Agency Models with Frequent Actions

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Abstract

The paper analyzes dynamic principal-agent models with short period lengths. The two main contributions are: (i) an analytic characterization of the values of optimal contracts in the limit as the period length goes to 0, and (ii) the construction of relatively simple (almost) optimal contracts for fixed period lengths. Our setting is flexible and includes the pure hidden action or pure hidden information models as special cases. We show how such details of the underlying information structure affect the optimal provision of incentives and the value of the contracts. The dependence is very tractable and we obtain sharp comparative statics results. The results are derived with a novel method that uses a quadratic approximation of the Pareto boundary of the equilibrium value set.

1 Introduction

We consider dynamic contracting problems in which a risk neutral principal interacts repeatedly with a risk averse agent under asymmetric information. These are benchmark models in labor economics, corporate finance (CEO compensation and optimal capital structure), and the literatures on optimal dynamic insurance and taxation. The questions of the optimal dynamic incentive design in those situations are central to both economic theory and the applications. In the paper we develop a novel discrete-time method that allows us to solve such problems analytically for a range of contracting environments.

We focus on settings with frequent decisions and information arrival ("short period length"). Importantly, the class of models we consider is permissive regarding the precise nature of information structure in each period. It embraces models in which the agent has private information about his own action only, as when devoting costly effort to develop a risky project (*pure hidden action*), ones in which the agent also has some partial information about the environment, for example own stochastic productivity (*private information*), and ones when the agent acts after all the uncertainty is resolved, as when diverting funds from the realized cash flows (*pure hidden information*). Aside from the degree of private information, models differ in distributions of signals and the effects of agent's action. On the one hand, this flexibility is crucial for applications; on the other, the details of the information structure are known to be paramount for the design of incentives.

Existing characterizations of optimal contracts for discrete time models do not provide manageable methods for explicitly constructing optimal contracts, or for performing comparative static analysis. In a pathbreaking paper, Sannikov [2008] introduced a continuous time agency model that is very tractable and can be solved with standard stochastic calculus techniques. However, the continuous time method cannot reflect any of the details of information structure mentioned above. It is an open question whether the continuous time solution provides a good approximate solution for any (or all) of those contracting situations in a standard, discrete time setting.

In this paper, for each information structure, we look at a sequence of discrete time models with shrinking period length. We develop a *quadratic approximation method* that allows us to solve each of those problems when the period length is short. More precisely, first, for any type of information structure we characterize the limit of the Pareto frontier of value sets achievable by incentive compatible contracts, as the period length shrinks. Second, we construct relatively simple suboptimal contracts, whose values converge to the Pareto frontier as the period length shrinks. Importantly, while the details of the information structure matter for the solutions, we show that they can all be summarized in a single function, the *variance of continuation values* (VCV) function. The VCV function is a parameter in the equation characterizing the limit of the frontiers, and its definition also contains the key information needed to design (almost) optimal discrete time contracts.

The method yields rich results about the incentive design, which we put here in three broad categories. First, Muller [2000] and Fudenberg and Levine [2009] demonstrated (in different settings) that no matter how short the period length, the solutions of discrete time models may depend on the details of the underlying information structure. We go beyond this result in our principal-agent framework and pin down exactly the relevant parameters. For example, restricting attention to pure hidden action models, the value of optimal contracts depends on a single parameter of the distribution of public signal, the *Fisher information quantity*, which measures its informativeness about the agent's action. The relevance of the Fisher information for the incentive design is, to the best of our knowledge, new. With private information the value depends also on parameters measuring cross-correlation of likelihood ratios of public signals given different private signals (see Example 3). Regarding the contracts, we prove an extreme result: there is no single contract that can "work" for two essentially different information structures (Proposition 3).

Second and crucially, despite this sensitivity, our uniform method yields the solutions for each information structure. Our method delivers (almost) optimal discrete time contracts without any parametric assumptions on the primitives (see Literature Review below). The contracts are fully dynamic, based on the agent's continuation value as a state variable. For example, in the pure hidden action case the continuation value evolves linearly in the likelihood ratio of the public signal (see Lemma 3).

Third, the dependence of the results on the information structure is particularly tractable, yielding a relatively easy comparative statics analysis. The analysis is reduced to the analysis of the novel VCV function. The problem is a simplified version of the static agency problem, with a risk neutral agent and a principal with a quadratic utility function.

We also believe our method sheds light on the continuous time approach. In particlar, we are able to provide two discrete time justifications (convergence results) for the optimal continuous time contracts. We show that for pure hidden action models with a particular value of the Fisher information quantity, as for the normal distribution, the optimal contracts converge in distribution to the optimal continuous time contract, and the same is true for their values. For "most" information structures the limits of values are different.[1] However, for a fixed variance of public signal, we show that the value of optimal continuous time contract is the lower bound on the limit of values for *any* information structure.

Let us outline our method for solving the contracting problems. It consists of two steps. From standard dynamic programming methods (see Abreu, Pearce, and Stacchetti [1986, 1990] and Spear and Srivastava [1987]), the contracts can be described with the agent's continuation value as a state variable. The continuation value promised last period fully determines the current period effort scheme (as a function of the private signal), as well as consumption and the new continuation value, contingent on revenue. In order to provide incentives to exert costly effort, the consumption and continuation value must respond to revenue and reward the agent for good outcomes. On the other hand, such volatility is inefficient and imposes a "cost of incentives", given the agent's risk aversion.

The first step consists in solving a family of static optimization problems and addresses exactly this issue. Given a mean effort and mean cost of effort, roughly, we look for an action scheme with those parameters and a continuation value function with minimal variance, which provides local (first-order) incentives for exerting effort. For a short period length, it turns out that the (appropriately rescaled) solutions to this problem are the optimal way of incentivizing the agent in each period of a dynamic contract. The solutions, and in particular the variance of continuation values, depend on the information structure: variance is low in the case when the public signal is statistically informative of the agent's effort.

The second step consists in solving a differential equation, with the variance of continuation values as a parameter. Its solution $F(w)$ is the limit of the principal's values for the optimal dynamic contracts (as the period shrinks to 0) that deliver a given value w to the agent. Overall, it is the first step that is peculiar to the contracting environment. It lets us reduce the principal's problem to a standard dynamic optimal control

[1] While the limits of values are always well defined, specifying the limits of contracts for models other than pure hidden action is much more delicate.

problem, in which the principal in every period chooses a mean effort and a mean cost of effort, and the incentive compatibility constraint is replaced by a condition on the law of motion of the state variable (continuation value). The differential equation of the second step is the standard HJB equation associated with the limit of such dynamic optimal control problems.[2] We also note that the condition replacing incentive compatibility is very different from the analogous one for the model stated directly in continuous time: it converges to this continuous time condition only for the pure hidden actions models with an additional *constraint* of linearity of continuation values in the public signal.

Literature Review. As mentioned above, our results rely on the parametrization of the dynamic contract by the agent's continuation value (Abreu, Pearce, and Stacchetti [1986, 1990], Spear and Srivastava [1987]). This insight leads to a method for computing the optimal contracts, or more generally Pareto efficient Perfect Public Equilibria, for models with a fixed period length based on the value iteration technique. Phelan and Townsend [1991] show a related method to compute optimal contracts based on the iteration of a linear programming problem. While those approaches are flexible and applicable to a wide variety of problems, they are computationally intense and do not yield analytical solutions.

One way to restore analytical tractability is to focus on models with patient players (Radner [1985], Fudenberg, Holmstrom, and Milgrom [1990]). This is equivalent to considering models with short period length, where the period length does not affect the information structure. While this simplifies the analysis, as the period length shrinks the informational frictions disappear. Abreu, Milgrom, and Pearce [1991] suggest a more realistic approach where increasing the frequency of actions also affects the information structure. In our case, as in the continuous time models, short periods come with a high variance of the public signal, which in particular exacerbates the informational problems and prevents the first-best outcome from being achieved in the limit.

On a technical level, Matsushima [1989] established efficiency results and Fudenberg, Levine, and Maskin [1994] the full Folk Theorem for patient players by decomposing continuation values on hyperplanes tangent to the (Pareto frontier of the) set of achievable values. Our method bears a close resemblance to this approach, where we use a quadratic instead of the linear approximation of the frontier. The more precise approximation is required by the richer class of processes of public signals we consider. Moreover, the curvature of the boundary is proportional to the efficiency cost of incentives; when the process of signals is such that the linear approximation is appropriate, we recover as a special case the Folk Theorem result for our restricted setting of principal-agent problems

[2] The sensitivity of solutions with respect to the information structure is inherent in the first step: even though two sequences of models converge to the same continuous time model, the corresponding sequences of the reduced dynamic optimal control problems, with the endogenous constraints on the laws of motion on the state variable (continuation value) - need not. In particular, for dynamic decision problems (such as portfolio selection) or dynamic arbitrage pricing equations, where the law of motion of a state variable (e.g. price of a risky asset) is exogenous, the weak convergence of models typically guarantees convergence of the solutions.

(see Section 3.1).

Our method provides an upper hemicontinuity result for the continuous time principal-agent model. Hellwig and Schmidt [2002] is among the first papers to provide such a result for the continuous-time principal-agent model by Holmstrom and Milgrom [1987], in which the agent has a CARA utility function, is compensated only at the end of the employment period and the "full dimension" assumption is satisfied. Biais, Mariotti, Plantin, and Rochet [2007] established upper hemicontinuity for the principal-agent model of diverting cash-flows by DeMarzo and Sannikov [2006], in which the agent is risk-neutral and the efficiency cost from diverting funds is linear. Sannikov and Skrzypacz [2007] considers a more general framework of games and limit processes that are an arbitrary mixture of Brownian diffusion and Poisson processes, and show convergence for games with normal noise and a pure hidden action structure, in the case of arbitrarily patient players. Our method lets us obtain general results making no recourse to CARA, infinite patience, or risk neutrality of the agent.

Earlier results established that the limits of the discrete time models might differ from the continuous time solutions and be sensitive to the information structure. Muller [2000] illustrated this in the context of the model by Holmstrom and Milgrom [1987], and Fudenberg and Levine [2009] did so for a reputation game with one long-lived player. In our setting, we establish not only that "details matter" but exactly what details matter. More importantly, our focus is not to point out the sensitivity but to deliver a uniform method for finding optimal dynamic contracts in the face of it, for a range of well known contracting settings. We expect that our quadratic approximation method is applicable in quite general settings.

2 Model

2.1 The Agency Problem

A risk neutral principal contracts with a risk-averse agent. The principal offers the agent a contract specifying a contingent payment for each period as a function of the public history (of reports by the agent and of outputs in previous periods). If the agent accepts it, the contract becomes legally binding and cannot be terminated by either party. In every period of length Δ, the timing is as follows. The agent observes a private signal about the output's random shock in the current period, and then sends a report to the principal and chooses an action (effort). The agent's action and the random shock determine the output, which is realized at the end of the period. The principal pays the agent after observing the output and the agent consumes his compensation (the agent can't save or borrow). Note that both the agent's signal and action are his private information, whereas output and the agent's report are publicly observed. Though the agent's actions are unobservable, the principal and the agent also implicitly agree to a full contingent action plan for the agent.

The agent's per period utility is given by $\Delta[u(c) - h(a)]$, where $a \in \mathcal{A}$ and $c \geq 0$

denotes his consumption. The agent's action and consumption are stated in flow units.[3] The consumption utility function $u : \mathbb{R}_+ \to \mathbb{R}$ is twice continuously differentiable, strictly increasing and strictly concave, with $u(0) = 0$ and $\lim_{c\to\infty} u(c) = \bar{u} < \infty$. The agent's action space is a closed interval $\mathcal{A} = [0, A]$. The cost of effort function $h : \mathcal{A} \to \mathbb{R}$ is strictly increasing, strictly convex and twice continuously differentiable, with $h(0) = 0$. We also assume that there exists $\gamma > 0$ such that $h(a) \geq \gamma a$ for all $a \in \mathcal{A}$. In addition, we assume that $h'(0_+) < u'(0)$, so absent asymmetric information it is efficient to have the agent exert positive effort.

The principal's per period payoff is $\Delta[x + a - c]$, where x is a random shock, a is the agent's action and c is the agent's compensation, again in flow units. We will interpret $y = \Delta[x + a]$ as the output realization. Both the principal and the agent discount future payoffs by the common discount factor $e^{-r\Delta}$, where $r > 0$ is the discount rate.

Let z_n denote the agent's private signal realization in period n. We assume that (x_n, z_n) are randomly distributed with a joint distribution $G^{\Delta}(x_n, z_n)$ and $\{(x_n, z_n)\}$ are i.i.d across periods. We also assume that $\mathbb{E}^{\Delta}[x_n] = 0$ and $\mathbb{V}^{\Delta}[x_n] = \sigma^2/\Delta$. The length of the period Δ parametrizes these densities because we assume that the quality of the signals (the inverse of their variances) increases with Δ. Later we make precise assumptions on how these distributions vary with Δ.

An example of a signal structure is a "hidden action" agency model in which z_n is completely uninformative about x_n. In a different example the agent has private information about the noise when taking an action: say, the agent observes the mean of a noise distribution. Private information can be interpreted either as the additional information about the environment, firm or market conditions, or as the private information about the agent's productivity shock (mean level of revenue produced with no additional effort, see e.g. Laffont and Tirole [1993], Chapter 2). In Section 4 we show that the results easily generalize to other specifications of private information about cost or productivity of effort, in which the shock also affects marginal values. We also extend the results to the "pure hidden information" model where the agent knows the "noise" realization before taking an action ($z_n \equiv x_n$), while the cost of effort may be expressed in monetary terms, as in the cash-flow diversion or dynamic insurance models.

2.2 The Principal's Problem and Important Curves

A *contract* is a process $\{c_n\}$ that for each period n specifies the agent's compensation c_n as a function of the public history, i.e., the history of reported signals and outputs $(\tilde{z}_0, y_0, \ldots, \tilde{z}_n, y_n)$[4]. A *reporting plan* $\{\widehat{z}_n\}$ and an *action plan* $\{a_n\}$ for the agent are processes that specify the agent's report of private signal and agent's action in each

[3] While our interpretation that consumption, which in principle depends also on the current period's output, flows during the duration of the period seems inconsistent, it is an indirect corollary to our results that consumption independent of current output is with no loss of generality when the period length is small - see the next Section.

[4] We invoke the revelation principle and restrict the set of reports to be the same as the set of signals.

period n as a function of the private history $(z_0, \tilde{z}_0, y_0, \ldots, z_{n-1}, \tilde{z}_{n-1}, y_{n-1}, z_n)$. Since the principal's contract does not depend on the agent's private signals and signals across periods are independent, there is no loss of generality in restricting the plans so that $\widehat{z}_n$ and a_n depend only on $(\tilde{z}_0, y_0, \ldots, \tilde{z}_{n-1}, y_{n-1}, z_n)$ and not on $(z_0, \ldots, z_{n-1})$.

The principal's expected discounted revenue for a contract-plan triple $(\{c_n\}, \{\widehat{z}_n\}, \{a_n\})$ is

$$\Pi(\{c_n\}, \{\widehat{z}_n\}, \{a_n\}) = \tilde{r}\, \mathbb{E}^{\Delta} \left[\sum_{n=0}^{\infty} e^{-r\Delta n} \left(y_n - \Delta c_n \right) \right] = \tilde{r} \Delta \mathbb{E}^{\Delta} \left[\sum_{n=0}^{\infty} e^{-r\Delta n} \left(a_n - c_n \right) \right],$$

while the agent's expected discounted utility is

$$U(\{c_n\}, \{\widehat{z}_n\}, \{a_n\}) = \tilde{r} \Delta \mathbb{E}^{\Delta} \left[\sum_{n=0}^{\infty} e^{-r\Delta n} \left(u\left(c_n\right) - h\left(a_n\right) \right) \right],$$

where the factor $\tilde{r}$ is such that $\tilde{r}\Delta = 1 - e^{-r\Delta}$, and normalizes the sums so that $\tilde{r}\Delta \sum e^{-r\Delta n} = 1$.[5]

Let $\{\widehat{z}_n^*\}$ be the *truthful* reporting plan, in which the agent honestly reveals his private signal. The action plan $\{a_n\}$ is *incentive compatible* (IC) for the contract $\{c_n\}$ if $(\{a_n\}, \{\widehat{z}_n^*\})$ maximizes the agent's utility: for any other plan $(\{a_n'\}, \{\widehat{z}_n'\})$, any N and any realization $(\tilde{z}_0, y_0, \ldots, \tilde{z}_{N-1}, y_{N-1}, z_N)$,

$$\begin{aligned} &\mathbb{E}^{\Delta} \left[\sum_{n=N}^{\infty} e^{-r\Delta n} \left(u\left(c_n\right) - h\left(a_n\right) \right) \Big| (\tilde{z}_0, y_0, \ldots, \tilde{z}_{N-1}, y_{N-1}, z_N), \{a_n\}, \{\widehat{z}_n^*\} \right] \\ \geq\ &\mathbb{E}^{\Delta} \left[\sum_{n=N}^{\infty} e^{-r\Delta n} \left(u\left(c_n\right) - h\left(a_n'\right) \right) \Big| (\tilde{z}_0, y_0, \ldots, \tilde{z}_{N-1}, y_{N-1}, z_N), \{a_n'\}, \{\widehat{z}_n'\} \right]. \end{aligned}$$

For a given agent's reservation utility w, the *principal's problem* consists of finding a contract-action plan $(\{c_n\}, \{a_n\})$ that maximizes his expected discounted revenue among all the incentive compatible plans that deliver an expected discounted utility w to the agent. For any $w \in [0, \bar{u})$, let $F^{\Delta}(w)$ be the principal's value from an optimal IC contract-action plan,

$$F^{\Delta}(w) = \sup \left\{ \Pi(\{c_n\}, \{\widehat{z}_n^*\}, \{a_n\}) \mid \{a_n\} \text{ is IC for } \{c_n\},\ U(\{c_n\}, \{\widehat{z}_n^*\}, \{a_n\}) = w \right\}.$$

We also define $\overline{F}\,(w)$ as the principal's value for an optimal feasible ("first best") contract-action plan (not necessarily IC). It is easy to see that such a plan is stationary, and so $\overline{F}\,(w)$ is just equal to the value of an optimal feasible one period contract-action pair:

$$\overline{F}\,(w) = \max_{a,c} \{ a - c \mid a \in \mathcal{A},\ c \geq 0,\ u\left(c\right) - h\left(a\right) = w \}.$$

[5] Note that $\tilde{r} \to r$ as $\Delta \downarrow 0$.

One can show that $\overline{F}$ satisfies the following ODE:

$$\overline{F}(w) = \max_{a,c}\{(a-c) + \overline{F}'(w)(w + h(a) - u(c))\}. \tag{1}$$

The main theorems of the paper will feature a very related ODE, which will involve an additional term capturing "cost of incentives" (see (6)).

Finally, let $\underline{F} : [0, \overline{u}) \to \mathbb{R}$ be the *retirement curve*. That is,

$$\underline{F}(w) = -u^{-1}(w).$$

Continuation values $(w, \underline{F}(w))$ with $w \in [0, \bar{u})$ are attained by wage contracts that pay the same $c_n = \underline{F}(w)$ in every period (regardless of history), when the agent chooses action $a_n = 0$ in every period. Since such wage contract-action plan is IC, $F^{\Delta} \geq \underline{F}$.

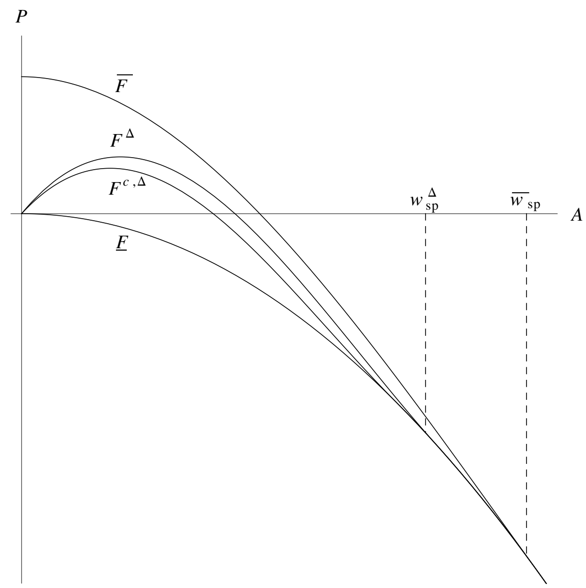

Figure 1: Value functions.

Notice that $F^{\Delta}(0) = 0$. This follows from the limited liability constraint $c \geq 0$ and $u(0) = h(0) = 0$: since the agent can always deviate to exerting no effort, the only way for the agent to receive an expected discounted utility of zero is for the contract to pay zero in every period.[6] Also, there exists $\overline{w}_{sp} \in [0, \bar{u})$ such that $\overline{F}(\overline{w}_{sp}) = \underline{F}(\overline{w}_{sp})$ and $\overline{F}(w) > \underline{F}(w)$ for all $w < \overline{w}_{sp}$. This is because if the agent must receive high expected utility, exerting any positive effort by the agent is too costly for the principal (see Spear

[6]Assumption (A2) below guarantees that if the agent gets a strictly positive expected continuation value when taking a strictly positive effort (which compensates him for the cost of effort), then he would also get a strictly positive expected continuation value from no effort.

and Srivastava [1987], Sannikov [2008]).[7]

Altogether, for any $\Delta > 0$ we have $\underline{F} \leq F^{\Delta} \leq \overline{F}$ (see Figure 1). In particular, this implies that there exist a minimal agent's value w_{sp}^{Δ}, $0 \leq w_{sp}^{\Delta} \leq \overline{w}_{sp}$, such that:

$$F^{\Delta}(w_{sp}^{\Delta}) = \underline{F}(w_{sp}^{\Delta}).$$

2.3 Frequent Actions: Parameterization and Assumptions

We are interested in solving the principal's problem when the period length Δ is small. We assume that while Δ decreases, (Z, X) are normalized signals generated by a fixed distribution (independent of Δ):

(A1) There exists a distribution function $G(x, z)$ with $\mathbb{E}[x] = 0$ and $\mathbb{V}[x] = \sigma^2$, such that for each $\Delta > 0$,

$$G^{\Delta}(x, z) = G(x\sqrt{\Delta}, z\sqrt{\Delta}).$$

Note that $\mathbb{E}^{\Delta}[x] = 0$, and $\mathbb{V}^{\Delta}[x] = \sigma^2/\Delta$. Consequently, the linear interpolation of the process $\{X_{k\Delta}\}_{k\in\mathbb{N}}$ where $X_{k\Delta} = \frac{\Delta}{\sigma}\sum_{n=1}^{k} x_n$, $x_n \sim G_X^{\Delta}$, converges in distribution to a Brownian Motion as $\Delta \to 0$ (Invariance Principle, see e.g. Theorem 4.20 in Karatzas and Shreve [1991]). This implies that the linear interpolations of revenue processes converge in distribution to the continuous time process $\{Y_t\}$ satisfying[8]

$$dY_t = \mathbb{E}^{\Delta}[a_t]dt + \sigma dB_t,$$

where $\{B_t\}$ is a Brownian Motion.

We also make some assumptions on the distribution of noise:

(A2) Z has a finite support $\mathcal{Z}$ and for any $z \in \mathcal{Z}$ the distribution of X conditional on $[Z = z]$ has density function $g(x|z)$. There exist $\bar{\delta}, \bar{M} > 0$ such that for all $z, z' \in \mathcal{Z}$ and $\delta : \mathbb{R} \to [0, \bar{\delta}]$, the three integrals

$$\int_{\mathbb{R}} \frac{g'(x - \delta(x)|z)^2}{g(x|z)} dx, \quad \int_{\mathbb{R}} \frac{g(x - \delta(x)|z')^2}{g(x|z)} dx \text{ and } \int_{\mathbb{R}} |g''(x - \delta(x)|z)|\, dx$$

are bounded above by $\bar{M}$.

[7] Formally, marginal cost of effort is bounded below by $\gamma > 0$ for positive actions while marginal utility of consumption converges to zero as consumption increases. This excludes any interior solution for $\overline{F}(w)$ for sufficiently high w since such solution must satisfy $h'(a) = u'(c)$, for $c > u^{-1}(w)$ such that $u(c) - h(a) = w$.

[8] For $y_n = \Delta[x_n + a(z_n)]$ and $B_{k\Delta} = \frac{1}{\sigma}\sum_{n=1}^{k}(y_n - \Delta\mathbb{E}^{\Delta}[a(z_n)])$, $(x_n, z_n) \sim G^{\Delta}$, the linear interpolation of the process $\{B_{k\Delta}\}_{k\in\mathbb{N}}$ converges in distribution to the Brownian Motion as $\Delta \to 0$. This is because $B_{k\Delta} = \frac{\Delta}{\sigma}\sum_{n=1}^{k}(x_n + \xi_n)$, with $\xi_n = a(z_n) - \mathbb{E}^{\Delta}[a(z_n)]$, and so each $\{B_{k\Delta}\}_{k\in\mathbb{N}}$ is a sum of two continuous path processes, one converging weakly to the Brownian Motion and the other to the process identically equal to zero (see Whitt [1980]).

