A Welfare Criterion for Models with Distorted Beliefs*

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Abstract

This paper proposes a welfare criterion for economies in which agents have heterogeneously distorted beliefs. Instead of taking a stand on agents' beliefs, our criterion asserts an allocation to be *belief-neutral* inefficient if it is inefficient under any convex combination of agents' beliefs. While this criterion gives an incomplete ranking of social allocations, it can identify negative-sum speculation in a broad range of prominent models with distorted beliefs. A common feature of these models is that each agent believes that he can make a profit at a greater expense of others.

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1 Introduction

The recent financial crisis and subsequent great recession have led to a flurry of new financial regulations. Many measures were undertaken to reduce incentive distortions caused by externalities. The most prominent externalities were moral hazard concerns that occur when private investors do not internalize the cost to governments for public bailouts. Equally important are fire-sale externalities. These occur when investors take leverage without internalizing that later they might be forced to fire-sell his assets. The depressed price is not only bad for him, but also causes negative spillover to others with similar positions. While incentive distortions played a prominent role in the build-up of the recent crisis, misconceptions and belief distortions were arguably even more important. Most investors believed a drop in house prices by more than 20% was unrealistic and they totally underestimated house price correlations across regions. Despite the importance of belief distortions in the formation of bubbles and the subsequent crises, belief distortion had virtually no impact on financial regulations. This is because economics does not offer a clear welfare criterion.

This paper tries to fill that gap by providing a welfare criterion for models in which individuals hold heterogeneously distorted beliefs. To illustrate the basic idea, we first consider a bet between Joe Stiglitz and Bob Wilson.[1] One day, Joe and Bob argued over the contents of a pillow. Joe maintained that the pillow had a natural down filling, while Bob thought a synthetic filling was more likely. Bob assessed with probability 0.1 that the filling was natural and Joe assessed the probability being 0.9. They decided to construct a bet about this: If the pillow had natural down, Bob would pay Joe $100, but if it had artificial down, Joe would pay Bob $100. They could only discover the truth by cutting the pillow open, which would destroy it. They agreed to share the cost of buying a new pillow ($50). It is clear that both Joe and Bob preferred the bet relative to no betting at all, as each expected to make a profit of $80 from the bet and a net profit of $55 after splitting the cost of replacing the pillow. While this bet was desirable from each individual's perspective, its outcome was socially inefficient—it led to a money transfer between Bob and Joe and a perfectly good pillow being destroyed.

It is useful to differentiate two distinct sources that may have caused the heterogeneous beliefs of Joe and Bob. On the one hand, a large body of the decision theory literature builds on the personalistic view of probability, advocated by Savage (1954), and holds that

[1] See Kreps (2012, page 193) for more details of the story.

beliefs reflect personal experience and risk attitude and are an integrated part of individuals' preferences under uncertainty.[2] According to this view, the beliefs assigned by Joe and Bob might have reflected their preferences for betting under the particular circumstance rather than their views of the likelihood of the outcome. On the other hand, overconfidence (which is commonly observed among successful individuals in experimental studies) or other psychological biases might have distorted the belief of one (or both) of them. As a result, they held drastically different beliefs, which, in turn, motivated them to take on the value-destroying bet.

We are particularly concerned by the second case in which belief distortions caused Joe and Bob to engage in value-destroying speculation. In general, people suffer from a range of well-documented behavioral biases that can distort their beliefs.[3] The presence of distorted beliefs motivates a benevolent social planner to use the correct belief to evaluate agents' welfare on their behalves. However, this task is challenging because the social planner may not observe the objective probability that determines the economic uncertainty. As common in many realistic economic situations, available data often does not allow the social planner to discriminate different beliefs. In fact, such an environment fosters belief distortions among agents in the first place. To cope with this challenge, we acknowledge the relevance of a set of reasonable beliefs and propose to evaluate social welfare based on *all* of the reasonable beliefs. The key insight of our welfare criterion is that if a social allocation is inefficient under any reasonable belief, then it is *belief-neutral* inefficient.

Specifically, we accept any convex combination of agents' beliefs as a reasonable belief and propose to extend the two standard welfare analysis approaches, the expected social welfare approach and the Pareto efficiency approach, based on all convex combinations of agents' beliefs. The expected social welfare approach directly compares two social allocations x and y for a given welfare function. Our welfare criterion posits that x is belief-neutral inferior to y if the expected total welfare from x is lower than that from y under *any* convex combination of the agents' beliefs. The Pareto efficiency approach assesses the efficiency of an allocation x by determining whether there exists an alternative allocation x' that improves the expected utility of some agents without hurting anyone else. Our criterion asserts that x is belief-neutral Pareto inefficient if it is Pareto inefficient under any convex combination of agents' beliefs. In implementing our welfare criterion, we strictly interpret agents' beliefs as their

[2]See Morris (1995) for extensive arguments that rationalize heterogeneous prior beliefs.

[3]See Hirshleifer (2001) and Barberis and Thaler (2003) for extensive reviews of the literature.

views of likelihood of economic outcomes, and incorporate all other aspects such as agents' risk-seeking preferences and heterogeneous prior beliefs by appropriate choice of their (state dependent) utility functions.

Back to the bet between Joe and Bob, it is difficult for the social planner to tell ex ante who was right. However, it is reasonable to assume that the objective probability that the pillow had natural down lied between their beliefs (i.e., between 0.1 and 0.9). Suppose that Joe and Bob were both risk neutral and that the social planner assign them equal weights in summing up their utilities in the social welfare function. Then, it is immediately clear that the bet is belief-neutral inferior to the status quo (no betting). This is because regardless of which reasonable belief the social planner adopts to evaluate expected utilities, the transfer of \$100 between Joe and Bob had no impact on the expected social welfare, but destroying the pillow led to a sure social loss of \$50. Without relying on any social welfare function, the bet is also belief-neutral Pareto inefficient. Suppose that the social planner adopts Joe's belief. Under this belief, the bet would lead to an expected profit of \$80 to Joe and an expected loss of \$80 to Bob, and the pillow to be destroyed. Obviously, a direct transfer of \$80 from Bob to Joe could improve everyone's expected utility by saving the pillow from being destroyed. Similarly, if the social planner adopts Bob's belief, a direct transfer of \$80 from Joe to Bob would improve everyone's expected utility again by saving the pillow. By this argument, the social planner can always find a suitable transfer without destroying the pillow to strictly improve everyone's expected utility under any convex combination of Bob and Joe's beliefs.

In this example, without taking a stand on which belief is correct, the social planner can categorically determine that the bet would lead to an inefficient social outcome. The key is that the bet is a negative-sum game. This attribute is also present in many other models with distorted beliefs—agents are willing to speculate against each other, as each believes he will win at the expense of the parties, even though the speculation is a negative-sum game. Our criterion is particularly useful in detecting this type of negative-sum speculation despite that it requires consistent identification of a social allocation being inefficient under different reasonable beliefs and is thus necessarily incomplete.

In Section 3, we apply our welfare criterion to a set of examples and show that it provides clear welfare ranking for almost all prominent models with heterogeneously distorted beliefs in the literature. Our first example concerns two risk-averse agents with deterministic endowments bet against each other due to heterogeneous beliefs. By making the agents' consumption more volatile, the bet is a negative-sum game in expected utility terms. While simplistic, this example highlights the welfare loss induced by excessive trading in a large number of asset pricing models with heterogeneous beliefs (e.g., Detemple and Murthy (1994), Kurz (1996), Zapatero (1998), Basak (2000), Buraschi and Jiltsov (2006), Jouini and Napp (2007), David (2008), Dumas, Kurshev and Uppal (2009), Xiong and Yan (2010), and Dumas, Lewis, and Osambela (2011)). Of course, in more general settings, agents may have risky endowment streams and some trading allows them to share endowment risk. In that case, there is a trade-off between the welfare gain from risk sharing and the welfare loss from speculative trading (e.g., Kubler and Schmedders (2011) and Simsek (2011)). Our criterion provides a tool to analyze such a trade-off.

The second example involves speculative bubbles. Even if all market participants are risk neutral, trading costs make trading a negative-sum game just like the bet between Joe and Bob. Still, agents trade because they believe they can off-load the asset at an excessively high price to whomever will be the optimistic trader at the time. This speculative motive underlies a number of bubble models (e.g., Harrison and Kreps (1978), Morris (1996), Scheinkman and Xiong (2003), Hong, Scheinkman and Xiong (2006), and Hong and Sraer (2011)). According to our welfare criterion, such activity is welfare reducing. Indeed, by analyzing a large sample of brokerage accounts held by individual households, Barber and Odean (2000) showed that trading costs led to severe under-performance of those who trade most often.

Our third example highlights overinvestment induced by price bubbles. The option to resell assets to new optimists at a later time leads to excessively high prices, which in turn induces overinvestment by firms (e.g., Bolton, Scheinkman and Xiong (2006), Gilchrist, Himmelberg, and Huberman (2005), and Panageas (2006)). For a firm with a decreasing return to scale, the resale option causes each agent to prefer an investment scale higher than the scale that maximizes his expected long-run value of the firm. Our welfare criterion identifies the welfare loss from such over-investment under any reasonable belief.

The fourth example builds on a recent model of Sims (2008) to analyze non-neutrality of heterogeneous beliefs about monetary policy. Those who believe high inflation is less likely to occur find nominal lending attractive, while those who regard high inflation as more likely find it inexpensive to borrow in nominal terms. Through such borrowing/lending, heterogeneous beliefs make the agents' consumption more volatile and cause them to either over- or under-invest in aggregate relative to what would be invested in the absence of heterogeneous beliefs. Our criterion can identify the welfare loss caused by these non-neutral effects and facilitate the analysis of the optimal monetary policy.

Finally, we analyze leverage cycles caused by heterogeneous beliefs (e.g., Geanakoplos (2003, 2009), Fostel and Geanakoplos (2008), Simsek (2010), and He and Xiong (2012)). In these models the forced liquidation of a position due to binding collateral constraints, not the option to voluntarily resell the asset, triggers a trade. Forced selling is associated with bankruptcy costs that make the initial transaction a negative-sum game. Initially, optimistic agents borrow too much because they underestimate the likelihood that they will be forced to liquidate their positions. To draw regulatory implications from such an analysis, a welfare analysis is needed. Our criterion provides that tool.

Our welfare criterion adds to the quickly growing literature on welfare behavioral economics. The advances in behavioral economics have convinced many economists that psychological biases may distort people's preferences and beliefs and cause them to make inefficient economic decisions. A challenge confronting welfare analysis in the presence of distortions in preferences/beliefs is how to uncover preferences/beliefs from observable choices. Different approaches have been proposed. For example, Koszegi and Rabin (2007) propose to first uncover people's systematic mistakes in a simple setting where the nature of some state-contingent preferences is obvious, and then use the uncovered mistakes to interpret preferences in more complicated situations. Bernheim and Rangel (2009) accommodate people's choice inconsistencies across different choice situations by advocating an unambiguous choice relation that is independent of choice situations. Like these papers, our welfare analysis also builds on the premise that (some) agents hold distorted beliefs. In contrast to these papers, our welfare criterion does not require taking a stand on any particular belief measure, and is neutral to a spectrum of reasonable beliefs. This is because our criterion is designed to capture negative-sum speculation induced by heterogeneously distorted beliefs between agents, rather than inefficient decisions made by an individual agent.

Our criterion extends the "externality view" to settings with distorted beliefs—all agents might agree that they face a negative sum game but they might still proceed, because each agent thinks that he can win at the expense of others. Interestingly, the Coase Theorem fails under these settings, as bargaining and market trading cannot lead to an efficient outcome even if there are no transaction costs and property rights are well defined. This is because in the presence of distorted beliefs, bargaining and market trading may exacerbate, rather than mitigate, value-destroying speculation. Our criterion is thus particularly relevant in the ongoing debate regarding the roles of financial innovations in facilitating hedging and speculation (e.g., Simsek (2011)).

The paper is organized as follows: Section 2 describes the welfare criterion in a generic setting. Section 3 provides a series of examples to demonstrate the capability of the criterion to generate clear welfare ranking in popular models with distorted beliefs. We conclude in Section 4. The technical proofs are provided in the Appendix.

2 The Welfare Criterion

We introduce the welfare criterion in a generic setting with T periods and $T+1$ dates: $t = 0, 1, ..., T$. The evolution of the state of the economy is represented by a binomial-tree process: $\{s_t\}_{t=0}^{T}$. In each period, the state variable can either increase or decrease by a discrete level. The tree is recombining and can take $t+1$ possible values on date t.

