - \hat{\beta}_0)}. \tag{1}$$

At round $n_0$, P can always jump to $(E_H(R_{n_0}), E_L(R_{n_0}))$, by Lemma 1. To sustain the PBE, therefore, the *net gain* from continuing the PBE and extracting some rent from $H$, compared to the immediate jump, must outweigh the *net loss* resulting from a negotiation breakdown at an inefficient contract. While the gain only pertains to $H$, the loss concerns both $L$ and $H$. The argument below exploits only the loss on $H$. (The loss on $L$ is exploited in later parts of the proof of Theorem 2.)

We now compute an upper bound on this gain and a lower bound on the loss. Comparing these bounds will yield an upper bound on P's posterior belief of facing $H$ after the first stage, following the pushdown choice sequence. To do so, we start with the following lemma:

LEMMA 4 *Along any choice sequence, H's continuation utility at round n,* $u_H(n)$ *is nondecreasing in n, and satisfies* $u_H(n+1) - u_H(n) \leq \eta \Delta_H$ *where* $\Delta_H = \max_{C \in \mathcal{C}} u_H(C) - \min_{C \in \mathcal{C}} u_H(C)$.

*Proof.* Given the current contract $R_n$ at round $n$, let $R_{n+1}$ denote any contract chosen by $H$ with positive probability among $R_n \cup \{M_n\}$. $H$'s utility satisfies the dynamic equation[13]

$$u_H(n) = \eta u_H(R_{n+1}) + (1-\eta) u_H(n+1). \tag{2}$$

Therefore, $u_H(n)$ is a convex combination of $u_H(R_{n+1})$ and $u_H(n+1)$. Because $H$ can always hold forever on to $R_{n+1}$, in all rounds $m \geq n$, $u_H(n+1)$ is bounded below by $u_H(R_{n+1})$. Combining these observations yields $u_H(n) \leq u_H(n+1)$. Moreover, we have $u_H(n+1) - u_H(n) = \eta(u_H(n+1) - u_H(R_{n+1})$, which implies that the second claim of the lemma. The intuition for this part is simple: if the utility jump was higher between two rounds, $H$ would prefer to wait until the next round rather than accept any contract today. ■

The net gain, between rounds $n_0$ and $n_1$, is bounded above by $\hat{\beta}_0(1-\mu_0)a(\hat{e}_0 - \hat{u}_0)$ for some Lipschitz constant $a > 0$. Indeed, $\hat{\beta}_0(1-\mu_0)$ is the probability that the agent is of type $H$ and that he accepts some $H$-efficient contract at some round of the first stage. Because $H$ accepts only $H$-efficient contract that give him at least his continuation utility,[14] and because that continuation utility is nondecreasing, by Lemma 4, the smallest utility that P can give $H$ when singling him out along that first stage, is $\hat{u}_0$. By contrast, $\hat{e}_0$ is an upper bound on the utility that P provides to $H$ if chooses the immediate jump. Therefore, the maximum rent that P can extract from $H$ is $\hat{e}_0 - \hat{u}_0$. The constant $a$ is a Lipschitz constant that bounds utility differences for $H$ along the $H$-efficient curve $\mathcal{E}_H$ in terms cost differences for P along that curve. That constant is based on the cost and utility functions $Q$ and $u_H$ along $\mathcal{E}_H$ and is derived in the Appendix (Lemma 10).

Similarly, the expected net gain made after round $n_1$, but seen from round $n_0$, is bounded above by $\hat{\beta}_0\mu_0 a(\hat{e}_0 - \hat{u}_1)$, because $\hat{\beta}_0\mu_0$ is the probability of facing $H$ and of reaching round $n_1$, and $\hat{u}_1$ is the smallest utility that P must provide to $H$ at any round following $n_1$.

[13]More generally, $H$'s utility satisfies the Bellman equation $u_H(n) = \max_{R \in \{R_n\} \cup M_n} \eta u_H(R) + (1-\eta) u_H(n+1)$. Equation (2) then follows for all contracts that are optimal for $H$ in round $n$.

[14]Indeed, by accepting such contract, $H$ reveals his type, and his continuation utility is exactly the one provided by the last accepted contract, by Proposition 1, Part i).

To get a lower bound on the net loss, the intuition is that, as long as $H$ accepts contracts in $\mathcal{H}$, he is getting contracts that are inefficient, and hence costly to P relative to the immediate jump to $E_H(R_{n_0})$ (indeed, those contracts are far away from $H$'s efficiency line, since they are to the right of $L$'s efficiency line, which is itself to the right of $H$'s efficiency line). The Appendix (Lemma 16) shows that there exists a constant $D > 0$ that gives a lower bound on this loss whenever the rent index at the beginning of each state is bounded above by $2\varepsilon$, which is shown to be without loss of generality (Remark 2 in the Appendix). We also need to compute the probability that a breakdown occurs between rounds $n_0$ and $n_1$. The key, here, is to observe that $H$'s utility can only jump upwards, at each round, by at most $\eta\Delta_H$, by Lemma 4. Therefore, there must be at least $\underline{n}(1) = \lfloor(\hat{u}_1 - \hat{u}_0)/\eta\Delta_H)\rfloor$ steps to get to $\hat{u}_1$, for any choice sequence.

Therefore, the breakdown probability is bounded below by[15]

$$1-(1-\eta)^{\underline{n}(1)} = 1-\exp\left(\underline{n}(1)\ln(1-\eta)\right) \geq -\underline{n}(1)\ln(1-\eta) - \frac{1}{2}\underline{n}(1)^2(\ln(1-\eta))^2.$$

Because the gain is of order $\varepsilon$, which is small, while the loss conditional on a breakdown is of order $D$, the probability of a breakdown must be of order $\varepsilon$, which means that $\underline{n}(1)\ln(1-\eta)$ must also be small. The quadratic term is therefore negligible. Moreover, because we are focusing on the case where $\eta$ is small, $\ln(1-\eta)$ can be approximated by $-\eta$. Combining these bounds on gains and losses yields[16]

$$\beta a[(\hat{e}_0 - \hat{u}_0)(1-\mu_0) + (\hat{e}_0 - \hat{u}_1)\mu_0] \geq \beta\mu_0 D\frac{\hat{u}_1 - \hat{u}_0}{\Delta_H}. \tag{3}$$

Recall that $\hat{u}_1$ was defined in terms of an undertermined threshold $t$. We now define $t$ by[17]

$$t^2 = \frac{a + D/\Delta_H}{a} > 1.$$

With this value of $t$, we have

$$\mu_0 \leq \frac{a}{a + D/\Delta_H}\frac{\hat{e}_0 - \hat{u}_0}{\hat{u}_1 - \hat{u}_0} = t^{-1}.$$

Combining this inequality with (1) implies that, upon observing the constructed choice sequence until round $n_1$, the posterior $\hat{\beta}_1$ satisfies

$$\hat{\beta}_1 \leq \frac{\mu_0\hat{\beta}_0}{\mu_0\hat{\beta}_0 + (1-\hat{\beta}_0)} \leq \hat{\beta}_0\frac{t^{-1}}{\hat{\beta}_0 t^{-1} + (1-\hat{\beta}_0)} = g\hat{\beta}_0.$$

[15] The inequality comes from the standard inequality $1 - \exp(x) \geq -x - x^2/2$, valid for all $x \leq 0$, which may be shown as follows. The function $x \mapsto \exp(x) - 1 - x - \frac{x^2}{2}$ vanishes at 0, as do its first and second derivatives. Since its third derivative is positive (equal to $\exp(x)$), its first derivative is convex and, from the previous observations, must have a minimum at zero. This implies that the function itself is increasing and, since it vanishes at 0, that it is negative for $x \leq 0$.

[16] For expositional simplicity, the "floor" operator is dropped. This change is negligible because $\underline{n}(1)$ is large, since $\hat{u}_1 - \hat{u}_0 = \frac{1}{t}(\hat{e}_0 - \hat{u}_0) \gg \eta\Delta_H$, for $\eta$ small. That observation applies to each stage $k$ constructed: see Footnote 19.

[17] $D$ is defined independently of $t$ (and of this entire stage construction), so there is no circularity in the definition.

where $g = \frac{t^{-1}}{\beta_0 t^{-1}+(1-\beta_0)}$. (We also use the inequality $\hat{\beta}_0 \leq \beta_0$.) Because $t^{-1} < 1$, $g$ is strictly less than 1. We have thus achieved our goal of guaranteeing that the posterior $\hat{\beta}_1$ drops by some fixed factor along the first stage, for some choice sequence.

To initiate the second stage, we use the value $\hat{u}_1$ that was defined as part of Stage 1.[18] The actual value of $u_H(n_1)$ may be slightly above $\hat{u}_1$, but by no more than $\Delta_H \eta$, by Lemma 4. The level $\hat{e}_1 = \max_{m \leq n_1}\{u_H(E_H(R_m))\}$ is the maximum value that $H$ gets if P jumps at any round $m \leq n_1$ along the particular choice sequence constructed so far. Having defined $\hat{u}_1$ and $\hat{e}_1$, we define $\hat{u}_2$, similarly to the first stage, by

$$\frac{\hat{e}_1 - \hat{u}_1}{\hat{u}_2 - \hat{u}_1} = t.$$

Let $\mu_1$ denote the probability, at round $n_1$, following the observation of the pushdown choice sequence used for Stage 1, that $H$ takes a contract in $\mathcal{H}$ at all rounds $n \geq n_1$ until $\hat{u}_2$ is reached. Repeating the previous analysis, there exists a pushdown choice sequence for Stage 2 such that, upon observing that sequence up to $\hat{u}_2$, the probability $\hat{\beta}_2$ of facing $H$ satisfies $\hat{\beta}_2 \leq \frac{\hat{\beta}_1 \mu_1}{\hat{\beta}_1 \mu_1 + (1-\hat{\beta}_1)}$. Let $n_2$ denote the round at which $\hat{u}_2$ is first exceeded. By a similar analysis to the first stage, we have

$$\hat{\beta}_2 \leq \frac{\mu_1 \hat{\beta}_1}{\mu_1 \hat{\beta}_1 + (1-\hat{\beta}_1)} \leq \hat{\beta}_1 \frac{t^{-1}}{\hat{\beta}_1 t^{-1} + (1-\hat{\beta}_1)} \leq g^2 \hat{\beta}_0.$$

The value of $\hat{e}_2$ is determined by the pushdown sequence of the second stage, by $\hat{e}_2 = \max_{m \leq n_2}\{u_H(E_H(R_m))\}$ along the pushdown sequence.

By induction, this defines a sequence of stages indexed by $k$. To each stage $k$ corresponds a terminal round, $n_k$, as well as values $\hat{u}_k, \hat{e}_k$ and $\hat{\beta}_k = \beta_{n_k}$, which is P's belief at the end of the $k^{th}$ stage following the pushdown sequences. Upon observing the pushdown choice sequence across stages 1 to $k$, we get

$$\hat{\beta}_k \leq g^k \hat{\beta}_0 \leq g^k \beta_0.$$

To determine the terminal stage, let $K$ denote the smallest $k$ such that $\hat{e}_k - \hat{u}_k < \bar{W}\eta$ for some constant $\bar{W}$ such that $\bar{W} > \max\{\frac{t-1}{t}(1+\Delta_H), \frac{\hat{W}}{t\Delta_H}\}$ where $\hat{W}$ is an arbitrarily large constant.[19] Such a stage must exist, because $\hat{w}_k = \hat{e}_k - \hat{u}_k$ converges to zero, by Lemma 14, part ii). Let $\rho$ be defined by $g^{-\rho} = \frac{t}{t-1}$. Since the ratio is greater than 1, $\rho$ is strictly positive. Also let $d = \frac{1}{2}\min\{\frac{1}{\rho}, 1\} \in (0, 1/2]$.

[18] The next stage is defined only following the pushdown choice sequence that we constructed in Stage 1: what matters to us is to understand what happens along a particular choice sequence constructed by piecing together pushdown sequences constructed for each stage.

[19] The number of rounds in each stage $k \leq K$ is bounded below by $\frac{\hat{u}_k - \hat{u}_{k-1}}{\Delta_H \eta} \geq \frac{1}{t\Delta_H \eta}(\hat{e}_{k-1} - \hat{u}_{k-1}) \geq \frac{\bar{W}\eta}{t\eta\Delta_H} = \hat{W}$, which can be made arbitrarily large by choosing $\hat{W}$ appropriately. The reason for choosing $\bar{W} > \max\{\frac{t-1}{t}(1+\Delta_H)$ is explained at the beginning of Part II.

As mentioned at the outset, the key to proving Theorem 2 is the following proposition, whose proof is the object of Parts II and III.

Proposition 3 *There exists $\tilde{\eta} > 0$ such that $\hat{\beta}_K > \eta^d$ for all $\eta < \tilde{\eta}$.*

Taking Proposition 3 as given for now, we compute an upper bound on the initial rent, by backward induction. At each stage $k \leq K$, we have

$$\hat{e}_K - \hat{u}_k = (\hat{e}_K - \hat{u}_{k+1}) + (\hat{u}_{k+1} - \hat{u}_k) \leq (\hat{e}_K - \hat{u}_{k+1}) + \frac{1}{t-1}(\hat{e}_k - \hat{u}_{k+1}) \leq \frac{t}{t-1}(\hat{e}_K - \hat{u}_{k+1}).$$

By construction, moreover, $\hat{e}_K - \hat{u}_K \leq \bar{W}\eta$, which implies that

$$\hat{e}_K - \hat{u}_0 \leq \left(\frac{t}{t-1}\right)^K \bar{W}\eta. \tag{4}$$

Since $\hat{\beta}_K \geq \eta^d$ and $\hat{\beta}_K \leq g^K \hat{\beta}_0 < 1$, we must also have

$$\frac{1}{g^K}\eta^d \leq 1.$$

Combining these inequalities yields

$$\hat{e}_K - \hat{u}_0 \leq \left(\frac{t}{t-1}\right)^K \bar{W}\eta = \bar{W}\eta \left(\frac{1}{g}\right)^{\rho K} \leq \eta \bar{W} \eta^{-\rho d} \leq \bar{W}\eta^{1/2}.$$

Since $\hat{e}_K \geq \hat{e}_0$, this shows that $\hat{e}_0 - \hat{u}_0 = O(\eta^{1/2})$ which contradicts the existence of the sequence of $\{\eta_m\}$, converging to zero, and corresponding PBEs for which $\hat{e}_0 - \hat{u}_0 \in (\varepsilon/2, \varepsilon)$.