3 Results

3.1 Solution to the Principal's Problem: Values

We now present a heuristic derivation of our results. The proofs of the corresponding lemmas, which in particular justify all the assumptions and simplifications made here, are postponed until Section A. Also, in this section we focus on calculating the principal's value for the optimal contract-action plan as the period length Δ shrinks to 0. The construction of the incentive compatible contract-action plans that achieve those values is postponed until Section 3.3.

Fix a period length Δ, and suppose that $f : [0, \bar{u}) \to \mathbb{R}$ represents a set of feasible continuation values in period 1. That is, for each $w_+ \in [0, \bar{u})$ in period 1, there is some incentive compatible contract-action plan with value w_+ for the agent and value $f(w_+)$ for the principal. Consider now the principal's problem in period 0, when he is constrained to deliver expected discounted utility $w \in [0, \bar{u})$ to the agent. Denoting $I = [0, \bar{u})$[9] the value $T_I^\Delta f(w)$ of this problem is

$$\sup_{a,c,W \in I} \ \mathbb{E}^\Delta \Big[\tilde{r}\Delta[a(z) - c(\Delta[x + a(z)], z)] + e^{-r\Delta} f(W(\Delta[x + a(z)], z)) \Big] \tag{2}$$

$$\text{s.t.} \quad w = \mathbb{E}^\Delta \Big[\tilde{r}\Delta[u(c(\Delta[x + a(z)], z)) - h(a(z))] + e^{-r\Delta} W(\Delta[x + a(z)], z) \Big] \quad \text{(PK)}$$

$$(z, a(z)) \in \arg \max_{\tilde{z} \in \mathcal{Z}, \hat{a} \in \mathcal{A}} \ \mathbb{E}^\Delta \Big[\tilde{r}\Delta[u(c(\Delta[x + \hat{a}], \tilde{z})) - h(\hat{a})] + e^{-r\Delta} W(\Delta[x + \hat{a}], \tilde{z}) \mid z \Big] \quad \text{(IC)}$$

Thus, computing the value $T_I^\Delta f(w)$ boils down to the optimal choice of three functions a, c and W, with (reported) signal z and observed revenue y as arguments. $a(z)$ is the recommended action and $c(y, z)$ is the agent's consumption in period 0, while $W(y, z)$ is the agent's continuation value from period 1 onward. It is also required that $W(y, z) \in I$ (the domain of f) for each (y, z). The *promise keeping* constraint (PK) in (2) requires that the expected discounted utility in period 0 is indeed w. The *incentive compatibility* constraint (IC) requires that for each signal z in period 0, it is optimal for the agent to report truthfully and then take the recommended action $a(z)$.

In the paper we show that when the period length Δ is short, both the objective function and the constraints of (2) can be simplified in several ways, only slightly altering the value of the problem. (i) We only consider constant consumption functions $c(y, z) \equiv c$. (ii) The incentive compatibility constraint is replaced by a weaker truthtelling constraint together with only local (first-order) incentives for the choice of action. (iii) The arguments "$\Delta[x + a(z)]$" in the formula above are approximated by simply "Δx ".[10] (iv) The function f is approximated by its second order Taylor expansion around w. (v) The feasibility constraint $W \in I$ is dropped.

[9] Later we will consider arbitrary intervals I for the definition of T_I^Δ.

[10] Note that the standard deviation of x is $1/\sqrt{\Delta} >> a(z)$. Also $a(z)$ is removed not in the (IC) but already in the (FOC) constraint.

The value $T^{\Delta,q}f(w)$ of the simplified problem is

$$\begin{aligned}
\sup_{a,c,W} \quad & \mathbb{E}^{\Delta}\Big[\tilde{r}\Delta[a(z)-c]+e^{-r\Delta}[f\left(w\right)+f'\left(w\right)\left(W\left(\Delta x,z\right)-w\right)+\frac{f''\left(w\right)}{2}(W(\Delta x,z)-w)^2]\Big] \\
\text{s.t.} \quad & w=\mathbb{E}^{\Delta}\left[\tilde{r}\Delta[u(c)-h(a(z))]+e^{-r\Delta}W(\Delta x,z)\right] \qquad (\mathrm{PK}_q) \qquad (3) \\
& \int_{\mathbb{R}} W(\Delta x,z)g^{\Delta}(x|z)dx \geq \int_{\mathbb{R}} W(\Delta x,\tilde{z})g^{\Delta}(x|z)dx \quad \forall z,\tilde{z} \qquad (\mathrm{TR}_q) \\
& \tilde{r}\Delta h'(a(z))=-e^{-r\Delta}\int_{\mathbb{R}} W(\Delta x,z)g^{\Delta\prime}(x|z)dx \quad \forall z \qquad (\mathrm{FOC}_q)
\end{aligned}$$

Given the quadratic approximation of f it is only the first two moments of W that matters to the principal. However, the first moment is fully pinned down by the promise keeping constraint (for a given choice of c and function a). Substituting this value, the objective function becomes approximately:

$$\mathbb{E}^{\Delta}\Big[\tilde{r}\Delta[a(z)-c]+f'(w)(w-u(c)+h(a(z)))+e^{-r\Delta}\Big[\frac{f''(w)}{2}\mathbb{V}^{\Delta}[(W(\Delta x,z)]+f\left(w\right)\Big]\Big].$$

The simplified objective function only depends on the second moment of the continuation value function. Assuming that $f''<0$, the principal would like to minimize it.

Finally, we split the principal's problem in two steps. First, he chooses an expected effort $\bar{a}$, an expected cost of effort $\bar{h}$ and consumption c. Second, he chooses an action scheme a with mean effort $\bar{a}$ and mean cost of effort $\bar{h}$, and continuation value function W with minimal variance among those satisfying the "relaxed incentive constraints" (TR_q) and (FOC_q). We will see that in the relevant range of continuation values, we may assume $\bar{a}>0$. Changing variables and substituting[11] the problem is approximately equal to

$$f(w)+\sup_{\bar{a}>0,\bar{h},c}\tilde{r}\Delta\left\{(\bar{a}-c)+f'\left(w\right)\left(w-u\left(c\right)+\bar{h}\right)+\frac{\tilde{r}f''\left(w\right)}{2}\Theta(\bar{a},\bar{h})-f(w)\right\} \qquad (4)$$

where

$$\begin{aligned}
\Theta(\bar{a},\bar{h}) \quad = \quad \inf_{a,v} \quad & \mathbb{E}\left[v(x,z)^2\right] \qquad (5) \\
\text{s.t.} \quad & \bar{a}=\mathbb{E}[a(z)], \quad \bar{h}=\mathbb{E}[h(a(z))], \\
& \int_{\mathbb{R}} v(x,z)g(x|z)dx \geq \int_{\mathbb{R}} v(x,\tilde{z})g(x|z)dx \quad \forall z,\tilde{z} \qquad (\mathrm{TR}_{\Theta}) \\
& h'(a(z))=-\int_{\mathbb{R}} v(x,z)g'(x|z)dx \quad \forall z. \qquad (\mathrm{FOC}_{\Theta})
\end{aligned}$$

We call Θ the Variance of Continuation Values (VCV) function.

Overall, equation (4) implies that the function F that solves the following equation:

$$F(w)=\sup_{\bar{a}>0,\bar{h},c}\left\{(\bar{a}-c)+F'(w)\left(w-u(c)+\bar{h}\right)+\frac{1}{2}F''(w)r\Theta(\bar{a},\bar{h})\right\}. \qquad (6)$$

[11] Substitute $v(x,z)=e^{-r\Delta}W(x/\sqrt{\Delta},z/\sqrt{\Delta})/\left[\tilde{r}\sqrt{\Delta}\right]$ and $g(x|z)=g^{\Delta}(x/\sqrt{\Delta}\,|\,z/\sqrt{\Delta})/\sqrt{\Delta}$.

is "almost" a fixed point of the Bellman operator T_I^Δ. On the other hand, the function F^Δ characterizing the principal's value from optimal IC contract-action plans is exactly a fixed point of T_I^Δ (see Abreu, Pearce, and Stacchetti [1986, 1990] and Spear and Srivastava [1987]). Using the fact that T_I^Δ is a contraction, this implies that F^Δ is "close" to F, and indeed F^Δ converges to F as the period length Δ shrinks to zero (see Proposition 5 and Lemma 6).

More precisely, the following is the first main result of the paper (its proof is in Section A). Consider the HJB equation (6) with the boundary conditions:

$$F(0) = 0 \tag{7}$$

and $F'(0)$ equal to the largest slope such that for some $w_{sp} > 0$

$$F\left(w_{sp}\right) = \underline{F}\left(w_{sp}\right) \quad \text{and} \quad F'\left(w_{sp}\right) = \underline{F}'\left(w_{sp}\right). \tag{8}$$

The first two conditions are analogous to the conditions that must be satisfied by F^Δ (see the end of Section 2.2); the last is the *smooth pasting condition.*

Theorem 1 *Equation (6) with the boundary conditions (7) and (8) has a unique solution F. For any agent's promised value $w \in [0, w_{sp}]$, $F(w)$ is the limit of the principal's value for an optimal contract as the period length Δ shrinks to zero:*

$$\lim_{\Delta \to 0} \sup_{w \in [0, w_{sp}]} \left(F(w) - F^\Delta(w)\right) = 0,$$

while for $w > w_{sp}$, $F(w)$ provides an upper bound:

$$F(w) \geq \lim_{\Delta \to 0} F^\Delta(w) \quad \textit{for all} \quad w > w_{sp}.$$

Theorem 1 shows that the limit of values of optimal contracts, as Δ shrinks to zero, can be characterized analytically using a two step solution procedure. In the first step one solves a family of static problems (5), parametrized by a pair $\left(\bar{a}, \bar{h}\right)$. In the second step one solves the differential equation (6).

The second step is relatively standard for characterizing the value function of a continuous-time optimal control problem. In fact, equation (6) is the Hamilton-Belman-Jacobi (HJB) equation for the optimal value function of the following problem:

$$\begin{aligned} F(w) \;&=\; \sup_{\{\bar{a}_t > 0, \bar{h}_t, c_t\}} \mathbb{E}\left[\int_0^\infty r[\bar{a}_t(W_t) - c_t(W_t)]e^{-rt}dt\right] \qquad (9) \\ \text{s.t.} \qquad & dW_t = r\left[W_t - u(c_t(W_t)) + \bar{h}_t(W_t)\right]dt + r\sqrt{\Theta(\bar{a}_t(W_t), \bar{h}_t(W_t))}\, dB_t, \quad W_0 = w. \end{aligned}$$

(with appropriate boundary conditions.)

The above is an unconstrained maximization problem for the principal. Intuitively, the constraint of incentive compatibility is reduced to deriving the VCV function Θ, i.e.

the first step in the procedure. We provide examples where we solve $\Theta(\bar{a}, \bar{h})$ in Section 3.2. Here we want to emphasize a couple of its properties.

First, notice that the VCV function Θ depends on the distribution G. Thus, different distributions of private signals z and/or "noise" x involve different costs of incentive provision. Second, $\Theta(\bar{a}, \bar{h})$ depends not only on the expected effort but also on the expected cost of effort. In the pure hidden action model, when z takes only one value, the only feasible choice of $\bar{h}$ is $h(\bar{a})$. When the agent has some private information, however, the variance might decrease for action schemes that condition on the private signal, for which $\bar{h} > h(\bar{a})$. Third, we want to stress that solving $\Theta(\bar{a}, \bar{h})$ is a purely static, and a fairly easy optimization problem (see Section 3.2).

Lastly, $\sqrt{\Theta(\bar{a}, \bar{h})}$ is the counterpart of the diffusion coefficient of the continuation value process in the continuous time models. One difference is crucial: in continuous time models, it is a direct consequence of the Martingale Representation Theorem that the continuation value increments must be linear in "noise". In our case, this would correspond to an additional restriction in problem (5) that v is a linear function. Thus, in the pure hidden action model, the (FOC_Θ) pins down $\Theta(\bar{a}, h(\bar{a})) = [h'(\bar{a})\,\sigma]^2$ (see Example 1). In our case we do not and cannot impose linearity restriction: the principal typically can do better by using nonlinear continuation value functions.

That Θ captures all the relevant information of the distribution $G(x, z)$ facilitates the comparative statics analysis of how the information structure affects the optimal values, as we illustrate in the next section. The following result is important for our analysis (the proof is in the online Appendix D).

For an arbitrary function $\Theta : \mathbb{R}^2_+ \to \mathbb{R}_+ \cup \{\infty\}$ define $D^\Theta_+ = \{(\bar{a}, \bar{h}) \mid \bar{a} > 0, \Theta(\bar{a}, \bar{h}) < \infty\}$.

Proposition 1 *Consider two functions Θ, $\underline{\Theta}$, and let F^Θ and $F^{\underline{\Theta}}$ be corresponding solutions to (6) with the boundary conditions (7) and (8).*
(i) If $\Theta \geq_{D^\Theta_+} \underline{\Theta}$ then $F^\Theta(w) \leq F^{\underline{\Theta}}(w)$ for all $w \in [0, w^\Theta_{sp}]$.
(ii) If $\Theta >_{D^\Theta_+} \underline{\Theta}$ then $F^\Theta(w) < F^{\underline{\Theta}}(w)$ for all $w \in (0, w^\Theta_{sp})$.

Recall that $\overline{F}$ is the value of an optimal feasible (first best) one period contract-action pair. The following proposition shows that when $r\Theta$ converges to zero, the function F in Theorem 1 converges to $\overline{F}$ for $w > 0$. In other words, the first best value $\overline{F}$ is achievable by the optimal contracts with short period length, when either the cost of incentives or the discount rate vanishes (see Figure 2; the proof is in the online Appendix D).[12]

Proposition 2 *Let F be a solution to (6) with the boundary conditions (7) and (8). Then for every $\delta > 0$ there is $\varepsilon > 0$ such that if $r\Theta \leq \varepsilon$ then*

$$F \geq \overline{F}(w) - \delta \quad \text{for all} \quad w \in [\delta, \overline{w}_{sp}].$$

[12]Note that for $w > 0$, the function $\overline{F}$ solves (1), which is the HJB equation (6) with $r\Theta \equiv 0$. But unlike F^Δ or F in the Theorem 1, $\overline{F}$ does not satisfy the boundary condition $\overline{F}(0) = 0$.

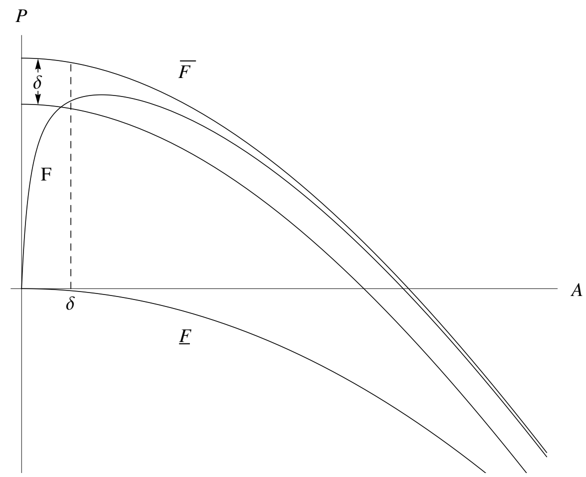

Figure 2: Proposition 3.

3.2 Examples and Comparative Statics

The following example shows that the value of the optimal contract-action plan formulated directly in continuous time (Sannikov [2008]) agrees with the limit of values of discrete time optimal action-plans for a particular signal structure.

Example 1 *Consider the pure hidden action case when X is normally distributed with mean 0 and variance σ^2. For any $\bar{a} > 0$,*[13]

$$\begin{aligned} \Theta(\bar{a}, h(\bar{a})) &= \min_v \int v(x)^2 g(x) dx, \\ s.t. \qquad & h'(\bar{a}) = -\int_{\mathbb{R}} v(x) g'(x) dx \end{aligned} \tag{10}$$

The optimal solution of this problem is $v(x) = h'(\bar{a})x$ and $\Theta(\bar{a}, h(\bar{a})) = [h'(\bar{a})\sigma]^2$. Also, $\Theta(\bar{a}, \bar{h}) = \infty$ for all $\bar{h} \neq h(\bar{a})$. Therefore the HJB equation (6) becomes

$$F(w) = \sup_{\bar{a}>0,c} \left\{ (\bar{a} - c) + F'(w)(w + h(\bar{a}) - u(c)) + \frac{1}{2} F''(w) r\sigma^2 h'(\bar{a})^2 \right\},$$

which is exactly Sannikov's equation (5).

The example shows that in the case of pure hidden action models with normal noise the value of the optimal contract depends on the single parameter of the distribution of noise, its variance. We generalize the example in the following way.

[13] In a pure hidden information model, when $\mathcal{Z} = \{z\}$, we write simply "$g(x)$" instead of "$g(x|z)$".

Lemma 1 *Consider a pure hidden action model with density $g_X(x)$. Then, for all $\bar{a} > 0$, $\Theta(\bar{a}, \bar{h}) = \infty$ if $\bar{h} \neq h(\bar{a})$ and*

$$\Theta(\bar{a}, h(\bar{a})) = \frac{h'(\bar{a})^2}{\mathcal{I}_g}, \quad \textit{where} \quad \mathcal{I}_g = \int \frac{g'(x)^2}{g(x)}\,dx.$$

Proof. That $\Theta(\bar{a}, \bar{h}) = \infty$ for all $\bar{h} \neq h(\bar{a})$ is clear. Just as in Example 1, the solution of problem (10) for $\bar{a} > 0$, as characterized by the necessary first order conditions, is $v(x) = C\frac{g'(x)}{g(x)}$, where the incentive compatibility constraint implies that $C = -\frac{h'(\bar{a})}{\mathcal{I}_g}$. Consequently, $\Theta(\bar{a}, h(\bar{a})) = \frac{h'(\bar{a})^2}{\mathcal{I}_g}$, when $\bar{a} > 0$. ■

The Lemma establishes that the variance of incentive transfers, and thus the value of optimal contract-action plans (when Δ shrinks to 0) depends on the single parameter $\mathcal{I}_g$ of the underlying noise distribution. The parameter $\mathcal{I}_g$ is the well known *Fisher information quantity* in Bayesian statistics. The relevance of the Fisher information quantity for contracting is, to the best of our knowledge, new. Yet the intuition behind its relevance is straightforward: it is a measure of informativeness of the public signal about the action of the agent. Its high value diminishes the information asymetry between the principal and the agent, and allows for the incentives to be provided more efficiently (by transfers with lower variance).

Consider the following example.

Example 2 *We study pure hidden action models for three cases of noise distribution, each with variance σ^2: (i) normal distribution, (ii) double exponential distribution and (iii) "linear" distribution, with corresponding densities:*[14]

$$\begin{aligned} g^n(x) &= \frac{1}{\sigma\sqrt{2\pi}}\,e^{-\left[\frac{x}{2\sigma}\right]^2}, \quad x \in \mathbb{R} \\ g^{2e}(x) &= \frac{\lambda}{2}\,e^{-\lambda|x|}, \quad x \in \mathbb{R} \\ g^l(x) &= c - c^2|x|, \quad |x| \leq 1/c, \end{aligned}$$

for $\lambda = \frac{\sqrt{2}}{\sigma}$ and $c = \frac{1}{\sigma\sqrt{6}}$. The corresponding Fisher information quantities are:

$$\mathcal{I}_{g^n} = 1/\sigma^2, \quad \mathcal{I}_{g^{2e}} = 2/\sigma^2, \quad \textit{and} \quad \mathcal{I}_{g^l} = \infty.$$

In particular, in the "linear" distribution case the incentives are costless and the first-best is achievable (Proposition 2). Intuitively, with bounded support of the noise, agent's defection from the prescribed action plan gives rise to signals that would not occur otherwise, and those signals have sufficiently high probability (density has sufficient mass at the extremes).

[14] Formally, the "linear" distribution does not satisfy our assumption (A2) as it results in infinite Fisher information quantity. But one may consider approximations with the density at the extremes of the support changed to, say, quadratic functions, resulting in finite but arbitrarily large Fisher information quantities.

A direct consequence of Lemma 1 and Proposition 1 is the following:

Corollary 1 *In the pure hidden action model with density $g(x)$, the limit value of the optimal contract-action plans as Δ shrinks to zero is increasing in the Fisher information quantity $\mathcal{I}_g$.*

Let us now consider a setting in which the agent has some private information about the environment.

Example 3 *Consider the case when the agent privately observes the mean of a normally distributed random component in revenue, and the private signal takes two values with equal probability. Formally, $\mathcal{Z} = \{l, h\} \in \mathbb{R}$, $l < h$, $X|Z \sim N(Z, 1)$, and $\mathbb{P}(Z = l) = \mathbb{P}(Z = h) = 1/2$. We claim that*

$$\Theta(\bar{a}, \bar{h}) = \min_{a_l, a_h} \frac{h'(a_h)^2 + Ch'(a_l)^2}{2} \quad s.t. \quad \bar{a} = \frac{a_l + a_h}{2} \text{ and } \bar{h} = \frac{h(a_l) + h(a_h)}{2},$$

where $C = \frac{1+e^{(h-l)^2}}{1+e^{(h-l)^2}-(h-l)^2} > 1$.

Indeed, consider the auxiliary problem:

$$\begin{aligned}
\hat{\Theta}(a_l, a_h) &= \inf_v \frac{1}{2}\left\{\int_{\mathbb{R}} v(x,h)^2 g(x|h)dx + \int_{\mathbb{R}} v(x,l)^2 g(x|l)dx\right\} \\
& \quad s.t.\ h'(a_z) = -\int_{\mathbb{R}} v(x,z) g'(x|z) dx \quad z = l, h \\
& \quad 0 = \int_{\mathbb{R}} \left(v(x,h) - v(x,l)\right) g(x|h) dx.
\end{aligned}$$

The auxiliary problem assumes that the "downward constraint" of (TR_Θ) is binding and the "upward constraint " of (TR_Θ) is redundant, which can be easily verified.

The optimal solution to the auxiliary problem is

$$v(x,l) = \lambda_l \frac{g'(x|l)}{g(x|l)} + \lambda \frac{g(x|h)}{g(x|l)} \quad \text{and} \quad v(x,h) = \lambda_h \frac{g'(x|h)}{g(x|h)} - \lambda,$$

where $(\lambda_l, \lambda_h, \lambda)$ are Lagrange multipliers for the corresponding constraints. Solving the system of three equations for $(\lambda_l, \lambda_h, \lambda)$ and substituting into the value function yields the result.[15]

[15] Since

$$\int \frac{g'(x|z)^2}{g(x|z)} = 1 \quad \text{for } z = l, h, \quad \int \frac{g'(x|l)}{g(x|l)} g(x|h) = -(h-l) \quad \text{and} \quad \int \frac{g(x|h)^2}{g(x|l)} = e^{-(h-l)^2},$$

In the example above, for a fixed expected cost of effort $\bar{a}$, the function $\Theta(\bar{a},\bar{h})$ is minimized for $\bar{h} > h(\bar{a})$, i.e. when $a_l \neq a_h$.[16]. Thus, the example illustrates that in a model with private information, in any period, for a fixed mean effort $\bar{a}$ there arises a nontrivial tradeoff between two sorts of implementation costs (see HJB equation (6)). The first is the *direct cost of effort*, which is proportional to $\bar{h} = \mathbb{E}[h(a(z))]$, and the second is the *cost of incentives*, which is proportional to $\Theta(\bar{a},\bar{h})$. On one hand, given a convex cost function, a "flat" effort scheme $a(z) \equiv \bar{a}$ minimizes the direct cost. On the other hand, the cost of incentives might be minimized by the effort scheme that uses private information and does not satisfy $a(z) \equiv \bar{a}$, which implies that $\bar{h} > h(\bar{a})$. How this is resolved depends on the relative "prices" of each cost in the differential equation (6), given by F' and F'' respectively. The tradeoff is implicit in the HJB equation (6), and absent in the continuous time counterpart (see Example 1).

Example 1 provides one justification for Sannikov's continuous-time model: its optimal value function F agrees with the limit of F^{Δ} (as Δ shrinks to zero) for a pure hidden action model with normal distribution. The following Lemma provides a different justification: F is the lower bound of the limit of F^{Δ} for any model with an arbitrary information structure that has the same variance of noise.

Lemma 2 *Let $G(x,z)$ be any distribution such that $\mathbb{V}[x] = \sigma^2$. Let Θ be its corresponding VCV function, and let Θ^n be the VCV function of the pure hidden action model with normal noise and $\mathbb{V}[x] = \sigma^2$. Then $\Theta \leq \Theta^n$ and $F^{\Theta^n} \leq F^{\Theta}$.*

Proof. The optimal policy function for $\Theta^n(\bar{a}, h(\bar{a}))$ with $\bar{a} > 0$ in the pure hidden action case with normal distribution has a linear incentive transfer function $v(x) = h'(\bar{a})x$, and $\Theta^n(\bar{a}, h(\bar{a})) = [h'(\bar{a})\sigma]^2$ (Example 1). This transfer function provides not only "ex-ante", but also "ex-post" incentives, thus inducing the agent to a constant effort scheme $a(z) = \bar{a}$ under any distribution of signals. Therefore (a,v) is feasible for $\Theta(\bar{a}, h(\bar{a}))$ and as long as the variance of noise X is σ^2, the variance of incentive transfers is $[h'(\bar{a})\sigma]^2$. Thus $\Theta \leq \Theta^n$ and so the Lemma follows from Proposition 1, part (i). ■

3.3 Solution to the Principal's Problem: Contract-Action Plans

In this section we show how to construct contract-action plans that are relatively simple yet approximately optimal as the period length is short.