There are N agents, indexed by $j \in \{1, 2, ..., N\}$. On each date, each agent holds a belief about the probability of the tree increasing in the following period, which we denote by $\pi_{t,i}^{j}$, where t is the date and i is a state on the date. As this belief can vary across dates and states, we summarize agent j's beliefs by $\Pi^j = \left\{\pi_{t,i}^{j}\right\}$. We restrict the agent's belief in each period and each state to be strictly positive: $\pi_{t,i}^{j} > 0$. One can determine the agent's probability assessments of all future states from Π^j.

Suppose that the agents consume only on the final date T. A social choice x represents a set of consumption allocations to the agents across the $T+1$ possible states s_T:

$$x = \left\{x_T^j\left(s_T\right)\right\}.$$

An allocation is called feasible if it satisfies the aggregate budget constraint in each final state.

Suppose that agent j has state-dependent utility function $u_j\left[s_T, x_T^j\left(s_T\right)\right]$, which in strictly increasing and concave with respect to consumption. This utility specification is sufficiently general to capture the standard utility functions used in most economic models and, as we will discuss later, to accommodate agents' heterogeneous prior beliefs. Also note that by specifying a utility function dependent on only consumptions in the final states, our framework ignores preferences that depend on ex ante uncertainty such as uncertainty aversion. Based on the utility specification and the agent's beliefs, his expected utility at time 0 is $E_0^j\left\{u_j\left[s_T, x_T^j\left(s_T\right)\right]\right\}$, where the superscript j denotes the expectation under agent j's beliefs.

2.1 Heterogeneous Beliefs

We allow agents to hold different beliefs, (i.e., $\Pi^j \neq \Pi^{j'}$ if $j \neq j'$). Before we dive into welfare analysis, it is useful to sort out different sources of heterogeneous beliefs, which matter in welfare analysis. Throughout our later analysis, we treat agents' state-dependent beliefs as given. It is straightforward to think of the beliefs as outcomes of the agents' learning processes. Suppose that an unobservable variable π determines the probability of the tree moving up each period. Each agent has a prior belief about the distribution of π, observes some information about π in each period, and uses Bayes' rule to update his belief about π. Through this learning process, three sources may lead to heterogeneous beliefs among agents: 1) different information, 2) different prior beliefs, and 3) different updating rules.

The widely used common prior assumption, which is also referred to as the Harsanyi doctrine, posits that agents hold the same prior beliefs. This assumption implies that two rational agents having access to the same information will necessarily come up with the same belief about π as they use the same Bayes' rule. Back in the 1970s, economists tended to link heterogeneous beliefs to agents' asymmetric information. Aumann (1976) clarified an important conceptual point by showing that if agents are rational with a common prior and if there is common knowledge of their posterior beliefs, then their posterior beliefs must be identical. This is because each agent would have used others' beliefs to infer their information and, as a result, their beliefs would have converged. Milgrom and Stokey (1982) and Sebenius and Geanakoplos (1983) extended Aumann's insight to establish the so called no-trade theorem. That is, in the absence of ex ante gains from trade, asymmetric information cannot generate trade among rational agents with a common prior. Taken together, asymmetric information alone cannot lead to common-knowledge heterogeneous beliefs and speculation.[4] The no-trade theorem also motivates us to ignore asymmetric information in our analysis because any trade would reveal asymmetric information in the absence of ex ante gains from trade.

[4]While asymmetric information alone cannot lead to trade, a large literature analyzes liquidity and trading in settings with both asymmetric information and random supply shocks following the classic analysis of Grossman and Stiglitz (1980) and Kyle (1985). The presence of random supply shocks creates ex ante gains from trade for rational speculators.

The decision theory literature that builds on Savage's (1954) notion of subjective probability treats probabilities separately for individual agents. Economics does not offer much guidance on how individuals form their prior beliefs. Instead, economists tend to agree that prior beliefs probably depend on individuals' background and experience. Morris (1995) summarized a series of arguments to advocate the view that rational agents may hold heterogeneous prior beliefs, just like heterogeneous risk preferences. Heterogeneous prior beliefs can lead agents to speculate against each other.

We emphasize another source of heterogeneous beliefs—different updating rules. A large branch of behavioral finance literature highlights that people suffer from a range of well-established psychological biases, such as overconfidence, limited attention, representativeness bias, and conservatism, in making financial decisions. See Hirshleifer (2001) and Barberis and Thaler (2003) for extensive reviews of the literature. These biases cause agents to use different updating rules in processing information. In particular, overconfidence causes agents to exaggerate the precision of noisy signals and thus to over-react to the signals. When agents over-react to different signals, they may end up with substantially different beliefs and, as a result, may speculate against each other.

Extensive evidence shows that excessive trading severely undercuts portfolio performance of individual investors in the US, Finland, and Taiwan, e.g., Odean (1999), Barber and Odean (2000), Grinblatt and Keloharju (2000), and Barber, Lee, Liu, and Odean (2009). For example, Barber and Odean (2000) analyzed trading records of a large sample of retail investors in the US and found that trading caused the returns of the most active 20 percent of investors to under-perform the market returns by over 5 percent per year. While both heterogeneous prior beliefs and beliefs distorted by psychological biases can induce investors to trade, it is difficult to fully explain such consistent poor trading performance in samples across different countries and different periods without relating it to certain belief distortions.

The presence of belief distortions prompts welfare concerns. Some agents may be unaware of his belief distortions and, as a result, take actions that hurt the welfare of themselves and others. Thus, it is important that a social planner evaluates their welfare on their behalves by using an appropriate measure of the probability that drives the economic uncertainty. Before we analyze welfare in the presence of distorted beliefs, we briefly discuss welfare analysis with heterogeneous priors.

2.2 Welfare Analysis with Heterogeneous Priors

Wilson (1968) adopts the Savage view of subjective probabilities to analyze social welfare in a setting with heterogeneous priors. We now describe his framework, which serves as an anchor for our analysis of social welfare in the presence of belief distortions. In this subsection, we adopt the model setting introduced earlier but emphasize that the heterogeneous beliefs among agents are purely due to the differences in their prior beliefs. As prior beliefs are part of each agent's preferences, the social planner uses each agent's own beliefs to determine his expected future utility.

Wilson (1968) adopts the notion of Pareto optimality in choosing an optimal social allocation among the agents. That is, an allocation is Pareto optimal if there is no alternative sharing rule which would increase the expected utility of some agents without decreasing the expected utility of any other agent. Pareto optimality is a necessary, but not a sufficient, condition for determination of the sharing rule. It is well known, nevertheless, that, given Pareto optimality, a necessary and sufficient condition for determination of the social allocation is an assignment of weights to the agents. The reasoning works as follows. The locus of the agents' expected utilities obtainable from all feasible social allocations is a convex set, and therefore every Pareto optimal allocation corresponds a tangent hyperplane to this set. Thus, corresponding to every Pareto optimal social allocation there exists a set of nonnegative weights $\{\lambda_j | j = 1, ..., N\}$ such that the allocation, $x = \left\{x_T^j(s_T)\right\}$, optimizes the following social welfare function:

$$\max_x \quad \sum_j \lambda_j E_0^j \left\{ u_j \left[s_T, x_T^j(s_T) \right] \right\}. \tag{1}$$

Varying the agents' weights $\{\lambda_j\}$ leads to the Pareto frontier.

In optimizing (1), the social planner aggregates all agents' expected utilities based on their own beliefs and the given weights. Suppose the social planner holds a probability measure of his own, which we denote by Π^{SP}. We can rewrite the social welfare function in (1) under Π^{SP} by transforming each agent's subjective beliefs into his utility function. Specifically, denote the probability of each final state in agent j's belief measure by $f^j(s_T)$

and in the social planner's measure by $f^{SP}(s_T)$. Then, (1) is equivalent to

$$\begin{aligned}\sum_j \rho_j E_0^j \left\{u_j\left[s_T, x_T^j(s_T)\right]\right\} &= \sum_j \lambda_j \sum_{s_T} f^j(s_T)\, u_j\left[s_T, x_T^j(s_T)\right] \\ &= \sum_j \lambda_j \sum_{s_T} f^{SP}(s_T) \frac{f^j(s_T)}{f^{SP}(s_T)} u_j\left[s_T, x_T^j(s_T)\right] \\ &= \sum_j \lambda_j E_0^{SP}\left[\frac{f^j(s_T)}{f^{SP}(s_T)} u_j\left[s_T, x_T^j(s_T)\right]\right],\end{aligned}$$

where $\frac{f^j(s_T)}{f^{SP}(s_T)}$ is the Radon-Nikodym derivative of agent j's probability measure with respect to the social planner's measure. The product of $\frac{f^j(s_T)}{f^{SP}(s_T)}$ with the agent's utility function u_j acts as his effective utility function in the social welfare function under the social planner's probability measure.

2.3 Welfare Analysis with Distorted Beliefs

In the presence of distorted beliefs, it is important that the social planner to use an appropriate probability measure to evaluate agents' expected utilities in welfare analysis. The challenge here is that the social planner may not observe the probability that drives economic uncertainty in the economy. Given the agents' different belief measures, whose measure is appropriate for welfare analysis? Is there an even more appropriate one outside those used by the agents? We now introduce a belief-neutral welfare criterion.

To focus on welfare consequences of belief distortions, we use an appropriate choice of the agents' state-dependent utility functions to incorporate their subjective prior beliefs. As we discussed in the previous subsection, we can incorporate an agent's subjective prior beliefs into his utility function.[5] Suppose that the remaining differences in agents' beliefs are purely driven by distortions induced by their psychological biases. Without taking a stand on which agent's beliefs are superior, it is reasonable to argue that the objective probability measure lies between the agents' beliefs. In other words, the objective probability measure coincides with either the beliefs of one of the agents or a convex combination of their beliefs. Thus, we consider the set of reasonable probability measures composed of all convex combinations of

[5] There are two caveats. First, state dependent utility functions may not capture preferences that concern agents' ex ante uncertainty, such as uncertainty aversion. As uncertainty aversion tends to make agents reluctant to trade against each other, it is not a particularly concern given our focus on analyzing negative-sum speculation. Second, subjective prior beliefs may interact with behavioral biases in agents' learning processes and become inseparable from belief distortions.

the agents' beliefs. Denote Π^{ρ} to be a convex combination of the agents' beliefs with weight $h = \left\{h^1, ..., h^N\right\}$:

$$\Pi^h = \sum_j h^j \Pi^j, \quad \text{where} \quad h^j \geq 0 \quad \text{and} \quad \sum h^j = 1.$$

The space spanned by $\left\{\Pi^h\right\}$ is sufficiently large and contains all reasonable belief measures based on the given environment. We are not concerned by the possibility that the objective probability measure might lie outside this space. This is because in this case the objective measure is irrelevant to the agents' choices.

The key insight of our welfare criterion is to analyze the efficiency of a social allocation across all of these reasonable probability measures. Specifically, we propose the following belief-neutral criterion:

A belief-neutral welfare criterion If a social allocation x is inefficient under any reasonable probability measure Π^h, then it is belief-neutral inefficient.

We can use two different approaches to implement this welfare criterion, one based on a given social welfare function and the other through the notion of Pareto efficiency. As we discussed earlier, in the absence of belief distortions these two approaches are internally consistent as any Pareto efficient social allocation corresponds an optimal allocation that maximizes the aggregate expected utilities of agents under a set of nonnegative weights.

2.3.1 Expected Social Welfare

We assume that the social planner uses a linear social welfare function with a set of nonnegative weights $\{\lambda_i\}$:

$$W\left[u_1, u_2, ..., u_N\right] = \sum_{i=1}^{N} \lambda_i u_i,$$

which is often called Bergsonian social welfare function. If the weights are all equal, it becomes the so-called utilitarian social welfare function:

$$W\left[u_1, u_2, ..., u_N\right] = \sum_{i=1}^{N} u_i.$$

In our later examples, we use the utilitarian social welfare function.

Given that these social welfare functions are linear and that the social planner uses the same probability measure to evaluate the expected utilities of all agents, the expected social welfare is independent of the order of aggregating welfare and computing expectation. In our analysis, we find it more convenient to first aggregate agents' welfare in each of the final states and then compare the expected social welfare under different probability measures.

Based on a set of nonnegative welfare weights, we can implement our criterion as follows.