## Part II: Micro Level

This part, and the next, prove Proposition 3 above. Suppose, by contradiction, that $\hat{\beta}_K < \eta^d$. By definition of $K$, the previous stage $K-1$ must satisfy $\hat{e}_{K-1} - \hat{u}_{K-1} > \bar{W}\eta$. This implies that[20]

$$\hat{w}_K = \hat{e}_K - \hat{u}_K \geq \underline{W}\eta \tag{5}$$

where $\underline{W} = \frac{t-1}{t}\bar{W}$. Since we chose $\bar{W} > \frac{t}{t-1}(1 + \Delta_H)$, we have $\underline{W} > 1 + \Delta_H$. Combining this with Lemma 4, we obtain, for the augmented index evaluated at round $n(K)$,[21]

$$\hat{w}_K = \hat{e}_K - u_H(n(K)) \geq (\underline{W} - \Delta_H)\eta \geq \eta. \tag{6}$$

[20] This inequality comes the fact that $\hat{e}_{K-1} - \hat{u}_{K-1} = t(\hat{u}_K - \hat{u}_{K-1})$, by construction of the stages in Part I, and the fact that $\hat{e}_K \geq \hat{e}_{K-1}$.

[21] The reason for using $u_H(n(K))$ instead of $\hat{u}_K$ is that $H$'s continuation utility at round $n(K)$ is need be exactly equal to $\hat{u}_K$: it is above it up to an increment that is bounded above by $\Delta_H\eta$.

If $\hat{w}_K \leq \frac{D\eta}{2a}$, Proposition 5 (Part III) implies that $\hat{w}_K \leq \hat{w}\eta^{1+d}$, which contradicts (6) for $\eta$ small enough, ruling out this case.

The objective of the present part is to solve the remaining case in which $\hat{w}_K \in \left(\frac{D\eta}{2a}; \bar{W}\eta\right)$. We will analyze the dynamics of $\beta_n$ and $\hat{w}_n$ along some judiciously chosen choice sequence between the levels $\hat{w}_{n(K)}$ and $\frac{\eta D}{2a}$, and establish the following result.

Proposition 4 *Let $N \geq n(K)$ denote the first round for which $\hat{w}_N \leq \frac{\eta D}{2a}$. Then,*
*1.* $\frac{\eta D}{2a} - \hat{w}_N = o(\eta)$
*2.* $\beta_N = O(\eta^d)$.

The objective of this part is, therefore, to build a bridge between Parts I and III, showing that if $\hat{\beta}_K < \eta^d$, there must exist a choice sequence and a round $N$, to which the contradiction argument of Part III can be applied.

To construct a choice sequence that yields 1. and 2., we start by expressing P's IC constraint, at each round $n$. For each $R_{n+1} \in M_n \cup \{R_n\}$, let $\mu_n^\theta(R_{n+1})$ denote the probability that $\theta$ accepts $R_{n+1}$. Also let $E_\theta(n) = E_\theta(R_n)$. Because P can always jump to $(E_L(n), E_H(n))$ by Lemma 1, P's IC constraint implies, as explained below, that

$$w_n a\beta_n \geq \sum_{R_{n+1}\in(M_n\cup\{R_n\})\cap\mathcal{H}} \beta_n\mu_n^H(R_{n+1})\eta D + (1-\beta_n)\mu_n^L(R_{n+1})\eta(Q(R_{n+1}) - Q(E_L(n))) \quad (7)$$

$$= \sum_{R_{n+1}\in(M_n\cup\{R_n\})\cap\mathcal{H}} \mu_n^L(R_{n+1})\left[\beta_n\mu_n(R_{n+1})\eta D + (1-\beta_n)\eta(Q(R_{n+1}) - Q(E_L(n)))\right], \quad (8)$$

where $\mu_n(R_{n+1}) = \mu_n^H(R_{n+1})/\mu_n^L(R_{n+1})$ and $D$ is the lower bound on the loss on $H$ given in Lemma 11.[22]

The left-hand side of (7) is an upper bound on the gain, relative to the immediate jump, made on the high type: given his continuation utility $u_H(n)$, the lowest achievable cost that provides this utility is the cost of the $H$-efficient contract that gives $u_H(n)$. From the technical Lemma 11, in the Appendix, this gain is bounded above by $a(u_H(E_H(n)) - u_H(n)) = aw_n$ (that bound is computed using a 'best-case scenario' for P, in which $H$ accepts with probability 1 the $H$-efficient contract $C_n$ providing $u_H(n)$).[23] The first term of the right-hand side is the net loss on $H$ if he accepts a

[22]We can assume without loss of generality that $\mu_n^L(R_{n+1})$ is strictly positive for all $R_{n+1} \in (M_n \cup \{R_n\}) \cap \mathcal{H}$: first, if any contract in that set is not chosen with any probability, we can construct an equilibrium in which those contracts are removed. And if any contract $R'_{n+1}$ in that set is chosen only by $H$ with positive probability, then Proposition 1 implies that $H$ gets the $H$-efficient contract $C$ that gives him the same utility as $R'_{n+1}$, so that the equilibrium can be modified by having P propose $C$ instead of $R'_{n+1}$. That change reduces P's cost and does not affect incentives.

[23]This is an upper bound on the gain, since $C_n$ is the cheapest way of providing $H$ with his continuation utility.

contract in the $H$-Rent configuration and, hence, far from efficient, in case a breakdown occurs at the end of round $n$. This loss is bounded by $D$ as long as $w_n \leq 2\varepsilon$, which will be true along the choice sequence that we consider. The last term is the net loss on $L$ in case of such a breakdown.

Proposition 4 is based on the following lemma. Fix any positive integer $\bar{N}$, positive constants $\bar{\beta}$ and $\bar{w}$, as well as a small positive $\bar{\varepsilon}$. Let $y_n = u_H(E_H(n)) - u_H(R_{n+1})$. The quantity $y_n$ represents $H$'s utility gap, for any choice $R_{n+1}$, between the immediate jump and his utility in case of a negotiation breakdown at round $n$ (the breakdown occurs *after* the agent has chosen the new contract, $R_{n+1}$, which explains the index). This quantity $y_n$ is important for the analysis, because it provides a control on the increments of $w_n$ and makes sure that we do not overshoot the threshold $\frac{\eta D}{2a}$ by too much. Indeed, subtracting $u_H(E_H(n))$ from (2) and rearranging – and recalling that $w_n = u_H(E_H(n)) - u_H(n)$ – leads, along any choice sequence, to[24]

$$w_{n+1} = w_n - \eta y_n + \eta w_{n+1} + (1-\eta)(u_H(E_H(n+1)) - u_H(E_H(n))).$$

These concepts are represented on Figure 3

Lemma 5 *Consider a round $\bar{n}$ such that $\beta_{\bar{n}} \leq \bar{\beta}\eta^d$ and $w_{\bar{n}} \leq \bar{w}\eta$, and fix some positive integer $\bar{N}$ and a small number $\bar{\varepsilon} > 0$, and let $\mathcal{S}$ denote the event that the agent chooses contracts at rounds $n \in \{\bar{n}+1, \ldots, \bar{n}+\bar{N}\}$ such that $y_n = O(\eta^{d/4})$, $\beta_n \leq \beta_{\bar{n}}\bar{\varepsilon}^{-(n-\bar{n})}$, and $w_n \leq \bar{W}(\bar{N})$ for all round $n \in \{\bar{n}, \ldots, \bar{n}+\bar{N}\}$, where $\bar{W}(\bar{N})$ is independent of $\eta$. Then, for $\eta$ small enough, the probability of $\mathcal{S}$ is greater than $1 - k(\bar{N})\bar{\varepsilon}$, where $k(\bar{N})$ is independent of $\bar{\varepsilon}$ and $\eta$.*

We will now modify the analysis of Part I to apply it to a new kind of "stage" consisting of rounds $\bar{n}+1$ to $\bar{n}+\bar{N}$, where $\bar{N}$ and $\bar{\varepsilon}$ will be determined shortly. The first such stage starts at $\bar{n} = n(K)$, the second of these stages starts at $\bar{n} = n(K) + \bar{N}$, etc. These stages are different from those of Part I, because the number $\bar{N}$ of rounds in each stage is fixed and, inversely, $H$'s utility at the end of each state is not pinned down.

The analysis of Part I is modified as follows. First, notice that P's IC constraint at round $\bar{n}$, looking ahead over the next $\bar{N}$ rounds, must satisfy

$$\beta_{\bar{n}} a \left\{(1-\mu_{\bar{n}})(e_{\bar{n}} - u_H(\bar{n})) + \mu_{\bar{n}}(e_{\bar{n}} - E[u_H(\bar{n}+\bar{N})])\right\} \geq \beta_{\bar{n}}\mu_{\bar{n}} D\eta\bar{N} - \beta_{\bar{n}} O(\varepsilon),$$

[Is the last, $O(\varepsilon)$ term needed?] where $\mu_{\bar{n}}$ is the probability, seen from round $\bar{n}$, that $H$ rejects all $H$-efficient contracts between rounds $\bar{n}$ and $\bar{n}+\bar{N}$. The argument for this equation is the same as before, the only difference being that we are now taking the expectation of $u_H(\bar{n}+\bar{N})$ because we do not know its value (before, we had precisely defined the end of the stage as the first time that $u_H$

[24]Equation (36) in the Appendix shows that $w_{n+1}(1 - b\beta_{n+1}) \geq w_n - \eta y_n$, which is simpler to work with.

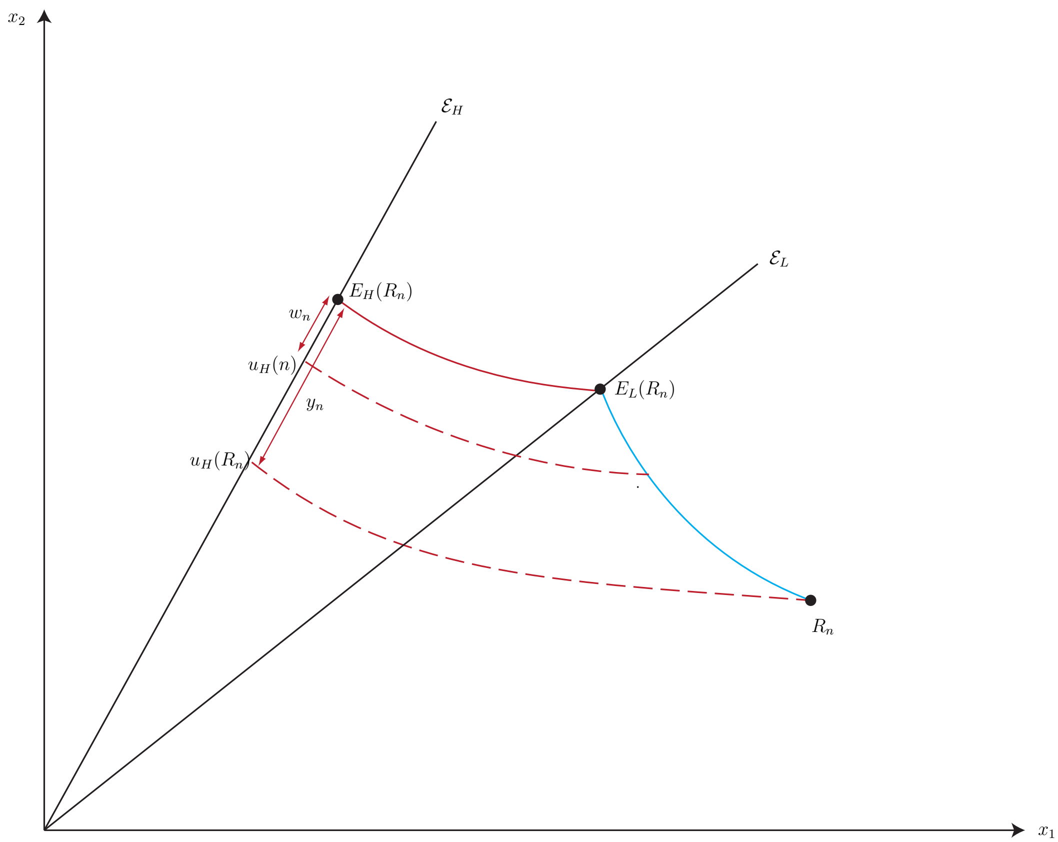


Figure 3: Concepts

crosses some level, but now $\bar{N}$ is exogenous). The loss $D$ is valid because $w_n$ is small throughout the stage with arbitrarily high probability, by Lemmas 5 and 16.

This implies that

$$\mu_{\bar{n}} \leq \frac{a(e_{\bar{n}} - u_H(\bar{n}))}{aE[u_H(\bar{n} + \bar{N}) - u_H(\bar{n})] + D\eta\bar{N}} \leq \frac{a\bar{W}}{D\bar{N}},$$

where the second inequality comes from the fact that $u_H$ is nondecreasing across all paths which implies, taking expectations, that $Eu_H(\bar{n} + \bar{N}) \geq u_H(\bar{n})$, and from the fact that $\hat{w}_{\bar{n}} \leq \bar{W}\eta$ (this inequality holds for all stages of Part II, without loss of generality, see Remark 1 below). Now let $\mu_{\bar{n}}^{\mathcal{S}}$ (resp. $\mu_{\bar{n}}^{\mathcal{B}}$) denote the probability that $H$ rejects all $H$-efficient contracts, *conditional* on the event $\mathcal{S}$ (resp. conditional on its complement, $\mathcal{B}$), and let $p_{\mathcal{S}}$ (resp. $p_{\mathcal{B}}$) the probability of $\mathcal{S}$ ($\mathcal{B}$).