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substituting the solution into the constraints, the Lagrange multipliers must satisfy

$$\lambda_l - \lambda(h-l) = -h'(a_l), \quad \lambda_h = -h'(a_h) \quad \text{and} \quad -\lambda + \lambda_l(h-l) - \lambda e^{-(h-l)^2} = 0.$$

Therefore,

$$\lambda = \lambda_l \frac{(h-l)}{1+e^{(h-l)^2}}, \; \lambda_l = -h'(a_l)\frac{1+e^{(h-l)^2}}{1+e^{(h-l)^2}-(h-l)^2}, \; \lambda_h = -h'(a_h)$$

[16] Note that $\frac{\partial}{\partial a_h}\frac{h'(a_h)^2 + Ch'(2\bar{a}-a_h)^2}{2}\Big|_{a_h=\bar{a}} = h'(\bar{a})h''(\bar{a})(1-C) < 0.$

Recall first the construction of fully optimal contract-action plans using agent's continuation value as a state variable (Abreu, Pearce, and Stacchetti [1986, 1990] and Spear and Srivastava [1987]). Fix a period length Δ and for each of the agent's feasible continuation values w in $[0, \bar{u})$, let (a_w, c_w, W_w) be an optimal policy for the problem $T^{\Delta}_{[0,\bar{u})}F^{\Delta}(w)$ (see (2)). Starting with the exogenous reservation utility of the agent w_0 as an initial state variable, in any period n, the agent's compensation is given by $c_{w^n}(y^n, \tilde{z}^n)$, he takes action $a_{w^n}(z^n)$, while the law of motion of the state variable is given by $w_{n+1} = W_{w^n}(y^n, \tilde{z}^n)$. Iterating, for any reservation utility w_0, the family of optimal policies generates a contract-action plan, which is incentive compatible and optimal, and provides utility w_0 for the agent.

Our approximately optimal contract-action plans are also generated by a family of policies. Instead of using optimal policies for $T^{\Delta}_{[0,\bar{u})}F^{\Delta}(w)$ as above, we use *simple* policies (Definition 1 below). Simple policies are not derived from the solutions of problem $T^{\Delta}_{[0,\bar{u})}$ (see (2)) applied to F^{Δ}, but from the solutions of the approximate problem $T^{\Delta,q}$ (see (4)) applied to F (as in Theorem 1).

We will need the following, stronger version of the VCV function. For any $\varepsilon > 0$, define the function $\Theta^{\varepsilon}(\bar{a}, \bar{h})$ just as in (5), but with the (TR_{Θ}) strengthened to

$$\int_{\mathbb{R}} v(x, z)g(x|z)dx \geq \int_{\mathbb{R}} v(x, z')g(x|z)dx + 3\varepsilon \quad \forall z \neq z' \quad (\mathrm{TR}_{\Theta,\varepsilon}).$$

Definition 1 *Fix a period length $\Delta > 0$ and an approximation error $\varepsilon > 0$, and let F be as in Theorem 1. For any agent's promised value $w \in [0, w_{sp}]$ we define a* simple policy *(a, c, W) as follows. Let $(\bar{a}, \bar{h}, c)$ be an ε-suboptimal policy of the HJB equation (6) at w, and let (α, v) be an ε-suboptimal policy for the corresponding problem $\Theta^{\varepsilon}(\bar{a}, \bar{h})$.*

If $w \in \left[\Delta^{1/3}, w_{sp} - \Delta^{1/3}\right]$ let (see Figure 3)

$$\begin{aligned} &c(y, z) = c, \\ &W(y, z) = C + \sqrt{\Delta}\tilde{r}e^{r\Delta} \times v(y\sqrt{\Delta}, z\sqrt{\Delta})\mathbf{1}_{|v|\leq M}, \\ &a(z) \text{ is an action that satisfies the (IC) constraint in (2) for the } (c, W) \text{ above,} \end{aligned} \tag{11}$$

where M is a (large) constant that depends only on ε (see Definition 3 in Appendix B) and C is chosen so that the promise keeping (PK) is satisfied.[17]

If $w \notin \left[\Delta^{1/3}, w_{sp} - \Delta^{1/3}\right]$ let

$$\begin{aligned} c(y) &= u^{-1}(w), \\ W(y) &= w, \\ a(z) &= 0. \end{aligned} \tag{12}$$

For any reservation utility $w \in [0, w_{sp}]$ for the agent, a simple contract-action plan *is that generated by the set of simple policies.*

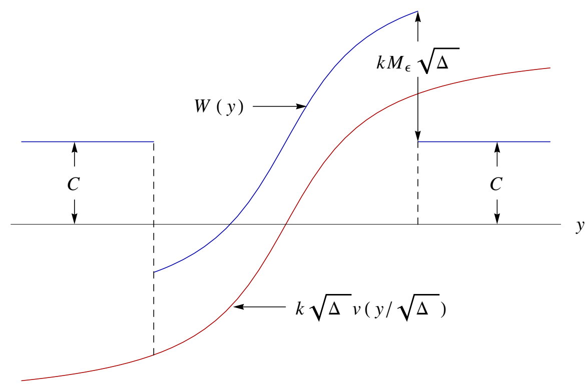

Figure 3: Continuation value function for a fixed z, where $k = \tilde{r}e^{r\Delta}$.

In a simple contract-action plan, as long as it stays within $\left[\Delta^{1/3}, w_{sp} - \Delta^{1/3}\right]$, the agents contintuion value changes from period to period, driven by the public signals and reports. Once it falls outside of this interval, the plan becomes stationary and pays the agent in every period a fixed wage $u^{-1}(w)$ and requires no effort, delivering a continuation value w to the agent (and $\underline{F}(w)$ to the principal).

The following Theorem is the second main result of the paper. It shows that for small approximation error and short period length any simple contract-action plan is fully incentive compatible and almost optimal (proof is in Appendix A).

Theorem 2 *Let F be as in Theorem 1 and fix a period length Δ and an approximation error $\varepsilon > 0$. For sufficiently small Δ and ε, for any agent's reservation utility $w \in [0, w_{sp}]$, a corresponding simple contract-action plan for w is incentive compatible and $[O(\varepsilon) + O(\Delta^{1/3})]$-suboptimal.*

In the case of pure hidden action, we follow up on Lemma 1 and Example 2 from the previous section but now in the context of contract-action plans. We saw there that the solution of $\Theta(\bar{a}, h(\bar{a}))$ is a function v that is linear in the likelihood ratio of the noise density. Therefore, in simple policies, the continuation value function will be a truncated linear function of the likelihood ratio.

Lemma 3 *Consider a pure hidden action model with density $g(x)$. Then, for each $w \in [0, w_{sp}]^{\Delta}$, a corresponding simple policy (a, c, W) is such that*

$$W(y) = C - \sqrt{\Delta}\tilde{r}e^{r\Delta}\lambda(y) \times \mathbf{1}_{|\lambda(y)| \leq M_\varepsilon} \quad \textit{where} \quad \lambda(y) = \frac{h'(\bar{a})}{\mathcal{I}_g} \times \frac{g'(y/\sqrt{\Delta})}{g(y/\sqrt{\Delta})}$$

and $\bar{a}$ and C are as in Definition (1).

[17] Lemma 14 in Section A shows that in the definition of the action scheme a above, if Δ is sufficiently small the global incentive constraint (IC) can be replaced by the local incentive constraint, for all z.

Example 4 *Consider again pure hidden action models for normal, double exponential and "linear" distribution, all with variance* σ^2*. Then, by Lemma 3, simple contract-action plans have continuation values processes given by*

$$\begin{aligned} W^n(y) &= C_1 + C_2 \times y \times \mathbf{1}_{|y| \leq C_3}, \\ W^{2e}(y) &= \begin{cases} \overline{C}, \text{ when } y \geq 0 \\ \underline{C}, \text{ when } y < 0 \end{cases}, \\ W^l(y) &= C_1 + C_2 \times \frac{sgn(y)}{(1-|y|)} \times \mathbf{1}_{(1-|y|)^{-1} \leq C_3}, \end{aligned}$$

for appropriate constants, as in Lemma 3.

In the following we ask the question whether, as the period length shrinks, the details of the signal structure matter for the design of approximately optimal contracts. Recall that in the case of values (Theorem 1) the dependence was fully captured by the VCV function Θ. However, note that the simple contract-action plans in Lemma 3 and Example 4 look very different for different noise distributions, even if they share the same function Θ: e.g. the case of the normal distribution with variance 1 and a double-exponential distribution with variance 2 (Theorem 1 and Example 4). It is not difficult to establish that, in the pure hidden action model, an optimal contract-action plans *requires* a continuation value process that is close to linear in likelihood ratios, as in Lemma 3. Thus, for example, while continuation values that are linear in revenue will work for the normal noise, they will be very suboptimal when the noise is double-exponential. One would like to conclude from this that there is no single contract that will work for two pure hidden action models with different noise structures.

The following Proposition establishes that this conclusion is in fact correct. We note that the conclusion requires a more elaborate argument than the discussion above suggests, as the continuation value process is defined endogenously, relative to the noise structure (the same contract gives rise to different processes, for different noise structures).

Consider two noise distributions with densities g and γ that have the same Fisher information quantity but linearly independent likelihood ratios:

$$\mathcal{I}_g = \mathcal{I}_\gamma, \quad \inf_C \int \left[\frac{g'(x_g)}{g(x_g)} - C\frac{\gamma'(x_\gamma)}{\gamma(x_\gamma)} \right]^2 g(x_g)\,\gamma(x_\gamma)\, dx_g dx_\gamma > 0. \tag{13}$$

Proposition 3 *Consider two pure hidden action models with noise densities* g *and* γ *that satisfy (13). For every* $w_g, w_\gamma \in (0, w_{sp})$ *there exists* $\delta > 0$*, such that for sufficiently small* Δ *there is no contract* $\{c_n\}$ *that is* δ*-suboptimal for the two distributions and delivers values* w_g *and* w_γ*.*

The proof is in the online Appendix E. The Proposition compares the contracts for the special case of signal structures with pure hidden action and the same values of the optimal contracts (as period length shrinks). While this is the most relevant case, as this is exactly when one would suspect the same contract to work, we also comment in the Appendix how to extend the proof to the case of arbitrary two signal structures.

A different way to interpret the result is to say that knowing the optimal continuous-time contract provides little guidance as to how the optimal discrete time contracts look like, no matter how short is the period length. Such contracts must depend on the distribution of noise in the discrete-time models, as in Theorem 2.

On the other hand, in the context of pure hidden action models with the same Fisher information quantities, the simple contract-action plans converge in distribution to a unique continuous-time contract, as we argue below. For any $\mathcal{I}_g > 0$, $w \in [0, \bar{u})$ and continuous time Brownian Motion process $\{B_t\}$, consider a continuous time process $\{W_t\}$ that starts at w and satisfies the stochastic differential equation:

$$dW_t = r\left(W_t - u\left(c\left(W_t\right)\right) + h\left(a\left(W_t\right)\right)\right)dt + r\frac{h'\left(a\left(W_t\right)\right)}{\sqrt{\mathcal{I}_g}}dB_t, \tag{14}$$

where $c\left(W_t\right)$ and $a\left(W_t\right)$ are the minimizers[18] in the solution of (6) with the boundary conditions (7) and (8), together with:

$$W_t = W_\tau, \text{ for } t \geq \tau,$$

where τ is a random time when W_t hits 0 or $w_{sp.}$ This process generates a pair of continuous time processes $(\{c_t\}, \{a_t\})$:

$$a_t = \begin{cases} a\left(W_t\right), \text{ for } t < \tau \\ 0, \text{ for } t \geq \tau \end{cases} \quad \text{and} \quad c_t = \begin{cases} c\left(W_t\right), \text{ for } t < \tau \\ -\underline{F}\left(W_\tau\right), \text{ for } t \geq \tau. \end{cases} \tag{15}$$

In the case when $\mathcal{I}_g = 1$, for any promised value to the agent $w \in [0, \bar{u})$, the pair $(\{c_t\}, \{a_t\})$ is the optimal continuous-time contract derived in Sannikov [2008].[19]. As in Section 2.3 it follows from the Invariance Principle (see e.g. Theorem 4.20 in Karatzas and Shreve [1991]) that the (linear interpolations of) the processes of continuation values for the simple contract-action plans (see Definition 1 and Lemma 3) converge in distribution to the continuous-time process defined in (14). Therefore the simple contract-action plans converge in distribution to their continuous-time analogue in (15).[20]

Lemma 4 *Consider a pure hidden action model with noise density g. For any ε, $\Delta > 0$ and $w \in [0, \bar{u})$, let $(\{c_n^{\Delta,\varepsilon}\}, \{a_n^{\Delta,\varepsilon}\})$ be a simple contract-action plan for F solving (6) with the boundary conditions (7) and (8). Then*

$$\lim_{\varepsilon \to 0} \lim_{\Delta \to 0} (\{c_t^{\Delta,\varepsilon}\}, \{a_t^{\Delta,\varepsilon}\}) = (\{c_t\}, \{a_t\}),$$

[18] Part (ii) of Lemma 19 shows that there is $\gamma > 0$ such that for any $w \in (0, w_{sp})$, the constraint $\bar{a} > 0$ in (6) can be replaced by $\bar{a} \geq \gamma$ without loss of generality. The existence of the minimizers thus follows from the compactness of $[\gamma, A]$ and $\left[0, u^{-1}\left(w_{sp}\right)\right]$ and the continuity of the right hand side of (6) in $\overline{a}$ and c, for the case when $\overline{h} = h\left(\overline{a}\right)$.

[19] We identify two processes that agree in distribution;

[20] Note that the minimizers $c\left(\cdot\right)$ and $a\left(\cdot\right)$ in the solution of the HJB equation (6) are bounded continuous functions.

where $(\{c_t\},\{a_t\})$ *is the process defined by (14) and (15) for* w*,* $(\{c_t^{\Delta,\varepsilon}\},\{a_t^{\Delta,\varepsilon}\})$ *is the linear interpolation of* $(\{c_n^{\Delta,\varepsilon}\},\{a_n^{\Delta,\varepsilon}\})$*, and the convergence is in distribution.*

4 Extensions

4.1 Changing signal structure

Suppose now that for a period length Δ the private signal z is distributed with cdf G_Z^Δ, while given the private signal z and action a, the revenue y is distributed with cdf $G_Y^\Delta(y|z,a)$. This extends the model in the paper along two dimensions. First, it generalizes the way the period length Δ parametrizes the distribution of signals. Second, it generalizes the way the agent's effort affects the distribution of public signal.

For any $\overline{a}$ and $\overline{h}\geq h(\overline{a})$, as well as $M>0$ and $\Delta>0$, consider the following problem:

$$
\begin{aligned}
\Theta_M^\Delta(\overline{a},\overline{h}) \;=\; & \inf_{a,|v|\leq M\sqrt{\Delta}} \int v^2(y,z)G_Y^\Delta(dy|z,a(z))G_Z^\Delta(dz), \qquad (16)\\
& \bar{a}=\int a(z)G_Z^\Delta(dz), \quad \bar{h}=\int h(a(z))G_Z^\Delta(dz)\\
& (z,a(z))\in\arg\max_{\tilde{z}\in\mathcal{Z},\hat{a}\in\mathcal{A}}\left\{-r\Delta h(\hat{a})+e^{-r\Delta}\int_{\mathbb{R}} v(y,\tilde{z})G_Y^\Delta(dy|z,\hat{a})\right\} \quad \forall z.
\end{aligned}
$$

Suppose that

$$
\lim_{\Delta\to 0,M\to\infty}\frac{\Theta_M^\Delta(\bar{a},\bar{h})}{\Delta}=\Theta(\bar{a},\bar{h}) \qquad (17)
$$

uniformly in $(\bar{a},\bar{h})$ for some function Θ. Then our results can be extended to this general case, in the following sense.

Unlike the case analyzed in the previous sections, when (A2) was satisfied, now there need not exist a unique solution to the HJB differential equation (6) with boundary conditions (7) and (8).[21] However, we can get around this problem by analyzing a perturbed equation, which always has a unique solution and that provides an arbitrarily good approximation to the solution of the principal's problem. For any $\zeta>0$ consider the following differential equation:

$$
F_\zeta(w)=\sup_{\bar{a},\bar{h},c}\left\{(\bar{a}-c)+F_\zeta'(w)(w+\bar{h}-u(c))+\frac{1}{2}F_\zeta''(w)r\max\{\zeta,\Theta(\bar{a},\bar{h})\}\right\}, \qquad (18)
$$

with the boundary conditions (7) and (8), where $w_{sp,\zeta}$ denotes now the point where F_ζ satisfies (8).

[21] Formally, under (A2) the function Θ was bounded away from zero in the relevant domain (see Lemma 16). This guaranteed that the HJB equation is *uniformly elliptic*, and therefore has a unique solution.

For F_ζ solving the HJB equation (18) on an interval I with $F''_\zeta < 0$, same techniques as in the proof of Proposition 6 establish that $|T_I^\Delta F_\zeta - F_\zeta|_{I^\Delta} = o(\Delta) + O(\zeta\Delta)$. Lemma 6 establishes then the following.[22]

Theorem 3 *For any $\zeta > 0$, equation (18) with the boundary conditions (7) and (8) has a unique solution F_ζ. The value $w_{sp} = \lim_{\zeta\to 0} w_{sp,\zeta}$ and the function $F = \lim_{\zeta\to 0} F_\zeta$ exist. For any agent's promised value $w \in [0, w_{sp}]$, $F(w)$ is the limit of the principal's value of an optimal contract as the period length Δ shrinks to zero:*

$$\lim_{\Delta\to 0} \left|F - F^\Delta\right|_{[0,w_{sp}]} = 0,$$

while for $w > w_{sp}$, $F(w)$ provides an upper bound:

$$F(w) \geq \lim_{\Delta\to 0} F^\Delta(w) \quad \text{for all} \quad w > w_{sp}.$$

More precisely, for fixed ζ we have

$$\left|F_\zeta - F^\Delta\right|^+_{[0,w_{sp,\zeta}]} = O(\zeta) + \frac{o(\Delta)}{\Delta} \quad \text{and} \quad \left|F^\Delta - F_\zeta\right|^+_{[0,\bar{u}]} = O(\zeta) + \frac{o(\Delta)}{\Delta}.$$

Below we provide several examples, in which Θ is defined explicitly by a single class of optimization problems. We also show how the (almost) optimal policies can be used to construct (almost) optimal discrete time contracts, analogously to Theorem 2.

4.1.1 Pure hidden information

We may investigate the version of the model in which the agent knows the noise realization before taking an action (see also Section 4.2). In this case we assume that the agent's actions are unbounded from below.[23] Formally, we replace assumphion (A2) with:

(A2') $X \equiv Z$ and X has a density function $g(x)$. The set of available actions is $\mathcal{A} = (-\infty, A]$ for some $A \in \mathbb{R}_+$, and $h(a) = 0$ for $a < 0$.

It is shown in Section F of the Online Appendix that the VCV function can be defined explicitly as:

$$\begin{aligned} \Theta(\bar{a},\bar{h}) \quad = \quad \inf_{a,v} \quad & \mathbb{E}\left[v(x)^2\right] \\ \text{s.t.} \quad & \bar{a} = \mathbb{E}[a(z)], \quad \bar{h} = \mathbb{E}[h(a(z))], \\ & h'(a(x)) = v'(x) \quad \forall x \qquad\qquad (\text{FOC}_\Theta\text{-PHI}) \end{aligned} \tag{19}$$

[22] The existence, uniqueness and strict concavity of solutions F_ζ in Theorem 3 follow in exactly the same way as for functions F in Theorem 1 (see section D). The existence of $w_{sp} = \lim_{\zeta\to 0} w_{sp,\zeta}$ and $F = \lim_{\zeta\to 0} F_\zeta$ follows immediately from the monotonicity in ζ: $\zeta < (\leq)\zeta'$ implies $F_\zeta > (\geq) F_{\zeta'}$ (Proposition 1).

[23] In a separate note we show that a pure hidden action model with compact action set results in the first best contracts as the period length shrinks to zero.

where the infimum is over piecewise continuously differentiable functions $a(\cdot)$ and continuous functions $v(\cdot)$, and the (FOC$_\Theta$-PHI) condition is required everywhere except for finitely many points of discontinuity of $a(\cdot)$. Notice two differences relative to the original definition: first, the transfer function v has only one argument and so the reporting of the signal is not necessary, and second, the marginal benefit of effort in the incentive constraint is simply the derivative of the transfer function (or, continuation value function). Thus in the pure hidden information case Theorem 3 applies with this explicit definition of the VCV function.

Section F also provides the corresponding definition of simple contract-action plans and shows that they are approximately optimal.

Finally, the following result allows us to rank the distributions of noise in terms of the cost of incentives that they impose, and thus, due to Proposition 1, the values of the optimal contracts (analogously to the ordering of Fisher information quantities in the pure hidden action case). The condition is a strong form of ranking of signal's dispersion.[24]

Lemma 5 *Consider two signal distributions $\overline{G}$ and G of noise for the pure hidden information case, with corresponding strictly positive densities $\overline{g}$ and g. Suppose that:*

$$\overline{G}(x) = G(x') \implies \overline{g}(x) \geq g(x'), \ \forall x, x'$$

Then, for the corresponding VCV functions, $\Theta_{\overline{G}} \leq \Theta_G$.

4.1.2 Additional private information

We may generalize the basic model by allowing the agent to have private information about his cost function and the efficiency of effort. The effect of effort might also depend on the noise. Thus, for example, if we interpret the private signal as reflecting productivity, the base model could deal with a case of production functions parametrized by a "benchmark" level of revenue requiring no effort and identical functions parametrizing any additional improvement (see e.g. Laffont and Tirole [1993], Chapter 2). The current model allows the productivity to also affect the cost/efficiency of marginal effort.

In particular, fix a twice continuously differentiable, bounded function ϕ and for any period length $\Delta > 0$ let

$$y = \Delta[x + \phi(a, x\sqrt{\Delta}, z\sqrt{\Delta})],$$

where the distribution of noise X and the private signal Z satisfy the assumptions in (A2). Also, let the cost of effort be $h(a, z\sqrt{\Delta})$, where each $h(\cdot, z\sqrt{\Delta})$ is as in the previous

[24]The condition implies lower variance, but is incomparable to SOSD: a SOS inferior distribution can either dominate or be dominated in terms of our ranking.

sections. With just slight changes in notation in the proofs one establishes that

$$\begin{aligned}
\Theta(\bar{a}, \bar{h}) &= \inf_{a,v} \quad \mathbb{E}\left[v(x,z)^2\right] \\
\text{s.t.} & \quad \bar{a} = \mathbb{E}[\phi(a(z),x,z)], \quad \bar{h} = \mathbb{E}[h(a(z),z)], \\
& \quad \int_{\mathbb{R}} v(x,z)g(x|z)dx \geq \int_{\mathbb{R}} v(x,z')g(x|z)dx \quad \forall z, z'. \\
& \quad h_1(a(z),z) = -\int_{\mathbb{R}} v(x,z)g'(x|z)\phi_1(a,x,z)dx \quad \forall z
\end{aligned}$$

4.1.3 Folk Theorem

Consider now the model in which for every period length Δ we have $y = \Delta s$, where s is a (public) signal with distribution that depends on action a and is independent of Δ. The analysis in this case coincides with the analysis of a discrete time model with fixed period length 1 and a fixed distribution of signals G_S, in which the per period discount factor converges to one. For simplicity let us consider the pure hidden action case and assume that the sets of available actions $\mathcal{A}$ is finite, with conditional densities $g(s|a)$.

In our principal-agent model the standard identifiability assumptions (see Fudenberg, Levine, and Maskin [1994]) reduce to $\{g(\cdot|a)\}_{a\in\mathcal{A}}$ being linearly independent, which implies that for any $a \in \mathcal{A}$ there exists a bounded function v_a such that:

$$\begin{aligned}
\int v_a(s)\, g(s|a)\, ds &= 0, & (20)\\
\int v_a(s)\, g(s|a')\, ds &\leq e^r r\left[h(a') - h(a)\right] \quad \forall a' \neq a.
\end{aligned}$$

In other words, for the period length 1 the policy (a, v_a) satisfies the constraints of the problem (16) for $(\overline{a}, \overline{h}) = (a, h(a))$ and so for M big enough $\Theta^1_M(a, h(a)) \leq \mathbb{E}_a[v_a(s)^2]$.

For any $\Delta > 0$, it is easy to verify that the function $v^\Delta_a(y) = \Delta e^{-r(1-\Delta)} v_a\left(\frac{y}{\Delta}\right)$ satisfies the constraints of the problem (16) for $(\overline{a}, \overline{h}) = (a, h(a))$, and so $\Theta^\Delta_M(a, h(a)) \leq \Delta^2 e^{-2r(1-\Delta)} \Theta^1_M(a, h(a)) = o(\Delta)$. Consequently, due to Proposition 2, in the limit the first best outcome is achievable. In other words, we recover the Folk Theorem result for the principal agent setting.