The welfare-function-based criterion Consider two social allocations, x and y under the Bergsonian social welfare function with a given set of nonnegative weights. If the expected social welfare of allocation x dominates that of allocation y in any reasonable probability measure Π^h,

$$E_0^h \left[W \left[u_1 \left(s_T, x_T^1 \left(s_T \right) \right), ..., u_N \left(s_T, x_T^N (s_T) \right) \right] \right] \geq E_0^h \left[W \left[u_1 \left(s_T, y_T^1 \left(s_T \right) \right), ..., u_N \left(s_T, y_T^N (s_T) \right) \right] \right],$$

with the inequality holding strictly under at least one reasonable measure, then allocation x is belief-neutral superior to allocation y.

To establish the superiority of one social allocation relative to another, a higher expected social welfare in any convex combination of the agents' beliefs is required. This proposed belief-neutral superiority is a partial ordering of social allocations. In the case of two social allocations x and y, x might dominate y in one measure and y might dominate x in another measure. In such cases, we would say that x and y are incomparable.

Despite its incompleteness, this criterion is nevertheless useful in detecting negative-sum speculation driven by distorted beliefs. We now apply this criterion to analyze the bet between Joe and Bob described in the introduction. Suppose that both Joe and Bob were risk neutral: $u_{Joe}(w) = w$ and $u_{Bob}(w) = w$, and that the social planner uses the utilitarian social welfare function:

$$W \left[u_{Joe}, u_{Bob} \right] = u_{Joe} + u_{Bob} = w_{Joe} + w_{Bob}.$$

Here, we assume that the difference between Joe and Bob's beliefs is driven by belief distortions of one or both of them rather than difference in their risk preferences and subjective priors. Consider the social allocation x resulted from the bet and the status quo y without any betting. It is obvious that without any betting, regardless of the probability measure the social planner adopts, the expected social welfare is simply the sum of Joe and Bob's initial wealth. The bet would cause a transfer of \$100 between them and the pillow being destroyed. The money transfer has no impact on the social welfare regardless of its direction and the probability measure the social planner adopts to evaluate the welfare. However, replacing the pillow would incur a sure cost of $50. Taken together, regardless of any reasonable probability measure the social planner uses, the expected social welfare from the bet is always lower than that from the status quo by 50, due to the cost of replacing the pillow. Thus, the status quo is belief-neutral superior to the bet.

2.3.2 Pareto Efficiency

Instead of relying on a particular social welfare function, we can also implement our welfare criterion by extending the notion of Pareto efficiency. The essence of Pareto efficiency is to determine whether there exists an alternative allocation that improves the welfare (i.e., expected utility) of some agents without hurting any other agent. If such an alternative exists, the allocation under evaluation is Pareto inefficient. In the presence of distorted beliefs, instead of letting the agents use their own beliefs to evaluate their expected utilities, the social planner uses the same probability measure (from the set of reasonable measures) to evaluate each agent's expected utility. As the social planner cannot identify which measure among all the reasonable measures is more appropriate, he shall use all of them. If a social allocation turns out to be Pareto inefficient among all of the reasonable measures, it is unambiguously inefficient. This logic leads to the following implementation of our criterion:

The belief-neutral Pareto inefficiency Consider a social allocation x. Suppose that under any reasonable probability measure Π^h, there always exists another allocation x' such that it improves some agents' expected utilities without reducing anyone's (i.e., $\forall i$, $E_0^h\left[u_i\left(s_T, x_T^i\left(s_T\right)\right)\right] \leq E_0^h\left[u_i\left(s_T, x_T'^i\left(s_T\right)\right)\right]$ with the inequality holding strictly for least one agent.) Then, this allocation is belief-neutral Pareto inefficient.

Back to the bet between Joe and Bob. We can show that the social allocation created by the bet is belief-neutral Pareto inefficient. We first evaluate this allocation under Joe's belief measure. It is immediately clear that instead of taking the bet, an alternative allocation by transferring $55 from Bob to Joe makes Joe indifferent but improves the expected utility of Bob. Similarly, under Bob's measure, an alternative allocation by transferring $55 from Joe to Bob makes Bob indifferent but improves the expected utility of Joe. More generally, under any convex combination of Joe and Bob's beliefs, there always exists an appropriate transfer that makes one of them indifferent but improves the expected utility of the other. The gain from such a transfer is exactly due to saving the pillow being destroyed, or, in other words, by avoiding the negative sum induced by the bet between Joe and Bob.

3 Examples

This section provides a series of examples that demonstrate that, despite its incompleteness, the simple welfare criterion we propose can produce surprisingly sharp welfare ranking in a wide range of prominent economic models with heterogeneous beliefs. The key is that heterogeneous beliefs can directly lead to negative-sum games between agents. As in the example of Joe and Bob, each agent believes that he can make a profit even though he also expects the losses incurred by others to be greater than his gain. In the case of Joe and Bob, the losses take the form of a destroyed pillow. More generally, the losses could include excessive risk taking, trading costs, inefficiently high investment, and bankruptcy costs in leveraged investments. The examples in this section use variants of well known models to illustrate the different sources of losses. They also demonstrate that our welfare criterion is able to detect the inefficiency in each case.

3.1 Speculative Trading Models and Excessive Risks

A large class of economic models analyzes trading between agents who hold heterogeneous beliefs regarding economic fundamentals and the impact of their trading on equilibrium asset price dynamics (e.g., Detemple and Murthy (1994), Kurz (1996), Zapatero (1998), Basak (2000), Buraschi and Jiltsov (2006), Jouini and Napp (2007), David (2008), Dumas, Kurshev and Uppal (2009), Xiong and Yan (2010), and Dumas, Lewis, and Osambela (2011)). A key insight of these models is that trading induced by heterogeneous beliefs can lead to endogenous fluctuations in agents' wealth distribution, which, in turn, amplifies asset price volatility and induces time-varying risk premia. More specifically, a positive shock increases the wealth of optimists more than that of pessimists, as optimists tend to take larger asset positions. The optimists' greater wealth increases allow them to take even larger positions and thus amplify the impact of the shock on equilibrium asset prices.

Despite the capability of these models to capture important dynamics of asset prices and risk premia, researchers tend to avoid making any welfare statement due to the lack of a well-specified welfare criterion. Our simple criterion can potentially fill this gap by offering useful insight for this type of models. The key point is that trading induced by heterogeneous beliefs makes agents' consumption excessively risky *according to any agent.* Each agent takes these risks because she expects to make high expected returns. However, each agent also recognizes that these returns will come in the form of a transfer from other agents with different beliefs. Thus, when agents are risk averse, trading reduces the social welfare according to any agent's belief, or more generally, any convex combination of their beliefs.

To illustrate this point, we consider a simple setting with two agents in an economy, A and B, and a single period. Each agent is endowed with half dollars and lives from $t = 0$ to $t = 1$. There is neither aggregate nor idiosyncratic endowment risk. Suppose that each agent consumes at $t = 1$ and has an increasing and strictly concave utility function $u(c_i)$. The two agents hold heterogeneous beliefs about a random variable, say $\widetilde{f}$, which can take two possible values, either H or L. One may interpret this random variable as weather, which is independent of the agents' endowment risk. Suppose agent A assigns a probability of π^A to state $\tilde{f} = H$, while agent B assigns π^B. The difference in beliefs causes the agents to engage in speculative trades against each other. We allow them to trade a contract that pays 1 if $\tilde{f} = H$ and 0 if $\tilde{f} = L$.

Suppose that the contract is traded at a price of p at $t = 0$. Agent i ($i \in \{A, B\}$) chooses the optimal amount k^i in the contract to maximize his expected utility:

$$\max_{k^i} \ \pi^i u\left(0.5 + k^i\left(1-p\right)\right) + \left(1-\pi^i\right) u\left(0.5 - k^i p\right).$$

The first order condition gives:

$$\left(1-p\right)\pi^i u'\left(0.5 + k^i\left(1-p\right)\right) = p\left(1-\pi^i\right) u'\left(0.5 - k^i p\right).$$

The market clearing condition requires that:

$$k^A + k^B = 0.$$

The standard results hold that there exist a market equilibrium allocation, $\left\{k^A, k^B, p\right\}$, which solve each agent's optimality condition and the market clearing condition.

The market equilibrium in this example is inefficient according to our criterion. To see this, first consider the welfare-function version of the criterion. Suppose the planner has a utilitarian social welfare function:

$$W\left[u_A\left(x\right), u_B\left(x\right)\right] = u\left(x_i^A\right) + u\left(x_i^B\right).$$

We compare the social welfare of agents from their equilibrium consumption:

$$x = \left\{\left(x_H^i, x_L^i\right)\right\}_{i\in\{A,B\}} = \left\{\left(0.5 + k^i\left(1-p\right), 0.5 - k^i p\right)\right\}_{i\in\{A,B\}},$$

with their welfare from the status quo allocation with no trading:

$$y = \left\{ \left(y_H^i, y_L^i \right) \equiv (0.5, 0.5) \right\}_{i \in \{A,B\}}.$$

As it is difficult to judge whose beliefs are superior, we use any convex combination of the two agents' beliefs to evaluate the social welfare: $\Pi^h = h\Pi^A + (1-h)\Pi^B$, $\forall h \in [0,1]$. According to this set of beliefs, the probability of state H is $h\pi^A + (1-h)\pi^B$. By varying h between 0 and 1, we implicitly assume that the correct probability measure lies between the two agents' beliefs. The following proposition shows a welfare ranking based on this evaluation.

Proposition 1 *If $\pi^A \neq \pi^B$ and the social planner has the utilitarian welfare function, then the social welfare of the status quo dominates that of the market equilibrium according to any convex combination of the two agents' beliefs.*

The mechanism that underlies Proposition 1 is simply that the trade induced by the agents' heterogeneous beliefs makes their consumption riskier than their endowments. Each agent is taking these additional risks because she thinks she will get a high expected return from the trade. Since this is an endowment economy, each agent also recognizes that the higher returns will come in the form of a transfer from the other agent. Put differently, each agent agrees that trade increases everyone's consumption risks without increasing expected returns in the aggregate. In view of risk aversion, the utilitarian social welfare falls according to each agent (as well as any convex combination of their beliefs).

One may object to Proposition 1 on the grounds that the result depends on the choice of the social welfare function. For example, if the planner uses a Bergsonian welfare function that puts all the weight on agent A, representing a desire for redistribution towards this agent, then Proposition 1 no longer applies. In particular, with this A-*biased* welfare function, one can no longer rank the market equilibrium and the status quo allocation regardless of the belief used for this calculation. The status quo dominates the market equilibrium according to B's beliefs, but it is dominated according to A's beliefs.

Nonetheless, there is a sense in which the market equilibrium is inefficient even for the A-biased welfare function. The market equilibrium looks attractive in this setting only because, under A's beliefs, it redistributes wealth to agent A. However, this is a very inefficient form of redistribution since it also increases the riskiness of both agents' consumption. In particular, one could accomplish the same redistribution more efficiently by preventing trade and transferring some of the initial endowment from B to A. Put differently, while the market equilibrium is not dominated by the status quo allocation, it is dominated by the status quo allocation combined with a transfer of initial endowments.

The second version of our criterion, belief-neutral Pareto inefficiency, allows for a transfer of endowments in welfare comparisons -the same way Pareto inefficiency does. Consequently, it detects the inefficiency of the market equilibrium in this example even for the A-biased welfare function. In fact, applying this criterion implies that the market equilibrium is inefficient for *any welfare function.* To state this result, define the status quo with transfer $T \in [-0.5, 0.5]$ from agent B to agent A as the allocation:

$$y(T) = \{(0.5 + T, 0.5 + T), (0.5 - T, 0.5 - T)\}$$

Proposition 2 *If $\pi^A \neq \pi^B$, then the market equilibrium is belief-neutral Pareto inefficient. In particular, under any convex combination of agents' beliefs, Π^h, the market equilibrium is Pareto dominated by the status quo allocation with some transfer $T \in [-0.5, 0.5]$.*

A corollary to this proposition is that the market equilibrium is inefficient for any welfare function that is increasing in agents' utilities. To see the intuition for the proposition, fix a belief, Π^h, and consider agents' certainty equivalent wealths under this belief. That is, consider the certain amount that an agent (with belief Π^h) would be indifferent between accepting in exchange for her equilibrium allocation. Since trade increases consumption risks, it also reduces the sum of these certainty equivalent wealths according to any belief, Π^h. It follows that the market equilibrium is dominated by the status-quo allocation combined with an appropriate transfer.