We have $\mu_{\bar{n}} = p_{\mathcal{S}}\mu_{\bar{n}}^{\mathcal{S}} + p_{\mathcal{B}}\mu_{\bar{n}}^{\mathcal{B}}$. Since $p_{\mathcal{S}} \geq 1 - k(\bar{N})\bar{\varepsilon}$, we conclude that

$$\mu_{\bar{n}}^{\mathcal{S}} \leq \frac{a\bar{W}(1 + k(\bar{N})\bar{\varepsilon})}{D\bar{N}}.$$

We now choose $\bar{\varepsilon}$ and $\bar{N}$ so that this ratio is less than 1/2: first choose $\bar{N}$ so that $\frac{a\bar{W}}{D\bar{N}} < 1/4$, then choose $\bar{\varepsilon}$ so as to get $\frac{a\bar{W}k(\bar{N})\bar{\varepsilon}}{D\bar{N}} < 1/4$.

Proceeding as in Part I, there must exist a pushdown choice sequence within $\mathcal{S}$ such that the ex post probability that $H$ has not chosen an $H$-efficient contract is weakly less than $\mu_{\bar{n}}^{\mathcal{S}}$. Therefore, along that sequence, we have i) $y_n$ small, and ii) $\beta_{\bar{n}+\bar{N}} \leq \frac{\beta_{\bar{n}}}{2}$: in other words, we have built a sequence over $\bar{N}$ rounds, starting from $\bar{n}$, such that $y_n$ and $\beta_n$ stay small, and ends up *smaller* than at the beginning.[25]

Now starting from round $n(K)$, we build a sequence of stages of this kind. Because $\hat{w}_n$ converges to zero (Lemma 14, part ii)), it will eventually cross $\frac{D\eta}{2a}$. Let $N$ denote the first round at which $\hat{w}_N$ drops below that threshold. From (36), in the Appendix, we have

$$w_{n+1}(1 - b\beta_{n+1}) \geq w_n - \eta y_n.$$

The stages were constructed in such a way that $y_n = O(\eta^{d/4})$ and $\beta_n$ remains $O(\eta^d)$ at each round of each stage. Applying these observations to round $N - 1$, we obtain

$$w_N - w_{N-1} \geq -o(\eta).$$

Finally, we have

$$\begin{aligned} \hat{w}_N - \hat{w}_{N-1} &= (w_N - w_{N-1}) + (\max\{e_k : k \leq N\} - e_N) - (\max\{e_k : k \leq N-1\} - e_{N-1}) \\ &\geq -o(\eta) - (e_N - e_{N-1}) \end{aligned}$$

The difference in parentheses is bounded above by $\frac{\alpha\beta_{N-1}}{1-\beta_{N-1}}w_{N-1} = o(\eta)$, from (32). Since $\hat{w}_{N-1} > \frac{\eta D}{2a}$, by construction of $N$, we conclude that

$$\hat{w}_N \geq \hat{w}_{N-1} - o(\eta) \geq \frac{\eta D}{2a} - o(\eta) \geq \frac{\eta D}{3a}.$$

This concludes the proof of Proposition 4, and implies that we have reached a round $N$ such that $\hat{w}_N$ is above $\hat{w}\eta$ for some $\hat{w} > 0$ independent of $\eta$ and $\beta_N = O(\eta^d)$. Part III will show that this is impossible.

[25] Remark $\beta_n$ can increase up to $\beta_{\bar{n}}\bar{\varepsilon}^{-\bar{N}}$ along such a stage. However, because $\bar{N}$ is fixed, it still remains of order $O(\eta^d)$ along the sequence, and has in any to drop below $\beta_{\bar{n}}/2$ at round $\bar{n} + \bar{N}$ for the pushdown sequence.

Remark 1 *It is a priori possible that $\hat{w}_n$ goes above $\bar{W}\eta$ at the end of some stage constructed in this section. If that happens, the bound $D = D(2\varepsilon)$ need not be valid. At the end of such stage, should it occur, $\beta_n$ is of order $\eta^d \leq \beta_0$. We can restart the stages of Part I as if $n$ were the initial round. Since $\beta$ decreases along the stages of Part I, we have to reach again a round at which $\hat{w}_n$ drops below $\bar{W}\eta$. At that point we necessarily have $\beta_n \leq \eta^d$. Because $\hat{w}_n$ converges to zero along any sequence (by Lemma 14), and thus also along the sequences constructed through Parts I and II, the back and forth between stages of Part I and Part II has to stop in finite time at some round $N$ of the type above, i.e., with $\hat{w}_N \in (\hat{w}\eta, \frac{\eta D}{2a})$ and $\beta_N \leq \eta^d$. Proposition 5 of Part III then shows a contradiction with such stage.*

## Part III: Asymptotic Level

The goal of this section is to show the following proposition:

Proposition 5 *If one reaches a round $N$ such that $\beta_N \leq \eta^d$ and $\hat{w}_N \leq \frac{\eta D}{2a}$, then $\hat{w}_N \leq \hat{w}\eta^{1+d}$, for some constant $\hat{w} > 0$ and $\eta$ small enough.*

The proof will proceed in three steps: 1) show that, starting from such a round, one can build a choice sequence along which $\beta_n$ is decreasing and a version of P's ex ante IC constraint is also satisfied ex post, at each round. 2) Show that along such sequence, one must necessarily have $w_n \leq \eta\beta_n c$ for all $n \geq N$, for some $c > 0$ that depends on $a, D$ but not on $\eta$. 3) Conclude with the desired inequality for $\hat{w}_N$ by showing that $\hat{w}_N - w_N = O(\eta^{1+2d})$ (Proposition 7). Combining the last two observations, along with the fact that $\beta_N \leq \eta^d$ will then prove Proposition 5.

The idea of the proof is again to derive, for a given PBE, an equation for the dynamics of the posterior belief $\beta_n$, based on an incentive compatibility condition for P. This time, however, there are no stages: the equation is used for each single round, and exploits the losses on both types, $H$ and $L$. For $\eta$ small enough, this equation is then showed to contradict the convergence of $w_n$ to zero, which has to hold for any PBE, by Proposition 2.

In the Appendix (Lemma 11), it is shown that

$$u_H(E_H(n+1) - u_H(E_H(n))) \geq -\hat{b}\beta_{n+1}w_{n+1} \tag{9}$$

for some constant $\hat{b} > 0$. That equation comes from two observations. First, $L$'s utility from the current contract $R_n$ cannot decrease by too much between consecutive rounds. Indeed, recall that $\beta_{n+1}$ is the probability of facing $H$ in round $n+1$, while $w_{n+1}$ is a measure of the maximum rent that P can extract from $H$ at round $n+1$. If the product $\beta_{n+1}w_{n+1}$ is small, it means that, comes round $n+1$, P has very little incentive to extract rents from $H$, which implies, intuitively, that his continuation strategy must be similar to what he would do if he only faced $L$, namely to jump to the $L$ efficient contract $E_L(R_{n+1})$, which gives $L$ utility $u_L(R_{n+1})$. Anticipating this, however, $L$ is willing to forgo the current contract $R_n$ only if $R_{n+1}$ gives him a utility that is not much lower than $R_n$. The second observation is that the $H$-efficient contracts $E_H(n)$ and $E_H(n+1)$ are constructed based on the utility that $L$ gets from $R_n$ and $R_{n+1}$. A simple Lipschitz property, established in the Appendix, then yields (9).

We also use the following relation between $w_n$ and $w_{n+1}$ (this is (37), in the Appendix):

$$w_{n+1} = w_n - \eta y_n + \eta w_{n+1} + (1-\eta)(u_H(E_H(n+1)) - u_H(E_H(n))).$$

Combining this inequality (35), in the Appendix, yields

$$(1-\eta)w_{n+1} \geq w_n - \eta y_n - \hat{b}\beta_{n+1}w_{n+1}. \tag{10}$$

In Part I, we focused on *pushdown* sequences, in order to relate P's ex ante incentive compatibility constraint with his ex post belief's about the agent. Here, similarly, we need to focus on particular choices by the agent that play a similar role.

To express P's IC constraint, recall from Part II that

$$w_n a\beta_n \geq \sum_{R_{n+1}\in(M_n\cup\{R_n\})\cap\mathcal{H}} \mu_n^L(R_{n+1})\left[\beta_n\mu_n(R_{n+1})\eta D + (1-\beta_n)\eta(Q(R_{n+1}) - Q(E_L(n)))\right], \tag{11}$$

where $\mu_n(R_{n+1}) = \mu_n^H(R_{n+1})/\mu_n^L(R_{n+1})$ and $D$ is the lower bound on the loss on $H$ given in Lemma 11.

In particular, the RHS of (11) is a convex combination of terms indexed by $R_{n+1}$, and there must exist $R_{n+1} \in (M_n \cup \{R_n\}) \cap \mathcal{H}$ such that

$$w_n a\beta_n \geq \beta_n\mu_n(R_{n+1})\eta D + (1-\beta_n)\eta(Q(R_{n+1}) - Q(E_L(n))). \tag{12}$$

Therefore, there exists a *continuation* of that choice sequence that satisfies (12) for all $n \geq n(K)$. In what follows we entirely focus on that sequence, which will be called a **regular** choice sequence. To simplify notation let $N = n(K)$.

We split the IC constraint (12) into two parts, in a way that allows us to modify P's belief $\beta_n$ to achieve tractability. The first step is to note that $Q(R_{n+1})-Q(E_L(n)) \geq Q(E_L(n+1))-Q(E_L(n)) \geq -k\beta_{n+1}w_{n+1}$, where the second inequality comes (34) of Lemma 11 in the Appendix. To simplify notation, let $\mu_n = \mu_n(R_{n+1})$. Equation (12) implies that

$$\beta_n w_n a \geq \beta_n\mu_n\eta D - \eta k\beta_{n+1}w_{n+1},$$

which may be re-expressed as

$$\mu_n \leq \frac{w_n a}{\eta D} + k \frac{\beta_{n+1}}{\beta_n D} w_{n+1}. \tag{13}$$

The first step, in order to exploit this equation, is to show that $\beta_n$ is small and decreasing along that sequence for $n \geq N$. More precisely, we will prove the following lemma.

LEMMA 6 *There exists $\hat{\eta} > 0$ and $\hat{w} > 0$ such that for $\eta < \hat{\eta}$ and $n \geq N$, i) $\beta_n$ is decreasing in $n$, ii) $\mu_n$ is bounded above by $3/4$, and iii) $w_n$ is bounded above by $\hat{w}\eta$.*

As seen in Lemma 6 $\frac{\beta_{n+1}}{\beta_n} \leq \mu_n(1+\epsilon) \leq \frac{3}{4}(1+\epsilon)$. Hence, the second term in the right-hand side of equation (13) is of order $w_{n+1}$, while its first term is of order $\frac{w_n}{\eta}$. Since $w_{n+1}$ is bounded above by $w_n\left(1 + \frac{\alpha\beta_n}{1-\beta_n}\right)$, from (33) in the Appendix, the last term is negligible compared to the first one. Therefore, by slightly increasing $a$, whose specific value does not matter in any case for the proof, we get

$$\mu_n \leq \frac{w_n a}{\eta D}, \qquad (IC_n^{LL}) \tag{14}$$

Moreover, (12) also implies that

$$\beta_n w_n a \geq (1-\beta_n)\eta(Q(R_{n+1}) - Q(E_L(n))) \qquad (IC_n^{LH}) \tag{15}$$

**Relaxed Problem**

We now introduce a relaxed problem, for rounds $n \geq N$, constructed in two steps. First, P's incentive constraint at each round $n \geq N$ is weakened to the coupled inequalities (14) and (15). Clearly, if the initial sequence of contracts proposed by P and choices made by the agent formed a PBE, then the problem in which P's IC constraint is replaced by these two weaker equations must also have a solution.

The second step is as follows. If, starting from round $N$, one decreases $\mu_N^L(R_{N+1})$ so that $\mu_N = \mu_N(R_{N+1}) = \frac{\mu_N^H(R_{n+1})}{\mu_N^L(R_{n+1})}$ is increased so as to satisfy (14) as an equality,[26] then then the posteriors $\beta_n$'s at all rounds $n \geq N+1$ are increased as a result, by Bayesian updating. Therefore, P's IC constraint (15) at future rounds is preserved (in fact, looser).[27] After this is done, one can increase $\mu_{N+1}(R_{N+2})$ (by decreasing $\mu_N^L(R_{N+2})$) so as to make $IC_{N+1}^{LL}$ tight, preserving IC constraints (15) for $n \geq N+2$, (without perturbing P's IC constraints at round $N$), etc. To simplify the notation,

[26]The interpretation would be that the principal erroneously becomes too optimistic, conditional on observing $R_{n+1}$, about the posterior probability of facing $H$.

[27]At this point, we totally ignore the IC constraints outside of the regular sequence. Our point is only to show that a contradiction along that specific choice sequence.

let $\mu_n = \mu_n(R_{n+1})$. Increasing $\mu_n$ inductively for all $n \geq N$, this shows that the regular choice sequence, along with the new mixing probabilities for $L$ and resulting belief sequence $\{\beta_n\}_{n\geq N+1}$ is a solution to the relaxed problem with

$$\mu_n = \frac{w_n a}{\eta D} \tag{16}$$

The resulting regular sequence, with the new mixing probabilities and beliefs, is called the *relaxed* version of the initial sequence. Since only the agent's mixing probabilities were changed, the contracts proposed, the sequence $w_n$, and the limits $\bar{C}_L, \bar{C}_H$ are unchanged. In particular, $H$'s and $L$'s incentives are unchanged. Since, also, $w_n$ goes to zero, (14) implies that $\mu_n$ goes to zero and, hence, that $\beta_n$ goes to zero along the relaxation.

As just argued, the relaxed problem, which consists of the incentive compatibility and dynamic equations arising along the relaxed regular sequence, must have a solution. The remainder of the proof shows that this is impossible, which will yield the desired contradiction. We do this by building a dynamic equation for $\beta_n$, in the relaxed problem, and show from that equation that $w_n$ cannot converge to zero along the relaxed regular sequence, contradicting Proposition 2.