More generally, in a pure moral hazard model with finitely many actions, whenever the conditional density $g^\Delta(y|a)$ can be written as $\Delta^\alpha g(y\Delta^\alpha|a)$ for some α and the conditional densities $\{g(\cdot|a)\}_{a\in\mathcal{A}}$ are linearly independent, the first best outcome is achievable in the limit. The above Folk Theorem result provides an example with $\alpha = -1$. In a different example, the agent's action a determines the volatility of the revenue process: for any period length Δ the revenue is, say, normally distributed with mean 0 and variance $\Delta(1-a)$.[25] In this case the conditional density $g^\Delta(y|a)$ can be written as $\Delta^\alpha g(y\Delta^\alpha|a)$ with $\alpha = -1/2$ and again the first best is achievable.

[25] The model is trivial in the case when the Principal is risk neutral. If we assume that the Principal has mean-variance preferences, and so for a given distribution of revenue his utility is equal to $E[y] - Var[y]$,

4.2 Changing payoff structure

The method can be used to tackle different payoff structures. For brevity we will focus on a particular model, in which the cost of effort to the agent is not independent of consumption, but is in fact expressed directly in monetary terms. An application is the problem of incentives to prevent cash-flow diversion (see DeMarzo and Fishman [2007], DeMarzo and Sannikov [2006], Biais, Mariotti, Plantin, and Rochet [2007]). Since our method allows for the risk averse agent, it can be more broadly applied to the insurance problems.[26]

The action $a \in \mathcal{A} = [0, \infty)$ of the agent will be interpreted as the amount of money diverted from the privately observed cash-flow (or income). Agent's benefit, in monetary terms, from withholding a is $h(a)$, where h is a concave function such that $h' \leq 1$ and $h'(a) = \gamma$ for $a \geq A$. For any Δ the stage game payoffs are thus:

$$\begin{aligned} u_P(a,c) &= \Delta(drift - a - c) + noise, \\ u_A(a,c) &= \Delta u(c + h(a)). \end{aligned}$$

We thus go beyond the "linear" approach in the literature and allow the h function to be nonlinear as well as agent to be risk averse.

As in the literature, we assume that in every period after observing the public signal the principal can break the contract, which will result in a continuation payoffs $w_P, w_A > 0$ for the principal and the agent.[27] One can show that the payoffs to the principal and the agent cannot fall below w_P and w_A (see DeMarzo and Fishman [2007], DeMarzo and Sannikov [2006], Biais, Mariotti, Plantin, and Rochet [2007]). Using arguments as in the proofs for Section 4.1.1, one shows that the values of the optimal contracts converge to F, where F is the maximal function that solves

$$\begin{aligned} F(w) &= \max_{\bar{a},\bar{u},c} \left\{ drift - (\bar{a} + c) + F'(w)(w - \bar{u}) + \frac{rF''(w)}{2}\Theta(\bar{a},\bar{u},c) \right\}, \\ F(w_A) &= w_P, \quad F \leq \overline{F}, \end{aligned}$$

where

$$\begin{aligned} \Theta(\bar{a},\bar{u},c) &= \inf_{a,v} \int v^2(x)g(x)dx, \\ \text{s.t.} \qquad & \bar{a} = \int a(x)g(x)dx, \quad \bar{u} = \int u(c + h(a(x)))g(x)dx \\ & v'(x) = u'(c + h(a(x)))h'(a(x)). \end{aligned}$$

his per period utility is equal to $\Delta(1 - a - c)$, and so (up to a constant) the same as considered in the paper.

[26] Existing cash-flow divesion models allow only for risk neutral agent, in which case the pure hidden information model can be formulated directly in continuous time.

[27] Interpreted as the insurance problem, the principal might decide to implement a costly perfect monitoring scheme so that the parties get the first best minus an exogenously determined cost of monitoring.

5 Conclusions

We study a rich family of dynamic agency problems that includes the standard hidden action and hidden information models as special cases. We develop a *quadratic approximation method* that, when the period length is short, allows us to characterize the upper boundary of the equilibrium value set by a differential equation, and to construct contracts that are both relatively simple and almost optimal. The quadratic approximation method developed here should be useful in many dynamic settings with asymmetric information (for example repeated partnerships and oligopoly games).

The solutions we derive depend on the information structure, including the corresponding densities of signals. Nevertheless our method is very tractable as it involves solving a family of simple static problems and a differential equation. The upper boundary of the equilibrium value set depends on a single parameter of the information structure, the variance of incentive transfers. The simple contracts are built from the optimal solutions of the static problems, which are functions of the likelihood ratios of public signals familiar from the static contracting literature.

In particular, while easy to construct, the contracts are sensitive to the details of the information structure, for any period length. The upper boundary of the value set of the continuous time model is the limit, as the period length shrinks to 0, of the boundaries for the discrete time models with particular information structures, whereas the optimal continuous time contract does not provide enough information to construct (approximately) optimal discrete time contracts.

A Proof of Theorems 1 and 2

In Section D in the Online Appendix we establish several properties of the differential equation (6). In particular, Corollary 2 establishes existence and uniqueness, and Lemma 18 the strict concavity of the solution F in the statement of the Theorem 1. Moreover, Lemma 19 shows that F satisfies

$$F(w) = \sup_{\bar{a},\bar{h},c} \left\{ (\bar{a} - c) + F'(w)\left(w + \bar{h} - u(c)\right) + \frac{1}{2}F''(w) r\Theta(\bar{a}, \bar{h}) \right\}, \tag{21}$$

which differs from (6) only in that the additional constraint "$\bar{a} > 0$" has been dropped.

For the remaining main part of the proofs recall the Bellman operator T_I^{Δ} associated with the principal's problem, i.e. the stage-game maximization problem parametrized by the agent's continuation value, defined in (2). In particular, $T_{[0,\bar{u})}^{\Delta}$ is the Bellman operator associated with the principal's optimization problem. The following Proposition is a direct consequence of *self-generation.*

Proposition 4 *F^{Δ} is the largest fixed point f of $T_{[0,\bar{u})}^{\Delta}$ such that $f \leq \overline{F}$.*

More generally, for any interval $I \subset \mathbb{R}$, let F_I^Δ be the largest fixed point f of T_I^Δ such that $f \leq \overline{F}$. Note that if $f : I \to \mathbb{R}$ and $J \subset I$ then $T_I^\Delta f \geq T_J^\Delta f$.[28]

Proposition 5 *Let $I \subset \mathbb{R}$ be any interval. Then for any two bounded functions $f_1, f_2 : I \to \mathbb{R}$, $|T_I^\Delta f_1 - T_I^\Delta f_2|_I^+ \leq e^{-r\Delta}|f_1 - f_2|_I^+$.*

Proof. The proof is analogous to that of Blackwell's theorem (Blackwell [1965]). ■

The rest of the proof follows from the crucial Proposition 6 and Lemma 6 below. Proposition 6 implies that, roughly, F in the statement of the Theorem 1 is almost a fixed point of the $T_{[0,\bar{u})}^\Delta$ operator, and that the simple policies are almost optimal in the problem $T_{[0,w_{sp}]}^\Delta F$. Lemma 6 implies that, roughly, if a function F is almost a fixed point of $T_{[0,\bar{u})}^\Delta$ then the largest fixed point of $T_{[0,\bar{u})}^\Delta$ is approximately bounded above by F, and on the other hand that the almost optimal policies for $T_{[0,w_{sp}]}^\Delta F$ generate contract-action plans that approximately achieve F.[29] To simplify notation, below $\Phi^\Delta(a, c, W; f)$ denotes the objective function in the problem $T_I^\Delta f(w)$ (see equation (2)):

$$\Phi^\Delta(a, c, W; f) = \mathbb{E}^\Delta\Big[\tilde{r}\Delta[a(z) - c(\Delta[x + a(z)], z)] + e^{-r\Delta} f(W(\Delta[x + a(z)], z))\Big]. \quad (22)$$

The proof of the following proposition is in Section B.[30] For any $I = [\underline{w}, \overline{w}]$ and $\Delta > 0$ (small), let $I^\Delta = \big[\underline{w} + \Delta^{1/3}, \overline{w} - \Delta^{1/3}\big]$.

Proposition 6 *Let F solve the HJB equation (21) on an interval I with $F'' < 0$. Then $|T_I^\Delta F - F|_{I^\Delta} = o(\Delta)$. Moreover, when $I = [0, w_{sp}]$, let (a, c, W) be a simple policy defined for $(F, \varepsilon, \Delta, w)$ by (11), for $\Delta, \varepsilon > 0$ and $w \in I^\Delta$. If Δ, ε are sufficiently small, then (a, c, W) satisfies the (IC) constraint, and $\Phi^\Delta(a, c, W; F) \geq F(w) - O(\varepsilon\Delta)$.*

For $\Delta > 0$, interval I and set of feasible policies $p = \{(a_w, c_w, W_w)\}_{w \in I}$ for the Bellman operator T_I^Δ, let $T_I^{\Delta,p}$ be the operator defined as $T_I^{\Delta,p} f(w) = \Phi^\Delta(a_w, c_w, W_w; f)$ and $F_I^{\Delta,p}$ be the value function associated with the contract-action plans generated by the policies p. Note that $F_I^{\Delta,p}$ is a fixed point of $T_I^{\Delta,p}$.

Lemma 6 *Consider a function $f : I \to \mathbb{R}$, $\varepsilon \geq 0$ and $J \subseteq I$. Then*

(i) $\left|T_I^\Delta f - f\right|_J^+ = o(\Delta) + O(\varepsilon\Delta)$ *and* $\left|F_J^\Delta - f\right|_J^+ < \infty$ *imply that* $\left|F_J^\Delta - f\right|_J^+ = O(\varepsilon) + o(\Delta)/\Delta$,

(ii) $\left|f - T_I^{\Delta,p} f\right|_J^+ = o(\Delta) + O(\varepsilon\Delta)$ *and* $\left|f - F_I^{\Delta,p}\right|_I^+ < \infty$ *imply that* $\left|f - F_I^{\Delta,p}\right|_I^+ = O(\varepsilon) + \left|f - F_I^{\Delta,p}\right|_{I \setminus J}^+ + o(\Delta)/\Delta$.

[28]For a function $f : I \to \mathbb{R}$, we define $|f|_I = \sup_{w \in I} |f(w)|$ and $|f|_I^+ = |\max\{0, f(w)\}|_I$.

[29]See also the proof below Lemma 7 in Biais, Mariotti, Plantin, and Rochet [2007].

[30]The proof in the Appendix establishes a more general result. Definition 3 in Section B extends the definition of simple policies to an arbitrary I, and the proof establishes the second half of the proposition without restricting I to be $[0, w_{sp}]$.

Proof. (i) Fix $\Delta > 0$. We have

$$\begin{aligned} \left|F_J^\Delta - f\right|_J^+ &\leq \left|F_J^\Delta - T_J^\Delta F_J^\Delta\right|_J^+ + \left|T_J^\Delta F_J^\Delta - T_J^\Delta f\right|_J^+ + \left|T_J^\Delta f - f\right|_J^+ \\ &\leq e^{-r\Delta}\left|F_J^\Delta - f\right|_J^+ + \left|T_J^\Delta f - f\right|_J^+, \end{aligned}$$

where $\left|F_J^\Delta - T_J^\Delta F_J^\Delta\right|_J^+ = 0$ by Proposition 4 and $\left|T_J^\Delta F_J^\Delta - T_J^\Delta f\right|_J^+ \leq e^{-r\Delta}\left|F_J^\Delta - F\right|_J^+$ by Proposition 5. Consequently, if $\left|F_J^\Delta - f\right|_J^+ < \infty$

$$\left|F_J^\Delta - f\right|_J^+ \leq \frac{\left|T_J^\Delta f - f\right|_J^+}{r\Delta} \leq \frac{\left|T_I^\Delta f - f\right|_J^+}{r\Delta} = O(\varepsilon) + \frac{o(\Delta)}{\Delta},$$

where the second inequality follows because $T_J^\Delta f \leq T_I^\Delta f$ for $J \subseteq I$. This establishes the first implication. The proof of part (ii) is analogous. ■

Given Proposition 6 and Lemma 6, the proof of Theorems 1 and 2 is as follows. For any $\Delta > 0$, consider the interval $I = [-\Delta^{1/3}, \bar{u} + \Delta^{1/3}]$. Let F be the solution on I of the HJB equation (21) satisfying the boundary conditions (7) and (8). Since $F'' < 0$ (see Lemma 18 in Section D), Proposition 6 implies that $|T_I^\Delta F - F|_{[0,\bar{u})} = o(\Delta)$. Also $\left|F^\Delta - F\right|_{[0,\bar{u})}^+ = \left|F^\Delta - F\right|_{[0,w_{sp}]}^+ < \infty$, and part (i) of Lemma 6 with $J = [0, \bar{u})$ imply that

$$\left|F^\Delta - F\right|_{[0,\bar{u})}^+ = \frac{o(\Delta)}{\Delta}.$$

On the other hand, let $I = [0, w_{sp}]$, $\Delta > 0$, $\varepsilon > 0$ be an approximation error, and $p = \{(a_w, c_w, W_w)\}_{w \in I}$ be a set of simple policies for T_I^Δ. For F as above but restricted to $I = [0, w_{sp}]$, by Proposition 6, we have that $\left|F - T_I^{\Delta,p} F\right|_{I^\Delta}^+ = o(\Delta) + O(\varepsilon\Delta)$. Thus, since $\left|F - F_I^{\Delta,p}\right|_{[0,w_{sp}]}^+ < \infty$, part (ii) of Lemma 6 implies that

$$\left|F - F_I^{\Delta,p}\right|_I^+ = O(\varepsilon) + \left|F - F_I^{\Delta,p}\right|_{I \setminus I^\Delta}^+ + \frac{o(\Delta)}{\Delta} = O(\varepsilon) + \frac{o(\Delta)}{\Delta}.$$

The last equality follows from the continuity of F, $F(0) = F_I^{\Delta,p}(0)$ and $F(w_{sp}) = \underline{F}(w_{sp}) \leq F_I^{\Delta,p}(w_{sp})$. This concludes the proof of both Theorems.

B Proof of Proposition 6

In order to prove Proposition 6 it will be useful to also consider other related Bellman operators with a modified objective function and/or constraints. If we restrict the consumption schedule $c(y, z)$ to be constant, we obtain the operator $T_I^{\Delta,c}$. Also recall the modified Bellman operator $T^{\Delta,q}$ with a quadratic objective function and simplified constraints defined in (3). To simplify notation, below $\Phi^{\Delta,q}(a, c, W; f, w)$ will denote the objective function of this simplified problem, equal to

$$\mathbb{E}^\Delta\left[\tilde{r}\Delta[a(z) - c] + e^{-r\Delta}[f(w) + f'(w)(W(\Delta x, z) - w) + \frac{f''(w)}{2}(W(\Delta x, z) - w)^2]\right] \quad (23)$$

The proof of Proposition 6 is established by a series of Lemmas that relate values of Bellman operators applied to a function F solving HJB equation (21), as well as their policy functions. Regarding the values, the line of argument can be illustrated as follows:

$$F \underset{\text{Lemma 10}}{\sim} T^{\Delta,q}F \underset{\text{Lemma 13}}{\sim} T_I^{\Delta,c}F \underset{\text{Lemma 15}}{\sim} T_I^{\Delta}F.$$

The following two Lemmas will be helpful in the rest of this section (the proofs are in Section C). Lemma 7 says that for any period length $\Delta > 0$ and any of the Bellman operators applied to a strictly concave function F, the continuation value policy function must have variance at most proportional to Δ. Intuitively, this must be the case in order to bound the efficiency loss, due to the high variance and strict concavity of F, by potential per-period gains, which are of order Δ. Lemma 8 says that strengthening the constraint associated with truthful reporting in the problem of minimizing variance of incentive transfers (thus solving Θ^ε instead of Θ) affects the problem in a negligible way.

Lemma 7 *Let $I = [\underline{w}, \overline{w}]$ and $F : I \to \mathbb{R}$ be twice continuously differentiable with $F'' < 0$. Let X be any of the Bellman operators T_I^{Δ}, $T_I^{\Delta,c}$ and $T^{\Delta,q}$. Suppose that the policy (a, c, W) is Δ-suboptimal for the problem $XF(w)$, with $w \leq e^{-r\Delta}\overline{w}$. Then for some V that depends on F only we have $\mathbb{V}^{\Delta}[W(\Delta[x + a(z)], z)] \leq V\Delta$.*

Lemma 8 *For any $\varepsilon > 0$ and $(\overline{a}, \overline{h})$, there exists $(\widetilde{a}, \widetilde{h})$ such that $|\overline{a} - \widetilde{a}| = O(\varepsilon)$, $|\overline{h} - \widetilde{h}| = O(\varepsilon)$ and*

$$\Theta^{\varepsilon}(\widetilde{a}, \widetilde{h}) \leq \Theta(\overline{a}, \overline{h}) + O(\varepsilon) \times \sqrt{\Theta(\overline{a}, \overline{h})} + O\left(\varepsilon^2\right).$$

The following Lemma shows the connection between the Bellman operator $T^{\Delta,q}$, for short period length Δ, and the HJB equation (6). The intuition for the Lemma is as discussed in Section 3.1.

Definition 2 *For a twice differentiable function $F : I \to \infty$ with $F'' < 0$, $\varepsilon > 0$, $\Delta > 0$, $w \in I$ and ε-suboptimal policies $\left(\overline{a}, \overline{h}, c\right)$ in (21) and (a, v) in the problem $\Theta^{\varepsilon}\left(\overline{a}, \overline{h}\right)$, the* quadratic simple policy *(a_q, c_q, W_q) is defined as*

$$\begin{aligned} c_q &= c. \\ W_q(y, z) &= w + \tilde{r}\Delta e^{r\Delta}[w + \bar{h} - u(c)] + \tilde{r}\sqrt{\Delta}v(y/\sqrt{\Delta}, \sqrt{\Delta}z) \\ a_q(z) &= a(z\sqrt{\Delta}), \end{aligned} \tag{24}$$

Lemma 9 *Consider any twice differentiable function $F : I \to \infty$ with $F'' < 0$. Then, $T^{\Delta,q}F(w)$ equals*

$$e^{-r\Delta}F(w) - \tilde{r}\Delta \sup_{\bar{a},\bar{h},c}\left\{(\bar{a} - c) + F'(w)[w + \bar{h} - u(c)] + e^{r\Delta}\frac{F''(w)}{2}\tilde{r}\Theta(\bar{a}, \bar{h})\right\} + O\left(\Delta^2\right).$$

Moreover, for fixed $\varepsilon > 0$, $w \in I$ and $\Delta > 0$, a quadratic simple policy is feasible and $O(\varepsilon\Delta)$-suboptimal for $T^{\Delta,q}F(w)$.

Proof. Fix $w \in I$ and any feasible policy (a, c, W) for $T^{\Delta,q}F(w)$, let $\bar{a} = \mathbb{E}^{\Delta}[a(z)]$, $\bar{h} = \mathbb{E}^{\Delta}[h(a(z))]$ and $\bar{W} = \mathbb{E}^{\Delta}[W(\Delta x, z)]$. The promise-keeping constraint (PK_q) for $T^{\Delta,q}F(w)$ implies that $\bar{W} - w = \tilde{r}\Delta e^{r\Delta}[w + \bar{h} - u(c)]$. Therefore, $T^{\Delta,q}F(w)$ equals

$$
\begin{aligned}
&= \sup_{a,c,W} \left\{ \tilde{r}\Delta(\bar{a} - c) + e^{-\Delta r}\mathbb{E}^{\Delta}[F(w) + F'(w)(W(\Delta x, z) - w) + \frac{1}{2}F''(w)(W(\Delta x, z) - w)^2] \right\} \\
&\approx \tilde{r}\Delta \sup_{a,c,W} \left\{ (\bar{a} - c) + F'(w)[w + \bar{h} - u(c)] + e^{-r\Delta}\frac{F''(w)}{2\tilde{r}\Delta}\mathbb{V}^{\Delta}[W(\Delta x, z)] \right\} + e^{-r\Delta}F(w) \\
&= \tilde{r}\Delta \sup_{\bar{a},\bar{h},c} \left\{ (\bar{a} - c) + F'(w)[w + \bar{h} - u(c)] + e^{r\Delta}\frac{F''(w)}{2}\tilde{r}\Theta(\bar{a}, \bar{h}) \right\} + e^{-r\Delta}F(w),
\end{aligned} \tag{25}
$$

where the approximation is of $O(\Delta^2)$ and the last line follows from the definition of $\Theta(\bar{a}, \bar{h})$, as we argue below.

For a given $(\bar{a}, \bar{h})$, since $F''(w) < 0$ and $\int g^{\Delta\prime}(x|z)dx = 0$, the above optimization problem involves the subproblem

$$
\begin{aligned}
\inf_{a,W_0} \quad & \mathbb{E}^{\Delta}[V(x,z)^2] \\
\text{s.t.} \quad & \bar{a} = \mathbb{E}^{\Delta}[a(z)], \quad \bar{h} = \mathbb{E}^{\Delta}[h(a(z))], \quad 0 = \mathbb{E}^{\Delta}[V(x,z)], \\
& \int_{\mathbb{R}} V(x,z)g^{\Delta}(x|z)dx \geq \int_{\mathbb{R}} V(x,z')g^{\Delta}(x|z)dx \quad \forall z, z' \qquad (\text{TR}_q) \\
& \tilde{r}h'(a(z)) = -\frac{e^{-r\Delta}}{\Delta}\int V(x,z)g^{\Delta\prime}(x|z)dx \quad \forall z \qquad (\text{FOC}_q)
\end{aligned}
$$

where $V(x, z) = W(\Delta x, z) - \bar{W}$. Note that the constraint $0 = \mathbb{E}^{\Delta}[V(x, z)]$ can be dropped since it will be satisfied by a solution (or infimum sequence) of the relaxed problem. Also recall that $G_Z^{\Delta}(z) = G_Z(z\sqrt{\Delta})$ and $g^{\Delta}(x|z) = \sqrt{\Delta}g(x\sqrt{\Delta} \,|\, z\sqrt{\Delta})$. Hence, if $v(x, z) = e^{-r\Delta}V(x/\sqrt{\Delta}, z/\sqrt{\Delta})/\left[\tilde{r}\sqrt{\Delta}\right]$ and $\tilde{a}(z) = a(z/\sqrt{\Delta})$, the subproblem becomes

$$
\begin{aligned}
\inf_{a,v} \quad & \tilde{r}^2\Delta e^{2r\Delta}\mathbb{E}[v(x,z)^2] \\
\text{s.t.} \quad & \bar{a} = \mathbb{E}[\tilde{a}(z)], \quad \bar{h} = \mathbb{E}[h(\tilde{a}(z))], \\
& \int_{\mathbb{R}} v(x,z)g(x|z)dx \geq \int_{\mathbb{R}} v(x,z')g(x|z)dx \quad \forall z, z' \qquad (\text{TR}_{\Theta}) \\
& h'(\tilde{a}(z)) = -\int v(x,z)g'(x|z)dx \quad \forall z \qquad (\text{FOC}_{\Theta})
\end{aligned}
$$

The value of this last problem is by definition $\tilde{r}^2\Delta e^{2r\Delta}\Theta(\bar{a}, \bar{h})$. This justifies the substitution of $\Theta(\bar{a}, \bar{h})$ in the equation above.

Lemma 8 establishes that substituting Θ^{ε} for Θ in the expression (25) affects the approximation by at most $O(\Delta\varepsilon)$. This together with the changes of variables detailed above prove that a quadratic simple policy defined in (24) is both feasible (in particular: satisfies the constraint (TR_q) corresponding to truthful reporting) and $O(\varepsilon\Delta)$-suboptimal for $T^{\Delta,q}F(w)$. ■

The following Lemma follows easily from the previous result, establishing that F is "almost" a fixed point of the Bellman operator $T^{\Delta,q}$. Recall that $\Phi^{\Delta,q}(a,c,W;f,w)$ denotes the objective function in the problem $T^{\Delta,q}f(w)$ (see 23).

Lemma 10 *Let F solve the HJB equation (21) on an interval I with $F'' < 0$. Then $|T^{\Delta,q}F - F|_I = o(\Delta)$. Moreover, for any $\varepsilon, \Delta > 0$, $w \in I$ and corresponding quadratic simple policy (a_q, c_q, W_q), $\Phi^{\Delta,q}(a_q, c_q, W_q; F, w) \geq F(w) - O(\Delta\varepsilon)$, uniformly in I.*

Proof. From Lemma 9 we have that $T^{\Delta,q}F(w) - F(w)$ is equal to

$$\tilde{r}\Delta\left[\sup_{\bar{a},\bar{h},c}\left\{(\bar{a}-c)+F'(w)[w+\bar{h}-u(c)]+e^{r\Delta}\frac{F''(w)}{2}\tilde{r}\Theta(\bar{a},\bar{h})\right\}-F(w)\right]+O\left(\Delta^2\right).$$

Since F satisfies the HJB equation (21), it follows that $|T^{\Delta,q}F - F|_I = O(\Delta^2)$. Lemma 9 also yields that $\Phi^{\Delta,q}(a_q, c_q, W_q; F, w) \geq F(w) - O(\Delta^2) - O(\Delta\varepsilon)$. ■

The following two technical Lemmas (proved in Section C) are crucial for the proofs of Lemmas 13 and 14. Lemma 11 will imply that the incentives provided by the tails of the continuation values are negligible, for short period length. Lemma 12 will imply in particular that the marginal benefit of action is almost constant in action, for short period length. This will be used to show the strict convexity of the agent's problem, and therefore that local incentives are sufficient, as well as that approximating the public signal by Δx hardly affects the incentives, and so agent's action.