One should also be cautious not to over-interpret this example to mean that trading always reduces social welfare. Richer economic settings will often feature a trade-off between welfare enhancing trading and speculation. To illustrate this trade-off, consider a variant of the earlier example in which there is also a risk sharing motive for trade. In particular, suppose the random variable, $\tilde{f}$, also affects agents' endowment risks. One may interpret this random variable as corresponding to a relatively price shock (e.g., the price of corn) that leads to a reallocation of wealth. In particular, if $\tilde{f} = H$, then agent A (e.g., the miller) incurs a loss of e, while agent B (e.g., the farmer) incurs a gain of e. If $\tilde{f} = L$ then the agents' endowments are the same as before. Thus, the status quo allocation is now given by:

$$y = \left\{ \left(y_H^A, y_L^A\right), \left(y_H^B, y_L^B\right) \right\} = \{(0.5 - e, 0.5), (0.5 + e, 0.5)\}.$$

The equilibrium allocation is characterized by the following first order condition for agent A,

$$(1-p)\,\pi^A u'\left(0.5 - e + k^A (1-p)\right) = p\left(1-\pi^A\right) u'\left(0.5 - k^A p\right),$$

a similar condition for agent B, and the market clearing condition $k^A + k^B = 0$.

When traders have common beliefs, $\pi^A = \pi^B \equiv \pi$, the equilibrium is given by

$$p = \pi \text{ and } k^{opt} \equiv \left(k^A = e, k^B = -e\right).$$

In particular, agents fully diversify their idiosyncratic risks. Agent A (resp. agent B) consumes a constant amount $0.5 - e\pi$ (resp. $0.5 + e\pi$) regardless of the state. On the other hand, when traders have different beliefs, $\pi^A \neq \pi^B$, their consumption is risky. This is because agents deviate from the optimal risky asset allocation, k^{opt}, in view of their pursuit from speculative gains. Our next result illustrates the inefficiency of this equilibrium by comparing it with the status quo allocation combined with the risky asset allocation, k^{opt} and a transfer $T \in [-0.5, 0.5]$:

$$y\left(T, k^{opt}\right) = \left\{(0.5 + T, 0.5 + T), (0.5 - T, 0.5 - T)\right\}.$$

Proposition 3 *If $\pi^A \neq \pi^B$, then the market equilibrium is belief-neutral Pareto inefficient. In particular, under any convex combination of agents' beliefs, Π^h, the market equilibrium is Pareto dominated by the status quo allocation combined with the optimal risky asset allocation, k^{opt}, and some transfer $T \in [-0.5, 0.5]$.*

The intuition for this result is the same as before. Note, however, that in this version it is not optimal to prevent trading completely. Rather, the planner would like to prevent *speculative trading*, that is, trading in excess of an optimal risk sharing benchmark. While this is a stylized example, this point (and thus, Proposition 3) applies more generally. In particular, consider any economy with complete financial markets in which agents have the same preferences, assumed to be separable over time and states, but potentially distorted beliefs. Suppose there are at least two agents, $\{A, B\}$, and at least two states, $\{H, L\}$. Regardless of the planner's belief, a Pareto efficient allocation features the equalization of agents' marginal utilities across states: $\frac{u'\left(x_H^A\right)}{u'\left(x_L^A\right)} = \frac{u'\left(x_H^B\right)}{u'\left(x_L^B\right)}$. Put differently, with complete markets, full risk sharing is a *belief-neutral feature* of any efficient equilibrium. In contrast, the equilibrium allocation features the equalization of agents' marginal utilities times their distorted beliefs, e.g., $\frac{\pi_H^A}{\pi_L^A}\frac{u'\left(x_H^A\right)}{u'\left(x_L^A\right)} = \frac{\pi_H^B}{\pi_L^B}\frac{u'\left(x_H^B\right)}{u'\left(x_L^B\right)}$. In particular, when agents disagree about the relative probabilities of states H and L, the equilibrium allocation is always belief-neutral Pareto inefficient. Speculative trading leaves its footprint in terms of deviations from optimal risk sharing, which is inefficient regardless of the planner's belief.

One may object to Proposition 3 on the grounds that it might not be feasible for the planner to implement the optimal risky asset allocation, k^{opt}, perhaps because the planner is not sufficiently informed about the nature of agents' endowment risks. Rather, the planner's options might be either to allow unrestricted trade in the risky asset or to prevent trade completely. Our next result shows that the planner would prefer to prevent trade as long as agents' endowment risks, e, are sufficiently small.

Proposition 4 *If $\pi^A \neq \pi^B$, then there exists $\bar{e} > 0$ (which depends on π^A, π^B) such that when the endowment risks are below this threshold, $e < \bar{e}$, and under any convex combination of agents' beliefs, π^P, the market equilibrium is Pareto dominated by the status quo allocation with some transfer $T \in [-0.5, 0.5]$.*

Intuitively, while the risk sharing motive for trade tends to decrease consumption risks, the speculation motive for trade tends to increase them. When endowment risks, e, are sufficiently small (or when belief disagreements are sufficiently large), the speculation motive dominates. Thus, in these cases trade increases consumption risks and reduces welfare. Importantly, our welfare criterion also captures the trade-off between risk sharing and speculation, and detects the inefficiency when the latter force dominates. Simsek (2011) and Kubler and Schmedders (2011) analyze richer settings that feature a similar trade-off between risk-sharing and speculation. Our welfare criterion can be useful to detect the inefficiency from speculative trading in these models.

3.2 Bubble Models and Trading Costs

Another segment of the literature emphasizes that when short sales are constrained, heterogeneous beliefs can lead to price bubbles as asset owners anticipate reselling their assets to other more optimistic agents in the future (e.g., Harrison and Kreps (1978), Morris (1996), Scheinkman and Xiong (2003), Hong, Scheinkman and Xiong (2006), and Hong and Sraer (2011)). In these models, heterogeneous beliefs lead agents to not only trade against each

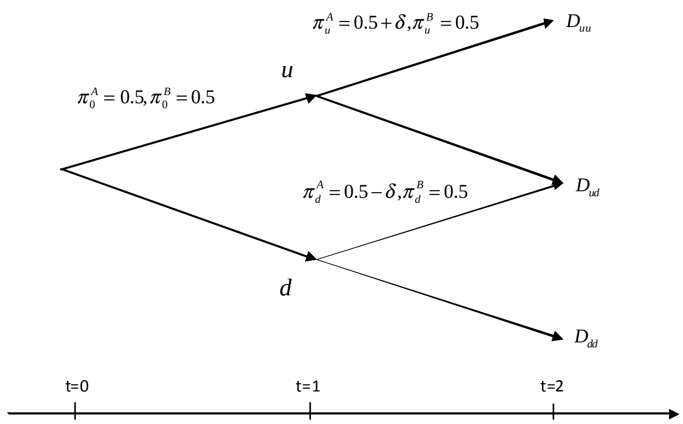

Figure 1: Belief dynamics in the bubble model and the overinvestment model.

other but also to over-value assets. Over-valuation does not reduce social welfare by itself as it is simply a welfare transfer across agents. However, in the presence of practical trading frictions such as brokerage fees and bid-ask spreads, excessive trading can reduce the social welfare of all investors even if they are risk neutral. Our criterion illustrates this point.

We focus on a simple binomial setting with three dates (i.e., $t = 0, 1, 2$), two risk-neutral agents (A and B), and a risky asset. The asset's final payoff across the four possible states (uu, ud, du, and dd) on $t = 2$ are D_{uu}, $D_{ud} = D_{du}$, and D_{dd}, respectively. Figure 1 depicts the dynamics of the fundamental state and the two agents' beliefs. We assume that the two agents have time-varying beliefs: they start with the same beliefs but hold different beliefs on date 1:

$$\pi_0^A = \pi_0^B = 1/2,\ \pi_u^A = 1/2 + \delta > \pi_u^B = 1/2,\ \text{and}\ \pi_d^A = 1/2 - \delta < \pi_d^B = 1/2. \tag{2}$$

In particular, agent A becomes more optimistic than agent B in state u of date 1 and less optimistic in state d. The parameter $\delta > 0$ determines the two agents' belief dispersion in both states u and d.

To facilitate our analysis, we also impose the following payoff structure:

$$\widetilde{R} = \{D_{uu}, D_{ud}, D_{dd}\} = \{R+1, R, R-1\},$$

where $R > 1$ is a constant. Based on this payoff structure, it is straightforward to verify that at $t = 0$ the two agents share the same expectation of the asset's final payoff:

$$E_0^A\left[\widetilde{R}\right] = E_0^B\left[\widetilde{R}\right] = R.$$

Suppose that at $t = 0$ the total supply of the asset is equally divided among the two groups. The reverse of the two agents' beliefs at $t = 1$ gives each agent an option to resell his holding to the other agent; more specifically, for agent A to sell to agent B in state d and for agent B to sell to agent A in state u. For simplicity, we assume that the trading price is determined by the buyer's reservation value. It should be clear that as long as the price is between the buyer and seller's reservation values, the asset owner's resale option is valuable. Then, it is straightforward to derive the following market price in state u:

$$p_u = R + 1/2 + \delta,$$

which is paid by agent A, and in state d:

$$p_d = R - 1/2,$$

which is paid by agent B. By backward induction, both agents on date 0 value the asset by

$$p_0 = R + \delta/2.$$

Despite that each agent's expectation of the asset payoff is R, their valuation of the asset is $R + \delta/2$. The difference is driven by the resale option, i.e., the speculative motive to resell the asset to the other agent at a price higher than his own valuation on date 1. This resale option contributes a non-fundamental component to asset prices in the aforementioned bubble models.

To make the welfare analysis meaningful, we also assume that the seller incurs a cost of k in selling each unit of asset. We allow the cost to be modest so that it does not prevent agents from trading:

$$k < \delta.$$

Because this cost goes to the seller, it does not affect the market prices derived earlier. However, in the presence of the trading cost, trading reduces the social welfare in each agent's beliefs, and, more generally, in any convex combination of the agents' beliefs.

Our welfare criterion illustrates that the equilibrium is inefficient also in this example. First, consider the welfare-function-based version of the criterion and suppose the planner has the utilitarian social welfare function. We will compare the social welfare function from the status quo with no trade with that of the market equilibrium, using any convex combination of the two agents' beliefs, $\Pi^h = h\Pi^A + (1-h)\ \Pi^B$, $\forall h \in [0,1]$. In the status quo, agents consume the payoff from their asset holdings. Since agents are risk-neutral and they agree (at date 0) about the asset's expected payoff, the utilitarian social welfare is given by:

$$E_0^h \left[W \left(u_A, u_B \right) \right] = E_0^h \left[\widetilde{R} \right] = R, \quad \forall h \in (0,1) .$$

In contrast, in equilibrium agents always trade half of the asset, either from agent B to agent A in state u or vice versa in state d. Trading transfers wealth across agents at some cost, $k/2$ (which will be incurred with certainty). Thus, the utilitarian welfare is now given by the asset payoff net of trading costs:

$$E_0^h \left[W \left(u_A, u_B \right) \right] = E_0^h \left[\widetilde{R} \right] - \frac{k}{2} = R - \frac{k}{2}, \ \forall h \in (0,1) .$$

It follows that the status quo of no trade is belief-neutral superior to the market equilibrium.

In fact, using the second version of our welfare criterion, this result can be generalized to any social welfare function. In particular, the market equilibrium in this example is belief-neutral Pareto inefficient. To see this, consider the status quo allocation with an initial transfer of $T \in [-R, R]$ from agent B to agent A. Given this allocation, agent A's expected payoff is given by $R+T$ while agent B's payoff is $R-T$. In contrast, agents' expected payoffs in equilibrium sum to $R - \frac{k}{2}$ under any convex combination of their beliefs. It follows that, for any belief, Π^h, the market equilibrium is Pareto dominated by the status quo allocation with some transfer $T \in [-R, R]$.

This analysis also illustrates that the key reason for the inefficiency of the equilibrium in this example is the trading cost, k. While trading costs do not seem large in liquid financial markets, they can nevertheless have a significant impact on investment performance, especially for those who trade frequently. Barber and Odean (2000) analyzed performance of a large sample of brokerage accounts held by individual households in the US. They showed that the average household under-performed the market return by 1.5% each year, with trading costs contributing to a majority of the under-performance. For those who traded most actively, trading costs caused under-performance of over 5% each year. Excessive trading is particularly worrisome during bubbles, because bubbles tend to occur with trading frenzies, e.g., Scheinkman and Xiong (2003) and Hong and Stein (2007).