Multiplying both sides of (10) by $\frac{a}{\eta D}$, and using (16), we obtain for $n \geq N$

$$\mu_{n+1} \geq (1-\eta)\mu_{n+1} = \mu_n - \frac{a}{D} y_n - \tilde{b}\beta_{n+1}. \tag{17}$$

for some constant $\tilde{b} > 0$ (also using that $w_n \leq \hat{w}$ for $n \geq N$, from Part iii) of Lemma 6).

The Bayesian updating equation

$$\beta_{n+1} = \frac{\beta_n \mu^H(R_{n+1})}{\beta_n \mu^H(R_{n+1}) + (1-\beta_n)\mu^L(R_{n+1})} = \frac{\beta_n \mu_n}{\beta_n \mu_n + (1-\beta_n)}$$

implies that[28]

$$\frac{\beta_{n+1}}{\beta_n} \geq \mu_n \geq \frac{\beta_{n+1}}{\beta_n} - \mu_n \beta_n + \mu_n O(\beta_n^2) \geq \frac{\beta_{n+1}}{\beta_n} - \beta_{n+1} + o(\beta_{n+1}). \tag{18}$$

In the Appendix (equation (47)), we will show that $y_n^2 \leq \frac{\bar{A}\beta_{n+1}}{1-\beta_0}$. Intuitively, this equation means that the loss on $L$ in round $n$, which is of order $\eta y_n^2$, must be smaller than the gain on $H$, which is of order $\beta_n w_n$ (i.e., the probability of facing $H$ times the maximum gain).[29]

Combining the upper bound on $y_n^2$ with (17) and (18), we obtain the following dynamic equation for $\beta_n$, for all $n \geq N$:

$$\frac{\beta_{n+2}}{\beta_{n+1}} \geq \frac{\beta_{n+1}}{\beta_n} - c\sqrt{\beta_{n+1}} - (1+\tilde{b})\beta_{n+1} \tag{19}$$

[28] We have $\frac{\beta_{n+1}}{\beta_n} = \mu_n \frac{1}{1-\beta_n(1-\mu_n)} = \mu_n(1+\beta_n(1-\mu_n)) + \mu_n O(\beta_n^2)$. Rearranging yields the second inequality.

[29] Dividing by $\eta$, we get $y_n^2 \leq C\beta_n w_n/\eta$ for some constant $C$. Since $w_n/\eta$ is proportional to $\mu_n$ and $\mu_n\beta_n$ is roughly equal to $\beta_{n+1}$, this gives some idea for how the equation was derived.

where $c = \frac{a}{D}\sqrt{\frac{\bar{A}}{1-\beta_0}}$. For $\beta_{n+1}$ small enough, the last term is negligible compared to the next to last term, because $\sqrt{\beta_{n+1}} \ll c$. Therefore, by slightly increasing the value of $c$, whose precise value does not affect the proof, we obtain

$$\frac{\beta_{n+2}}{\beta_{n+1}} \geq \frac{\beta_{n+1}}{\beta_n} - c\sqrt{\beta_{n+1}}. \tag{20}$$

Let $q_n = \frac{\beta_{n+1}}{\beta_n}$. We have $\prod_0^n q_k = \frac{\beta_{n+1}}{\beta_0}$. (20) may be rewritten as

$$q_{n+1} \geq q_n - c'\sqrt{\Pi_0^n q_k} \tag{21}$$

where $c' = \sqrt{\beta_0}c$. Because $q_n$ is proportional to $w_n$ and hence must converge to zero as $n$ goes to $\infty$. The rest of this section shows, however, that $\{q_n\}_{n\in\mathbb{N}}$ cannot converge to zero. Let $\hat{c} = \frac{4c^2 D}{a}$.

Proposition 6 *Along the regular choice sequence, we have $w_n \leq \hat{c}\eta\beta_n$ for all $n \geq N$.*

*Proof.* The proposition is based on the following two lemmas, which are proved in the Appendix.

Lemma 7 *Suppose that there exist $\hat{N} > N$ such that*

$$\beta_{\hat{N}+1} \geq 4c^2\beta_{\hat{N}}^2, \tag{22}$$

$$\beta_{\hat{N}}^{1/4} \leq \frac{1}{2\sqrt{c}}, \tag{23}$$

*Then,*

$$\lim \inf_{n\to+\infty} \frac{q_{n+1}}{q_n} \geq 1.$$

Lemma 8 *Suppose that $\{q_n\}$ is a strictly positive sequence such that*

$$q_n - q_{n+1} \leq c'\sqrt{\Pi_0^n q_k}$$

*and* $\liminf_n q_{n+1}/q_n \geq 1$. *Then, $\{q_n\}$ does not converge to zero.*

To conclude the proof of Proposition 6, suppose that there exists $\hat{N} \geq N$ such that $w_{\hat{N}} > \hat{c}\eta\beta_{\hat{N}}$. From (16), this implies that $\mu_{\hat{N}} > \frac{a}{D}\hat{c}\beta_{\hat{N}}$, and from the first inequality of (18), this implies (using the definition of $\hat{c}$), that equation (22) holds for $\hat{N}$. Moreover, from Lemma 6, $\beta_{\hat{N}}$ clearly satisfies (23), for $\eta$ small enough. Therefore the hypotheses of Lemma 7 are satisfied and, hence, $\liminf_{n\to+\infty} \frac{q_{n+1}}{q_n} \geq 1$. Combining this with Lemma 8 then implies that $w_n$ cannot converge to zero, which contradicts Proposition 2, since $w_n$ converges to zero along any choice sequence. ■

Proposition 7 *There exist $\hat{w} > 0$ and $\bar{\eta} > 0$ such that $\hat{w}_N - w_N \leq \hat{w}\eta^{1+2d}$ for all for $\eta \leq \bar{\eta}$.*

*Proof.* Recalling the definition of $\bar{C}_H$ as the $H$-efficient contract that provides $H$ with its asymptotic utility $\lim_n u_H(n)$, we have

$$u_H(\bar{C}_H) - u_H(E_H(N)) = \sum_{n \geq N} u_H(E_H(n+1)) - u_H(E_H(n)) \leq 2\alpha \sum_{n \geq N} \beta_n w_n,$$

where the last inequality comes from (32). From Proposition 6, we have $w_n \leq \hat{c}\eta\beta_n$ for all $n \geq N$. Therefore,

$$u_H(\bar{C}_H) - u_H(E_H(N)) \leq \tilde{K}\eta \sum_{n \geq N} \beta_n^2,$$

where $\tilde{K} = 2\alpha\hat{c}$.

We have $\beta_{n+1} \leq 2\mu_n\beta_n = \frac{2aw_n}{D\eta}$, by (16). Using again the inequality $\frac{w_n}{\eta} \leq \hat{c}\beta_n$, which holds for all $n \geq N$, we have, letting $\hat{K} = 2a\hat{c}/D$,

$$\beta_n \leq \beta_N \prod_{k=N+1}^{n-1} (\hat{K}\beta_k) \leq \beta_N (\hat{K}\beta_N)^{n-N}.$$

For $\beta_N < \frac{1}{\sqrt{2}\hat{K}}$, this implies that

$$\sum_{n \geq N} \beta_n^2 \leq \sum_{n \geq N} \beta_N^2 2^{-(n-N)} = 2\beta_N^2.$$

We then obtain

$$u_H(\bar{C}_H) - u_H(E_H(N)) \leq 2\tilde{K}\eta\beta_N^2 \leq 2\tilde{K}\eta^{1+2d}, \tag{24}$$

where the last inequality comes from the fact that $\beta_N \leq \eta^d$.

To conclude, note that $\hat{w}_N - w_N = \max\{e_k : k \leq N\} - e_N = \max\{u_H(E_H(k)) : k \leq N\} - u_H(E_H(N))$. Since $\max\{u_H(E_H(k)) : k \leq N\} \leq u_H(\bar{C}_H)$, by an argument that is similar to the proof of Lemma 14), (24) yields the result. ∎

# 5 Discussion

TBA

# Appendices

## A Proof of Theorem 1

It suffices to prove the result for $R_0$ in the $H$-Rent configuration and $\beta_0 \in (0,1)$: the degenerate prior and No-Rent cases obtain as direct consequences of Proposition 1, and the $L$-Rent case obtains by symmetry of the $H$-Rent case. The proof proceeds in two steps:

- Step 1 - Prove the existence of an equilibrium in an auxiliary game played by P and $H$.
- Step 2 - Construct a strategy profile of the original game based on the equilibrium established in Step 1, and verify that it defines a PBE of the original game.

### Step 1: Auxiliary game

The game starts with a contract $R_0 \in \mathcal{H}$, where $\mathcal{H} \subset \mathcal{C}$ is the set of contracts in the $H$-Rent configuration, and a belief $\beta \in (0,1)$. For this auxiliary game, $\beta$ is just a parameter of the payoff functions and devoid of its interpretation as a belief.

The auxiliary game is a dynamic game with infinitely many rounds. At each round $n$, starting with the state $R_n$, P proposes new contracts $R_{n+1} \in \mathcal{H}$ and $C_n \in \mathcal{E}_H$ subject to the constraints

$$u_L(R_{n+1}) \geq u_L(R_n) \tag{25}$$

$$u_H(C_n) \geq u_H(R_n) \tag{26}$$

$H$ then chooses a number $\mu_n \in [0,1]$. (The interpretation is that $H$ accepts $R_{n+1}$ with probability $\mu_n$ and $C_n$ with probability $(1-\mu_n)$. For this auxiliary game, however, $\mu_n$ is just an action deterministically affecting payoffs.)

The principal's cost, for a given strategy pair $\{R_n, C_n\}$ and $\{\mu_n\}$ is

$$\begin{aligned}\mathcal{Q}(\{R_n, C_n\}, \{\mu_n\}) = &\sum_{n\geq 0} Q(C_n)\beta(1-\eta)^n(1-\mu_n)\Pi_{k=0}^{n-1}\mu_k \\ &+ \sum_{n\geq 0} Q(R_{n+1})\left(\beta(1-\eta)^n\eta\Pi_{k=0}^{n}\mu_j + (1-\beta)(1-\eta)^n\eta\right)\end{aligned} \tag{27}$$

$H$'s payoff is

$$\mathcal{V}(\{R_n, C_n\}, \{\mu_n\}) = \sum_{n\geq 0} u_H(C_n)(1-\eta)^n(1-\mu_n)\Pi_{k=0}^{n-1}\mu_k + \sum_{n\geq 0} u_H(R_{n+1})(1-\eta)^n\eta\Pi_{k=0}^{n}\mu_n \tag{28}$$

These payoffs correspond to the expected cost and utility that P and $H$ would obtain in an equilibrium of the original game in which P proposes two contracts at each round, the breakdown probability is $\eta$, $\{\mu_n\}$ is the mixing strategy of $H$, $L$ always accepts $R_{n+1}$, and the initial probability of facing $H$ is equal to $\beta$.

LEMMA 9 *For any initial $R_0$ and $\beta \in (0,1)$, there exists a perfect equilibrium of the auxiliary game*

*Proof.* To apply Theorem 1 of Harris (1985), we need to check Assumptions 1–5 of that theorem. The payoff function of the principal is simply the negative of his cost, $\mathcal{Q}$. P's (unconstrained) action set in round $n$ is $S_{Pn} = \mathcal{H} \times \mathcal{E}_H$, while $H$'s action space is $S_{Ln} = [0,1]$ which are both compact and Hausdorff spaces. Hence, Assumptions 1 and 2 are satisfied. P's feasible set at each round $n$, as defined by the constraints (25) and (26), is closed and depends continuously on the current state. Therefore, the set $\mathcal{L}$ of feasible sequences is closed in $\mathcal{S} = \times_n (S_{Pn} \times S_{Ln})$ endowed with the product topology, and the set of feasible actions in round $n$ depends continuously on past play. Thus, Assumptions 3 and 4 are satisfied. Finally, the payoffs $-\mathcal{Q}$ and $\mathcal{V}$ are clearly continuous on their domain $\mathcal{L}$, so Assumption 5 is satisfied as well. The result follows. ■

**Step 2: Equilibrium of the original game**

Starting from $R_0 \in \mathcal{H}$ and a belief $\beta_0 \in (0,1)$, consider the following strategies

At each round $n$:

- P proposes the contracts $(C_n, R_{n+1})$ corresponding to the auxiliary game
- In equilibrium, $L$ accepts $R_{n+1}$ with probability 1, while $H$ accepts $R_{n+1}$ with probability $\mu_n$ and $C_n$ with probability $(1-\mu_n)$
- If P deviates by proposing a contract $R_{n+1}$ such that $u_L(R_{n+1}) < u_L(R_n)$, $L$ rejects that contract with probability 1 and if the agent accepts $\beta_n$ jumps to 1. $H$ randomizes between $C_n$ and $R_n$ as if he had been offered $C_n$ and $\tilde{R}_{n+1} = R_n$ in the auxiliary game
- If the agent deviates by rejecting both $C_n$ and $R_{n+1}$ (hence, holding on to $R_n$), $\beta_n$ jumps to 1
- If P deviates by proposing more than two contracts, $L$ picks with probability 1 the contract $R_{n+1}$ that gives him the highest utility, if that utility is weakly greater than $u_L(R_n)$ and, among those, the on that is closest to $L$-efficiency, and rejects everything otherwise. $H$ randomizes between $R_{n+1}$ (if it exists) and the contract $C_n \neq R_{n+1}$ that gives him the highest utility and is closest to $H$-efficiency among those, according to the same mixing distribution as the one obtained under the auxiliary equilibrium if P had only proposed $R_{n+1}$ and the

contract $\tilde{C}_n$ that is $H$ efficient and gives $H$ the same utility as $C_n$. Continuation play is the same as if P had only proposed two contracts, $C_n, R_{n+1}$ as in the auxiliary equilibrium. Finally, if the agent picks any contract other than the two contracts described here, P assigns probability 1 to $H$.