Lemma 11 *For any $\varepsilon > 0$ there exist M such that for all $v : \mathbb{R}^2 \to \mathbb{R}$ with $\mathbb{E}^{\Delta}\left[v(x,z)^2\right] \leq 1$ the following inequalities hold*

$$(i)\ \mathbb{P}_Z\left[\left|\int_{|v|>M} v(x,z)\,g'(x|z)\,dx\right| \leq \varepsilon\right] \geq 1-\varepsilon,$$

$$(ii)\ \mathbb{P}_Z\left[\left|\int_{|v|>M} v(x,z)\,g(x|z')\,dx\right| \leq \varepsilon,\ \forall z'\right] \geq 1-\varepsilon.$$

Note that the Lemma is somewhat more general than needed for our results as it does not restrict Z to have finite support. With finite support when ε is sufficiently small the probabilistic statements can be replaced by "for every z".

Lemma 12 *For any $\varepsilon > 0$ and M there exists $\hat{\delta} > 0$ such that for all $\delta(\cdot)$ with $0 \leq \delta(\cdot) \leq \hat{\delta}$ and $v : \mathbb{R}^2 \to \mathbb{R}$ the following holds*

$$(i)\ \left|\int_{|v|\leq M}[v(x,z)-v(x+\delta(z),z)]g'(x|z)\,dx\right| \leq \varepsilon \quad \forall z \tag{26}$$

$$(ii)\ \left|\int_{|v|\leq M}[v(x,z)-v(x+\delta(z),z)]g(x|z')\,dx\right| \leq \varepsilon \quad \forall z,z'$$

$$(iii)\ \left|\int_{|v|\leq M}[v(x,z)^2-v(x+\delta(z),z)^2]g(x|z)\,dx\right| \leq \varepsilon \quad \forall z.$$

The next lemma shows that the effect of simplifications implicit in the definition of $T^{\Delta,q}$ - quadratic approximation of F, possibly unbounded values of W, only local incentive constraints for the effort choice, approximating public signal with just Δx - is negligible when the period length Δ is short. (We deal with constant consumption in Lemma 15.) *Simple* policies (Definitions 1 and 3) differ from quadratic simple policies (Definition 2) in that, essentially, they undo those simplifications: the tails of the continuation values are truncated, the local IC implicit in the quadratic simple policies is replaced by the global IC and the public signal is $\Delta[x + a(z)]$.

Definition 3 *For a twice differentiable function $F : I \to \infty$ with $F'' < 0$, $\varepsilon > 0$, $\Delta > 0$, $w \in I^{\Delta}$ and quadratic simple policies (a_q, c_q, W_q) in the problem $T^{\Delta,q}F(w)$ based on (a, v), define the* simple policy (a, c, W) *for $T_I^{\Delta,c}F(w)$ as*

$$
\begin{aligned}
& c = c_q, \\
& W(y,z) = C + W_q(y,z)\mathbf{1}_{|W_q(y,z) - \mathbb{E}^{\Delta}[W_q(\Delta x, z)]| \le \sqrt{\Delta} M} \qquad \text{(AC)} \\
& a(z) \text{ is an action that satisfies the (IC) constraint in (2),}
\end{aligned}
$$

where M is the constant that depends only on ε defined in Lemma 11[31].

Note that in the case when F solves the HJB equation (6) and $I = [0, w_{sp}]$ the above definition of a simple policy agrees with Definition 1.

Simple policies achieve similar values to quadratic simple policies for the following reasons. First, the truncation of continuation values has little effect on the incentives. Given that F'' is negative and bounded away from zero, using W far away from its mean is costly. Under assumption (A2), there are hardly any revenue realizations that are so informative of agent's effort to be worth the cost of such extreme continuation values. Thus the truncation affects agents incentives only slightly.[32] On the other hand, given the truncation, the quadratic approximation of F has little effect on the value of the problem and continuation values are included in I.

Second, since the effect of the agent's effort on the distribution of public signal is small, the optimal incentivizing scheme W under local IC for action choice only is such that the agent faces incentives almost constant in his own action (the expected continuation value is almost linear in his own action). Given strict convexity of the cost of effort, the agent's problem is strictly convex and local incentives are sufficient. Also, approximating public signal by just Δx affects incentives, and so the agent's action only slightly.

Lemma 13 *Let $F : I \to \mathbb{R}$ be twice continuously differentiable with $F'' < 0$. Then $|T_I^{\Delta,c}F - T^{\Delta,q}F|_{I^{\Delta}} = o(\Delta)$. Moreover, for fixed $\varepsilon > 0$ consider quadratic simple policies (a_q, c_q, W_q) for $T^{\Delta,q}F(w)$, $\Delta > 0$, $w \in I^{\Delta}$. Then for Δ and ε sufficiently small corresponding simple policies (a, c, W) satisfy the (IC) constraint and $\Phi^{\Delta}(a, c, W; F) \ge \Phi^{\Delta,q}(a_q, c_q, W_q; F, w) - O(\varepsilon\Delta)$, uniformly in w.*

[31] We assume, without loss of generality, that $\mathbb{V}[v(x)]] \le 1$ - else work with rescaled v (see Lemma 7).
[32] See also Sannikov and Skrzypacz [2007].

Proof. Fix $\varepsilon > 0$ such that $\mathbb{P}_Z(z) > \varepsilon$, for each z, $\Delta > 0$ such that $\sqrt{\Delta} < \delta/A$, for δ as in Lemma 12 and $w \in I^{\Delta}$. Fix a quadratic simple policy (a_q, c_q, W_q) and the corresponding simple policy (a, c, W).

Step 1: In this step we show that the simple policy (a, c, W) satisfies the incentive constraint for truthful reporting, and so the full (IC).

Recall that $W_q^{\Delta}(y, z) = Const + \tilde{r}\sqrt{\Delta}v(y/\sqrt{\Delta}, z\sqrt{\Delta})$ for a function v that satisfies (TR_Θ), i.e.,

$$\int v(x,z)g(x|z)dx \geq \int v(x,z')g(x|z)dx + 3\varepsilon \quad \forall z \neq z'$$

or

$$\int \sqrt{\Delta}v(\sqrt{\Delta}x, \sqrt{\Delta}z)g^{\Delta}(x|z)dx \geq \int \sqrt{\Delta}v(\sqrt{\Delta}x, \sqrt{\Delta}z')g^{\Delta}(x|z)dx + 3\sqrt{\Delta}\varepsilon \quad \forall z \neq z'. \tag{27}$$

Lemma 7 implies that $\mathbb{V}^{\Delta}\left[\sqrt{\Delta}v(\sqrt{\Delta}x, \sqrt{\Delta}z)\right] \leq V\Delta$, or $\mathbb{V}[v(x,z)] \leq V$, and so Lemma 11 applied to $v(x)$ yields that for sufficiently large M that depends only on ε

$$\left|\int_{|v|>M} \sqrt{\Delta}v(\sqrt{\Delta}x, \sqrt{\Delta}z')g^{\Delta}(x|z)dx\right| = \sqrt{\Delta}\left|\int_{|v|>M} v(x,z')g(x|z)dx\right| \leq \sqrt{\Delta}\varepsilon \quad \forall z, z'.$$

On the other hand, by Lemma 12 it follows that for small $\Delta > 0$ and every z, z' and a

$$\begin{aligned} &\left|\int_{|v|\leq M} \sqrt{\Delta}\big[v(\sqrt{\Delta}x, \sqrt{\Delta}z') - v(\sqrt{\Delta}[x+a], \sqrt{\Delta}z')\big]g^{\Delta}(x|z)dx\right| \\ = &\left|\int_{|v|\leq M} \sqrt{\Delta}\big[v(x,z') - v(x+\sqrt{\Delta}a, z')\big]g(x|z)dx\right| \leq \sqrt{\Delta}\varepsilon. \end{aligned}$$

The last three inequalities together imply that

$$\begin{aligned} &\int_{|v|\leq M} \sqrt{\Delta}v(\sqrt{\Delta}[x+a(z)], \sqrt{\Delta}z)g^{\Delta}(x|z)dx \\ \geq &\int_{|v|\leq M} \sqrt{\Delta}v(\sqrt{\Delta}[x+a], \sqrt{\Delta}z')g^{\Delta}(x|z)dx \quad \forall z \neq z',\ \forall a, \end{aligned}$$

and so the agent has incentives to report his signal truthfully. This, together with the definition of the effort function a implies that the simple policy (a, c, W) satisfies the (IC) constraint.

Step 2: In this step we show that $\Phi^{\Delta}(a, c, W; F) \geq \Phi^{\Delta,q}(a_q, c_q, W_q; F, w) - O(\varepsilon\Delta)$, uniformly in w. Since ε is arbitrary, in view of Lemma 9, this establishes that $|T^{\Delta,q}F - T_I^{\Delta,d}F|_{I^{\Delta}}^{+} = o(\Delta)$.

By Taylor series expansion

$$F(W(y,z)) = F(w) + F'(w)(W(y,z) - w) + \frac{1}{2}F''(w)(W(y,z) - w)^2 + o([W(y,z) - w]^2).$$

(PK) implies that $w - \mathbb{E}^{\Delta}[W(y,z)] = O(\Delta)$ and $\left|W(y,z) - \mathbb{E}^{\Delta}[W(y,z)]\right| = O(\sqrt{\Delta})$ by construction. Hence $|w - W(y,z)| = O(\sqrt{\Delta})$ for all y. Therefore, for Δ small enough, the policy (a, c, W) is feasible when $w \in I^{\Delta}$ and

$$\begin{aligned}\Phi^{\Delta}(a,c,W;F) \quad &\geq \quad \tilde{r}\Delta(\mathbb{E}^{\Delta}\left[a(z)\right] - c) + e^{-r\Delta}\Big[F(w) + F'(w)\mathbb{E}^{\Delta}[W(y,z) - w] \\ &\quad + \frac{F''(w)}{2}\mathbb{E}^{\Delta}[(W(y,z)-w)^2]\Big] + o\left(\Delta\right) \qquad (28)\end{aligned}$$

Let us bound from below the terms in the second line of the above expression by the corresponding terms in $\Phi^{\Delta,q}(a_q, c_q, W_q; F, w)$.

Given the definition of W, the necessary local version of (IC) takes the following form:

$$\begin{aligned}-h'(a(z)) \quad &= \quad \frac{e^{-r\Delta}}{\tilde{r}\Delta}\int_{\mathbb{R}} W(\Delta[x + a\left(z\right)], z) g^{\Delta\prime}(x|z)dx \\ &= \quad \frac{e^{-r\Delta}}{\Delta}\int_{|v|\leq M_{\varepsilon}} \sqrt{\Delta}v(\sqrt{\Delta}[x + a(z)], \sqrt{\Delta}z) g^{\Delta\prime}(x|z)dx, \qquad (29)\end{aligned}$$

whereas the definition of W_q and (FOC$_q$) imply

$$-h'(a_q(z)) = \frac{e^{-r\Delta}}{\tilde{r}\Delta}\int_{\mathbb{R}} W(\Delta x, z) g^{\Delta\prime}(x|z)dx = \frac{e^{-r\Delta}}{\Delta}\int_{\mathbb{R}} \sqrt{\Delta}v(\sqrt{\Delta}x, \sqrt{\Delta}z) g^{\Delta\prime}(x|z)dx.$$

As in step 1, Lemma 7 implies that $\mathbb{V}^{\Delta}\left[\sqrt{\Delta}v(\sqrt{\Delta}x, \sqrt{\Delta}z)\right] \leq V\Delta$, or $\mathbb{V}\left[v(x,z)\right] \leq V$, and so Lemma 11 applied to $v(x)$ yields that for every z

$$\frac{1}{\Delta}\left|\int_{|v|>M} \sqrt{\Delta}v(\sqrt{\Delta}x, \sqrt{\Delta}z) g^{\Delta\prime}(x|z)dx\right| = \left|\int_{|v|>M} v(x,z) g'(x|z)dx\right| \leq \varepsilon.$$

On the other hand, from Lemma 12 it follows that for sufficiently small Δ and every z

$$\begin{aligned}&\frac{1}{\Delta}\left|\int_{|v|\leq M} \sqrt{\Delta}\left[v(\sqrt{\Delta}x, \sqrt{\Delta}z) - v(\sqrt{\Delta}[x + a\left(z\right)], \sqrt{\Delta}z)\right] g^{\Delta\prime}(x|z)dx\right| \\ = \quad &\left|\int_{|v|\leq M} \left[v(x,z) - v(x + \sqrt{\Delta}a(z), z)\right] g'(x|z)dx\right| \leq \varepsilon.\end{aligned}$$

Consequently, for every z, $|h'(a_q(z) - h'\left(a(z)\right)| \leq 2\varepsilon$, and so

$$|a_q(z) - a(z)| \leq \frac{|h'(a_q(z) - h'(a(z))|}{\inf h''} \leq \frac{2\varepsilon}{\inf h''}. \qquad (30)$$

Since also $c(y,z) \equiv c_q$ we have

$$\tilde{r}\Delta\left|(\mathbb{E}^{\Delta}\left[a_q(z)\right] - c_q) - (\mathbb{E}^{\Delta}\left[a(z)\right] - c)\right| = O(\varepsilon\Delta). \qquad (31)$$

Subtracting (PK_q) for problem $T^{\Delta,q}F$ from (PK) for problem $T^{\Delta,d}F$ and using (30), we obtain

$$e^{-r\Delta}F'(w)\left|\mathbb{E}^{\Delta}\left[W(y,z)\right]-\mathbb{E}^{\Delta}\left[W_q(\Delta x,z)\right]\right|=O(\varepsilon\Delta). \tag{32}$$

Finally

$$\begin{aligned}\mathbb{E}^{\Delta}[(W(y,z)-w)^2] &= \tilde{r}^2e^{2r\Delta}\mathbb{E}^{\Delta}[\Delta v^2(\sqrt{\Delta}[x+a(z)],\sqrt{\Delta}z)\mathbf{1}_{|v|\le M}]+O\left(\Delta^2\right)\\ &\le \tilde{r}^2e^{2r\Delta}\mathbb{E}^{\Delta}[\Delta v^2(\sqrt{\Delta}x,\sqrt{\Delta}z)\mathbf{1}_{|v|\le M}]+O\left(\varepsilon\Delta\right)\\ &\le \tilde{r}^2e^{2r\Delta}\mathbb{E}^{\Delta}[\Delta v^2(\sqrt{\Delta}x,\sqrt{\Delta}z)]+O\left(\varepsilon\Delta\right)\\ &= \mathbb{E}^{\Delta}[(W_q(\Delta x,z)-w)^2]+O\left(\varepsilon\Delta\right).\end{aligned} \tag{33}$$

The first inequality follows from Lemma 11. The equalities follow from the definitions of W_q and W and the fact that $\mathbb{E}^{\Delta}[W(\Delta[x+a(z)],z)-w]=O(\Delta)$ and $\mathbb{E}^{\Delta}[W_q(\Delta x,z)-w]=O(\Delta)$, from (PK) and ($PK_q$). Inequalities (31)–(33) together with (28) establish the proof of this step.

Step 3: In this step we show that $|T_I^{\Delta}F(w)-T^{\Delta,q}F(w)|^{+}_{I^{\Delta}}=o(\Delta)$. The proof is almost analogous to the previous steps. First, for a policy (a,c,W) that is $\varepsilon\Delta$-suboptimal in the problem $T_I^{\Delta}F(w)$, given construction as in Lemma 8, one can assume that truthful reporting is at least $3\sqrt{\Delta}\varepsilon$ more profitable than the best deviation (see equation 27). Then, given (a,c,W), define (a_q,c_q,W_q) as in Definition 1: $c_q=c$, $W_q(\Delta x,z)=C+W(\Delta x,z)$ if $\left|W(\Delta x,z)-\mathbb{E}^{\Delta}\left[W(y,z)\right]\right|\le\sqrt{\Delta}M_{\varepsilon}$ and $W_q(\Delta x,z)=C$ otherwise, while $a_q(z)$ is defined by the (FOC_q) condition and C is chosen to satisfy (PK_q). As in step 1, (a_q,c_q,W_q) satisfies the (TR_q), and it satisfies (FOC_q) by construction. We prove as in step 2 that $\Phi^{\Delta,q}(a_q,c_q,W_q;F,w)\ge\Phi^{\Delta}(a,c,W;F)-O(\varepsilon\Delta)$. ■

The next Lemma shows that the actions in the definition of simple policies are pinned down by the local version of the (IC) only.

Lemma 14 *Consider any simple policy (a,c,W) defined in (11) for I,F,ε,Δ,w. For sufficiently small Δ, $a(z)$ is the unique action that satisfies the local version of (IC), for all z.*

Proof. Fix $w\in I$ and a simple policy (a,c,W). Consider the necessary local version of (IC) in (29). Since h is strictly convex, Lemma 12 implies that for sufficiently small Δ and any z there is a unique solution $a(z)$ to this equation. ■

The last Lemma needed to establish the proof of Proposition 6 shows that the restriction to wage contract-action schemes is without loss of generality. Intuitively, since with short periods the signal about agent's action is weak, in order to provide nonegligible incentives the variation in utility from signal-contingent payments must be of high order $\sqrt{\Delta}$. While the continuation value function may provide such incentives, the direct money payments are only of order Δ. Thus, changing consumption to be constant affects the incentives only slightly.

Lemma 15 *Assume $F : I \to \mathbb{R}$ is twice continuously differentiable and $F'' < 0$. Then* $\left| T_I^{\Delta,c} F - T_I^{\Delta} F \right|_{I^{\Delta}} = o(\Delta)$.

Proof. Fix $\varepsilon, \Delta > 0$ and any $w \in I^{\Delta}$, and let (a, c, W) be a policy function that is $\varepsilon\Delta-$suboptimal in the problem $T_I^{\Delta} F(w)$. Using Lemma 11 and arguments as in the proof of Lemma 13, we may assume without loss of generality that for every public signal $y = \Delta[x + a(z)]$, $\left|W(y,z) - \mathbb{E}^{\Delta}[W(y,z)]\right| = O(\sqrt{\Delta})$, and that

$$\begin{aligned} \Phi^{\Delta}(a, c, W; F) &= \tilde{r}\Delta(\mathbb{E}^{\Delta}[a(z)] - \mathbb{E}^{\Delta}[c(y)]) + e^{-r\Delta}\big[F(w) \\ &\quad + F'(w)\left(\mathbb{E}^{\Delta}[W(y,z)] - w\right) + \frac{F''(w)}{2}\mathbb{V}^{\Delta}[W(y,z)]\big] + o(\Delta). \end{aligned} \tag{34}$$

Let the policy (a_c, c_c, W_c) with constant consumption be defined so that $a_c \equiv a$, $\mathbb{E}^{\Delta}[u(c(y,z))] = u(c_c)$ and

$$e^{-r\Delta} W_c(y,z) = \Delta\tilde{r}\left[u(c(y,z)) - \mathbb{E}^{\Delta}[u(c(y,z))]\right] + e^{-r\Delta} W(y,z).$$

We will compare the terms in (34) with the analogous terms for the policy (a_c, c_c, W_c). We have $\mathbb{E}^{\Delta}[a(z)] = \mathbb{E}^{\Delta}[a_c(z)]$, $\mathbb{E}^{\Delta}[W(y,z)] = \mathbb{E}^{\Delta}[W_c(y,z)]$ and, from concavity of u, $c_c \leq \mathbb{E}^{\Delta}[c(y,z)]$. Letting $\zeta(y,z) := \tilde{r}e^{r\Delta}[u(c(y,z)) - u(c_c)]$, we have

$$\begin{aligned} &\mathbb{V}^{\Delta}[W_c(y,z)] - \mathbb{V}^{\Delta}[W(y,z)] \\ =\ &\mathbb{E}^{\Delta}\big[\left(W(y,z) + \Delta\zeta(y) - \mathbb{E}^{\Delta}[W(y,z)]\right)^2\big] - \mathbb{E}^{\Delta}\big[\left(W(y,z) - \mathbb{E}^{\Delta}[W(y,z)]\right)^2\big] \\ =\ &\Delta^2\mathbb{E}^{\Delta}\left[\zeta^2(y,z)\right] + \Delta\mathbb{E}^{\Delta}\left[\left(W(y,z) - \mathbb{E}^{\Delta}[W(y,z)]\right)\zeta(y,z)\right] \\ \leq\ &\Delta^2(\tilde{r}e^{r\Delta}\overline{u})^2 + \Delta^{3/2}\tilde{r}e^{r\Delta}\overline{u} = o(\Delta). \quad \blacksquare \end{aligned}$$

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Online Appendix for “Agency Models with Frequent Actions”

Tomasz Sadzik and Ennio Stacchetti

C Additional Proofs for Section B

Proof. (Lemma 7) In each of the above problems the policy $(a, c, W) = (0, 0, e^{r\Delta} w)$ is an available policy that satisfies all the constraints and delivers a value of at least $F(w) + [\min F'] \left(e^{r\Delta} - 1\right) \overline{w} = F(w) + O(\Delta)$. Let $\hat{h} = \mathbb{E}^{\Delta}[h(a(z))]$, $\hat{u} = \mathbb{E}^{\Delta}[u(c(\Delta[x + a(z)]))]$ and $\hat{W} = \mathbb{E}^{\Delta}[W(\Delta[x + a(z)], z)]$. The promise-keeping constraint implies that

$$\hat{W} - w = \tilde{r}\Delta e^{r\Delta}[w + \hat{h} - \hat{u}] = O(\Delta),$$

since $w \in [\underline{w}, \overline{w}]$, $\hat{h} \in [0, h(A)]$ and $\hat{u} \in [0, \bar{u}]$. Therefore, $W(\Delta[x + a(z)], z) - w = (W(\Delta[x + a(z)], z) - \hat{W}) + (\hat{W} - w)$ implies

$$\mathbb{E}^{\Delta}[(W(\Delta[x + a(z)], z) - w)^2] = \mathbb{V}^{\Delta}\left[W(\Delta[x + a(z)], z)\right] + O(\Delta^2).$$

Consequently, for Y either $\Phi^{\Delta,q}(a, c, W; F, w)$ or $\Phi^{\Delta}(a, c, W; F)$ we have $Y \geq F(w) + O(\Delta)$ and

$$Y \leq \tilde{r}\Delta A + e^{-r\Delta}\Big(F(w) + \tilde{r}\Delta e^{r\Delta} F'(w)[w + \hat{h} - \hat{u}] + \frac{\max F''}{2}\mathbb{V}^{\Delta}\left[W(\Delta[x + a(z)], z)\right]\Big) + O(\Delta^2),$$

which after rearranging terms gives the result for an appropriate V. ■

Proof. (Lemma 8) Since $G_X(\cdot|z)$ are linearly independent, let $\phi_z(x)$ be the functions bounded by some B such that

$$\int \phi_z(x)\, g_{X|Z}(x|z) = 0, \quad \int \phi_z(x)\, g_{X|Z}(x|z') < -1. \quad \forall z, z'$$

Fix some $(\overline{a}, \overline{h})$ and consder the optimal policy $a(\cdot), v(\cdot,\cdot)$ for the problem $\Theta(\overline{a}, \overline{h})$. We define $v^*(x, z) = v(x, z) + \varepsilon\phi_z(x)$ and let $a^*(\cdot)$ be defined by the (FOC$_\Theta$). Note that for all z

$$\int_{\mathbb{R}} 2\varepsilon\phi_z(x)\, g'_{X|Z}(x|z) dx = O(\varepsilon),$$

and so, from (FOC$_\Theta$), $|a(z) - a^*(z)| = O(\varepsilon)$. This implies that for $\widetilde{a} = \mathbb{E}_Z[a^*(z)]$ and $\widetilde{h} = \mathbb{E}_Z[h(a^*(z))]$ $|\overline{a} - \widetilde{a}|$, $\left|\overline{h} - \widetilde{h}\right| = O(\varepsilon)$. On the other hand,

$$\begin{aligned}\mathbb{E}[|v^*(x,z)^2 - v(x,z)^2|] &\leq \varepsilon^2 M^2 + 2\mathbb{E}[|\varepsilon\phi_z(x)\, v(x,z)|] \leq \\ &\leq \varepsilon^2 M^2 + 2\varepsilon M\sqrt{\mathbb{E}[v(x,z)^2]} = \varepsilon^2 M^2 + 2\varepsilon M\sqrt{\Theta(\overline{a},\overline{h})}.\end{aligned}$$

■

Proof. (Lemma 11) (i) Fix $\varepsilon > 0$ and consider a function v that satisfies $\mathbb{E}[v(x,z)^2] \leq 1$. For any $\delta > 0$ pick M_δ big enough so that (from Lebesgue's Monotone Convergence Theorem)

$$\int\left[\int_{|v|>M_\delta} v^2(x,z)\, g_{X|Z}(x|z)\, dx\right] dG_Z(z) \leq \delta. \tag{35}$$

From the Tschebyshev's inequality,

$$\mathbb{P}_Z\left[\int_{|v|>M_\delta} v^2(x,z)\, g_{X|Z}(x|z)\, dx > \gamma\right] \leq \frac{\delta}{\gamma}. \tag{36}$$

Therefore, for all z for which $\int_{|v|>M_\delta} v^2(x,z)\, g_{X|Z}(x|z)\, dx \leq \gamma$,

$$\int_{|v|>M_\delta} \left|v(x,z) g'_{X|Z}(x|z)\right| dx \leq \left[\int_{|v|>M_\delta} v^2(x,z) g_{X|Z}(x|z) dx \times \int \frac{g'_{X|Z}(x|z)^2}{g_{X|Z}(x|z)} dx\right]^{\frac{1}{2}} \leq \sqrt{\gamma \bar{M}}.$$

The result thus follows by picking $\gamma = \varepsilon^2/\bar{M}$ and $\delta = \varepsilon\gamma$.