3.3 Bubble Models and Over-investment

A severe consequence of asset price bubbles is over-investment. Several recent papers built on the bubble models previously discussed to analyze over-investment (e.g., Bolton, Scheinkman and Xiong (2006), Gilchrist, Himmelberg, and Huberman (2005), and Panageas (2006)). The key idea of these papers is that even in the absence of any governance failure, a firm may choose to over-invest at the expense of its long-run fundamental value in order to maximize its current market value, which contains not only the long-run fundamental value but also the resale option value. Our criterion can highlight welfare losses of such over-investment.

We extend the two-period setting from the bubble example to incorporate firm investment. We remove the trading cost by setting k to be zero, but include firm investment. Suppose that the risky asset is equity issued by a firm. The firm chooses its investment at date 0. Suppose that the firm's investment is cost free but the investment return has a decreasing returns to scale. If the firm chooses to establish a production capacity of n units, the dollar return to per unit of capacity across the three states on date 2 is

$$\widetilde{R} = \{D_{uu}, D_{ud}, D_{dd}\} = \{R + 1 - n, R - n, R - 1 - n\},$$

where $R > 1$ is a constant. Due to the firm's decreasing return to scale, a larger investment scale n reduces the per unit return by n across all states on date 2.

Suppose that the firm issues one share of equity for each unit of production capacity. We denote the market price of each share on date 0 by p_0. Following the beliefs specified from the bubble example in equation (2) for the two risk neutral agents A and B, it is straightforward to derive that at date 1 agent A will acquire all the shares at state u at a price of

$$\begin{aligned} p_u &= (1/2 + \delta)(R + 1 - n) + (1/2 - \delta)(R - n) \\ &= R - n + (1/2 + \delta), \end{aligned}$$

and agent B will acquire all the shares at state d at a price of

$$\begin{aligned} p_d &= 1/2\,(R - n) + 1/2\,(R - 1 - n) \\ &= R - n - 1/2. \end{aligned}$$

By backward induction, at date 0 both agents A and B value each share at the same price of

$$p_0 = R - n + \delta/2,$$

which is higher than their expectation of the share's long-run fundamental $E^A\left[\widetilde{R}\right]=E^B\left[\widetilde{R}\right]=R-n$. The difference is exactly due to the value of each asset owner's resale option in anticipation of the reverse of the two agents' beliefs on date 1.

Since both A and B agree about the initial share price, the firm chooses its production capacity, n, to maximize its market value given by:

$$n \cdot p_0 = n \cdot (R - n + \delta/2).$$

Thus, the optimal investment level is given by:

$$n^* = \frac{1}{2}\left(R + \frac{\delta}{2} - 1\right).$$

Note that the optimal investment depends on δ, the magnitude of the two agents' belief dispersion at date 1.

Is this investment decision socially efficient? Suppose the planner uses the utilitarian social welfare function along with a convex combination of the two agents' beliefs, $\Pi^h = h\Pi^A + (1-h)\,\Pi^B$, $\forall h \in (0,1)$. As in the previous section, the utilitarian social welfare is equal to the firm's date 2 (or long-run) value, given by:

$$n \cdot E^h\left[\widetilde{R}\right] = n\,(R-n).$$

This expression is maximized for:

$$n^{**} = (R-1)/2 < n^*.$$

This implies that the firm over-invests in the market equilibrium relative to the level that maximizes the utilitarian social welfare (or the firm's long-run fundamental value) under any convex combination of the agents' beliefs.

As before, this result can be generalized to any social welfare function because the market equilibrium is in fact belief-neutral Pareto inefficient. In particular, for any belief, Π^h, it can be checked that the market equilibrium with investment n^* is Pareto dominated by the alternative allocation with investment $n^{**} < n^*$ combined with some initial transfer, $T \in [-n^*(R-n^*), n^*(R-n^*)]$, from agent B to agent A.

The driving force behind the inefficient over-investment is exactly the value of the resale option at the firm's date-0 market valuation. Anticipating the possibility of reselling the share to the other agent at date 1 at a profit, each agent over-values the share at date 0 relative to his own expectation of the share's long-run fundamental value. This in turn induces the firm to over-invest. Note that each agent recognizes that this level of investment reduces the firm's long run value. However, each agent also thinks that these losses will be borne by the other agent. Thus, our criterion applies and shows that the over-investment decision is belief-neutral Pareto inefficient. This result illustrates that, even in the absence of any governance failure, investors' heterogeneous beliefs can induce firms to make welfare reducing investment decisions. Consistent with this view, Gilchrist, Himmelberg, and Huberman (2005) provide evidence that firms tend to increase investment in response to increased heterogeneous beliefs proxied by dispersion in analysts' earnings forecasts.

3.4 Macroeconomic Models and Consumption-Savings Distortions

The previous subsection illustrated a channel by which belief disagreements generate over-investment. Belief disagreements might further distort investment through individuals' consumption-savings decision. In particular, as illustrated in Section 3.1, individuals with belief disagreements perceive a greater expected returns from their investments. This affects their savings decision the same way an increase in the real interest rate does. It creates a substitution effect which tends to increase savings, but also an income effect which tends to increase current consumption, and thus, reduce savings. Depending on which effect dominates, individuals might save too much or too little relative to a common beliefs benchmark.

We next illustrate this source of inefficiency using a recent model by Sims (2008), who shows that heterogeneous beliefs about monetary policy may lead to non-neutral effects. The model features agents who hold heterogeneous beliefs about inflation. Those who believe high inflation is less likely find nominal lending attractive, while those who believe high inflation is more likely find nominal borrowing a cheap source of financing. Heterogeneous beliefs motivate inflation optimists to borrow in nominal terms from inflation pessimists. This makes the consumption of both types of agents more volatile, just like in Section 3.1. However, heterogeneous beliefs in this model might also affect agents' savings. If the agents have rates of relative risk aversion less (or higher) than one, then the substitution effect (resp. the income effect) dominates, and each type chooses to save more (or less) than what would have been in the absence of heterogeneous beliefs. The net saving in turn leads to over- (or under-) investment. These non-neutral effects of heterogeneous beliefs about monetary policy prompt attention from policy makers. To analyze issues related to optimal monetary policies, it is important to have a well-specified welfare criterion. Our criterion fills this gap.

As the setting used by Sims is simple enough, we adopt it in full. The setting has two dates and two types of agents. We normalize the size of the population to one. Each agent starts with an endowment of B_0 dollars of nominal bonds issued by the government and an endowment of Y units of goods. At the initial date, he can consume part of the goods endowment in the first period and invest the rest either in the nominal bonds or in a real asset.

There are two possible states of the world on the second date $j \in \{f, m\}$. In state j, the government fixes the lump-sum tax rate on each agent to be τ_j and the gross nominal interest rate to R. In state f, the tax backing for bonds is low and hence prices are high, while in state m, taxes are high and prices are therefore lower. Thus, the government's second date budget constraints determine the bond price:

$$P_{2j} = \frac{RB_0}{\tau_j}, \quad \text{where} \quad j = f, m.$$

The economy has a representative firm, which produces at the second date according to a decreasing return to scale production function: $g(S) = AS^{1-\alpha}$, where S is the capital input and A is a constant. The firm has to rent capital from individual agents at a market rental rate of ρ. We normalize the firm's ownership to one share, which is equally divided among the agents. Thus, the firm's profit per unit of ownership is

$$\delta = AS^{1-\alpha} - \rho S.$$

The firm's profit optimization requires that

$$\rho = A(1-\alpha)S^{-\alpha}.$$

There are two types of agents: $i \in \{a, b\}$. Type i agents believe that the probability of state f is $p_i \in (0,1)$. Each type contributes to half of the population. Each agent maximizes his aggregate utility across the two dates:

$$\max \; U(C_{i1}) + \beta\left[p_i U(C_{if}) + (1-p_i) U(C_{im})\right]$$

where C_{i1}, C_{if}, and C_{im} are a type i agent's consumption on date 1 and in states f and m of date 2, and β is the agent's time discount rate. On the first date, the agent can allocate his initial good endowment Y to consumption C_{i1}, renting capital to the firm S_i, and buying more nominal bonds $B_i - B_0$ at a nominal price of P_1:

$$C_{i1} + S_i + \frac{B_i - B_0}{P_1} = Y.$$

Note that the agent can take a short position in the capital, which is equivalent to borrowing in real terms at a rate of ρ. He can also take a short position in the nominal bonds, which is equivalent to borrowing in nominal terms at a rate of R. His consumption in state j of the second date is given by

$$C_{ij} = \rho S_i + \frac{RB_i}{P_{2j}} - \tau_j + \frac{\delta}{2}$$

where P_{2j} is the nominal bond price in the state. Suppose that both types of agents have a power utility function: $U(C) = \frac{C^{1-\sigma}}{1-\sigma}$ with σ as the rate of relative risk aversion.

The first order condition for the agent with respect to S_i gives

$$C_{i1}^{-\sigma} = \beta\rho \left[p_i C_{if}^{-\sigma} + (1-p_i)\, C_{im}^{-\sigma}\right], \quad i \in \{a, b\}$$

and with respect to B_i gives

$$\frac{1}{P_1} C_{i1}^{-\sigma} = \beta R \left[\frac{p_i C_{if}^{-\sigma}}{P_{2f}} + \frac{(1-p_i)\, C_{im}^{-\sigma}}{P_{2m}}\right], \quad i \in \{a, b\}.$$

The market clearing condition for the capital gives

$$S = S_a + S_b$$

and for the nominal bonds gives

$$B_0 = B_a + B_b.$$

These conditions allow us to determine a unique equilibrium represented by $\{S_a, S_b, B_a, B_b, P_1\}$.

While analytical solution of the equilibrium is not available, it is numerically tractable. We adopt the same parameter values used by Sims to illustrate the equilibrium:

$$Y = 1.60, R = 1.10, \tau_f = 1.10, \tau_m = 1.65, \alpha = 0.30, \beta = 0.90, A = 1.20, \sigma = 0.50, B_0 = 1.50. \quad (3)$$

We compare the equilibrium outcomes for three sets of beliefs: Two homogeneous beliefs benchmarks, $\{p_a = 0.3, p_b = 0.3\}$ and $\{p_a = 0.7, p_b = 0.7\}$, and a heterogeneous beliefs economy in which each agent has the belief in one of the benchmarks, $\{p_a = 0.3, p_b = 0.7\}$.

$\{p_a, p_b\}$	S_a	S_b	S	B_a	B_b	P_1	C_{a1}	C_{af}	C_{am}	C_{b1}	C_{bf}	C_{bm}
$\{0.3, 0.3\}$	0.51	0.51	1.03	1.50	1.50	0.84	1.09	0.61	0.61	1.09	0.61	0.61
$\{0.7, 0.7\}$	0.51	0.51	1.03	1.50	1.50	0.98	1.09	0.61	0.61	1.09	0.61	0.61
$\{0.3, 0.7\}$	−2.19	3.30	1.12	3.94	−0.94	0.89	1.04	0.20	1.09	1.04	1.09	0.20

Table I: Equilibrium under homogeneous and heterogeneous beliefs.

Table I lists the equilibrium quantities in the three settings. First note that the two homogeneous beliefs equilibria have some common (belief-neutral) properties. In particular, beliefs about inflation affect the nominal bond price, P_1, but they have no effect on real allocations. In contrast, the equilibrium with heterogeneous beliefs has two main differences in terms of real allocations. First, with heterogeneous beliefs, agents have more volatile consumption across the two states of the second date. This variability is due to the speculation between the agents about the nominal price inflation. The type a agents (the inflation pessimists) invest more in nominal bonds and at the same time short-sell the capital (i.e., borrow in real terms). These positions allow them to profit from in state m. In contrast, the type b agents choose the opposite positions and profit in state f. Second, with heterogeneous beliefs, agents also save more (and consume less at the first date). Intuitively, belief disagreements induce agents to perceive a greater expected return from their investments, which creates substitution and income effects. Given the elasticity of intertemporal substitution, $1/\sigma = 2 > 1$, the substitution effect dominates. Thus, in this case agents' save more to engage in more speculation. This leads to a greater aggregate investment ($S = 1.23$) than in homogeneous belief benchmarks ($S = 1.14$).[6]

Taken together, this setting with heterogeneous beliefs exhibits two types of inefficiency: more volatile consumption and distorted savings (and investment). To discuss welfare implications of heterogeneous beliefs, we compare the social welfare in the two settings. As in the previous examples, we start by considering the utilitarian social welfare function. Instead of taking a stance on their beliefs, we evaluate the social welfare using any convex combination of the two types of beliefs: $p \in [p_a, p_b]$. The left panel of Figure 2 depicts the social welfare based on the equilibrium consumption of the two types of agents in the two settings as p varies between p_a and p_b. Heterogeneous beliefs reduce the social welfare regardless of the

[6] In contrast, if $1/\sigma < 1$, then the income effect from speculation dominates and agents save less with heterogeneous beliefs relative to the case with homogeneous beliefs.