We verify that this construction generates a PBE of the original game. Consider, first, the optimality of $L$'s strategy. It is clearly suboptimal for $L$ to accept $C_n$: that results in $\beta_n$ jumping to 1 and in $L$ getting utility $u_L(C_n)$, which is strictly less than his equilibrium payoff (given that $R_n$ is in the $H$-Rent configuration). From (25), $L$'s continuation utility is weakly increasing along the equilibrium path. If $L$ rejects $R_{n+1}$ and holds on to $R_n$, his continuation payoff is bounded above by $u_L(R_n)$, which cannot be a strict improvement.[Explain last point] If $P$ proposes more than two contracts $L$'s strategy is optimal: if he picks any other contract $C_n$, P will believe him to be type $H$ and jump to the $H$ efficient contract that gives $H$ the same utility as $C_n$, and thus a lower utility to $L$ than the one maximizing his utility among all the contracts being proposed.

From (26), $u_H(C_n) \geq u_H(R_n)$. Therefore, if $H$ holds on to $R_n$ his continuation utility is equal to $u_H(R_n)$, which is weakly dominated by taking $C_n$. Moreover, given that $H$ randomizes between $C_n$ and $R_{n+1}$, his expected payoff is given by (28), and by perfection of the auxiliary equilibrium, the strategy $\{\mu_n\}$ is a best response to the sequence of contracts. If P proposes more than 2 contracts, then $H$ cannot benefit from choosing a third contract $C_n$, other than the two identified in the strategy prescribed to him in such case. Indeed, any such choice would lead P to believe that he faces $H$, and cause him to propose the $H$-efficient contract that gives $H$ the same utility as $C_n$. This is weakly dominated by accepting the contract that gives $H$ his highest utility and reveals his type.

Finally, consider the optimality of P's strategy. By construction of the auxiliary equilibrium, P's strategy is optimal among all strategies that propose contracts $(R_{n+1}, C_n)$ satisfying (25) and (26). If P proposes more than two contracts with at least one contract that gives $L$ weakly more than $u_L(R_n)$, the result, given $L$ and $H$'s response in that case is clearly equivalent to proposing only two contracts (only, proposing a contract $C_n$ that is $H$-efficient is actually better), and that deviation cannot be profitable by definition of the auxiliary equilibrium.

If P deviates by proposing a contract $R_{n+1}$ such that $u_L(R_{n+1}) < u_L(R_n)$, it is optimal for $L$ to reject, since if he accepts $\beta_n$ jumps to 1 and $L$ gets a continuation payoff bounded above by $u_L(R_{n+1})$. No matter what other contract $C_n$ $H$ chooses with positive probability, there is another deviation which consists in proposing $\tilde{R}_{n=1} = R_n$ and the $H$-efficient contract $\tilde{C}_n$ that gives $H$ utility $u_H(C_n)$, which weakly reduces P's immediate cost and is consistent with the auxiliary equilibrium. The continuation play is assumed to be the one following that proposal in the auxiliary equilibrium.[Conclude]

## B Proofs of Section 3

PROOF OF PROPOSITION 1

Part i) Let $\bar{u}$ denote the agent's maximal expected payoff, given his type $\theta$, over all possible continuation PBEs starting from $R_0$ at which P puts probability 1 on type $\theta$, and let $u = u_\theta(R_0)$. By time homogeneity, $\bar{u}$ will be the same in the next round if the agent rejects new offers from P in round 0 and the renegotiation stage is not interrupted, by Assumption 1. Suppose by contradiction that $u < \bar{u}$. If the agent rejects any given offer, his continuation payoff is bounded above by $\tilde{u} = \eta u + (1-\eta)\bar{u} < \bar{u}$. Therefore, the agent is willing to accept anything above $\tilde{u}$, showing that $\bar{u} \leq \tilde{u}$, a contradiction (as is clear, a deviation where P proposes the $\theta$-efficient contract that gives utility in $(\tilde{u}, \bar{u})$ is strictly beneficial for P, given the concavity of the agent's utility). Let $\underline{Q}$ denote the cost of the $\theta$-efficient contract, $\underline{C}$, that gives $\theta$ utility $u$. Clearly, any PBE must cost exactly $\underline{Q}$, otherwise P has a profitable deviation which is to propose the $\theta$-efficient contract that gives $\theta$ slightly more than $u$ and costs less than the PBE. Moreover, the only way of achieving $\underline{Q}$ is to propose $\underline{C}$ in the first round and have it accepted with probability one.

Part ii) Suppose without loss that $\theta = L$ (the opposite case is treated identically). Let $u_L = u_L(R_0)$ and $u_H = u_H(R_0)$. Also let $\bar{u}_H(\beta)$ denote the supremum utility that $H$ can achieve over any continuation PBE starting from $R_0$ when P assigns probability $\beta$ to $H$, and let $\bar{u}_H = \sup_{\beta\in[0,1]} \bar{u}_H(\beta)$. Suppose by contradiction that $\bar{u}_H > u_H$. Then, for any small $\varepsilon > 0$, there exists $\bar{\beta}$ and an associated PBE for which $H$'s continuation utility is above $\bar{u}_H - \varepsilon > u_H$. For that PBE, because $L$ gets at least $u_L$ and $C$ is $L$-efficient, $\bar{Q}_L \geq Q$, where $Q = Q(R_0)$, and $\bar{Q}_L$ is the expected cost under that PBE conditional on facing $\theta_L$. Since not proposing any new contract is always feasible for P, and costs $Q$, the continuation cost $\bar{Q}_H$ conditional on facing $H$ must satisfy $\bar{Q}_H \leq Q$. Suppose that P deviates from that PBE by proposing the $H$-efficient contract that gives $\theta_H$ utility $\bar{u}_H - \varepsilon - \epsilon$, for arbitrarily small $\epsilon$. Because, for small enough $\varepsilon$ and $\epsilon$, $\bar{u}_H - \varepsilon - \epsilon > \eta u_H + (1-\eta)\bar{u}_H$, $H$ accepts this proposal with probability 1. For any strategy that $\theta_L$ chooses and continuation equilibrium, this proposal strictly reduces P's expected cost, yielding a contradiction. This shows that $\bar{u}_H(\beta) = u_H$ for all $\beta$.[WHAT IF $\beta = 0$?] To conclude, suppose that P proposes the $H$-efficient contract that gives $H$ utility $u_H + \epsilon$, for $\epsilon$ arbitrarily small. $H$ accepts that contract regardless of $L$ strategy, and P achieves at least (and hence, exactly) the optimal cost under full commitment.

Part iii) Suppose without loss that $Q_L \geq Q_H$, where $Q_\theta = Q(E_\theta(R_0))$(the other case is proved symmetrically). Let $\bar{Q}$ denote the maximal expected cost incurred by P over all PBEs and beliefs

$\beta \in [0,1]$, starting from $R_0$. We start by showing that $\bar{Q} \leq Q_L$. Suppose by contradiction that $\bar{Q} > Q_L$ and consider any PBE that achieves $\bar{Q}$.[30] Now suppose that P deviates by proposing the pair $\tilde{C}_L, \tilde{C}_H$ of contracts such that $\tilde{C}_\theta$ is efficient for $\theta$ and costs $\bar{Q} - \varepsilon$ for some $\varepsilon$ arbitrarily small compared to $\eta$. Those contracts maximize each type's utility subject to costing P at most $\bar{Q} - \varepsilon$. Because these contracts are efficient and incentive compatible, Part ii) guarantees that no type ever chooses the contract meant for the other type. Moreover, no matter what belief and continuation PBE follows rejection of these contracts, the continuation cost is by construction less than $\bar{Q}$. This means that there is one type of the agent who must be getting a lower payoff if he rejects the contract $\tilde{C}_\theta$ meant for him, because P has to be spending weakly no more on his contract than under $\tilde{C}_\theta$ (up to $\varepsilon$, which is negligible compared to $\eta$). The contract $\tilde{C}_\theta$ maximizes that type's utility subject to that cost constraint. Since rejection also leads to a renegotiation breakdown with a probability $\eta$, which gives that type a strictly lower utility than $\tilde{C}_\theta$, accepting $\tilde{C}_\theta$ is strictly more profitable than rejection for that type, and thus he accepts $\tilde{C}_\theta$ with probability 1. As a result, a rejection fully reveals that the agent is of the other type. From Part i), that agent gets $u_\theta(C)$ after rejection, which is strictly less than the utility he gets from $\tilde{C}_\theta$ (since that contract maximizes the agent's utility subject to a higher cost than what P incurs with $C$). Therefore, both types accept their contract, and this reduces the cost of the principal strictly below $\bar{Q}$, showing that this is a profitable deviation. This shows that, necessarily, $\bar{Q} \leq Q_L$.

Since $L$ cannot get utility less than $u_L(C)$, under any PBE, and $Q_L$ is the cheapest way of providing that utility, this means that in all PBEs starting with $\beta \in (0,1)$, P must spend weakly less than $\bar{Q}$ on the high type, in order to guarantee that $\bar{Q} \leq Q_L$. Let $\bar{u}_H$ denote the highest expected utility that $H$ gets over all PBEs and beliefs $\beta > 0$. Since the principal spends less than $Q_L$ on $H$, $\bar{u}_H$ is bounded by the utility $\hat{u}_H$ obtained from the $H$-efficient contract $\hat{C}_H$ that costs $Q_L$. We will show that $\bar{u}_H = u_H(C_H)$. Suppose by contradiction that $\bar{u}_H > u_H(C_H)$, and consider a PBE that achieves $\bar{u}_H$ (again, the proof is easily adapted if the maximum is not achieved, by considering a PBE that gets very close to providing $\bar{u}_H$). The expected cost $Q$ from that PBE must be above $\beta Q(\bar{C}_H) + (1-\beta)Q_L$, where $\bar{C}_H$ is the $H$-efficient contract that gives utility $\bar{u}_H$ to $H$. Suppose that P deviates by proposing the contracts $\tilde{C}_L, \tilde{C}_H$ such that $\tilde{C}_L$ is $L$-efficient and gives utility $u_L(C) + \varepsilon^2$ to $L$, and $\tilde{C}_H$ is $H$ efficient and gives utility $\bar{u}_H - \varepsilon$ to $H$, for $\varepsilon$ small compared to $\eta$. $H$ accepts $\tilde{C}_H$, since rejection leads to a continuation utility bounded above by $\bar{u}_H$, and a strictly lower payoff in case of a breakdown. Given that, $L$ also accepts, since rejection will reveal his type, and, by Part i), result in a utility of $u_L(C)$. The cost gain on the high type is of order $\varepsilon$ compared to $Q(\bar{C}_H)$ and the cost loss on the low type is of order $\varepsilon^2$ compared to $Q_L$. Therefore, this deviation

[30] If the supremum $\bar{Q}$ is not achieved, the argument below can easily be adapted by considering a PBE whose expected cost is arbitrarily close to $\bar{Q}$.

is strictly profitable for $\varepsilon$ small enough. This shows that $\bar{u}_H = u_H(C_H)$. This immediately implies that $L$'s maximal utility across all PBEs for $\beta \in (0,1)$ is $u_L(C_L)$. The proof is a straightforward modification of the end of the proof of Part i).

Part iv) The argument is similar to the proof of Part iii). Let $\bar{Q}$ denote P's maximal expected cost over all PBEs and beliefs, starting from $C$. We will start by showing that $\bar{Q} \leq Q(E_L)$, where $E_L = E_L(R_0)$. Suppose by contradiction that $\bar{Q}$ is strictly greater than $Q(E_L)$ and achieved for some PBE and belief,[31] and consider the following deviation: P proposes the contracts $\tilde{C}_\theta$ that are efficient for each type and cost $\bar{Q} - \varepsilon$ for $\varepsilon$ arbitrarily small. It is easily shown that these contracts are IC, and by a similar argument as in Part ii), rejecting those contracts is always a strictly dominated strategy for one of the two types, and hence for both types. This is a strictly profitable deviation for P, yielding a contradiction. Hence, $\bar{Q} \leq Q(E_L)$. Since $L$ gets an expected utility of at least $u_L(R_0)$ in all PBEs, and providing that utility costs at least $Q_L = Q(E_L)$ to P, this means that P spends at most $Q_L$ on $H$, in all PBEs, and for all initial beliefs $\beta > 0$. This implies that $u_H(0)$ is bounded above by the utility that $H$ achieves with the $H$-efficient contract that costs $Q_L$. We now show that $u_H(0)$ is bounded above by $u_H(E_L)$. Suppose not, and consider a PBE that gives $H$ its highest utility, across PBEs and beliefs, $\bar{u}_H > u_H(E_L)$. The expected cost $Q$ from that PBE must be above $\beta Q(\bar{C}_H) + (1-\beta)Q_L$, where $\bar{C}_H$ is the $H$-efficient contract that gives utility $\bar{u}_H$ to $H$. Suppose that P deviates by proposing the contracts $\tilde{C}_L, \tilde{C}_H$ such that $\tilde{C}_L$ is $L$-efficient and gives utility $u_L(C) + \varepsilon^2$ to $L$, and $\tilde{C}_H$ is $H$-efficient and gives utility $\bar{u}_H - \varepsilon$ to $H$, for $\varepsilon$ arbitrarily small. Because $\tilde{C}_H$ gives strictly more to $H$ than $\bar{u}_H$, $H$ will accept $\tilde{C}_H$ and, hence $L$ will accept $\tilde{C}_L$. Repeating the proof of Part ii), one can show that this deviation is strictly profitable, establishing the desired contradiction. The only difference with that earlier proof lies in showing that the proposed contracts are incentive compatible. This is indeed true, for $\varepsilon$ small enough, because $\bar{u}_H > u_H(E_L)$ so $H$ does not want to mimic $L$.[32]

Proof of Lemma 2

Consider any PBE starting with $R_0$ in the $H$-Rent configuration. Consider, by contradiction, the first round $n$ such that i) $R_n$ is the $H$-Rent configuration and ii) $L$ accepts with positive probability a contract $R_{n+1}$ that is in a different configuration. Suppose that $R_{n+1}$ is in the No-Rent configuration. Then $u_L(n) = u_L(R_{n+1})$, by Part iii) of Proposition 1. This immediately implies that $u_L(R_n) \leq u_L(R_{n+1})$: $R_{n+1}$ is on a weakly higher iso-utility curve of $u_L$ than $R_n$. Moreover, because $H$ can always accept $R_{n+1}$, $u_H(n) \geq u_H(R_{n+1}) \geq u_H(E_H(R_n))$, where the last inequality

[31] As before, one can use a PBE that yields a cost arbitrarily close to $\bar{Q}$, in case it is not exactly achieved.