(ii) Let γ and δ be as in (i) and M_δ be such that (35) holds. For any z for which $\int_{|v|>M_\delta} v^2(x,z)\, g_{X|Z}(x|z)\, dx \leq \gamma$ and any z' we have

$$\int_{|v|>M_\delta} \left|v(x,z) g_{X|Z}(x|z')\right| dx \leq \left[\int_{|v|>M_\delta} v^2(x,z) g_{X|Z}(x|z) dx \times \int \frac{g_{X|Z}(x|z')^2}{g_{X|Z}(x|z)} dx\right]^{\frac{1}{2}} \leq \sqrt{\gamma \bar{M}},$$

where the last inequality follows from the assumption (A3). The proof then follows from (36). ■

Proof. (Lemma 12) (i) For every x and z, $|g'_{X|Z}(x|z) - g'_{X|Z}(x - \delta(z)|z)| \leq \delta |g''_{X|Z}(x - \xi(x,z)|z)|$ for some $\xi(x,z) \in [0, \delta(z)] \subset [0, \hat{\delta}]$. Therefore, with $\bar{\delta}$ and $\bar{M}$ the constants in (A2), for every $\delta \leq \min\left\{\bar{\delta}, \frac{\varepsilon}{M\bar{M}}\right\}$ we have that

$$\begin{aligned}\int_{|v|\leq M} \left|v(x,z)\left[g'_{X|Z}(x|z) - g'_{X|Z}(x-\delta(z)|z)\right]\right| dx &\leq \delta M \int_{\mathbb{R}} \left|g''_{X|Z}(x - \xi(x,z)|z)\right| dx \\ &\leq \delta M \bar{M} \leq \varepsilon,\end{aligned}$$

which establishes (26). The proof of (ii) is analogous and is omitted.

(iii) Similarly, for any $\delta \leq \min\{\bar{\delta}, \varepsilon/[M^2\sqrt{\bar{M}}]\}$ we have that

$$\begin{aligned} & \int_{\mathbb{R}} \int_{|v| \leq M} \left| v(x,z)^2 \left(g(x,z) - g(x - a(z), z) \right) \right| dx dz \\ \leq \;& \delta M^2 \int_{\mathbb{R}} \int_{\mathbb{R}} \left| g'_{X|Z}(x - \xi(x,z)|z) g_Z(z) \right| dx dz \\ \leq \;& \delta M^2 \int_{\mathbb{R}} \left[\int_{\mathbb{R}} \frac{g'_{X|Z}(x - \xi(x,z) \,|\, z)^2}{g_{X|Z}(x|z)} \, dx \right]^{\frac{1}{2}} g_Z(z) dz \leq \delta M^2 \sqrt{\bar{M}} \leq \varepsilon, \end{aligned}$$

with the second inequality following from the Cauchy-Schwarz inequality, which establishes the Lemma. ■

D The HJB Equation

The following Lemma establishes a property of the variance of incentive transfers function Θ that will be crucial to all the following results on the properties of the HJB equation.

Lemma 16 *Suppose (A2) holds. Then the variance of incentive transfers function is bounded away from zero for strictly positive expected effort levels,*

$$\Theta\left(\bar{a}, \bar{h}\right) \geq \underline{\theta} > 0. \quad \forall \bar{a} > 0, \bar{h} \tag{37}$$

Proof. Consider function Θ^n that is defined just as Θ except that the condition (TR_Θ) is dropped. On the one hand, trivially, $\Theta \geq \Theta^n$. On the other hand, from Lemma 1 it follows that

$$\Theta^n \geq \frac{\gamma^2}{\min_z \mathcal{I}_{g_{X|Z}(\cdot|z)}} \geq \frac{\gamma^2}{\bar{M}} > 0,$$

where γ is such that $h'(a) \geq \gamma$ for $a > 0$ and $\bar{M}$ is from assumption (A2). ■

The following Lemma establishes some basic properties of the solution of the HJB equation

Lemma 17 *Suppose* $\Theta\left(\bar{a}, \bar{h}\right) \geq \underline{\theta} > 0$.

(i) For any initial conditions $F(\underline{w})$ *and* $F'(\underline{w})$ *the HJB equation (6) has a unique solution* F *in any interval* $[\underline{w}, \bar{w}] \subset \mathbb{R}$.

(ii) F *is twice continuously differentiable and* (F, F') *depends continuously on the initial conditions.*

(iii) F' *is monotone with respect to* $F'(\underline{w})$. *That is, if* F_1 *and* F_2 *are two solutions of the HJB equation in an interval* $[\underline{w}, \bar{w}] \subset \mathbb{R}$ *with* $F_1(\underline{w}) = F_2(\underline{w})$ *and* $F_1'(\underline{w}) > F_2'(\underline{w})$, *then* $F_1'(w) > F_2'(w)$ *(and hence* $F_1(w) > F_2(w)$*) for all* $w > \underline{w}$.

Proof. See Sannikov [2008]. ■

Corollary 2 *The HJB equation (6) with the boundary conditions (7) and (8) has a unique solution F.*

The Corollary follows immediately from Lemma 17. Note also that the continuity and monotonicity in the initial slope suggest the natural procedure for computing F.

Lemma 18 *Suppose $\Theta\left(\bar{a},\bar{h}\right) \geq \underline{\theta} > 0$. The solution F of the HJB equation (6) with the boundary conditions (7) and (8) is strictly concave.*

Proof. See Sannikov [2008]. ■

Part (i) of the next Lemma establishes that the function F in the statement of the Theorem 1 satisfies the HJB equation (21), with the constraint "$\bar{a} > 0$" dropped. Part (ii) shows a related result for the general case from Section 4, which will be used in Section F below.

Lemma 19

(i) The function F in Theorem 1 solves HJB equation (21).

(ii) For any $[\underline{w},\overline{w}] \subset (0, w_{sp})$ there exists $\gamma > 0$ such that for all sufficiently small ζ, the F_ζ as in Theorem 3 solves equation (18) on $[\underline{w},\overline{w}]$ with an additional constraint $\bar{a} \geq \gamma$.

Proof. (i) For any $\lambda \in \mathbb{R}$ let H_λ be the linear function tangent to the retirement curve $\{(w, \underline{F}(w)) : w \in [0,\bar{u})\}$ with the slope λ (if $\lambda \geq \underline{F}'(0)$, $H_\lambda(w) = \lambda w$). On the one hand, since F and $\underline{F}$ are concave and $F \geq \underline{F}$, for any $w \in I$ we have $F(w) \geq H_{F'(w)}(w)$. On the other hand, for any $w \in I$, the value of the maximization problem in the expression above under constraint $\bar{a} = 0$ is at most $\max_c\{-c+F'(w)(w-u(c))\} = \underline{F}(w')+\underline{F}'(w')(w-w') = H_{F'(w)}(w)$, where w' is such that either $\underline{F}'(w') = F'(w)$ or $w' = 0$ in case $F'(w) > \underline{F}'(0)$. Consequently, choosing $\bar{a} = 0$ in the maximization problem above can never be strictly optimal. Equivalently, since F satisfies the the HJB equation (6), it also satisfies the equation (21) with the constraint "$\overline{a} > 0$" dropped.

(ii) We may assume $w_{sp} > 0$. Note also that for any $\zeta > 0$ and F_ζ as in Theorem 1 we have

$$\overline{F}'\left(\overline{w}_{sp}\right) \leq F'_\zeta\left(w\right) \leq \overline{F}\left(\underline{w}\right)/\underline{w},$$

for all $w \in [\underline{w},\overline{w}]$. We will establish that there is $\alpha > 0$ such that for any ζ and $w \in [\underline{w},\overline{w}]$, $F_\zeta(w) - H_{F'_\zeta(w)}(w) \geq \alpha$. If not, then let $\{w_n\}, \{w'_n\}, \{\zeta_n\}$ and $\{\alpha_n\}$ with $w_n \in [\underline{w},\overline{w}]$, $w'_n \leq w_{sp}$, $\zeta_n \downarrow 0$, $\alpha_n \downarrow 0$ be such that $F_{\zeta_n}(w_n) - H_{F'_{\zeta_n}(w_n)}(w_n) \leq \alpha_n$ (where w'_n is such that $\underline{F}'(w'_n) = F'_{\zeta_n}(w_n)$). We consider three cases, and in each derive a contradiction.

(Case 1) Suppose that for some $\delta > 0$ and all n, $w'_n \in [\delta, w_{sp} - \delta]$. The concavity of F_{ζ_n} and $\underline{F}$ imply that $F_{\zeta_n}(w_n) - H_{F'_{\zeta_n}(w_n)}(w_n) \geq F_{\zeta_n}(w'_n) - H_{F'_{\zeta_n}(w_n)}(w'_n) = F_{\zeta_n}(w'_n) - \underline{F}(w'_n)$. But, since F_{ζ_n} is increasing as $\zeta_n \downarrow 0$ (Proposition 1, part (i)), $F_{\zeta_n}(w'_n) - \underline{F}(w'_n) \geq \inf_{w\in[\delta,w_{sp}-\delta]} F_{\zeta_1}(w) - \underline{F}(w) > 0$, a contradiction.

(Case 2) If $w_n' \downarrow 0$ (we might assume so by choosing a subsequence), then we would have $F_{\zeta_n}(w_n) \to H_{F'_{\zeta_n}(w_n)}(w_n) \to \underline{F}'(0) \times w_n$. By concavity of all F_{ζ_n} this would imply that, first, $F_{\zeta_n}(w) \to \underline{F}'(0) \times w$ for all $w \in [0, w_n]$, and second, that there is a sequence $\{w_n''\}$, $w_n'' \in [0, w_n]$, such that $F'_{\zeta_n}(w_n'') \to \underline{F}'(0)$ and $F''_{\zeta_n}(w_n'') \to 0$. But then

$$\begin{aligned} F_{\zeta_n}(w_n'') &\to \max_{a,c} \{(a-c) + \underline{F}'(0)(w_n'' + h(a) - u(c))\} \\ &= \max_a \{a + \underline{F}'(0)(w_n'' + h(a))\} > \underline{F}'(0)\, w_n'', \end{aligned}$$

where the equality follows from the fact that $\underline{F}'(0) = \frac{1}{u'(0)}$ and strict concavity of u, while the inequality follows from $h_+'(0) < u'(0)$. This establishes the required contradiction.

(Case 3) If $w_n' \uparrow w_{sp}$, we derive the contradiction in the analogous way as in case 2.

We have established that for all ζ and $w \in [\underline{w}, \overline{w}]$, $F_\zeta(w) - H_{F'_\zeta(w)}(w) \geq \alpha > 0$. On the other hand, for any ζ and $w \in [\underline{w}, \overline{w}]$, if we restrict the policy on the right hand side of equation (21) to satisfy $\bar{a} \leq \gamma$, for sufficiently small $\gamma > 0$, then

$$\begin{aligned} &\sup_{\bar{a} \leq \gamma, \bar{h}, c} \left\{ (\bar{a} - c) + F'_\zeta(w)\left(w + \bar{h} - u(c)\right) + \frac{1}{2} F''_\zeta(w) r \max\{\zeta, \Theta(\bar{a}, \bar{h})\} \right\} \leq \\ &\max_c \{-c + F'_\zeta(w)(w - u(c)) + \frac{1}{2} F''_\zeta(w) r\zeta\} + \frac{\alpha}{2} \leq H_{F'_\zeta(w)}(w) + \frac{\alpha}{2} \leq F_\zeta(w) - \frac{\alpha}{2}, \end{aligned}$$

where the first inequality follows because F'_ζ are uniformly bounded on $[\underline{w}, \overline{w}]$ and $\bar{h} \leq \frac{\bar{a}}{A} h(A)$. This establishes the Lemma. ∎

D.1 Proof of Proposition 1

The Proposition is based on the following "single crossing" lemma.

Lemma 20 *Consider two functions $\Theta \geq_{D_+^\Theta} \underline{\Theta} \geq 0$, and suppose that $F^\Theta, F^{\underline{\Theta}} : I \to \mathbb{R}$ solve the corresponding HJB equations (6) with $F^{\Theta\prime\prime} \leq 0$.*

(i) If for some w, $F^\Theta(w) = F^{\underline{\Theta}}(w)$ and $F^{\Theta\prime}(w') > F^{\underline{\Theta}\prime}(w')$ in a right neighborhood of w, then $F^{\Theta\prime}(w') > F^{\underline{\Theta}\prime}(w')$ for all $w' > w$.

(ii) Assume $\Theta >_{D_+^\Theta} \underline{\Theta}$. If for some w, $F^\Theta(w) = F^{\underline{\Theta}}(w)$ and $F^{\Theta\prime}(w) \geq F^{\underline{\Theta}\prime}(w)$, then $F^{\Theta\prime}(w') > F^{\underline{\Theta}\prime}(w')$ for all $w' > w$.

Note that the precondition of part (i) is implied by (but is not equivalent to) $F^\Theta(w) = F^{\underline{\Theta}}(w)$ and $F^{\Theta\prime}(w) > F^{\underline{\Theta}\prime}(w)$.

Proof. (Lemma 20) We prove only part (i) (the proof of part (ii) is analogous). First, by assumption, $F^{\Theta\prime}(w') > F^{\underline{\Theta}\prime}(w')$ for all $w' > w$ sufficiently close to w. Suppose now that there exists $w' > w$ with $F^{\Theta\prime}(w') \leq F^{\underline{\Theta}\prime}(w')$ - we now assume that w' is the smallest with this property. Since $F^{\Theta\prime} >_{(w,w')} F^{\underline{\Theta}\prime}$, we have that $F^\Theta(w') > F^{\underline{\Theta}}(w')$. Therefore, it must be the case that $F^{\Theta\prime\prime}(w') > F^{\underline{\Theta}\prime\prime}(w')$: otherwise, since $F^{\Theta\prime\prime}(w') \leq 0$ and $\Theta \geq_{D_+^\Theta} \underline{\Theta}$, every policy $(\bar{a}, \bar{h}, c)$ would yield a weakly higher value of the right-hand side of HJB equation (6) for $F^{\underline{\Theta}}(w')$ than for $F^{\Theta}(w')$. But then $F^{\Theta\prime\prime}(w') > F^{\underline{\Theta}\prime\prime}(w')$ implies that $F^{\Theta\prime}(w'') < F^{\underline{\Theta}\prime}(w'')$ for w'' in a left neighborhood of w', contradicting the minimality of w'. ■

Given the Lemma, the proof of part (i) of Proposition 1 proceeds as follows. Applying part (i) of Lemma 20 to $w = 0$, if $F^{\Theta\prime}(0) > F^{\underline{\Theta}\prime}(0)$ then $F^{\Theta\prime}(w') > F^{\underline{\Theta}\prime}(w')$ for all $w' > 0$. Therefore $F^{\underline{\Theta}} \geq \underline{F}$ would imply $F^{\Theta}(w') > \underline{F}(w')$ for all $w' > 0$, violating the boundary conditions for F^{Θ}. Using the analogous argument, $F^{\underline{\Theta}\prime}(w) \geq F^{\Theta\prime}(w)$ for all $w \in [0, w^{\Theta}_{sp}]$, and so $F^{\underline{\Theta}}(w) \geq F^{\Theta}(w)$, for all $w \in [0, w^{\Theta}_{sp}]$, establishing part (i) of the Proposition. The proof of part (ii) is analogous.

We note that part (i) of the Proposition 1 is immediately applicable to the limit values for the general case defined in Theorem 3 (as it is applicable to the functions F_{ζ} and weak inequalities are preserved in the limit). The following Lemma shows that under an additional mild constraint part (ii), i.e., strict monotonicity, is applicable to the general case as well.

Consider the following assumption:

$$\text{(Cont) } \Theta\left(\bar{a}, \bar{h}\right) \geq \delta\left(\bar{a}\right) \text{ for a continuous } \delta \text{ with } \delta\left(\bar{a}\right) > 0 \text{ when } \bar{a} > 0.$$

Lemma 21 *Assume (Cont) holds. Then F as in Theorem 3 solves the HJB equation (6) with boundary conditions (7) and (8).*

Proof. (Lemma 21) Choose any $[\underline{w}, \overline{w}] \subset (0, w_{sp})$. Part (ii) of Lemma 19 guarantees that for sufficiently small ζ all F_{ζ} satisfy the constraint $\bar{a} \geq \gamma$ on $[\underline{w}, \overline{w}]$, for some $\gamma > 0$. Therefore, for sufficiently small ζ all F_{ζ} satisfy on $[\underline{w}, \overline{w}]$:

$$F''(w) = \inf_{\bar{a} \geq \gamma, \bar{h}, c} \left\{ \frac{F(w) - (\bar{a} - c) - F'(w)\left(w + \bar{h} - u(c)\right)}{r\Theta\left(\bar{a}, \bar{h}\right)/2} \right\},$$

with the right-hand side Lipschitz continuous in $(w, F(w), F'(w))$, since $\Theta \geq \delta(\gamma) > 0$ for $\bar{a} \geq \gamma$.

Part (i) of Proposition 1 guarantees that F_{ζ} converge in the supremum norm as $\zeta \downarrow 0$ to a function F. Since F'_{ζ} are uniformly bounded on $[\underline{w}, \overline{w}]$, it follows that all F''_{ζ} and F'_{ζ} are Lipschitz continuous with the same Lipschitz constant, and so F'_{ζ} converge to F' not only in L^1 but in the supremum norm, by the Arzela-Ascoli Theorem. Uniform Lipschitz continuity guarantees also that $F' = \frac{d}{dw}F$, that $F'' := \lim_{\zeta \downarrow 0} F''_{\zeta}$ exists and F satisfies the above equation (all on $[\underline{w}, \overline{w}]$). Since the set $[\underline{w}, \overline{w}]$ is arbitrary, this proves that F solves (6) in $(0, w_{sp})$, and so establishes proof of the lemma. ■

D.2 Proof of Proposition 2

The proof follows from the following Lemma.

Lemma 22 *For any $\delta > 0$ there is $\varepsilon > 0$ sufficiently small and $\widetilde{w} \in [0, \overline{w}_{sp}]$ such that the following holds: If $r\Theta \leq \varepsilon$ then the solution F of the HJB equation (6) with initial conditions*

$$F(\widetilde{w}) = \overline{F}(\widetilde{w}) - \delta, \;\; F'(\widetilde{w}) = \overline{F}'(\widetilde{w})$$

satisfies

$$F'' \leq_{[0,\overline{w}_{sp}]} -\frac{2\delta}{\varepsilon}.$$

Proof. For any $\lambda \in [\overline{F}'(\overline{w}_{sp}), \infty)$ let G_λ be the linear function tangent to the first-best frontier $\left\{(w, \overline{F}(w)) : w \in [0, \overline{w}_{sp}]\right\}$ with the slope λ. We will show that if for an arbitrary $w \in [0, w_{sp}]$

$$G_{F'(w)}(w) - F(w) \geq \delta, \tag{38}$$

then $F''(w) \leq -\frac{2\delta}{\varepsilon}$. Note that then as long as $-\frac{2\delta}{\varepsilon} \leq \min_{w\in[0,\overline{w}_{sp}]} \overline{F}''(w)$ the above condition will be satisfied over the whole interval $[0, \overline{w}_{sp}]$, which will establish the Lemma.

The HJB equation (6) takes the form

$$F''(w) \leq \min_{a,h,c} \frac{2}{r\Theta(a,h)} \left\{F(w) - (a-c) - F'(w)(w+h-u(c))\right\}. \tag{39}$$

Let w' be such that $F'(w) = \overline{F}'(w')$. For the policy $(a(w'), c(w'))$ in the problem (1) at w' we have:

$$\begin{aligned} &F(w) - (a(w') - c(w')) - F'(w)(w + h(a(w')) - u(c(w'))) \;= \\ &\overline{F}(w') - (a(w') - c(w')) - \overline{F}'(w')(w' + h(a(w')) - u(c(w'))) \\ &+ \left[F(w) - \overline{F}(w') + F'_\zeta(w)(w' - w)\right] \;=\; \left[F(w) - \overline{F}(w') + F'(w)(w' - w)\right] \leq -\delta, \end{aligned}$$

where the last equality follows from (1), while the last inequality follows from (38). Since $(a(w'), h(a(w')), c(w'))$ is an available policy in the problem (39) and $r\Theta \leq \varepsilon$, this establishes that $F''(w) \leq -\frac{2\delta}{\varepsilon}$. ■

Given the Lemma, for any $\delta > 0$ and sufficiently small $\zeta > 0$ the solution F of the HJB equation (6) with initial conditions $F(\widetilde{w}) = \overline{F}(\widetilde{w}) - \delta$, $F'(\widetilde{w}) = \overline{F}'(\widetilde{w})$ with $\widetilde{w} \in [\delta, \overline{w}_{sp}]$ will satisfy $F(\underline{w}) = \underline{F}(\underline{w})$ and $F(\overline{w}) = \underline{F}(\overline{w})$ for some $0 < \underline{w} < \overline{w} < \overline{w}_{sp}$. This together with Proposition 6 and part (ii) of Lemma 6 establishes the proof of the Proposition.

E Proof of Proposition 3

Fix period length $\Delta > 0$, densities g_X and γ_X satisfying (13) and any $w_g, w_\gamma \in [0, \bar{u})$. Consider the problem of finding a contract $\{c_n\}$ and action plans $\{a_{g,n}\}, \{a_{\gamma,n}\}$ that maximize the sum of principal's expected discounted revenues under noise densities g_X and γ_X, such that $\{c_n\}, \{a_{g,n}\}$ is incentive compatible under g_X and $\{c_n\}, \{a_{\gamma,n}\}$ is incentive compatible under γ_X, and they deliver expected discounted utilities w_g and w_γ to the agent.[33] Let $F^{\Delta}_{g,\gamma}(w_g, w_\gamma)$ be the value to the principal from the optimal contract:

$$\begin{aligned} F^{\Delta}_{g,\gamma}(w_g, w_\gamma) &= \sup \Big\{\Pi_g(\{c_n\},\{a_{g,n}\}) + \Pi_\gamma(\{c_n\},\{a_{\gamma,n}\}) \;\Big| \\ &\qquad \{a_{g,n}\} \text{ is IC for } \{c_n\},\ U_g(\{c_n\},\{a_{g,n}\}) = w \text{ under density } g_X, \\ &\qquad \{a_{\gamma,n}\} \text{ is IC for } \{c_n\},\ U_\gamma(\{c_n\},\{a_{\gamma,n}\}) = w \text{ under density } \gamma_X \Big\} \end{aligned}$$

To establish the Proposition we show that if $w_g, w_\gamma \in (0, w_{sp})$ then there is $\delta > 0$ such that for sufficiently small Δ $F^{\Delta}_{g,\gamma}(w_g, w_\gamma) + \delta \le F(w_g) + F(w_\gamma) =: F_2(w_g, w_\gamma)$, where F is as in Theorem 1.