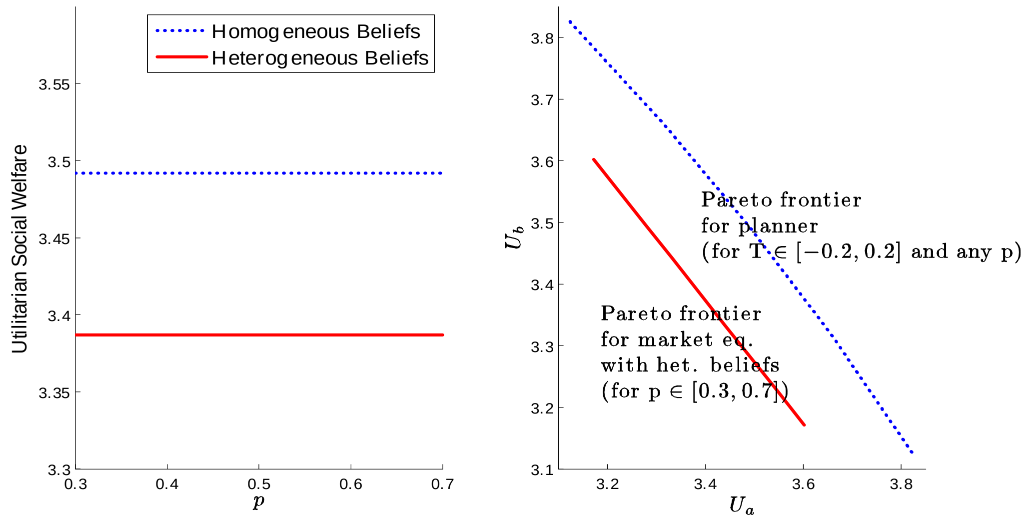

Figure 2: The left panel plots the utilitarian social welfare based on any convex combination of the two types of beliefs. The right panel plots the Pareto frontiers respectively for the market equilibrium and a belief-neutral planner.

belief measure one uses to evaluate the utilitarian social welfare.

As before, this result can be generalized to any social welfare function because the market equilibrium is in fact belief-neutral Pareto inefficient. To illustrate this result, define $y(T)$ as an initial allocation in which a fraction, T, of all of agent B's endowments (bonds, goods, and shares of the representative firm) are transferred to agent A. For each T, consider the common beliefs equilibrium starting with this initial allocation -which are feasible allocations available to the planner.[7] The second panel of Figure 2 plots the Pareto frontier corresponding to these allocations as the transfer, T, varies. The same panel also plots the Pareto frontier for the equilibrium with heterogeneous beliefs as p varies between p_a and p_b. The figure shows that, for any belief $p \in [p_a, p_b]$, the equilibrium with heterogeneous beliefs is Pareto dominated. The intuition is the same as in the earlier sections: In this economy, more volatile consumption and distorted savings is sub-optimal *according to any agent's belief*. A planner who corrects these inefficiencies can always redistribute wealth to improve over the market equilibrium. Thus, this example demonstrates that our criterion is able to give clear welfare ranking in Sims' model regarding non-neutrality of heterogeneous

[7]The case $T = 0$ corresponds to the homogeneous beliefs benchmarks displayed in Table I. The cases with transfer, $T \neq 0$, are scaled versions of these real allocations. In view of the homotheticity of preferences, Agent A's allocations are multiplied by $1 + T$, agent B's allocations are multiplied by $1 - T$, and their sum is unaffected. Thus, the aggregate allocations are also unaffected by the transfer, T.

beliefs about monetary policy.

3.5 Leverage Cycle Models and Bankruptcy Costs

A quickly growing strand of literature analyzes leverage cycles based on agents' heterogeneous beliefs (e.g., Geanakoplos (2003, 2009), Fostel and Geanakoplos (2008), Simsek (2010), and He and Xiong (2010)). The key feature of those models is that optimism can motivate cash-constrained optimists to use collateralized short-term debt to finance their asset acquisition. The leverage initially fuels the price boom but later forces the optimists to deleverage after bad shocks, resulting in a leverage cycle. This framework nicely integrates the optimists' leverage cycle with the asset price cycle. Both of these cycles are important for understanding historical episodes of financial crises, including the recent one. To use this framework to analyze relevant policy issues such as regulation over financial institutions' leverage, it is important to discuss welfare implications. We now illustrate the capability of our criterion to generate clear welfare ranking in this framework. The key insight is that, under appropriate assumptions, optimists choose to borrow with risky debt contracts despite the bankruptcy costs that this might generate in the future. Optimists undertake these costly investments purely for speculative reasons, i.e., because they think the asset is underpriced. Our criterion detects the inefficiency of these equilibria.

Consider a setting with 3 dates, i.e., $t = 0, 1, 2$, and two types of risk-neutral agents (A and B). Figure 3 depicts the asset payoff and the beliefs of the two types. Suppose that the final payoff of a risky asset across the three final states at date 2 is $\widetilde{R} = \{1, 1, \theta\}$, where $\theta \in (0, 1)$. The asset gives a low payoff of θ after two negative fundamental moves and gives 1 in other final states. We normalize the net supply of the asset to one unit and the risk-free interest rate to zero. Each type holds a constant belief about the probability of the fundamental state rising on the tree in the following period. We denote the two groups' beliefs by π^A and π^B with $\pi^A > \pi^B$. Suppose $\pi^A, \pi^B \in (0, 1)$ so that both agents assign a positive probability to both positive and negative moves.

Given the difference in agents' beliefs, it is desirable for the optimists (type-A agents) to acquire all of the asset. However, they face a practical problem in that they may not have sufficient cash endowments to make the purchases. To highlight this problem, we assume that there is one unit of optimists, each with an initial cash endowment of $c > 0$. They can use asset holdings as collateral to raise debt financing. The standard collateralized

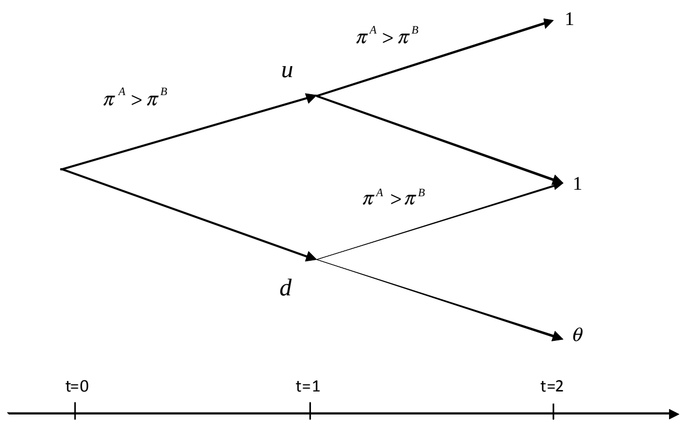

Figure 3: Asset payoff and belief structure of the leverage cycle model.

debt contracts allow a borrower to pledge the asset to raise debt. If he is unable to make the promised debt payment, the creditor can seize the collateral. This in turn makes the availability and cost of the borrower's debt financing dependent on the future value of the collateral. On the other hand, the availability of debt financing directly determines how much the optimists can bid up the asset price beyond the pessimists' asset valuation.

For simplicity, suppose that the pessimists (type-B agents) initially own all of the asset at $t = 0$ and are thus able to finance any credit need of the optimists.

In deciding how much to borrow, type-A agents face two sources of costs. First, as the creditors (likely type-B agents) are more concerned about the potential default risk than the borrowers, higher leverage tends to be more costly. Second, if type-A agents sell their assets at dates $t = 1$ or $t = 2$, they face a personal liquidation cost, α. In particular, if the agent fails to pay her debt, then she is forced to liquidate and incurs the cost, α. One can interpret this liquidation cost as the cost of vacating a house if a homeowner defaults on his mortgage. At the end of the section, we describe a version of the model in which the liquidation costs are incurred by creditors, which can be interpreted as foreclosure costs. The two versions have similar welfare implications.

Our setting maintains several key features used by Geanakoplos (2009), which include the same binomial payoff structure and the same collateralized debt contract. We add liquidation costs, which is a realistic feature that seemed especially relevant for the recent subprime crisis. Since this complicates the analysis, we allow for only two types of beliefs rather than a continuum. This derivation follows He and Xiong (2010), who analyzed equilibrium debt financing in a setting with two types of agents whose beliefs vary over time but without liquidation costs.

There are two relevant debt contracts in equilibrium. One contract promises a payment of θ at date 1 collateralized by one unit of the asset. Because the asset's fundamental value in the worst state of date 2 is able to cover θ, this debt contract is riskless throughout and can thus give the borrower an initial credit of θ. The second contract promises a payment at date 1 equal to type-B agents' (the creditors') asset valuation in state d of date 1:

$$K_d \equiv E_d^B[\widetilde{R}] = \pi^B + \left(1 - \pi^B\right)\theta > \theta.$$

As the creditors value the collateral for at least K_d at date 1, this debt is also riskless and thus allows a borrower to borrow at the risk-free interest rate for the initial period. However, to refinance this debt in state d of date 1, the borrower has to make a greater promise of paying 1 at date 2. This new promise allows him to raise K_d from type-B agents to pay off his initial debt, but exposes him to default risk if the asset's fundamental value eventually turns out to be θ on date 2. Relative to the first contract, the second one gives higher leverage at the expense of a higher refinancing cost in state d of date 1 as well as the possibility of incurring liquidation costs at date 2. We prove in the appendix that these two debt choices dominate the other alternatives.

We assume that the liquidation cost, α, is modest so in some scenarios the type-A agents will choose the higher leverage (i.e., the contract with a promise of K_d) and thus face the risk of default:

$$\alpha < \frac{\pi^A \pi^B (\pi^A - \pi^B)}{(1-\pi^A)^2[1-(1-\pi^B)^2]}(1-\pi^B)(1-\theta). \tag{4}$$

Under this assumption, the analysis in the appendix shows that there exists a price threshold $p_0^* \in \left(E_0^B\left[\tilde{R}\right], E_0^A\left[\tilde{R}\right]\right)$, such that type-A agents choose the risky debt if and only if $p_0 < p_0^*$. Intuitively, when the price is low, type-A agents see a bargain in the asset price. Thus, they are induced to leverage more by taking the risky debt, despite the refinancing and liquidation costs this entails.

The remaining question is whether $p_0 < p_0^*$ is observed in equilibrium. Proposition 5 in the appendix characterizes the equilibrium and shows that the asset price satisfies, $p_0 < p_0^*$, whenever type-A agents' initial cash is below a threshold,

$$c < p_0^* - K_d. \tag{5}$$

In these cases, type-A agents take risky debt and use all of their purchasing capacity, $c+K_d$, to buy assets. The asset price is given by the upper envelope of type-A agents' purchasing capacity and type-B agents' valuation:

$$p = \max\left(c + K_d, E_0^B\left[\tilde{R}\right]\right),$$

which is indeed less than p_0^*. Intuitively, if type-A agents' initial cash is very low, $c < E_0^B\left[\tilde{R}\right] - K_d$, then they cannot buy all of the assets. Some assets are left in the hands of the type-B agents and the price is equal to their valuation, $E_0^B\left[\tilde{R}\right]$. If type-A agents' initial cash is higher, $c \geq E_0\left[\tilde{R}\right] - p$, then they buy all of the assets and the price is determined by their purchasing capacity. When their purchasing capacity is not too large, i.e., under condition (5), the price remains below the threshold, p_0^*.