[32] It is straightforward to show that $L$ does not want to mimic $H$, since P spends less on $H$ than on $L$, and $L$ is already getting his maximal utility given the cost that P incurs conditional on facing $L$.

comes from the fact that $u_H$ is increasing along the iso-utility curve of $u_L$ in the direction of $\mathcal{E}_H$. This implies that the continuation cost for P is strictly above $\beta_n Q(E_H(R_n)) + (1-\beta_n)Q(E_L(R_n))$, which contradicts Lemma 1. Now suppose that $R_{n+1}$ is in the $L$-Rent configuration. Part iv) of Proposition 1 applied to the $L$-Rent configuration implies that, by choosing $R_{n+1}$, $L$ gets a continuation utility of at most $u_L(E_L(R_{n+1})$. Therefore, $u_L(E_L(R_{n+1})$ must be weakly greater than $u_L(R_n)$. This, along with the single-crossing property, implies that $u_H(R_{n+1})$ is strictly greater than $u_H(E_H(R_n))$ and contradicts Part iv) of Proposition 1 applied to $H$. ■

Proof of Proposition 2

i) Observe, first, that negotiation cannot end endogenously at a finite round $N$ such that $\beta_n = \beta_N > 0$ and $R_n = R_N$ for all $n \geq N$. Indeed, P could strictly reduce his cost at round $N$ by proposing the $H$-efficient contract $E_H(R_N)$ and have it accepted by $H$ with probability 1, by Part iv) of Proposition 1. Hence, consider the case in which P keeps proposing new contracts until renegotiation is exogenously interrupted, and suppose by contradiction that there is a choice sequence and corresponding belief subsequence $\{\beta_{n(k)}\}_{k\in\mathbb{N}}$ that converges to $\beta^* > 0$. Let $u_H^* = \sup\{u_H(R_n)\}$. For $H$ to accept $R_n$ with positive probability infinitely often, $u_H(R_n)$ must converge to $u_H^*$, including along the subsequence $\{n(k)\}$.[33] However, that implies that proposing the $H$-efficient contract $C_H$ that gives $u_H^*$ to $H$ is again a strictly profitable deviation: it does not change P's cost conditional on facing $L$, but it strictly reduces the cost conditional on facing $H$ (by an amount arbitrarily close to $Q(C_L) - Q(C_H)$, where $C_\theta$ is the $\theta$-efficient contract that provides $H$ with utility $u_H^*$), which happens with a probability arbitrarily close to $\beta^* > 0$.

ii) Suppose that there exists $\varepsilon > 0$ and a subsequence $\{R_m\}$ for which $Q(R_m) - Q(E_L(m)) \geq \varepsilon$. For $m$ large enough, $\beta_m$ is bounded above by $\eta\varepsilon/2\Delta_Q$, where $\Delta_Q = \max_{C\in\mathcal{C}} Q(C) - \min_{C\in\mathcal{C}} Q(C)$. Therefore, P can deviate by proposing $E_L(m), E_H(R_m)$ and make an immediate gain $\eta\varepsilon$ on $L$ and lose at most $\eta\varepsilon/2$ on $H$, which is profitable. This shows that the limit points of $\{R_n\}$ are all $L$-efficient. Let $u_L^* = \sup\{u_L(R_n)\}$. Since $L$ can always hold on to $R_n$, $u_L(R_n)$ must converge to $u_L^*$. Combining these observations, $\{R_n\}$ must converge to the $L$-efficient contract $\bar{C}_L$ such that $u_L(\bar{C}_L) = u_L^*$. ■

[33] Otherwise, there must exist a subsequence of rounds for which $u_H(R_{m+1})$ is bounded above away from $u_H^*$ by some gap $\delta u > 0$, but the continuation utility $u_H(m)$ gets arbitrarily close to $u_H^*$, say within $\varepsilon\eta$ for some $\varepsilon$ arbitrarily small for $m$ large. This implies that accepting $R_{m+1}$ causes a loss of order $\eta\delta u$ (due to the probability of breakdown) and contradicts the fact that $u_H(m)$ is within $\varepsilon\eta$ of $u_H^*$.

## C Inequalities

LEMMA 10 (REGULARITY BOUNDS) *There exist positive constants* $\underline{a}, a, b$ *such that for any* $C, \hat{C} \in \mathcal{E}_H$ *such that* $u_H(C) < u_H(\hat{C})$*, we have*

$$\underline{a}(u_H(\hat{C}) - u_H(C)) \leq Q(\hat{C}) - Q(C) \leq a(u_H(\hat{C}) - u_H(C)) \tag{29}$$

$$Q(E_H(R_n)) - Q(E_H(R_{n+1})) \leq b(Q(E_L(R_n)) - Q(E_L(R_{n+1}))) \tag{30}$$

*Proof.* The efficiency curve $\mathcal{E}_H$ can be parameterized by a univariate parameter $\lambda$. Consider two contracts $C$ and $\hat{C}$ on $\mathcal{E}_H$ as in the statement, and set $\lambda = 0$ for $C$ and $\lambda = 1$ for $\hat{C}$. We have $Q(\hat{C}) - Q(C) = \int_0^1 \frac{\partial Q}{\partial \lambda} d\lambda$, where $Q(\lambda)$ denotes $P$'s cost along $\mathcal{E}_H$, and is continuously differentiable, by our assumptions. Because $u_H$ is strictly increasing and continuously differentiable and $\mathcal{E}_H$ is compact as a closed subset of $\mathcal{C}$, there exists $a > 0$ such that $\frac{\partial Q}{\partial \lambda} \leq a \frac{\partial u_H}{\partial \lambda}$ for all $\lambda$, where $u_H(\lambda)$ similarly denotes the utility of $H$ along $\mathcal{E}_H$. This implies that

$$Q(\hat{C}) - Q(C) \leq a \int_0^1 \frac{\partial u_H}{\partial \lambda} d\lambda = a(u_H(\hat{C}) - u_H(C)).$$

The first inequality in (29) is shown similarly.

Let $\tau : \mathcal{E}_H \to \mathcal{E}_L$ denote the one-to-one, increasing map which to any $H$-efficient contract $C$ associates the $L$-efficient contract which gives $H$ the same utility as $C$. The map $\tau$ is continuously differentiable, and hence Lipschitz on its (compact) domain. Since $Q$ is also continuously differentiable and strictly increasing, there exists $b > 0$ such that

$$Q(E) - Q(E') \leq b(Q(\tau(E)) - Q(\tau(E'))$$

for all $E, E'$ on $\mathcal{E}_H$. Applying the result to $E = E_H(R_n)$ and $E' = E_H(R_{n+1})$ yields the desired in equality.

[PUT QUADRATIC BOUND HERE? IS THAT GLOBAL?] ■

Let $Q_\theta$ denote P's expected continuation cost at round $n$, conditional on facing type $\theta$.

LEMMA 11 (INCENTIVE BOUNDS) *Given any PBE and choice sequence* $\{R_n\}$*, there exist positive* *constants $\alpha$, $\gamma$, $\hat{b}$, and $b$, such that*

$$Q_L \le Q(E_L(n)) + \frac{\beta_n}{(1-\beta_n)} a w_n, \tag{31}$$

$$u_H(E_H(n+1)) - u_H(E_H(n)) \le \frac{\alpha\beta_n}{1-\beta_n} w_n, \tag{32}$$

$$w_{n+1} \le w_n \left(1 + \frac{\alpha\beta_n}{1-\beta_n}\right), \tag{33}$$

$$u_L(R_n) - u_L(R_{n+1}) \le \gamma\beta_{n+1} w_{n+1}, \tag{34}$$

$$u_H(E_H(n+1) - u_H(E_H(n))) \ge -\hat{b}\beta_{n+1} w_{n+1}, \tag{35}$$

$$w_{n+1}(1 - b\beta_{n+1}) \ge w_n - \eta y_n. \tag{36}$$

*Proof.* Lemma 1 implies that

$$(1-\beta_n)Q_L + \beta_n Q_H \le \beta_n Q(E_H(n)) + (1-\beta_n)Q(E_L(n)).$$

Moreover, $Q_H$ is bounded below (by convexity of the cost function) by the cost of the $H$-efficient contract $C_H(n)$ that provides utility $u_H(n)$ to $H$, since that is the cheapest way of providing $H$ with his continuation utility. This implies that $Q_L \le Q(E_L(n)) + \frac{\beta_n}{1-\beta_n}(Q(E_H(n)) - Q(C_H(n))$. The contracts $E_H(n)$ and $C_H(n)$ both lie on $\mathcal{E}_H$. Equation (29) implies that $Q(E_H(n)) - Q(C_H(n)) \le a(u_H(E_H(n)) - u_H(n)) = a w_n$. This shows (31).

From (31), $R_{n+1}$ cannot give $L$ a utility greater than the $L$-efficient contract that costs $Q(E_L(n)) + \frac{a\beta_n}{1-\beta_n} w_n$. This implies that $Q(E_L(n+1)) - Q(E_L(n))$ is bounded above by $\frac{a\beta_n}{1-\beta_n} w_n$. Combining this with (30) yields

$$Q(E_H(n+1)) - Q(E_H(n)) \le \frac{ab\beta_n}{1-\beta_n} w_n$$

Applying the argument used to prove (30), this time to the inverse of $\tau$, we have

$$u_H(E_H(n+1)) - u_H(E_H(n)) \le \alpha''(Q(E_H(n+1)) - Q(E_H(n)))$$

for some $\alpha'' > 0$. Combining these inequalities yields (32). We have

$$\begin{aligned} w_{n+1} &= u_H(E_H(n+1)) - u_H(n+1) = [u_H(E_H(n+1)) - u_H(E_H(n))] + u_H(E_H(n)) - u_H(n+1) \\ &\le [u_H(E_H(n+1)) - u_H(E_H(n))] + u_H(E_H(n)) - u_H(n) \\ &\le w_n \left(\frac{\alpha\beta_n}{1-\beta_n} + 1\right) \end{aligned}$$

where the first inequality comes from the monotonicity of $u_H(n)$ in $n$, and the second inequality comes from (32). This shows (33).

Because $L$ can hold on forever to $R_n$, his continuation utility $u_L(n)$ is bounded below by $u_L(R_n)$. At round $n+1$, P's expected cost conditional on facing $L$ is bounded above by $Q(E_L(n+1)) + \frac{\beta_{n+1}}{1-\beta_{n+1}} a w_{n+1}$, from (31). Repeating an earlier differential argument, we have $u_L(E) - u_L(E') \leq \alpha_L(Q(E) - Q(E'))$ for all $E, E' \in \mathcal{E}_L$, for some $\alpha_L > 0$. Therefore, the highest utility which may be achieved at that cost is bounded above by $u_L(R_{n+1}) + \hat{a}\beta_{n+1}/(1-\beta_{n+1})w_{n+1}$, for some proportionality constant $\hat{a}$, and

$$u_L(R_n) \leq u_L(n) \leq u_L(n+1) \leq u_L(R_{n+1}) + \hat{a}\beta_{n+1}/(1-\beta_{n+1})w_{n+1},$$

which yields (34).

In general, $u_H(E_H(n+1) - u_H(E_H(n)))$ may be negative. From (34), we have

$$u_L(R_n) - u_L(R_{n+1}) \leq \hat{a}\beta_{n+1}/(1-\beta_{n+1})w_{n+1}.$$

Using that the map $\tau$ introduced earlier has a continuously differentiable inverse, this implies that

$$u_H(E_H(n+1) - u_H(E_H(n))) \geq -\hat{b}\beta_{n+1}/(1-\beta_{n+1})w_{n+1},$$

for some $\hat{b} > 0$, proving (35).