First, consider the following Bellman operator:

$$\begin{aligned} T^{\Delta}_{g,\gamma} f(w_g, w_\gamma) &= \sup_{a_g, a_\gamma, c, W_g, W_\gamma} \Phi^{\Delta}_g(a_g, c, W_g; f) + \Phi^{\Delta}_\gamma(a_\gamma, c, W_\gamma; f) \\ \text{s.t.} \quad & a_\phi \in \mathcal{A}, \quad c(y) \ge 0 \quad \text{and} \quad W_\phi(y) \in [0, \bar{u}) \quad \forall y \\ & w_\phi = \mathbb{E}^{\Delta}_\phi\Big[\tilde{r}\Delta[u(c(\Delta[x + a_\phi])) - h(a_\phi)] + e^{-r\Delta} W_\phi(\Delta[x + a_\phi])\Big] \qquad (\text{PK}_2) \\ & a_\phi \in \arg\max_{\hat{a}\in\mathcal{A}} \mathbb{E}^{\Delta}_\phi\Big[\tilde{r}\Delta[u(c(\Delta[x + \hat{a}])) - h(\hat{a})] + e^{-r\Delta} W_\phi(\Delta[x + \hat{a}])\Big] \quad (\text{IC}_2) \end{aligned}$$

where the supremum is taken over measurable functions and $\Phi^{\Delta}_\phi(a, c, W; f)$ is as in (22), for $\phi \in \{g, \gamma\}$. The following is an analogue of Proposition 4:

Proposition 7 *$F^{\Delta}_{g,\gamma}$ is the largest fixed point f of $T^{\Delta}_{g,\gamma}$ such that $f(w_g, w_\gamma) \le \overline{F}(w_g) + \overline{F}(w_\gamma)$.*

For a set of feasible policies p for the Bellman operator $T^{\Delta}_{g,\gamma}$, where

$$p = \left\{\left(a_{g,(w_g,w_\gamma)}, a_{\gamma,(w_g,w_\gamma)}, c_{(w_g,w_\gamma)}, W_{g,(w_g,w_\gamma)}, W_{\gamma,(w_g,w_\gamma)}\right)\right\}_{(w_g,w_\gamma)\in[0,\bar{u})^2},$$

let $T^{\Delta,p}_{g,\gamma}$ be the operator defined as $T^{\Delta,p}_{g,\gamma} f(w) = \Phi^{\Delta}_g(a_g, c, W_g; f) + \Phi^{\Delta}_\gamma(a_\gamma, c, W_\gamma; f)$, and let $F^{\Delta,p}_{g,\gamma}$ parametrize the values achieved by the contract-action plans generated by the policies p. Note that $F^{\Delta,p}_{g,\gamma}$ is a fixed point of $T^{\Delta,p}_{g,\gamma}$. Also, policies p together with an initial point $(w_g, w_\gamma) = (w^p_{g,0}, w^p_{\gamma,0})$ determine a stochastic process $\left\{\left(w^p_{g,n}, w^p_{\gamma,n}\right)\right\}$ of continuation values.

For the proof of the Proposition we use the following five claims. Claim 1 is related to Lemma 6. It shows that for a fixed set of policies p for the Bellman operator $T^{\Delta}_{g,\gamma}$, how far the value of the contract generated recursively from those policies falls short of

[33] Since we look at the pure hidden action models, in the rest of this section we simplify notation by ignoring the trivial reporting strategies.

F_2 $(F_2 - F_{g,\gamma}^{\Delta,p})$ can be expressed as a discounted expected sum of how far each policy applied to F_2 falls short of F_2 $(F_2 - T_{g,\gamma}^{\Delta,p} F_2)$.

The idea behind the construction in the remaining four claims is as follows. For any $\varepsilon > 0$ consider the set $S_\varepsilon = \{(w_g, w_\gamma) \in [\varepsilon, w_{sp} - \varepsilon]^2 : |w_g - w_\gamma| > \varepsilon, \max\{w_g, w_\gamma\} > w_0 + \varepsilon\}$, where w_0 is such that $F'(w_0) = \underline{F}'(0) = -\frac{1}{u'(0)}$. Claim 2 shows that once the two continuation values are in this set, $F_2 - T_{g,\gamma}^{\Delta,p} F_2$ must be negative: The reason is that to achieve $F_2(w_g, w_\gamma) = F(w_g) + F(w_\gamma)$ the wages paid in the separate two optimal policies for each continuation value must be different (such that $-1/u'(c_g) = F'(w_g)$, and $-1/u'(c_\gamma) = F'(w_\gamma)$), whereas $T_{g,\gamma}^{\Delta,p}$ restricts the wage to be the same.

Claim 3 shows that if $F_2 - T_{g,\gamma}^{\Delta,p} F_2$ is to remain small, it must be that the variances of continuation values W_g, W_γ and $W_g - W_\gamma$ must be bounded away from zero, and not too big. This follows from the results in the paper: for the policy p to fare well, the continuation values for each noise must be approximately linear in likelihod ratio. Also, since the likelihood ratios are linearly independent by assumption, $W_g - W_\gamma$ can't be too small. Using Claim 3, Claim 4 shows that under policies p once the process of continuation values (w_g, w_γ) enters set S_ε, it must stay there for a while with nonnegligible probability; Claim 5 shows that starting at any interior point of continuation values the process enters S_ε in finite time with nonnegligible probabilities. Those results, together with Claim 2 establish the Proposition.

Fix a set of policies p for the Bellman operator $T_{g,\gamma}^{\Delta}$.

Claim 1 *Consider function $F : [0, \bar{u})^2 \to \mathbb{R}$ and $(w_{g,0}^p, w_{\gamma,0}^p) \in [0, \bar{u})^2$. Then for any $N \in \mathbb{N}$*

$$\begin{aligned} F_2(w_g, w_\gamma) - F_{g,\gamma}^{\Delta,p}(w_g, w_\gamma) &= \mathbb{E}_{g,\gamma}^{\Delta}\Big[\sum_{n=0}^{N} e^{-rn\Delta}(F_2(w_{g,n}^p, w_{\gamma,n}^p) - T_{g,\gamma}^{\Delta,p} F_2(w_{g,n}^p, w_{\gamma,n}^p)) + \\ &\quad e^{-r(N+1)\Delta}(F_2(w_{g,N+1}^p, w_{\gamma,N+1}^p) - F_{g,\gamma}^{\Delta,p}(w_{g,N+1}^p, w_{\gamma,N+1}^p))\Big]. \end{aligned}$$

Proof. For any $(w_g, w_\gamma) \in [0, \bar{u})^2$ we have

$$\begin{aligned} F_2(w_g, w_\gamma) - F_{g,\gamma}^{\Delta,p}(w_g, w_\gamma) &= F_2(w_g, w_\gamma) - T_{g,\gamma}^{\Delta,p} F_{g,\gamma}^{\Delta,p}(w_g, w_\gamma) = \\ = \quad & F_2(w_g, w_\gamma) - T_{g,\gamma}^{\Delta,p} F_2(w_g, w_\gamma) + T_{g,\gamma}^{\Delta,p} F_2(w_g, w_\gamma) - T_{g,\gamma}^{\Delta,p} F_{g,\gamma}^{\Delta,p}(w_g, w_\gamma) \\ = \quad & \mathbb{E}_{g,\gamma}^{\Delta}\Big[F_2(w_g, w_\gamma) - T_{g,\gamma}^{\Delta,p} F_2(w_g, w_\gamma) + e^{-r\Delta}(F_2(w_{g,1}^p, w_{\gamma,1}^p) - F_{g,\gamma}^{\Delta,p}(w_{g,1}^p, w_{\gamma,1}^p))\Big], \end{aligned}$$

and using the equality recursively yields the proof. ■

Claim 2 *Fix $\varepsilon > 0$ and $(w_g, w_\gamma) \in S_\varepsilon$. Then there is δ_1 such that for sufficiently small $\Delta > 0$:*

$$F_2(w_g, w_\gamma) - T_{g,\gamma}^{\Delta} F_2(w_g, w_\gamma) > \delta_1 \Delta.$$

Proof. In analogy to $T^{\Delta,q}$ we also define a simplified "quadratic" operator $T^{\Delta,q}_{g,\gamma}$:

$$\begin{aligned} T^{\Delta,q}_{g,\gamma} f(w_g, w_\gamma) &= \sup_{a_g, a_\gamma, c, W_g, W_\gamma} \Phi^{\Delta,q}_g(a_g, c, W_g; f, w_g) + \Phi^{\Delta,q}_\gamma(a_\gamma, c, W_\gamma; f, w_\gamma) \\ \text{s.t.} \quad & a_\phi(z) \in \mathcal{A}, \quad c \geq 0, \quad \text{and} \quad W_\phi(y) \in \mathbb{R} \quad \forall y \\ & w_\phi = \mathbb{E}^\Delta_\phi \Big[\tilde{r}\Delta[u(c(\Delta[x + a_\phi])) - h(a_\phi)] + e^{-r\Delta} W_\phi(\Delta[x + a_\phi]) \Big] \quad (\text{PK}_2 q) \\ & \tilde{r} h'(a_\phi) = -\frac{e^{-r\Delta}}{\Delta} \int_{\mathbb{R}} W(\Delta x) \phi^{\Delta\prime}_X(x) dx \qquad (\text{FOC}_2 q\text{-AC}) \end{aligned}$$

where the supremum is taken over measurable functions and $\Phi^{\Delta,q}_\phi(a, c, W; f, w_\phi)$ is defined as in (23), for $\phi \in \{g, \gamma\}$. Using analogues to Lemmas 13 and 15 we establish:

$$|T^{\Delta,q}_{g,\gamma} F_2 - T^{\Delta}_{g,\gamma} F_2|_{[0,\bar{u})^2} = o(\Delta).$$

Fix $\varepsilon > 0$ and $(w_g, w_\gamma) \in [\varepsilon, w_{sp} - \varepsilon]^2$ such that $|w_g - w_\gamma| \geq \varepsilon$. In view of the above bound, it is sufficient to establish that $F_2(w_g, w_\gamma) - T^{\Delta,q}_{g,\gamma} F_2(w_g, w_\gamma) > \delta_1 \Delta$, and so, due to Proposition 6 and Lemmas 13 and 15, it is sufficient to show that

$$T^{\Delta,q}_g F(w_g) + T^{\Delta,q}_\gamma F(w_\gamma) - T^{\Delta,q}_{g,\gamma} F_2(w_g, w_\gamma) > \delta_1 \Delta,$$

where $T^{\Delta,q}_g$ and $T^{\Delta,q}_\gamma$ stand for operator $T^{\Delta,q}$ under the respective noise densities.

We have

$$\begin{aligned} T^{\Delta,q}_\phi F(w_\phi) &= \sup_c -\tilde{r}\Delta \{c + F'(w_\phi) u(c)\} + \sup_{a,W} \Psi^\Delta_\phi (a, W; F, w_\phi), \\ T^{\Delta,q}_{g,\gamma} F_2(w_g, w_\gamma) &= \sup_c -\tilde{r}\Delta \{2c + F'(w_g) u(c) + F'(w_\gamma) u(c)\} + \\ & \sup_{a_g, W_g} \Psi^\Delta_g (a_g, W_g; F, w_g) + \sup_{a_\gamma, W_\gamma} \Psi^\Delta_\gamma (a_\gamma, W_\gamma; F, w_\gamma), \end{aligned}$$

where

$$\begin{aligned} \Psi^\Delta_\phi (a, W; F, w_\phi) = e^{-\Delta r} F(w_\phi) &+ \tilde{r}\Delta \{a + F'(w_\phi)[w_\phi + h(a_\phi)]\} \\ &+ e^{-\Delta r} \mathbb{E}^\Delta_\phi [\frac{1}{2} F''(w_\phi) (W(\Delta x) - w_\phi)^2], \end{aligned}$$

$\phi \in \{g, \gamma\}$. The proof follows from the fact that F'' is bounded away from 0 on $[0, w_{sp})^2$ and so $|F'(w_g) - F'(w_\gamma)| > \varepsilon_1$ for some $\varepsilon_1 > 0$, which implies that for some δ_1:

$$\begin{aligned} & \sup_c - \{c + F'(w_g) u(c)\} + \sup_c - \{c + F'(w_\gamma) u(c)\} \\ > & \sup_c - \{2c + F'(w_g) u(c) + F'(w_\gamma) u(c)\} + \delta_1. \quad \blacksquare \end{aligned}$$

Claim 3 *Fix* $\varepsilon > 0$ *and* $(w_g, w_\gamma) \in [\varepsilon, w_{sp} - \varepsilon]^2$. *Then there is* $\delta_2 > 0$ *such that for sufficiently small* Δ *and any feasible policy* $(a_g, a_\gamma, c, W_g, W_\gamma)$ *for* $T^{\Delta}_{g,\gamma} F_2(w_g, w_\gamma)$ *if*

$$\Phi^{\Delta}_g(a_g, c, W_g; F_2) + \Phi^{\Delta}_\gamma(a_\gamma, c, W_\gamma; F_2) > F_2(w_g, w_\gamma) - \delta_2\Delta$$

then

$$\begin{aligned} \mathbb{V}^{\Delta}_g[W_g(\Delta(x_g + a_g))], \mathbb{V}^{\Delta}_\gamma[W_\gamma(\Delta(x_\gamma + a_\gamma))] &> \delta_2\Delta, \\ \mathbb{V}^{\Delta}_{g,\gamma}[W_g(\Delta(x_g + a_g)) - W_\gamma(\Delta(x_\gamma + a_\gamma))] &> \delta_2\Delta. \end{aligned}$$

On the other hand,

$$\begin{aligned} & F_2(w_g, w_\gamma) - \Phi^{\Delta}_g(a_g, c, W_g; F_2) + \Phi^{\Delta}_\gamma(a_\gamma, c, W_\gamma; F_2) \\ > \; & \delta_2\big(\mathbb{V}^{\Delta}_g[W_g(\Delta(x_g + a_g))] - \Delta\frac{(rh'(A))^2}{\mathcal{I}_{gX}}\big) + \delta_2\big(\mathbb{V}^{\Delta}_\gamma[W_\gamma(\Delta(x_\gamma + a_\gamma))] - \Delta\frac{(rh'(A))^2}{\mathcal{I}_{\gamma_X}}\big) \end{aligned}$$

Proof. Lemmas 21 part (i) and 9 imply that for certain $\delta_2 > 0$ and sufficiently small Δ if $\Phi^{\Delta}_g(a_g, c, W_g; F_2) + \Phi^{\Delta}_\gamma(a_\gamma, c, W_\gamma; F_2) > F_2(w_g, w_\gamma) - \delta_2\Delta$, then $a_g, a_\gamma > \gamma > 0$. But then Lemmas 9 and 1 imply that $\mathbb{V}^{\Delta}_\phi[W_\phi(\Delta(x_\phi + a_\phi))] \approx \Delta\frac{(rh'(a_\phi))^2}{\mathcal{I}_{\phi X}}$, for $\phi \in \{g, \gamma\}$, which yields the first inequality. The same Lemmas imply that $W_\phi(\Delta(x_\phi + a_\phi)) \approx \mathbb{E}^{\Delta}_\phi[W_\phi(\Delta(x_\phi + a_\phi))] + \sqrt{\Delta}D\frac{g'(x_\phi)}{g(x_\phi)}$ (in $L_1(\phi^{\Delta}_X)$), for $\phi \in \{g, \gamma\}$, and so the second inequality follows from the linear independence of likelihood ratios (13). Finally, F'' bounded away from zero immediately implies the third inequality. ■

Fix a set of policies p for the Bellman operator $T^{\Delta}_{g,\gamma}$, $T > 0$, $\varepsilon_1 > \varepsilon_2 > 0$.

Claim 4 *Fix an initial point* $(w_g, w_\gamma) \in S_\varepsilon$. *Then there are* $\delta_3, T > 0$ *such that for sufficiently small* Δ

$$\mathbb{E}^{\Delta}_{g,\gamma}\Big[\sum_{n=0}^{T/\Delta} e^{-rn\Delta}(F_2(w^p_{g,n}, w^p_{\gamma,n}) - T^{\Delta,p}_{g,\gamma}F_2(w^p_{g,n}, w^p_{\gamma,n}))\Big] \le \delta_3$$

implies

$$\mathbb{P}^{\Delta}_{g,\gamma}[(w^p_{g,n}, w^p_{\gamma,n}) \in S_{\varepsilon/2}, n = 0, ..., T/\Delta] > \delta_3.$$

Proof. If the precondition is satisfied, then Claim 3 implies that, for $\phi \in \{g, \gamma\}$,

$$\begin{aligned} \mathbb{V}^{\Delta}_\phi[(w^p_{\phi,t} - w^p_{\phi,0})] &\le T\frac{(rh'(A))^2}{\mathcal{I}_{gX}} + \delta/\delta' =: C_{T,\delta}, \text{ for } t \le T/\Delta, \\ \mathbb{E}^{\Delta}_\phi[\mathbb{V}^{\Delta}_\phi[(w^p_{\phi,T/\Delta} - w^p_{\phi,t})] | w^p_{\phi,0}] &\le C_{T,\delta}, \text{ for } t \le T/\Delta, \end{aligned}$$

with $C_{T,\delta} \to 0$ as $T, \delta \to 0$. We also have

$$\mathbb{E}^{\Delta}_\phi[(w^p_{\phi,t} - w^p_{\phi,t'}) | w^p_{\phi,t'}] \le D_{T,\delta}, \text{ for } t' < t' \le T/\Delta,$$

with $D_{T,\delta} \to 0$ as $T, \delta \to 0$. It therefore follows that for $\alpha = \frac{\varepsilon}{4} > 0$ and τ the stopping time of reaching the set $[\alpha, \infty)$

$$\begin{aligned} & \mathbb{P}^{\Delta}_{\phi}[\max_{t \leq T/\Delta} \{w^p_{\phi,t} - w^p_{\phi,0}\} \geq \alpha] \\ = & \mathbb{P}^{\Delta}_{\phi}[\max_{t \leq T/\Delta} \{w^p_{\phi,t} - w^p_{\phi,0}\} \geq \alpha, w^p_{\phi,T/\Delta} - w^p_{\phi,0} \geq \alpha/2] + \\ & \mathbb{P}^{\Delta}_{\phi}[\max_{t \leq T/\Delta} \{w^p_{\phi,t} - w^p_{\phi,0}\} \geq \alpha, w^p_{\phi,T/\Delta} - w^p_{\phi,0} < \alpha/2] \\ \leq & \mathbb{P}^{\Delta}_{\phi}[w^p_{\phi,T/\Delta} - w^p_{\phi,0} \geq \alpha/2] + \mathbb{P}^{\Delta}_{\phi}[w^p_{\phi,T/\Delta} - w^p_{\phi,\tau} < -\alpha/2] \ \leq \ 2\frac{C_{T,\delta}}{(\alpha/2 - D_{T,\delta})^2} \to 0, \end{aligned}$$

as $T, \delta \to 0$. This establishes the proof. ■

Claim 5 *Fix an initial point $(w_g, w_\gamma) \in [\varepsilon, w_{sp} - \varepsilon]^2$. Then there are $\delta_4, T > 0$ such that for sufficiently small Δ*

$$\mathbb{E}^{\Delta}_{g,\gamma}\Big[\sum_{n=0}^{T/\Delta} e^{-rn\Delta}(F_2(w^p_{g,n}, w^p_{\gamma,n}) - T^{\Delta,p}_{g,\gamma} F_2(w^p_{g,n}, w^p_{\gamma,n}))\Big] \leq \delta_4$$

implies

$$\mathbb{P}_{g,\gamma}[(w^p_{g,T/\Delta+1}, w^p_{\gamma,T/\Delta+1}) \in S_\varepsilon] > \delta_4.$$

Proof. The proof is similar to the proof of the previous claim and so is omitted. ■

Given the claims, the rest of the proof is as follows. If $(w_g, w_\gamma) \in S_\varepsilon$ then for the constants as in the claims

$$\begin{aligned} F_2(w_g, w_\gamma) - F^{\Delta,p}_{g,\gamma}(w_g, w_\gamma) \ & \geq \ \mathbb{E}^{\Delta}_{g,\gamma}\Big[\sum_{n=0}^{T/\Delta} e^{-rn\Delta}(F_2(w^p_{g,n}, w^p_{\gamma,n}) - T^{\Delta,p}_{g,\gamma} F_2(w^p_{g,n}, w^p_{\gamma,n}))\Big] \\ & \geq \ \min\left\{\delta_3, \frac{1 - e^{-rT}}{1 - e^{-r\Delta}}\delta_3\delta_1\right\}, \end{aligned}$$

where the first inequality follows from Claim 1 and the second inequality follows from Claims 2 and 4.

If on the other hand $(w_g, w_\gamma) \in [\varepsilon_1, w_{sp} - \varepsilon]^2 \setminus S_\varepsilon$ then

$$\begin{aligned} & F_2(w_g, w_\gamma) - F^{\Delta,p}_{g,\gamma}(w_g, w_\gamma) \ \geq \ \mathbb{E}^{\Delta}_{g,\gamma}\Big[\sum_{n=0}^{T/\Delta} e^{-rn\Delta}(F_2(w^p_{g,n}, w^p_{\gamma,n}) - T^{\Delta,p}_{g,\gamma} F_2(w^p_{g,n}, w^p_{\gamma,n})) \\ + \ & e^{-r(T+\Delta)}(F_2(w^p_{g,T/\Delta+1}, w^p_{\gamma,T/\Delta+1}) - T^{\Delta,p}_{g,\gamma} F_2(w^p_{g,T/\Delta+1}, w^p_{\gamma,T/\Delta+1}))\Big] \\ \geq \ & \min\left\{\delta_4, e^{-r(T+\Delta)}\delta_4 \min\left\{\delta_3, \frac{1 - e^{-r(T+\Delta)}}{1 - e^{-r\Delta}}\delta_3\delta_1\right\}\right\}, \end{aligned}$$

where the first inequality follows from Claim 1 and the second inequality follows from Claim 5 and the inequalities above. This establishes the proof of the Proposition.

We note that the proof can be extended beyond the pure hidden action case and $\mathcal{I}_{g_X} = \mathcal{I}_{\gamma_X}$. As regards the equality of Fisher information quantities, this guaranteed that the limits of the values of contracts F_g and F_γ for two noise distributions are the same function F (Lemma 1). Because of that, as long as the continuation values w_g and w_γ are not the same the derivatives $F_g'(w_g)$ and $F_\gamma'(w_\gamma)$ differ as well, which is crucial for Claim 2. Dropping the assumption $\mathcal{I}_{g_X} = \mathcal{I}_{\gamma_X}$ the proof would be analogous, yet the computation of the set of continuation values (w_g, w_γ) for which $F_g'(w_g) \neq F_\gamma'(w_\gamma)$ would be cumbersome.

On the other hand, the assumption of pure hidden action models was also not crucial for the proof: For two different information structures the proof will work as long as, roughly, the optimal policies in the problem of minimizing variance of incentive transfers are sufficiently different (see Claim 3).

F Proofs for Section 4.1.1

Throughout this section we assume that the following assumption holds:

(A2') $X \equiv Z$ and X has a density function $g_X(x)$. The set of available actions is $\mathcal{A} = (-\infty, A]$ for some $A \in \mathbb{R}_+$.

In this section we establish Theorem 3 and the analogue of Theorem 2, which takes the following form (see the definition of simple contract action plan below):

Theorem 4 *For $\zeta > 0$ let F_ζ be as in Theorem 3 and fix period length Δ, agent's promised value $w \in [0, \bar{u})$ and an approximation error $\varepsilon > 0$. A corresponding simple contract-action plan is incentive compatible by construction and $[O(\varepsilon) + O(\Delta^{1/3}) + O(\zeta)]$-suboptimal.*

The proof of the theorems follows just as in Section A from Lemma 6 and the following version of Proposition 6, which is proven in Section F.1.

Proposition 8 *Fix $\zeta \geq 0$ and F_ζ solving the HJB equation (18) on an interval I with $F_\zeta'' < 0$. Then $|T_I^\Delta F_\zeta - F_\zeta|_{I^\Delta} = o(\Delta) + O(\zeta\Delta)$. Moreover, for any $\varepsilon > 0$, $\Delta > 0$ and $w \in I^\Delta$, $\Phi^\Delta(a, c, W; F_\zeta) \geq F_\zeta(w) - O(\varepsilon\Delta) - O(\zeta\Delta)$, where (a, c, W) is a simple policy defined for $(F_\zeta, \varepsilon, \Delta, w)$ by (11) and (12).*

The simple contract-action plans are defined almost identically to those in Section 3.3 as follows. First, let us define the appropriate Bellman operators as in Section 3.3. For an interval $I \subset \mathbb{R}$ and any function $f : I \to \mathbb{R}$, define the new function $T_I^\Delta f : I \to \mathbb{R}$ by

$$T_I^\Delta f(w) = \sup_{a,c,W} \Phi^\Delta(a, c, W; f) \tag{40}$$

$$\text{s.t. } a(z) \in \mathcal{A} \quad \forall z, \quad c(y) \geq 0 \quad \text{and} \quad W(y) \in I \quad \forall y$$

$$w = \mathbb{E}^\Delta\Big[\tilde{r}\Delta[u(c(\Delta[x + a(z)])) - h(a(z))] + e^{-r\Delta}W(\Delta[x + a(z)])\Big] \tag{PK}$$

$$a(x) \in \arg\max_{\hat{a}\in\mathcal{A}} \tilde{r}\Delta\big[u(c(\Delta[x + \hat{a}])) - h(\hat{a})\big] + e^{-r\Delta}W(\Delta[x + \hat{a}]) \quad \forall x \tag{IC-PHI}$$

We note that the Belman operator T_I^Δ excludes reporting by the agent. However, in the pure hidden information case this is without loss of generality: With reporting, there may not exist two different noise realizations resulting in the same signal in equilibrium (as incentive compatibility would be violated). Thus, reporting is redundant.

Consider the following definition of simple policies (compare Definitions 1 and 3).

Definition 4 *For any $\zeta \geq 0$ and F_ζ solving (18) on an interval I, period length $\Delta > 0$, agent's promised value $w \in I$ and an approximation error $\varepsilon > 0$, define a* simple policy *(a, c, W) as follows. Let $(\bar{a}, \bar{h}, c)$ be an ε-suboptimal policy of (18) at w, and for the corresponding $(\bar{a}, \bar{h})$, let (a, v) be an ε-suboptimal policy of (19).*

If $w \in I^\Delta$ let

$$c(y) = c,$$

$$W(y) = C + \sqrt{\Delta}\tilde{r}e^{r\Delta} \times \begin{cases} v(-M) & \text{if } \; y/\sqrt{\Delta} < -M \\ v(y/\sqrt{\Delta}) & \text{if } \; |y/\sqrt{\Delta})| \leq M \\ v(M) & \text{if } \; y/\sqrt{\Delta} > M \end{cases},$$

$a(z)$ *is an action that satisfies the (IC) constraint in (40),*

where M is such that $\mathbb{P}_X([-M, M]) \geq 1-\varepsilon$ and C is chosen to satisfy the (PK) constraint in (40). If $w \notin I^\Delta$ define the policy as in (12).