We have thus characterized an equilibrium in which type-A agents borrow with risky debt with face value K_d, despite the liquidation costs that this entails. They are doing this purely for speculative reasons, that is, because they perceive the asset to be significantly underpriced, $p_0 < p_0^* < E_0^A\left[\tilde{R}\right]$. We next apply our welfare criterion to illustrate that this equilibrium is inefficient. To see this, first suppose the planner has the utilitarian welfare function. We use a convex combination of the two types' beliefs, $\Pi^h = h\Pi^A + (1-h)\,\Pi^B$, $\forall h \in (0,1)$, to calculate welfare. The risk neutrality of both types of agents implies that the social welfare is given by the expectation of the asset's fundamental value plus optimists' cash, c, and minus expected liquidation costs:

$$E_0^h\left[W\left(u_A, u_B\right)\right] = c + E_0^h\left[\widetilde{R} - \alpha I_{\tilde{R}=\theta}\right],$$

where $I_{\tilde{R}=\theta}$ denotes the indicator function for the realization of the sate $\tilde{R} = \theta$. Since $\pi_A, \pi_B > 0$, both agents assign a positive probability to this state, which implies:

$$E_0^h\left[W\left(u_A, u_B\right)\right] < c + E_0^h\left[\tilde{R}\right].$$

In contrast, the social welfare in the status quo allocation with no trade is given by $c + E_0^h\left[\widetilde{R}\right]$. Thus, our criterion identifies strict welfare losses under condition (5) due to the liquidation costs incurred by borrowers. As before, the result can be generalized to any social welfare function because the equilibrium is in fact belief-neutral Pareto inefficient.

As an alternative, we could consider a version of this model in which liquidation costs are borne by creditors, instead of borrowers. We next briefly describe this version of the model, which approximates more closely the bankruptcy costs in practice (e.g., foreclosure costs). Suppose there are only two dates, $t \in \{0, 1\}$, but three continuation states, $\{H, M, L\}$, in which the asset price will be either high, medium, or low. The agents agree about the probability of the low payoff state, π_L, but they disagree about the probabilities of the remaining states. In particular, type-A agents are more optimistic about the high state, i.e., $\pi_H^A > \pi_H^B$ (and thus, $\pi_M^A < \pi_M^B$). As before, type-A agents borrow from type-B agents using collateralized debt contracts. If a type-A agent defaults, then creditors have to liquidate the asset. Suppose a fraction, $\kappa \in (0, 1)$, of the value of the asset is lost in a forced liquidation, which is the main difference from the earlier version of the model. In this case, it can be seen that type-A agents face a trade-off between choosing a safe debt contract with face value L, and a risky debt contract with face value M. The risky debt enables them to borrow a larger amount,

$$\pi_L (1 - \kappa) L + (1 - \pi_L) M.$$

However, risky debt is expensive (i.e., it has high yield) because it leads to bankruptcy costs in some states of the world, which are priced by the creditors. As before, under appropriate conditions, the speculative motive for trade induces the type-A agents to finance their purchases with risky debt. This arrangement generates expected bankruptcy costs according to both agents. Thus, the equilibrium is inefficient according to our criterion.

Of course, in more general settings, agents acquire assets not just for speculative purposes but also for consumption. For example, people buy houses not only because they expect housing prices to appreciate but also because they enjoy living in the homes. It is important to incorporate both speculative incentives and consumption values in evaluating the welfare consequences of leverage cycles. Our criterion provides a useful tool for such an evaluation.

4 Conclusion

This paper proposes a simple welfare criterion for models in which agents have heterogeneously distorted beliefs. The criterion rules that a new allocation is inferior if for a given welfare functional the expected social welfare is reduced based on any convex combination of individual agents' beliefs. This criterion gives incomplete welfare ranking, but is nevertheless useful in identifying negative-sum speculation in which each agent believes that he can make a profit even though he also expects the expenses incurred by others to be greater than his gain. Through a series of examples, we show that this criterion is capable of generating a clear welfare ranking in a wide range of prominent models that build on heterogeneous beliefs, including speculative trading models, speculative bubble models, over-investment models, macroeconomic models with heterogeneous beliefs, and leverage cycle models.

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Appendix A Proofs of Propositions

A.1 Proof of Proposition 1 and 2

First, we establish that if $\pi^A \neq \pi^B$, the two agents will take a non-zero position in the speculative contract. The first order condition implies that

$$\frac{\pi^A}{1-\pi^A}\frac{u'\left(0.5+k^A\left(1-p\right)\right)}{u'\left(0.5-k^A p\right)} = \frac{\pi^B}{1-\pi^B}\frac{u'\left(0.5+k^B\left(1-p\right)\right)}{u'\left(0.5-k^B p\right)}.$$

Suppose that $k^A = k^B = 0$. Then, we must have

$$\frac{\pi^A}{1-\pi^A} = \frac{\pi^B}{1-\pi^B},$$

which contradicts $\pi^A \neq \pi^B$. Thus, k^A and k^B cannot be both zero, which in turn implies that both are nonzero.

To prove the propositions, consider a convex combination of the two agents' beliefs, $\Pi^h = h\Pi^A + (1-h)\Pi^B$, for some $h \in [0,1]$. First consider agent's utilitarian social welfare in equilibrium, given by:

$$U^h = \begin{array}{l} \pi^h\left[u\left(0.5+k^A\left(1-p\right)\right)+u\left(0.5-k^A\left(1-p\right)\right)\right] \\ +\left(1-\pi^h\right)\left[u\left(0.5-k^A p\right)+u\left(0.5+k^A p\right)\right]. \end{array}$$

The strict concavity of $u(\cdot)$ implies that

$$\begin{aligned} u\left(0.5+k^A\left(1-p\right)\right)+u\left(0.5-k^A\left(1-p\right)\right) &< 2u\left(0.5\right) \\ u\left(0.5-k^A p\right)+u\left(0.5+k^A p\right) &< 2u\left(0.5\right) \end{aligned}$$

Thus,

$$U^h < \pi^h \cdot 2u\left(0.5\right) + \left(1-\pi^h\right)\cdot 2u\left(0.5\right) = 2u\left(0.5\right)$$

which is the utilitarian social welfare under the status quo. This completes the proof of Proposition 1.

Next, consider each agent's certainty equivalent wealth, $w^{i,h}$, given by the solution to:

$$u\left(w^{i,h}\right) = \pi^h u\left(0.5+k^i\left(1-p\right)\right) + \left(1-\pi^h\right)u\left(0.5-k^i p\right), \text{ for each } i \in \{A,B\}.$$

The strict concavity of $u(\cdot)$ (along the fact that $k^i \neq 0$) implies that:

$$u\left(w^{i,h}\right) < u\left(\pi^h\left(0.5+k^i\left(1-p\right)\right)+\left(1-\pi^h\right)\left(0.5-k^i p\right)\right).$$

Since $u(\cdot)$ is strictly increasing, this further implies:

$$w^{i,h} < \pi^h \left(0.5 + k^i (1-p)\right) + \left(1-\pi^h\right) \left(0.5 - k^i p\right), \text{ for each } i \in \{A, B\}.$$

Adding these inequalities and using market clearing, $k^A + k^B = 0$, we have:

$$w^{A,h} + w^{B,h} < 1.$$

It follows that the status quo with an appropriate transfer Pareto dominates the equilibrium, completing the proof of Proposition 2.

A.2 Proof of Proposition 3, and 4

Recall that the optimal risky asset allocation is given by $k^{opt,A} = e$ and $k^{opt,B} = -e$. An analysis as in the previous proof shows that, when $\pi^A \neq \pi^B$, agents deviate from the optimal risky asset allocation, that is, $k^A \neq k^{opt,A}$. Next fix a belief, Π^h, and consider each agent's certainty equivalent wealth, $w^{i,h}$, given by the solution to:

$$\begin{aligned} u\left(w^{A,h}(e)\right) &= \pi^h u\left(0.5 - e + k^A (1-p)\right) + \left(1-\pi^h\right) u\left(0.5 - k^A p\right), \\ u\left(w^{B,h}(e)\right) &= \pi^h u\left(0.5 + e + k^B (1-p)\right) + \left(1-\pi^h\right) u\left(0.5 - k^B p\right). \end{aligned}$$

Since $k^A \neq k^{opt,A}$, an agent's consumption is not constant across the states. Then, the strict concavity of $u(\cdot)$ implies:

$$\begin{aligned} w^{A,h}(e) &< \pi^h \left(0.5 - e + k^A (1-p)\right) + \left(1-\pi^h\right) \left(0.5 - k^A p\right), \\ w^{B,h}(e) &< \pi^h \left(0.5 + e + k^B (1-p)\right) + \left(1-\pi^h\right) \left(0.5 - k^B p\right). \end{aligned}$$

Adding these inequalities and using market clearing, we have:

$$w^{A,h}(e) + w^{B,h}(e) < 1. \tag{6}$$

On the other hand, the status quo allocation combined with the optimal risky asset allocation, k^{opt}, gives each agent a constant consumption of $\frac{1}{2}$. It follows that this allocation combined with an appropriate transfer, T, Pareto dominates the equilibrium allocation, proving Proposition 3.

To prove Proposition 4, consider agents' certainty equivalent wealth from the status quo allocation with no trade. This is found by solving:

$$\begin{aligned} u\left(w^{status,A,h}(e)\right) &= \pi^h u(0.5 - e) + \left(1-\pi^h\right) u(0.5), \\ u\left(w^{status,B,h}(e)\right) &= \pi^h u(0.5 + e) + \left(1-\pi^h\right) u(0.5). \end{aligned}$$

A similar analysis to above shows that the sum, $w^{status,A,h}(e) + w^{status,B,h}(e)$, is also less than 1. Nonetheless, we claim that there exists $\overline{e} > 0$ such that:

$$w^{A,h}(e) + w^{B,h}(e) < w^{status,A,h}(e) + w^{status,B,h}(e).$$

Once we show this claim, it follows that the status quo allocation with an appropriate transfer, T, dominates the equilibrium allocation.

To prove the claim, first note that the left hand-side of Eq. (6) is a continuous function of e. Moreover, its limit is also strictly less than 1, because this limit is equal to the sum, $w^{A,h} + w^{B,h}$, calculated in the previous proof. In particular,

$$\lim_{e \to 0} w^{A,h}(e) + w^{B,h}(e) < 1.$$

In contrast, as $e \to 0$, the status quo approximates riskless consumption, which implies:

$$\lim_{e \to 0} w^{status,A,h}(e) + w^{status,B,h}(e) = 1.$$

Combining the last two expressions proves the claim, and thus, also Proposition 4.

A.3 Characterization of Equilibrium in Section 3.5

The following proposition summarizes the market equilibrium:

Proposition 5 *Depending on type-A agents' cash endowment c, the following five cases can emerge in equilibrium.*

- *Case 1: $c < c_1$, where $c_1 = E_0^B[\widetilde{R}] - K_d$. In this case, type-A agents acquire the asset at $t = 0$ by using a one-period debt contract with a promise of K_d. However, their purchasing capacity is insufficient to lift the asset price, p_0, above type-B agents' expectation of the asset's fundamental value. Consequently, $p_0 = E_0^B[\widetilde{R}]$.*

- *Case 2: $c \in [c_1, c_2)$, where $c_2 = p_0^* - K_d$ and*

$$p_0^* = \frac{\left(2-\pi^A\right)\pi^A[\pi^B + (1-\pi^B)\theta](1-\theta) - [\pi^A(1-\pi^B)(1-\theta) - (1-\pi^A)^2\alpha]\theta}{\left(2-\pi^A\right)\pi^A(1-\theta) - \pi^A(1-\pi^B)(1-\theta) + (1-\pi^A)^2\alpha}. \quad (7)$$

  In this case, type-A agents acquire the asset at $t = 0$ by using one-period debt contract with a promise of K_d. The asset price p_0 is given by type-A agents' aggregate purchasing capacity: $p_0 = c + K_d$.

- *Case 3: $c \in [c_2, c_3)$, where $c_3 = p_0^* - \theta$. In this case, type-A agents acquire the asset at $t = 0$ and are indifferent to use debt contracts with promises of θ and K_d. The asset price p_0 remains at a constant level $p_0 = p_0^*$. The fraction of borrowers who choose to use debt face value K_d is given by equation (8) below.*

- *Case 4: $c \in [c_3, c_4)$, where $c_4 = E_0^A[\widetilde{R}] - \theta$. In this case, type-A agents acquire the asset by using riskless debt with a promise of θ. The asset price p_0 is determined by their aggregate purchasing capacity: $p_0 = c + \theta$.*

- *Case 5: $c \geq c_4$, where $c_4 = E_0^A[\widetilde{R}] - \theta$. In this case, type-A agents have ample cash endowments to support their asset acquisition at a price equal to their expectation of the asset's fundamental value, $p_0 = E_0^A[\widetilde{R}]$, by using debt with a promised payment less than θ.*

We prove this proposition in two steps. First, we characterize type-A agents' optimal debt contract. We show that the relevant debt contracts are short-term debt with face value θ and K_d, and we characterize the choice between these two contracts. Second, we consider market clearing and characterize the equilibrium price for cases 1-5. In each case, we also show that (unlike in Geanakoplos, 2009) type-A agents do not have an incentive to hold cash to buy assets in state d of date 1. In particular, type-A agents use all of their purchasing power to buy the assets at date 0.