For the last equation, subtracting $u_H(E_H(n))$ from (2) and rearranging (recalling that $w_n = u_H(E_H(n)) - u_H(n)$) leads, along any choice sequence, to

$$w_{n+1} = w_n - \eta y_n + \eta w_{n+1} + (1-\eta)(u_H(E_H(n+1)) - u_H(E_H(n))). \tag{37}$$

Combining this with 35 yields

$$w_{n+1} - w_n \geq \eta w_{n+1} - \eta y_n - b\beta_{n+1}w_{n+1},$$

and the hence (36). ■

Lemma 12 *There exists $\underline{q} > 0$ such that for any $C$ on $\mathcal{E}_L$ and $R \in \mathcal{H}$ such that $u_L(R) = u_L(C)$,*

$$Q(R) - Q(C) \geq \underline{q}(u_H(C) - u_H(R))^2.$$

*Proof.* For each $C \in \mathcal{E}_L$, we parameterize the contract set $\mathcal{U}_L(C) = \{R \in \mathcal{H} : u_L(\tilde{C}) = u_L(C)\}$ as $\{C_\lambda\}$ where $\lambda \in [0,1]$ and $C_0 = C$. Let $Q(\lambda) = Q(C_\lambda)$ and $u_H(\lambda) = u_H(C_\lambda)$. By efficiency of $C$, we have $Q'(0) = 0$. By strict concavity of $u_L$ and by convexity of $Q$, there exists $\hat{q} > 0$ such that $Q(\lambda) - Q(0) \geq \hat{q}\lambda^2$ for $\lambda$ in a right neighborhood of 0. By compactness and convexity of $\mathcal{U}_L(C)$, moreover, $\hat{q}$ may be chosen small enough so that the inequality holds for all $\lambda \in [0,1]$. Since $u_H$ has bounded derivatives, there must exists $\bar{u} > 0$ such that $|u_H(\lambda) - u_H(0)| \leq \bar{u}\lambda$ (the single crossing property between $u_H$ and $u_L$ imply that $u_H(\lambda) \leq u_H(0)$ for all $\lambda \in [0,1]$). Combining these inequalities, there exists $\underline{q}(C) > 0$ such that

$$Q(C_\lambda) - Q(C) \geq \underline{q}(C)\left(u_H(C) - u_H(C_\lambda)\right)^2.$$

Moreover, $\underline{q}(C)$ can clearly be chosen to vary continuously in $C \in \mathcal{E}_L$. By compactness of $\mathcal{E}_L$, $\underline{q} = \min_{C \in \mathcal{E}_L} \underline{q}(C)$ is strictly positive and yields the desired inequality. [Show picture?] ■

Lemma 13 *There exist positive constants* $k_2, k_3$ *such that*

$$y_n^2 \leq k_2[Q(R_{n+1}) - Q(E_L(n))] + k_3\gamma\beta_{n+1}w_{n+1} \tag{38}$$

*Proof.* We have

$$\begin{aligned} y_n^2 &= [(u_H(E_H(n) - u_H(E_H(n+1)))) + (u_H(E_H(n+1)) - u_H(R_{n+1}))]^2 \\ &\leq 2[u_H(E_H(n) - u_H(E_H(n+1)))]^2 + 2[u_H(E_H(n+1)) - u_H(R_{n+1})]^2 \\ &\leq k_1[Q(E_H(n)) - Q(E_H(n+1))]^2 + k_2[Q(R_{n+1}) - Q(E_L(n+1))] \\ &\leq k_2[Q(R_{n+1}) - Q(E_L(n))] + k_3[Q(E_L(n)) - Q(E_L(n+1))]. \end{aligned}$$

The first inequality is standard ($(a+b)^2 \leq 2a^2 + 2b^2$). The second inequality comes from the linear relation between $Q(E_H(n))$ and $u_H(E_H(n))$ (see Lemma 11) and Lemma 12 applied to the contracts $C = E_L(R_{n+1})$ and $R = R_{n+1}$. The last inequality comes from (30) and the fact that $[Q(E_L(n)) - Q(E_L(n+1))]^2$ is bounded above by $\bar{Q}(Q(E_L(n)) - Q(E_L(n+1)))$ for some constant $\bar{Q}$, by compactness.

The difference $Q(E_L(n)) - Q(E_L(n+1))$ is proportional to $u_L(R_n) - u_L(R_{n+1})$ (by a simple transposition of the proof of (30)), which is bounded above by $\gamma\beta_{n+1}w_{n+1}$, from (34) [Also put in Regular bounds]. Therefore, we get

$$y_n^2 \leq k_2[Q(R_{n+1}) - Q(E_L(n))] + k_3\gamma\beta_{n+1}w_{n+1}$$

which yields the result. ■

# D Proofs for Parts I and II

Proof of Lemma 3 Fix any choice sequence and let $n_0$ denote the first round along that sequence such that $\bar{w}_{n_0} \leq \varepsilon$. By construction, $\bar{w}_{n_0-1} > \varepsilon$. From Lemma 4 (whose proof, in the main text, is independent of this lemma), we have $u_H(n_0) \leq u_H(n_0 - 1) + \eta\Delta_H$. Therefore,

$$\bar{w}_{n_0} \geq \bar{w}_{n_0-1} + u_H(n_0 - 1) - u_H(n_0) \geq \varepsilon - \eta\Delta_H.$$

Since we can always select a choice sequence along which $\beta_n$ is weakly decreasing, we also get $\beta_{n_0} \leq \beta_0$. ■

Lemma 14 *i) For any round $n_0$, $\check{\varepsilon} > 0$, and choice sequence, there exists a round $n > n_0$ such that $u_H(R_n) \geq \max\{u_H(E_H(R_m)) : m \leq n_0\} - \check{\varepsilon}$. ii) The augmented rent index $\hat{w}_n = \max\{u_H(E_H(R_m)) : m \leq n\} - u_H(n)$ converges to zero as $n$ goes to infinity, along any choice sequence.*

*Proof.* i) Fix $\check{\varepsilon} > 0$. Proposition 2 guarantees that, along any choice sequence, $R_n$ converges to an $L$-efficient $\bar{C}_L$. Continuity of $u_H(\cdot)$ implies that there exists a round $\check{n}$ such that $u_H(R_n) \geq u_H(\bar{C}_L) - \check{\varepsilon}$ for all $n \geq \check{n}$. Therefore, it suffices to show that $u_H(\bar{C}_L) \geq \max\{u_H(E_H(R_m)) : m \leq n_0\}$ for all $n_0$. Equivalently, we must show that $u_H(\bar{C}_L) \geq \max\{u_H(E_L(R_m)) : m \leq n_0\}$ for all $n_0$ since, by construction, $E_H(R)$ and $E_L(R)$ give the same utility to $H$ for any $R \in \mathcal{H}$. For contracts $C, C'$ on the $L$-efficiency line $\mathcal{E}_L$, $u_H(C) \leq u_H(C')$ if and only if $u_L(C) \leq u_L(C')$. Therefore, it suffices to show that $u_L(\bar{C}_L) \geq \max_{m \in \mathbb{N}}\{u_L(E_L(R_m))\}$. By construction, $u_L(E_L(R)) = u_L(R)$ for all $R \in \mathcal{H}$, since $E_L(R)$ is the $L$-efficient contract that gives $L$ the same utility as $R$. Therefore, we have reduced the problem to showing that

$$u_L(\bar{C}_L) \geq \max_{m \in \mathbb{N}}\{u_L(R_m)\}.$$

We recall that for all $n$, $u_L(n) \geq u_L(R_n)$ (since holding on to $R_n$ is always a feasible strategy for $L$, and that $u_L(n)$ is nondecreasing in $n$ for all choice sequences (see Lemma 4, the argument also applies to $L$). Since $R_n$ converges to $\bar{C}_L$ and $u_L(n)$ must converge to $u_L(\bar{C}_L)$. Since $u_L(n)$ is nondecreasing, we get

$$u_L(R_n) \leq u_L(n) \leq u_L(\bar{C}_L),$$

which concludes the proof of i). To prove ii), it suffices to notice that $\max\{u_H(E_H(R_m)) : m \leq n\}$ and $u_H(R_n)$ both converge to $u_H(\bar{C}_L)$, from the previous reasoning. ■

Lemma 15 *There exists a pushdown sequence at Stage 1.*

*Proof.* Let $\mu^\theta(\{\tilde{R}_n\})$ denote the probability, conditional on facing type $\theta$, of observing choice sequence $\{\tilde{R}_n\}$ until $\hat{u}_1$ is reached. By definition, summing over all choice sequence with elements in $\mathcal{H}$ and truncated at the first round when $\hat{u}_1$ is reached, we have $\sum_{\{\tilde{R}_n\}} \mu^H(\{\tilde{R}_n\}) = \mu_0$. Because $L$ always chooses contracts in $\mathcal{H}$, we also have $\sum_{\{\tilde{R}_n\}} \mu^L(\{\tilde{R}_n\}) = 1$. These two equations immediately imply that there exists a choice sequence $\{R_n^0\}$ such that $\mu^H(\{R_n^0\})/\mu^L(\{R_n^0\}) \leq \mu_0$. Conditional on observing that choice sequence, the posterior is given by Bayesian updating

$$\hat{\beta}_1 = \frac{\mu^H(\{R_n^0\})\hat{\beta}_0}{\mu^H(\{R_n^0\})\hat{\beta}_0 + \mu^L(\{R_n^0\})(1-\hat{\beta}_0)}.$$

Dividing by $\mu^L(\{R_n^0\})$ and using that $\mu^H(\{R_n^0\})/\mu^L(\{R_n^0\}) \leq \mu_0$ yields the result. ■

For any $\tilde{\epsilon}$, let

$$D(\tilde{\epsilon}) = \min\{Q(C) - Q(E) : C \in \mathcal{H}, E \in \mathcal{E}_H : u_H(E) \leq u_H(E_H(C)) + \tilde{\epsilon}\}. \quad (39)$$

$D(\tilde{\epsilon})$ is nonincreasing in $\tilde{\epsilon}$. Moreover, because $\mathcal{E}_H$ and $\mathcal{E}_L$ are smooth and disjoint curves of the compact domain $\mathcal{C}$, $D(\tilde{\epsilon})$ is strictly positive for $\tilde{\epsilon}$ small enough. For such values of $\tilde{\epsilon}$, $D(\tilde{\epsilon})$ defines a lower bound on the inefficiency of contracts in $\mathcal{H}$ conditional on facing $H$.

LEMMA 16 *If at the beginning of any stage $k$, $w_{n(k-1)} \leq \varepsilon$, then for all rounds $n$ of stage $k$,*

$$Q(R_n) \geq Q(E_H(R_{n(k-1)})) + D(\varepsilon)$$

*Proof.* Let $C$ denote the $L$-efficient contract that gives $H$ utility $u_H(n(k-1))$. Since $u_H(n)$ is nondecreasing, we have for any round $n$ of stage $k$, $u_H(n(k-1)) \leq u_H(n)$. From part iv) of Proposition 1, this implies that $R_n$ must cost weakly more than $C$: otherwise, we would have $u_L(E_L(R_n)) < u_L(C)$, which would imply that $u_H(n) \leq u_H(E_H(R_n)) < u_H(E_H(C)) = u_H(n(k-1))$, a contradiction. By assumption, we have $u_H(E_H(R_{n(k-1)})) - u_H(C) = w_{n(k-1)} \leq \varepsilon$. By definition of $D(\varepsilon)$, this implies that $Q(C) \geq Q(E_H(R_{n(k-1)})) + D(\varepsilon)$. Since $Q(R_n) \geq Q(C)$, this proves the lemma. ■

We will consider $\varepsilon$ such that $D(2\varepsilon) > 0$, we let $D = D(2\varepsilon)$ denote the lower bound on the loss that is used throughout the proof.

REMARK 2 *In principle, one could reach a stage $k$ for which $w_{k-1}$, and hence $\hat{w}_{k-1}$, is greater than $2\varepsilon$, which would imply that the lower bound $D$ on the loss is not guaranteed to hold for that stage. If that is the case, however, Lemma 3 guarantees that one can find a later round for which $\hat{w}_n$ is in $(\varepsilon/2, \varepsilon)$, and one can restart the analysis from that round (i.e., this is our new "$n_0$"). Moreover, $\hat{\beta}_{k-1} \leq \beta_0$, so the two conclusions of Lemma 3 hold. Re-starting Part I from the new round $n_0$, one may encounter a stage for which this problem arises again, in which case one re-initialize the analysis again, starting from a yet later round. However, because $\hat{w}_n$ converges to zero along any choice sequence as $n$ goes to infinity, by Lemma 14, there can only be finitely many initializations: there must be a round $n_0$ such that i) $\hat{w}_{n_0} \in (\varepsilon/2, \varepsilon)$, ii) $\beta_{n_0} \leq \beta_0$, and iii) $\hat{w}_k$ remains below $2\varepsilon$ for all stages constructed from $n_0$.*

PROOF OF LEMMA 5

We fix throughout some small $\bar{\varepsilon} > 0$. Consider some round $n$ contract $R_{n+1}$ in the menu $M_n \cup \{R_n\}$. If $\mu_n^L(R_{n+1}) \geq \bar{\varepsilon}$, then $\mu_n(R_{n+1}) \leq \frac{1}{\bar{\varepsilon}}$, and hence $\beta_{n+1} \leq \frac{\beta_n}{\bar{\varepsilon}}$. The set of contracts $R_{n+1}$ for which

$\mu_n^L(R_{n+1}) < \tilde{\varepsilon}$ has probability at most $G\tilde{\varepsilon}$, where $G$ is the upper bound on the size of the menu. Therefore, with probability $1 - G\bar{\varepsilon}$.

$$\beta_{n+1} \leq \frac{\beta_n}{\bar{\varepsilon}}$$

From (33) in the Appendix, $w_{n+1} \leq w_n \left(1 + \frac{\alpha\beta_n}{1-\beta_n}\right)$.

At round $\bar{n}$, we have $\beta_{\bar{n}} \leq \eta^d$ and $w_{\bar{n}} \leq \bar{w}\eta$. Therefore, $\beta_{\bar{n}+1} \leq \eta^d/\bar{\varepsilon}$ and $w_{\bar{n}+1} \leq k_1\eta$ for some constant $k_1$. This implies in particular that with probability $1 - G\bar{\varepsilon}$, the lower bound $D = D(2\varepsilon)$ on the loss is valid for round $\bar{n}+1$, because $w_{n+1} \leq 2\varepsilon$. The previous reasoning can be applied by induction to rounds $n = \bar{n}, \ldots, \bar{n} + \bar{N} - 1$. It implies that with probability $1 - k(\bar{N})\tilde{\varepsilon}$, we have

$$\beta_n \leq (\bar{\varepsilon})^{-\bar{N}} \bar{\beta}\eta^d \tag{40}$$

$$w_n \leq \bar{W}(\bar{N})\eta \tag{41}$$

for all $n \in \{\bar{n}, \ldots, \bar{n} + \bar{N}\}$, for some constants $k(\bar{N})$ and $\bar{W}(\bar{N})$ independent of $\bar{\varepsilon}$ and $\eta$.