The definition differs from the one in Section 3.3 in that: (i) argumnet function is F_ζ not F, (ii) reporting is ignored, (iii) continuation value function must be non-decreasing, (iv) range of signals for which incentives are provided (or M_ε) is readjusted. Given the above definition, simple contract-action plans are defined as in Definition 1.

Notice that, unlike in the model analyzed in the paper, there is no additional incentive compatibility constraint associated with truthful reporting, and so, by construction, simple policies are fully incentive compatible. Also, as before, (PK) is satisfied by construction, and $W(y) \in I$ if Δ is sufficiently small. Thus, simple policies are feasible for the problem (40), and so Proposition 8 verifies only that they are close to optimal.

F.1 Proof of Proposition 8

As in the paper, define $T_I^{\Delta,c}$ by restricting the consumption schedule $c(y)$ to be constant. Let us also define $T_I^{\Delta,d} f(w)$ as $T_I^{\Delta,c} f(w)$ with the additional constraints that

$a(\cdot)$ is piecewise continuously differentiable and $W(\cdot)$ is continuous. Finally, we modify the simplified operator $T^{\Delta,q}$ defined in (3) by replacing the local (first-order) incentive constraint (FOC$_q$) by [34]

$$\tilde{r}h'(a(x)) = e^{-r\Delta}W'(\Delta x) \quad \forall x \qquad \text{(FOC}_q\text{-PHI)}$$

The proof of Proposition 8 is established by a sequence of Lemmas, similar as in Section A. Regarding the values, the line of the argument can be illustrated as follows:

$$F \underset{\text{Lemma 23}}{\sim} T^{\Delta,q}F \underset{\text{Lemma 24}}{\sim} T_I^{\Delta,d}F \underset{\text{Lemma 26}}{\sim} T_I^{\Delta,c}F \underset{\text{Lemma 15}}{\sim} T_I^{\Delta}F.$$

Note that the last equivalence follows form the same Lemma as in the paper. Here we focus on the other three.

First, the Lemma 9 extend readily to the current pure hidden information case. Likewise, we extend the definition of *quadratic simple.*policies (see Definition 2)[35]

Remark 1 *In the pure hidden information case, the v in the definition of a quadratic simple policy at w is continuous and piecewise twice continuously differentriable (see the definition of Θ). We assume that for any $\varepsilon > 0$, there is a common finite set D such that the set of functions v'' for all $w \in I$ are equicontinuous outside of D, which is without loss of generality.*

The following is essentially a corollary of Lemma 9.

Lemma 23 *Fix $\zeta \geq 0$ and F_ζ solving the HJB equation (18) on an interval I with $F''_\zeta < 0$. Then $|T^{\Delta,q}F_\zeta - F_\zeta|_I = o(\Delta) + O(\zeta\Delta)$. Moreover, for any $\varepsilon, \Delta > 0$, $w \in I$ and corresponding quadratic simple policy (a_q, c_q, W_q), $\Phi^{\Delta,q}(a_q, c_q, W_q; F_\zeta, w) \geq F_\zeta(w) - O(\Delta\varepsilon) - O(\zeta\Delta)$, uniformly in I.*

Proof. From Lemma 9 we have

$$\begin{aligned} & T^{\Delta,q}F_\zeta(w) - F_\zeta(w) \\ = & \sup_{\bar{a},\bar{h},c} \tilde{r}\Delta\left\{(\bar{a} - c) + F'_\zeta(w)[w + \bar{h} - u(c)] + e^{r\Delta}\frac{F''_\zeta(w)}{2}\tilde{r}\Theta(\bar{a},\bar{h}) - F_\zeta(w)\right\} + O\left(\Delta^2\right) \\ = & O(\zeta\Delta) + O\left(\Delta^2\right). \end{aligned}$$

The last equality follows because F_ζ satisfies the HJB equation (18). Lemma 9 also yields that $\Phi^{\Delta,q}(a_q, c_q, W_q; F_\zeta, w) \geq F_\zeta(w) - O(\Delta^2) - O(\Delta\varepsilon) - O(\zeta\Delta)$, establishing the proof. ■

We establish now the crucial Lemma 24. First, we extend the general definition of simple policies to the pure hidden information case (compare Definition 3 in the paper).

[34] When $a(z) = 0$ or $a(z) = A$, at an optimum the inequalities in the (IC) constraint are attained with equality (see e.g. Edmans and Gabaix [2011]).

[35] Note that since the reporting is suppressed, the continuation value functions v in the definition of Θ and W^q_Δ in the definition of quadratic simple policies depend only on a single variable y.

Definition 5 *For a twice differentiable function $F : I \to \infty$ with $F'' < 0$, $\varepsilon > 0$, $\Delta > 0$, $w \in I^{\Delta}$ and quadratic simple policies (a_q, c_q, W_q) in the problem $T^{\Delta,q}F(w)$ based on (a, v), define the* simple policy (a, c, W) *for* $T_I^{\Delta,c}F(w)$ *as*

$$c = c_q,$$

$$W(y) = C + \begin{cases} W_q(-\sqrt{\Delta}M_\varepsilon) & \text{if } \Delta x < -\sqrt{\Delta}M_\varepsilon \\ W_q(\Delta x) & \text{if } |\Delta x| \le \sqrt{\Delta}M_\varepsilon \\ W_q(\sqrt{\Delta}M_\varepsilon) & \text{if } \Delta x > \sqrt{\Delta}M_\varepsilon \end{cases},$$

$a(z)$ is an action that satisfies the (IC) constraint in (40).

where M_ε is such that $\mathbb{P}_X([-M_\varepsilon, M_\varepsilon]) \ge 1 - \varepsilon$ and C is chosen to satisfy the (PK) constraint in (40).

Lemma 24 *Let $F : I \to \mathbb{R}$ be twice continuously differentiable with $F'' < 0$. Then $|T_I^{\Delta,c}F - T^{\Delta,q}F|_{I^\Delta} = o(\Delta)$. Moreover, for fixed $\varepsilon > 0$ consider quadratic simple policies (a_q, c_q, W_q) for $T^{\Delta,q}F(w)$, $\Delta > 0$, $w \in I^\Delta$. Then for Δ sufficiently small, $w \in I^\Delta$ and the corresponding simple policies (a, c, W), $\Phi^\Delta(a, c, W; F) \ge \Phi^{\Delta,q}(a_q, c_q, W_q; F, w) - O(\varepsilon\Delta) - o(\Delta)$, uniformly in w.*

Proof. (Lemma 13) Fix $\varepsilon > 0$, $\Delta > 0$ such that $\sqrt{\Delta} < \delta/A$, for δ as in Lemma 12 (with $M = M_\varepsilon$), and $w \in I^\Delta$.

Step 1: In this step we show that $\Phi^\Delta(a, c, W; F) \ge \Phi^{\Delta,q}(a_q, c_q, W_q; F, w) - O(\varepsilon\Delta)$, uniformly in w. Since ε is arbitrary, by of Lemma 9, this establishes $|T^{\Delta,q}F - T_I^{\Delta,d}F|^+_{I^\Delta} = o(\Delta)$.

First, the inequality (28) holds by the same arguments as before. It will thus be enough to establish (31), (32) and (33).

Given the definition of W, the necessary local version of (IC) take the following form:[36]

$$\tilde{r}h'(a(x)) = e^{-r\Delta}W'(y) = \tilde{r}v'(\sqrt{\Delta}[x + a(x)]), \tag{41}$$

whereas, given the definition of W_q and (FOC$_q$-PHI), we have

$$\tilde{r}h'(a_q(x)) = e^{-r\Delta}W_q'(\Delta x) = \tilde{r}v'(\sqrt{\Delta}x).$$

Let D be the finite set of points such that each v in the definition of the policy is twice continuously differentiable on $\mathbb{R}\backslash D$ (see Remark 1) and consider the set

$$N_\varepsilon^\Delta = [-M_\varepsilon/\sqrt{\Delta}, M_\varepsilon/\sqrt{\Delta} - A]\backslash \bigcup_{d \in D}\{d/\sqrt{\Delta} + \zeta : \zeta \in [0, A]\}.$$

[36]Recall that the W function, just as W_q, is constant in the second argument.

For sufficiently small Δ, $\mathbb{P}^{\Delta}\left[N_{\varepsilon}^{\Delta}\right] \geq 1-\varepsilon$. Moreover, for any $x \in N_{\varepsilon}^{\Delta}$, v' is continuously differentiable on $[\sqrt{\Delta}x, \sqrt{\Delta}[x+a(x)]]$. Consequently, for all such x $|h'(a_q(x)-h'(a(x))| \leq \sqrt{\Delta}\max v''$, where the maximum is taken over the set $[-M_{\varepsilon}, M_{\varepsilon}]$, and hence

$$|a_q(x)-a(x)| \leq \frac{\sqrt{\Delta}\max v''}{\inf h''}.$$

Since $\mathbb{P}^{\Delta}\left[N_{\varepsilon}^{\Delta}\right] \geq 1-\varepsilon$, we have that the inequalities (31) and (32) hold. Moreover, by taking the maximum over $\max v''$ over $[-M_{\varepsilon}, M_{\varepsilon}]$ for all w (which is well defined, due to the assumption of equicontinuity) we establish that the bounds in those inequalities are uniform in $w \in I^{\Delta}$. Finally, (33) follows from Lemma 11 just as in the previous case. This establishes the proof.

Step 2: In this step we show that $\left|T_I^{\Delta,d}F(w) - T^{\Delta,q}F(w)\right|_{I^{\Delta}}^{+} = o(\Delta)$.

Case 2: $X \equiv Z$.[37] For a policy (a,c,W) that is $\varepsilon\Delta$-suboptimal in the problem $T_I^{\Delta,d}F(w)$ define (a_q, c_q, W_q) as follows. Let $c_q = c$, $a_q(x) = a(x)$ for $x \in [-M_{\varepsilon}/\sqrt{\Delta}+1, M_{\varepsilon}/\sqrt{\Delta}-1]$, $a_q(x) = 0$ for $x \notin [-M_{\varepsilon}/\sqrt{\Delta}, M_{\varepsilon}/\sqrt{\Delta}]$ and a_q piecewise continuously differentiable. W_q is constant in the second argument and is defined by the local IC in (3), continuity and (PK). The policy (a_q, c_q, W_q) is feasible by construction, and we must prove that $\Phi^{\Delta,q}(a_q, c_q, W_q; F, w) \geq \Phi^{\Delta}(a,c,W;F) - O(\varepsilon\Delta)$.

On the one hand, $\mathbb{P}^{\Delta}[a_q(x) = a(x)] \geq 1-2\varepsilon$ for sufficiently small Δ, which implies the analogues of (31) and (32). On the other hand, for all $\underline{x}, \overline{x} \in [-M_{\varepsilon}/\sqrt{\Delta}, M_{\varepsilon}/\sqrt{\Delta}]$

$$\begin{aligned}
W_q(\Delta\bar{x}, z) - W_q(\Delta\underline{x}, z) &= \tilde{r}e^{r\Delta}\int_{\underline{x}}^{\bar{x}} \Delta h'(a_q(x))dx = \tilde{r}e^{r\Delta}\int_{\underline{x}}^{\bar{x}} \Delta h'(a(x))dx \\
&= \tilde{r}e^{r\Delta}\Big[\int_{\underline{x}}^{\overline{x}} \Delta h'(a(x))(1+a'(x))dx - \Delta(h(a(\overline{x})) - h(a(\underline{x})))\Big] \\
&= W(\Delta[\overline{x}+a(\overline{x})], \overline{x}) - W(\Delta[\underline{x}+a(\underline{x})], \underline{x}) + O(\Delta),
\end{aligned}$$

where the last inequality follows from the local neccesary version of (IC-PHI). Consequently $\mathbb{V}^{\Delta}[W_q(\Delta x, x)] \leq \mathbb{V}^{\Delta}\left[W(\Delta[x+a(x)], x)\mathbf{1}_{|x|\leq M_{\varepsilon}/\sqrt{\Delta}}\right] + O(\Delta^2)$. Moreover, since $\mathbb{V}^{\Delta}[W(\Delta[x+a(x)], x)] \leq V\Delta$ (Lemma 7) and $W' \in [0, h'(A)]$, there is K_{ε} such that for any Δ, $|x| \leq M_{\varepsilon}/\sqrt{\Delta}$ implies $y \in B$, where $B = \{y \mid |W(y) - \mathbb{E}^{\Delta}[W(\Delta[x+a(x)], x)]| \leq \sqrt{\Delta}K_{\varepsilon}\}$. Altogether, $\Phi^{\Delta,q}(a_q, c_q, W_q; F, w)$ is equal to

$$\begin{aligned}
&\tilde{r}\Delta(\mathbb{E}^{\Delta}[a(x)] - c) + e^{-r\Delta}\Big[F(w) + F'(w)\mathbb{E}^{\Delta}[W(\Delta[x+a(x)], x) - w] \\
&+ \frac{1}{2}F''(w)\,\mathbb{V}^{\Delta}\left[W(\Delta[x+a(x)], x)\mathbf{1}_B\right]\Big] + O(\varepsilon\Delta) \leq \Phi^{\Delta}(a,c,W;F) + O(\varepsilon\Delta),
\end{aligned}$$

[37] In Step 1 we used the fact that the quadratic simple policies, for all Δ, are based on the same set of v functions from the definition of Θ. In particular, the W_q functions have the same number of points of discontinuity, for all Δ. In this Step, without additional proofs we cannot assume such uniformity, and so the construction is different.

which establishes the Lemma. ∎

We move on to establish "$T_I^{\Delta,d}F \underset{26}{\sim} T_I^{\Delta,c}F$". The following Lemma 25 is related to the standard results in the static mechanism design.

Lemma 25 *Suppose $X \equiv Z$. For any $\Delta > 0$ and $w \in I^\Delta$, if (a, c, W) satisfies (IC) in $T_I^{\Delta,c}F(w)$ then $x + a(x)$ is nondecreasing. Conversely, if (a, c, W) satisfies the local version of (IC) almost everywhere and $x + a(x)$ is nondecreasing, then (a, c, W) satisfies the IC.*

Proof. The proof is standard, but we provide it for completeness. Suppose first that (a, c, W) is incentive compatible. Therefore for any $x' > x$

$$-\tilde{r}h(a(x')) + e^{-r\Delta}W(\Delta[x' + a(x')], x') \geq -\tilde{r}h(a(x) - (x' - x)) + e^{-r\Delta}W(\Delta[x + a(x)], x),$$
$$-\tilde{r}h(a(x)) + e^{-r\Delta}W(\Delta[x + a(x)], x) \geq -\tilde{r}h(a(x') + (x' - x)) + e^{-r\Delta}W(\Delta[x' + a(x')], x').$$

Hence,

$$h(a(x')) - h(a(x) - (x' - x)) \leq h(a(x') + (x' - x)) - h(a(x)).$$

Since h is convex, this implies that $a(x') \geq a(x) - (x' - x)$.

Conversely, we argue by contradiction. Assume that (a, c, W) satisfies the local IC and $x + a(x)$ is nondecreasing. Let

$$V(x, x') = -\tilde{r}h(a(x') + (x' - x)) + e^{r\Delta}W(\Delta[x' + a(x')], x').$$

By local IC, $V_2(x, x) = 0$ for all x. Suppose that for some $x' > x$ we have $0 < V(x, x') - V(x, x)$. Then

$$0 < \int_x^{x'} V_2(x, s)ds = \int_x^{x'} [V_2(x, s)) - V_2(s, s)]ds = -\int_x^{x'}\int_x^s V_{12}(z, s)dzds.$$

But

$$V_{12}(z, s) = \tilde{r}h''(a(s) + (s - z))(1 + a'(s)) \geq 0.$$

which is a contradiction. The case $V(x, x') > V(x, x)$ with $x' < x$ is analogous. ∎

Lemma 26 *Let $Z = X$, and let $F : I \to \mathbb{R}$ be twice continuously differentiable with $F'' < 0$. Then $|T_I^{\Delta,d}F = T_I^{\Delta,c}F|_{I^\Delta} = o(\Delta)$.*

Proof. Fix $\Delta, \varepsilon > 0$ and consider any Δ-suboptimal policy (a, c, W) for $T^{\Delta,c}F(w)$. Let M_ε be such that $\mathbb{P}_X^\Delta[[-M_\varepsilon/\sqrt{\Delta}, M_\varepsilon/\sqrt{\Delta}]] \geq 1 - \varepsilon$. We construct a policy (a_d, c_d, W_d) as follows. Below the function $a_d(\cdot)$ is derived from the function $a(\cdot)$ so that $a_d(\cdot)$ is piecewise continuously differentiable and $x + a_d(x)$ is nondecreasing. Then we let $c_d = c$, and W_d be such that it satisfies the local version of (IC):

$$\tilde{r}h'(a_d(x)) = e^{-r\Delta}W_d'(\Delta[x + a_d(x)]),$$

is continuous and the constant of integration is adjusted so that it satisfies the PK condition. By Lemma 25, the policy (a_d, c_d, W_d) is feasible by construction.

Below we will define a_d so that $a_d(x) = 0$ if $x \notin [-M_\varepsilon/\sqrt{\Delta}, M_\varepsilon/\sqrt{\Delta} + A]$, $x + a_d(x)$ is nondecreasing and

$$\int_{-M_\varepsilon/\sqrt{\Delta}}^{M_\varepsilon/\sqrt{\Delta}} |a_d(x) - a(x)|\, dx \leq \varepsilon \quad \text{and} \quad \int_{-M_\varepsilon/\sqrt{\Delta}}^{M_\varepsilon/\sqrt{\Delta}} |a_d'(x) - a'(x)|\, dx \leq \varepsilon. \tag{42}$$

Recall that if f is nondecreasing, then f is differentiable a.e. and $\int_a^b f'(x)dx \leq f(b) - f(a)$.[38] Since

$$\begin{aligned} & h'(a_d(x))(1 + a_d'(x)) - h'(a(x))(1 + a'(x)) \\ = \;& h'(a_d(x))(a_d'(x) - a'(x)) + (h'(a_d(x)) - h'(a(x)))(1 + a'(x)), \end{aligned}$$

(42) implies that for any $\underline{x}, \overline{x} \in [-M_\varepsilon/\sqrt{\Delta}, M_\varepsilon/\sqrt{\Delta}]$,

$$\begin{aligned} & W_d(\Delta[\overline{x} + a_d(\overline{x})]) - W_d(\Delta[\underline{x} + a_d(\underline{x})]) = \tilde{r}e^{r\Delta}\Delta \int_{\underline{x}}^{\overline{x}} h'(a_d(x))(1 + a_d'(x))dx \\ \leq \;& W(\Delta[\overline{x} + a(\overline{x})]) - W(\Delta[\underline{x} + a(\underline{x})]) + \tilde{r}e^{r\Delta}\Delta \left[h'(A)\varepsilon + \max h'' \left[\frac{2M_\varepsilon}{\sqrt{\Delta}} + a(\overline{x}) - a(\underline{x}) \right] \right] \end{aligned}$$

The rest of the proof will follow as in last step of Lemma 13 to establish that $\Phi^\Delta(a_d, c_d, W_d; F) \geq \Phi^\Delta(a, c, W; F) - O(\varepsilon\Delta)$.

We now construct an a_d satisfying (42) and $x + a_d(x)$ is nondecreasing. First, note that since for any $y > x$ we have $a(x) \geq a(y) - \frac{y-x}{\Delta}$, a may not discontinuously decrease. Therefore, the set of points $D \subset [-M_\varepsilon/\sqrt{\Delta}, M_\varepsilon/\sqrt{\Delta}]$ at which a may be discontinuous is at most countable. Moreover, if $J = \sum_{x \in D}(a(x_+) - a(x_-))$, then

$$J + \int_{-M_\varepsilon/\sqrt{\Delta}}^{M_\varepsilon/\sqrt{\Delta}} (1 + a'(x))dx = \frac{2M_\varepsilon}{\sqrt{\Delta}} + a(\overline{x}) - a(\underline{x}) \leq A + \frac{2M_\varepsilon}{\sqrt{\Delta}}.$$

Since $1 + a'(x) \geq 0$, this implies that $J \leq A + \frac{2M_\varepsilon}{\sqrt{\Delta}}$. Let D_f be a finite set of points where a is discontinuous such that $\sum_{x \in D_f}(a(x_+) - a(x_-)) \geq J - \varepsilon/2$, and let $\delta = \min_{x \in D_f}(a(x_+) - a(x_-))$.

For any $n \in \mathbb{N}$ and $x \in [-M_\varepsilon/\sqrt{\Delta}, M_\varepsilon/\sqrt{\Delta}]$ let

$$a_n'(x) = \frac{n}{2} \int_{x-1/n}^{x+1/n} a'(s)\, ds.$$

The function a_n' is differentiable and for any x, $a_n'(x) \geq -1$ (since $a'(x) \geq -1$). From the Lebesgue's Density Theorem it follows that for sufficiently large n, $\int_{-M_\varepsilon/\sqrt{\Delta}}^{M_\varepsilon/\sqrt{\Delta}} |a_n'(x) - a'(x)|dx \leq \delta$.

[38] See, for example, Theorem 2 in Chapter 5 of Royden [1988].

Finally, for $D_f = \{d_1, ..., \dot{d}_n\}$, $d_0 = -M_\varepsilon/\sqrt{\Delta}$, $d_{n+1} = M_\varepsilon/\sqrt{\Delta}$, and for any $x \in [d_i, d_{i+1})$ let

$$a_d(x) = a(d_i) + \int_{d_i}^{x} a_n'(s)\,ds.$$

The function a_d satisfies (42) and $x + a_d(x)$ is nondecreasing by construction, which establishes the proof. ■

F.2 Proof of Lemma 5

Fix $(\bar{a}, \bar{h})$ in the domain of Θ_Γ, any $\varepsilon > 0$ and let (a_Γ, v_Γ) be an ε-suboptimal policy for $\Theta_\Gamma(\bar{a}, \bar{h})$. We may assume that $\mathbb{E}_\Gamma[v_\Gamma(x)] = 0$. We will define a policy (a_G, v_G) that is feasible for the problem $\Theta_G(\bar{a}, \bar{h})$ and such that $\mathbb{E}_G[v_G^2(x)] \leq \Theta_\Gamma(\bar{a}, \bar{h})$.

For any x let $a_G(x) = a_\Gamma(x')$ where x' is such that $G(x) = \Gamma(x')$. Since both G and Γ are strictly increasing between 0 and 1, a_G is well defined. We have:

$$\int a_G(x)g(x)dx = \int a_G(G^{-1}(\Gamma(x')))\gamma(x')dx' = \int a_\Gamma(x')\gamma(x')dx',$$

where we have used the change of variables $x = G^{-1}(\Gamma(x'))$, so $\frac{dx}{dx'} = \frac{\gamma(x')}{g(x)}$. Similarly we get that $\int h(a_G(x))g(x)dx = \int h(a_\Gamma(x'))\gamma(x')dx'$.

The incentive transfer function v_G is defined via the (FOC$_\Theta$-PHI) condition:

$$v_G'(x) = h'(a_G(x)),$$

except for the finitely many points of discontinuity of $a_G(x)$, where it is extended continuously, together with the condition $\mathbb{E}_G[v_G(x)] = 0$. Choose any points $\overline{x} > \underline{x}$, $\overline{x}' > \underline{x}'$ such that $G(\overline{x}) = \Gamma(\overline{x}')$ and $G(\underline{x}) = \Gamma(\underline{x}')$. We have:

$$\begin{aligned} v_G(\overline{x}) - v_G(\underline{x}) &= \int_{\underline{x}}^{\overline{x}} h(a_G(x))dx \\ &= \int_{\underline{x}'}^{\overline{x}'} h(a_\Gamma(x'))\frac{\gamma(x')}{g(x)}dx' \leq \int_{\underline{x}'}^{\overline{x}'} h(a_\Gamma(x'))dx' = v_\Gamma(\overline{x}') - v_\Gamma(\underline{x}'). \end{aligned}$$

This means that the random variable $v_G(X)$, $X \sim G$, is *less dispersed* than $v_\Gamma(X)$, $X \sim \Gamma$. Since $\mathbb{E}_\Gamma[v_\Gamma(x)] = \mathbb{E}_G[v_G(x)] = 0$, this implies that for the concave function $\phi(x) = x^2$ we have $\mathbb{E}_G[v_G^2(x)] \leq \mathbb{E}_\Gamma[v_\Gamma^2(x)] \leq \Theta_\Gamma(\bar{a}, \bar{h}) + \varepsilon$ (see Theorem 3.B.2 in Shaked and Shanthikumar [2007], which is taken from Landsberger and Meilijson [1994]). Since $\varepsilon > 0$ is arbitrary, this concludes the proof.