Step 1. First consider type-A agents' debt contract choice. We start with short-term debt with maturity at $t = 1$. It can be seen that the face value of short-term debt should lie in the range of $[\theta, 1]$, i.e., between the two possible payoffs of the collateral. If the agent chooses to borrow short-term debt at $t = 0$, he has to roll over his debt at $t = 1$. If he fails to obtain refinancing, he will default and incur a personal liquidation cost of α. In state u, the subsequent asset payoff is surely 1; thus there is no problem rolling over the debt. In state d, the maximum debt financing the borrower can obtain from the pessimistic creditors is

$$K_d = E_d^B[\widetilde{R}] = \pi^B + \left(1 - \pi^B\right)\theta.$$

Thus, the borrower is able to structure a new debt contract with creditors if his initial debt promise is not higher than K_d. By making a new promise of F_d, he can obtain the following credit to repay his initial debt:

$$C(F_d)=\begin{cases} F_d & \text{if } F_d\le\theta \\ \pi^B F_d+(1-\pi^B)\theta & \text{if } \theta<F_d\le 1 \end{cases}$$

Note that the new debt is risk-free if $F_d \le \theta$ or risky if $\theta < F_d \le 1$. In the latter case, the lender will be paid with F_d in the good du state but receive the asset in the bad dd state. Thus, if the borrower's initial debt promiseF_0 is lower than or equal to K_d, he can obtain refinancing even in the lower state d at $t = 1$; and if F_0 is higher than K_d, he will have to default in the lower state d.

We now discuss the borrower's debt promise choice in using short-term debt. First consider the range, $[\theta,\ K_d]$. If the borrower promises $F_0 = \theta$, he can obtain an initial credit of θ, which allows him to establish an initial position of $c/\left(p_0-\theta\right)$ units of asset. The expected return on his cash is

$$R_0^\theta=\frac{(2-\pi^A)\pi^A(1-\theta)}{p_0-\theta}.$$

If he chooses a promise $F_0 \in (\theta,\ K_d]$, he can obtain an initial credit of F_0. The expected return on his cash after accounting for the possible liquidation cost α is

$$\begin{aligned} R_0^S &= \frac{\pi^A(1-F_0)+(1-\pi^A)\pi^A(1-F_d)+(1-\pi^A)^2(-\alpha)}{p_0-F_0} \\ &= \frac{\pi^A(1-F_0)+(1-\pi^A)\frac{\pi^A}{\pi^B}[\pi^B+(1-\pi^B)\theta-F_0]+(1-\pi^A)^2(-\alpha)}{p_0-F_0}. \end{aligned}$$

Note that while he can refinance his initial debt in state d on date 1, he will eventually default in state dd on date 2. It is straightforward to verify that $\frac{dR_0^S}{dF_0}<0$ if and only if

$$p_0>\widetilde{p}_0^*\equiv\frac{\pi^A+(1-\pi^A)\frac{\pi^A}{\pi^B}[\pi^B+(1-\pi^B)\theta]-(1-\pi^A)^2\alpha}{\pi^A+(1-\pi^A)\frac{\pi^A}{\pi^B}}.$$

Thus, if $p_0 > \widetilde{p}_0^*$, $F_0 = \theta$ is the optimal choice. If $p_0 = \widetilde{p}_0^*$, any $F_0 \in (\theta,\ K_d]$ would yield the same expected return. If $p_0 < \widetilde{p}_0^*$, $F_0 = K_d$ is superior to any promise in $(\theta,\ K_d)$. But we still need to compare this choice with $F_0 = \theta$ debt. Suppose that at a critical level p_0^*, the expected returns from $F_0 = \theta$ and K_d are equal:

$$\frac{\pi^A\left(1-K_d\right)+(1-\pi^A)^2(-\alpha)}{p_0^*-K_d}=\frac{(2-\pi^A)\pi^A(1-\theta)}{p_0^*-\theta}$$

which gives

$$p_0^*=\frac{[1-(1-\pi^A)^2][\pi^B+(1-\pi^B)\theta](1-\theta)-[\pi^A(1-\pi^B)(1-\theta)-(1-\pi^A)^2\alpha]\theta}{[1-(1-\pi^A)^2](1-\theta)-\pi^A(1-\pi^B)(1-\theta)+(1-\pi^A)^2\alpha}<\widetilde{p}_0^*.$$

Therefore, if $p_0 < p_0^*$, $F_0 = K_d$ is the optimal face value; if $p_0 > p_0^*$, $F_0 = \theta$ dominates; when $p_0 = p_0^*$, the borrower is indifferent between $F_0 = K_d$ and θ.

We now consider short-term debt with promise higher than K_d. For such a choice, the debt is no longer riskless as the borrower cannot refinance it in state d on date 1 and has to turn over the asset to the creditor. Anticipating this possibility, the creditor is willing to grant the following credit on date 0:

$$C_0(F_0) = \pi^B F_0 + (1-\pi^B)[\pi^B + (1-\pi^B)\theta].$$

Then, the expected return to the borrower is

$$R_0^S = \frac{\pi^A(1-F_0) + (1-\pi^A)(-\alpha)}{p_0 - \pi^B F_0 - (1-\pi^B)[\pi^B + (1-\pi^B)\theta]}.$$

It is straightforward to verify that $\frac{dR_0^S}{dF_0} < 0$ iff

$$p_0 > p_0^* \equiv 1 - (1-\pi^B)^2 + (1-\pi^B)^2\theta - \frac{\pi^B}{\pi^A}(1-\pi^A)\alpha.$$

Note that the asset price p_0 is bounded from below by the asset valuation of pessimists

$$E_0^B[\widetilde{R}] \equiv 1 - (1-\pi^B)^2 + (1-\pi^B)^2\theta.$$

As $E_0^B[\widetilde{R}] > p_0^*$, it is not optimal for the borrower to choose a debt promise above K_d.

It is also straightforward to verify that under condition (4), $p_0^* > E_0^B[\widetilde{R}]$. Therefore, the borrower's optimal short-term debt promise at $t = 0$ is

$$F_0 = \begin{cases} K_d, & \text{if } p_0 \in [E_0^B[\widetilde{R}], p_0^*); \\ \theta \text{ or } K_d, & \text{if } p_0 = p_0^*; \\ \theta, & \text{if } p_0 \in (p_0^*, E_0^A[\widetilde{R}]]. \end{cases}$$

Step 2. We now discuss different cases based on group-A agents' cash endowment c from high to low, in reverse order from those cases listed in Proposition 5

- Case 5: $c \geq c_4$.

In this case, the asset price is determined by type-A agents' beliefs at each date. Moreover, at these prices, type-A agents are able to finance their asset acquisition by using debt with promise less than θ. In fact, each type-A agent is indifferent between acquiring or not acquiring the asset. To ensure this case holds true, c has to satisfy

$$c \geq c_4 \equiv E_0^A\left[\widetilde{R}\right] - \theta.$$

- Case 4: $c_3 \leq c < c_4$.

In this case, type-A agents use debt with promise θ to finance their asset acquisition. However, their aggregate purchasing power is unable to sustain the price at their asset valuation. Instead, at $t = 0$, the price is determined by their purchasing power:

$$p_0 = c + \theta.$$

Going forward, in state d of date 1, type-A agents can still refinance their debt and thus keep the asset price at their valuation, i.e., $p_d = E_d^A\left[\widetilde{R}\right]$. To ensure that optimists' debt contract choice is optimal, we need to ensure that $p_0 > p_0^*$, which is equivalent to

$$c > c_3 \equiv p_0^* - \theta.$$

We next check type-A agents' incentive to save cash to date 1 in this case. First consider their return from buying at date 0 (and holding until date 2), which is given by:

$$\frac{[\pi^A + (1-\pi^A)\pi^A](1-\theta)}{p_0 - \theta} > 1,$$

where the inequality follows since $p_0 \in [p_0^*,\ E_0^A[\widetilde{R}])$. If instead they save cash to date 1, they will have to buy the asset from another type-A agent (who hold all the assets in the conjectured equilibrium). In view of liquidation costs, other type-A agents would sell at a price $E_d^A[\widetilde{R}] + \alpha$. Thus, the return from saving cash is given by:

$$\pi^A + (1-\pi^A)\frac{\pi^A(1-\theta)}{E_d^A[\widetilde{R}] + \alpha - \theta} < 1.$$

Thus, type-A agents have no incentive to save cash.

- Case 3: $c_2 \leq c < c_3$.

In this case, type-A agents are indifferent to using debt with promises of θ and K_d to purchase asset at price p_0^*. The expected return is

$$\begin{aligned}
& \frac{\left[\pi^A + (1-\pi^A)\pi^A\right](1-\theta)}{p_0^* - \theta}. \\
= \; & \frac{[1-(1-\pi^A)^2](1-\theta) - \pi^A(1-\pi^B)(1-\theta) + (1-\pi^A)^2\alpha}{\pi^B(1-\theta)} \\
> \; & \frac{[1-(1-\pi^A)^2](1-\theta) - \pi^A(1-\pi^B)(1-\theta)}{\pi^B(1-\theta)} = \pi^A + (1-\pi^A)\frac{\pi^A}{\pi^B},
\end{aligned}$$

where the equality follows from the definition of p_0^* in (7).

Next consider a type-A agent, which we refer to as an arbitrageur, and consider his incentive to save cash to date 1. If the state goes to u at $t = 1$, the arbitrageur cannot profit from his cash. If the state goes to d, he can potentially profit. He has three options. First, he could buy the asset from type-A agents who initially purchased with a debt contract with face value θ. To buy from these agents, the arbitragues would have to pay $p_d^{liq} = \alpha + E_d^A[\tilde{\theta}]$, which exceeds her valuation. Second, he could buy from type-A agents who initially purchased with a debt contract with face value K_d. These agents are distressed in the sense that they have collateralized all of their asset in exchange for K_d. At the same time, they incur a liquidation cost, α, from selling the asset at date 1. If instead they wait until date 2, then they incur the liquidation cost only if state dd is realized. Thus, they would be willing to sell the asset to the arbitraguer at a price:

$$p_d^{liq} = K_d - (1 - \pi^A)\alpha + \alpha.$$

Third, instead of buying the asset, the arbitrageurs could also refinance the debt contract of other optimists. This gives a payoff of K_d. The expected return to holding cash at date $t = 0$ is:

$$\pi^A + (1 - \pi^A)\frac{\pi^A(1-\theta)}{K_d - \theta} = \pi^A + (1 - \pi^A)\frac{\pi^A}{\pi^B}.$$

This shows that taking an asset position at $t = 0$ dominates saving cash.

Next consider the fraction of optimists, γ, that use debt with promise K_d. By market clearing, γ is determined as the solution to:

$$(1-\gamma)\frac{c}{p_0^* - \theta} + \gamma\frac{c}{p_0^* - K_d} = 1. \tag{8}$$

At the lower end of the region c_2, $\gamma = 0$, i.e., all optimists use short-term debt with promise K_d. Thus,

$$c_2 = p_0^* - K_d.$$

- Case 2: $c_1 \leq c < c_2$.

In this case, each optimist uses debt with promise K_d to finance his asset acquisition at $t = 0$, and the asset price is determined by the aggregate purchasing power of the optimists:

$$p_0 = c + K_d < p_0^*.$$

As the asset price is even lower than the previous case, the expected return to an optimist from taking a levered position with debt promise K_d is at least $\pi^A + (1 - \pi^A)\frac{\pi^A}{\pi^B}$. However, the expected return from saving cash is at most $\pi^A + (1 - \pi^A)\frac{\pi^A}{\pi^B}$. Thus, there is no incentive for any optimist to save cash at $t = 0$.

Once the optimists' cash endowment drops to a critical level c_1, the asset price becomes the pessimists' asset valuation: $E_0^B[\widetilde{R}]$. This determines c_1:

$$c_1 = E_0^B[\widetilde{R}] - K_d.$$

- Case 1: $c < c_1$.

In this case, each optimist acquires the asset by using debt with promise K_d, but their aggregate purchasing power is insufficient to maintain a level above the pessimists' valuation. The low price implies a high expected return, which makes it undesirable for any optimist to save cash at $t = 0$. This completes the proof of Proposition 5.