Consider choice sequences such that $\beta_n$ and $w_n$ satisfy the inequalities above throughout the stage, which occur with probability $1 - k(\bar{N})\tilde{\varepsilon}$. We now show that, at round $n = \bar{n}$, $Q(R_{n+1}) - Q(E_L(n))$ must be of order $O\left(\frac{\beta_n}{\mu_n^L(R_{n+1})}\right)$. If each term in the sum entering P's IC constraint (11) is nonnegative, this result comes from the inequality[34]

$$w_n a\beta_n \geq \mu_n^L(R_{n+1})[\beta_n\mu_n(R_{n+1})\eta D + (1-\beta_n)\eta(Q(R_{n+1}) - Q(E_L(n)))]$$

which implies that

$$Q(R_{n+1}) - Q(E_L(n)) \leq \frac{aw_n}{\eta(1-\beta_n)} \frac{\beta_n}{\mu_n^L(R_{n+1})}. \tag{42}$$

In general, while some terms $\mu_n^L(R_{n+1})(1-\beta_n)\eta(Q(R_{n+1}) - Q(E_L(n)))$ of that sum may be negative, they can only be very slightly so: we have

$$Q(R_{n+1}) - Q(E_L(n)) \geq Q(E_L(n+1)) - Q(E_L(n)) \geq -k\beta_{n+1}w_{n+1},$$

where the second inequality comes (34) of Lemma 11 in the Appendix. Moreover, $w_{n+1} \leq w_n \left(1 + \frac{\alpha\beta_n}{1-\beta_n}\right)$, from equation (33) in the Appendix, and $\mu_n^L(R_{n+1})\beta_{n+1}$ is of order $\beta_n$. Therefore, the lower bound is of order $\eta \times w_n\beta_n$, for each term of the sum that is negative. Since there are at most $G$ of them, we conclude that (42) holds up to a term of order $\eta\beta_n$, which is negligible.

This shows that for all $R_{n+1}$ such that $\mu_n^L(R_{n+1}) \geq \sqrt{\beta_n}$, the difference $Q(R_{n+1}) - Q(E_L(n))$ is at most of order $\sqrt{\beta_n}$. By Lemma 13 of the Appendix, this implies that $y_n$ is $O(\eta^{d/4})$, because $y_n^2$ is bounded above $Q(R_{n+1}) - Q(E_L(n)) + b\beta_{n+1}w_{n+1}$, and the first term is of order $\beta_n^2$, while the latter

[34] The inequality holds, because each term in the sum is nonnegative, and $w_n a\beta_n$ is bigger than the sum.

is of order $\beta_n\eta$. Moreover, the set of contracts $R_{n+1}$ for which $\mu_n^L(R_{n+1}) < \sqrt{\beta_n}$, is negligible: it arises with probability at most $G\sqrt{\beta_n}$. Since $\sqrt{\beta_n} = O(\eta^{d/2})$ is small compared to $\bar{\varepsilon}$ for $\eta$ small enough, we conclude that with probability $1 - O(\tilde{\varepsilon})$, $\beta_n = O(\eta^d)$, $w_n = O(\eta)$ and $y_n = O(\eta^{d/4})$ for round $\bar{n}$ and, by induction, for all rounds of the stage. ■

# E Proofs for Part III

Proof of Lemma 6

By assumption, $\beta_N \leq \eta^d$ so $\beta_N$ becomes arbitrarily small as $\eta$ gets small. Recall equation (33) from Lemma 11

$$w_{n+1} \leq w_n \left(1 + \frac{\alpha\beta_n}{1-\beta_n}\right)$$

where $\alpha > 0$. By Bayesian updating, we have $\beta_{n+1} = \frac{\mu_n\beta_n}{\mu_n\beta_n + (1-\beta_n)}$. Since $\beta_N$ is arbitrarily small, the denominator is arbitrarily close to 1 for $n = N$. More generally we have, for $\eta$ small enough,

$$\beta_{n+1} \leq \mu_n\beta_n(1+\epsilon) \tag{43}$$

where $\epsilon$ is a small positive constant, as long as $\beta_n$ remains small. At $N$, we have $\beta_N \leq \eta^d$ and $w_N = e_N - u_N \leq \bar{e}_N - u_N \leq \frac{\eta D}{2a}$, which implies that $w_{N+1} \leq \frac{\eta D}{2a}(1 + (1+\varepsilon)\alpha\beta_n)$. From (13), this implies that $\mu_N \leq \frac{1}{2} + O(\eta) \leq \frac{3}{5}$.

Consider the first round $M > N$ for which $\mu_M \geq 3/4$. The probability $\beta_n$ is decreasing[35] until at least round $M$. Proceeding by induction, from round $N$ to round $M$, the previous inequalities imply that

$$w_{N+m} \leq w_N \prod_{i=1}^{m} (1 + \alpha(1+\varepsilon)\beta_{N+i}) \tag{44}$$

and

$$\beta_{N+i} \leq \beta_N \prod_{j=0}^{i-1} \left(\mu_{N+j}(1+\epsilon)\right), \tag{45}$$

and (43) is valid for all rounds $n \in \{N, \ldots, M\}$. From (45), we have

$$\beta_{N+i} \leq \left(\frac{3(1+\epsilon)}{4}\right)^i \beta_N$$

Therefore, (44) implies that

$$w_M \leq w_N \prod_{i=1}^{M-N} \left(1 + \alpha(1+\varepsilon)\eta^d \left(\frac{3(1+\epsilon)}{4}\right)^i\right)$$

[35] This comes from the Bayesian updating equation $\beta_{n+1} = \frac{\mu_n\beta_n}{\mu_n\beta_n + (1-\beta_n)}$, which is nondecreasing in $\mu_n$. Taking $\mu_n = 1$ shows that $\beta_{n+1} \leq \beta_n$ as long as $\mu_n \leq 1$.

The product

$$\prod_{i=1}^{\infty}\left(1+\alpha(1+\varepsilon)\eta^d\left(\frac{3(1+\epsilon)}{4}\right)^i\right) \tag{46}$$

is finite for $\eta$ small enough, and converges to 1 as $\eta$ goes to zero.[36] Therefore, for $\eta$ small, $w_M$ is bounded above by $\frac{5}{4}w_N \leq \frac{5\eta D}{8a}$. From (13), this implies that $\mu_M$ is bounded above by $5/8+O(\eta) < 3/4$, so $M$ cannot be finite. This shows that for $\eta$ below some threshold $\hat{\eta}$, $\mu_n$ is bounded above by $3/4$ for all $n \geq N$ and, from (43), that $\beta_n$ is decreasing. Since $w_n$ is bounded above by $\frac{3}{2}w_N$ and $w_N \leq \frac{\eta D}{2a}$, the last claim follows easily. ■

Lemma 17 *There exists a positive constant $\bar{A}$ such that*

$$y_n^2 \leq \frac{\bar{A}\beta_{n+1}}{1-\beta_0} \tag{47}$$

*Proof.* Equation (15) implies that $Q(R_{n+1}) - Q(E_L(n)) \leq \frac{\beta_n w_n a}{\eta(1-\beta_0)}$, since $\beta_n \leq \beta_0$. This, along with (16), yield[37]

$$Q(R_{n+1}) - Q(E_L(n)) \leq \frac{D\beta_{n+1}}{1-\beta_0}.$$

Combining this inequality with Lemma 13, we get

$$y_n^2 \leq k_2\frac{D\beta_{n+1}}{1-\beta_0} + k_4\gamma\beta_{n+1}w_{n+1}.$$

Since $w_{n+1} \leq \frac{\eta D}{2a} \ll 1$, taking $\bar{A}$ slightly greater than $k_2 D$ proves the lemma. ■

Proof of Lemma 7

Taking the square root of (22) and multiplying the result by $\frac{\sqrt{\beta_{\hat{N}+1}}}{\beta_{\hat{N}}}$, we get

$$\frac{\beta_{\hat{N}+1}}{\beta_{\hat{N}}} \geq 2c\sqrt{\beta_{\hat{N}+1}}.$$

Combining this with (20) yields

$$\frac{\beta_{\hat{N}+2}}{\beta_{\hat{N}+1}} \geq c\sqrt{\beta_{\hat{N}+1}}$$

and, taking the square root of this expression,

$$\frac{\sqrt{\beta_{\hat{N}+2}}}{\beta_{\hat{N}+1}} \geq \frac{\sqrt{c}}{\beta_{\hat{N}+1}^{1/4}} \tag{48}$$

[36] Indeed, taking the logarithm of that product, we obtain a sequence that is approximately geometric with geometric factor 3/4 and, hence converges, uniformly in $\eta$. Moreover, each term of the sequence is of order $\eta^d$, which converges to 0 as $\eta$ goes to zero. This implies that all partial sums converge to zero and, by uniform convergence, that the sequence converges to zero as well. By continuity of the exponential function, the product itself thus converges to 1 as $\eta$ goes to zero.

[37] We are using $\beta_{n+1} \geq \mu_n\beta_n$, which comes from the first inequality of (18).

Combining this with (23) shows that (22) holds at round $\hat{N}+1$. Since $\beta_n$ is non-increasing in $n$ for $n \geq \hat{N}$ and hence satisfies (23) for all $n \geq \hat{N}$, we can apply the previous argument by induction to conclude that (22) and (48) hold for all $n \geq \hat{N}$.

Multiplying (20) by $\frac{\beta_n}{\beta_{n+1}}$, we obtain

$$\frac{q_{n+1}}{q_n} \geq 1 - \frac{c\beta_n}{\sqrt{\beta_{n+1}}}.$$

From (48) applied to round $n$ (instead of $\hat{N}+1$), the last term is bounded above by $\frac{c\beta_n^{1/4}}{\sqrt{c}}$, which converges to zero as $n$ goes to infinity. ∎

Proof of Lemma 8

Suppose by contradiction that $\{q_n\}$ converges to zero. This, along with the second assumption of the lemma, implies the existence, for any $\varepsilon > 0$, of an integer $\bar{N}$ such that i) $\frac{q_{n+1}}{q_n} \geq 1-\varepsilon$ and ii) $q_n \leq q_{\bar{N}} \leq \varepsilon$ for all $n \geq \bar{N}$.[38] Convergence of $\{q_n\}$ to zero also implies that $\max_N \Pi_0^N q_k$ is bounded above by some constant $\bar{\Pi}$. Letting $\tilde{\varepsilon} = \sqrt{q_{\bar{N}}}$, we have $\Pi_{\bar{N}+1}^{\bar{N}+k} q_k \leq \tilde{\varepsilon}^{2k}$ for all integers $k \geq 1$. Therefore, for any integer $K \geq 1$, we have

$$q_{\bar{N}+K} = q_{\bar{N}+K} - q_\infty = \sum_{n \geq \bar{N}+K} (q_n - q_{n+1}) \leq \tilde{c}\tilde{\varepsilon}^K \sum_{k \geq 0} \tilde{\varepsilon}^k,$$

where $\tilde{c} = c'\sqrt{\bar{\Pi}}$. Taking $K = 3$, this yields

$$q_{\bar{N}+3} \leq \frac{c'}{1-\tilde{\varepsilon}} q_{\bar{N}}^{3/2} \leq 2c' q_{\bar{N}}^{3/2}.$$

Applying inequality i) above to $n = \bar{N}$, $\bar{N}+1$, and $\bar{N}+2$, yields

$$q_{\bar{N}+3} \geq q_{\bar{N}}(1-\varepsilon)^3.$$

Combining these two inequalities yields $(1-\varepsilon)^3 \leq 2c' q_{\bar{N}}^{1/2} \leq 2c'\varepsilon^{1/2}$, which is impossible if we choose $\varepsilon$ small enough. This yields the desired contradiction. ∎

# F Proof of Theorem 2, Part B

Let $Q(u, \theta, \varepsilon)$ denote the cost of the cheapest contract that provides $\theta$ with utility at least $u$ and that is *not* within a distance $\varepsilon$ of the $\theta$-efficient contract $E_\theta(u)$ that provides $\theta$ utility with that

[38]Indeed, there exist $N_1$ such that i) holds for all $n \geq N_1$ and $N_2$ such that $q_n \leq \varepsilon$ for all $n \geq N_2$. Letting $N = \max\{N_1, N_2\}$, any $\bar{N} \in \arg\max_{n \geq N}\{q_n\}$ satisfies conditions i) and ii).

utility. By a standard Taylor approximation argument (using convexity of $Q$ and strict concavity of $u_\theta$, see proof of Lemma 12), it is easy to show that $Q(u, \theta, \varepsilon) \geq Q(E_\theta(u)) + \varepsilon^2$.

From Part A, there exists a threshold $\tilde{\eta}(\varepsilon^4)$ such that P must leave an expected utility to each type $\theta$ that is bounded below by $u_\theta(E_\theta(R_0)) - \varepsilon^4$ for all $\eta$'s below that threshold. The cheapest contract $E_\theta = E_\theta(u_\theta(E_\theta(R_0)) - \varepsilon^4)$ that provides this utility costs $Q(E_\theta(R_0)) - O(\varepsilon^4)$. From Lemma 1, P's expected cost is bounded above by $\beta_0 Q(E_H(R_0)) + (1 - \beta_0)Q(E_L(R_0))$ for any $\eta$ and PBE. These observations imply Part B): if $\theta$ accepted contracts that are $\varepsilon$ away from $E_\theta(R_0)$, those would be $\varepsilon + O(\varepsilon^4)$ away from $E_\theta$, and incur an excess cost of order $\varepsilon^2$ above $Q(E_\theta(R_0))$. Therefore, if $\theta$ accepted such contracts with probability at least $\varepsilon$, the efficiency loss, compared to accepting $E_\theta(u)$ would be of order $\varepsilon^3$, compared to a maximal potential gain of order $\varepsilon^4$. Setting $\bar{\eta}(\varepsilon) = \tilde{\eta}(\varepsilon^4)$ then yields the statement of Part B. ■

# G Notation

- $u_\theta(n)$: type $\theta$'s continuation utility at the beginning of round $n$.
- $E_\theta(R)$: If $R$ is the $H$-Rent configuration, $E_L(R)$ is the $L$-efficient contract that gives $L$ the same utility as $R$ and $E_H(R)$ is the $H$-efficient contract that gives $H$ the same utility as $E_L(R)$ (see the definition preceding Theorem 2).
- $E_\theta(n) = E_\theta(R_n)$.
- $w_n = E_H(n) - u_H(n)$.
- $\hat{w}_n = \max\{E_H(m) : m \leq n\} - u_H(n)$.
- $y_n = E_H(n) - u_H(R_{n+1})$.
- $\bar{C}_\theta = \lim_{n\to+\infty} u_\theta(n)$.

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