Arbitrageurs, bubbles, and credit conditions*

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Abstract

This paper studies a pure exchange economy populated by three types of agents: constrained agents who are subject to participation constraints, unconstrained agents who are only subject to a standard solvency constraint, and arbitrageurs who, in addition to being unconstrained, may incur transitory losses that are bounded by a state-dependent credit limit. We show that this credit possibility is valuable when there are asset pricing bubbles, which arise endogenously due to the presence of constrained agents, and that the bubble on the stock vanishes in the limit of infinite credit. In contrast to previous results in the literature, we show that the presence of risky arbitrage trading makes the stock more volatile and generates a leverage effect.

Keywords: Rational bubbles; Exchange economy; Leverage effect; Limits of arbitrage; Wealth constraints.

JEL Classification: D51, D52, G11, G12.

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1 Introduction

The existence of asset pricing bubbles in equilibrium models with continuous trading is closely related to the question: If the asset is overpriced, why is its price not corrected by short sales?. The answer has to do with trading constraints that limit the ability of investors to profit from the arbitrage opportunity that a bubble provides.

This is the case of the commonly used wealth constraints. These types of (solvency) constraints are effectively bounds on credit and were proposed in Harrison and Kreps (1979) and further explored in Dybvig and Huang (1988), as a mechanism to preclude doubling strategies[1]. A model of continuous trading in which agents prefer more to less requires some constraint to make doubling strategies infeasible, for otherwise there can be no equilibrium. Wealth constraints are useful in ruling out these strategies, but limit the scale at which an investor can sustain a strategy that would exploit the presence of bubbles[2], removing downward selling pressure on the mispriced assets, and making asset pricing bubbles compatible with equilibrium.

The goal of this paper is to characterize the role of wealth constraints in a model with endogenous mispricing. In particular, we are interested in understanding when and how they improve the capacity of the agent to exploit risky arbitrage strategies and what is their impact on equilibrium prices.

We answer these questions by building an exchange economy model with endogenous mispricing that nests Hugonnier (2010). That model considers a riskless asset in zero net supply, a dividend paying risky asset in positive supply and two types of price-taking agents. Constrained agents are subject to a participation constraint in the stock market which force them to keep a long position in the riskless asset, whereas unconstrained agents are free to choose the composition of their portfolio. Both types of agents face a standard nonnegativity constraint on wealth. We add a third class, 'arbitrageurs', who are unconstrained agents which benefit from credit conditions that allow them to withstand transitory marked-to-market losses, i.e., states with negative net worth. Trading in these states may be sustained through credit implicitly extended by the other agents in the economy. Hugonnier (2010) is a good starting point because when agents have logarithmic utility and face a standard nonnegativity constraint on wealth, the stock and the riskless asset contain bubbles, i.e., there is endogenous mispricing in both assets. The intuition

[1] Doubling strategies are essentially sequences of bets that win for sure in finite time by doubling up after a loss.

[2] An arbitrage strategy that exploits a bubble involves short selling the higher-cost asset and buying the lower-cost replicating portfolio. This strategy requires no initial wealth and provides positive payoffs, but may not be feasible at all scales due to the investor's wealth constraint.

for this result is best explained using the restricted stock market participation model of Basak and Cuoco (1998), a limiting case in Hugonnier (2010). This model features a constrained agent who cannot participate in the stock market, and an unconstrained agent, or stockholder, who holds a leveraged position that amounts to the whole supply of the risky security. The stockholder finances his equity position by borrowing from the non-stockholder, and this short position in the riskless bond has to be large enough to clear the market. In agreement with intuition, the interest rate decreases and the Sharpe ratio increases with respect to a frictionless economy, however, because these shifts are not sufficient to reach an equilibrium, the stock and the riskless asset must include bubbles to incite the stockholder to hold positions that are compatible with market clearing[3].

Our equilibrium model adds several insights that shed light on asset pricing puzzles and policy questions. In particular, we highlight the following results:

First, the ability of the arbitrageur to exploit risky arbitrages improves when the wealth constraint depends on the size of the market, that is, when the credit conditions are procyclical, in line with the empirical literature. We show that the mispricing on both assets decreases with improved credit conditions, to the extent that the bubble on the stock vanishes in the limit of infinite credit.

Second, in contrast with models that feature multiple agents with homogenous logarithmic utility and frictions[4], the stock volatility is strictly higher than the volatility of dividends in all states. This follows from the fact that the volatility of the price dividend ratio, which determines the volatility of the stock in excess of the volatility of dividends, is positive as a result of the arbitrageur's trading activity.

Third, the equity premium and the volatility are higher in bad times, generating a 'leverage effect', i.e., the well-established fact according to which volatility increases when the stock price falls. The model generates also a low interest rate. These results are of interest because it shows how, by introducing heterogeneity in the agents' credit conditions, a stylized model with logarithmic preferences and participation constraints may help in explaining empirical regularities.

Fourth, recent work by Fahri and Tirole (2010) left unanswered a question on whether authorities could regulate the likelihood and the size of bubbles by relaxing securitization (or collateralizability) standards. The model we propose sheds light on this question. In our model, arbitrageurs take advantage of improved credit conditions to better exploit

[3]The existence of bubbles in models with portfolio constraints is reminiscent of Caballero and Krishnamurthy (2009), who point to the emergence of bubbles as an equilibrium result due to asset shortages and portfolio imbalances.

[4]See Detemple and Murthy (1997), Basak and Cuoco (1998) and Basak and Croitoru (2000), among others.

mispriced assets. Therefore, bubble sizes depend negatively on the credit availability. Our analysis also explores the welfare implications of this form of policy intervention. In particular, we show that constrained and unconstrained agents may be made worse off by relaxing collateralizability standards as the arbitrageur's trading activity impact prices in a way that might be unfavorable to them.

There are only a few papers where arbitrage opportunities arise endogenously in equilibrium due the presence of portfolio constraints. We briefly review the most relevant for our work.

Gromb and Vayanos (2002) and Basak and Croitoru (2000) study economies where all agents are subject to portfolio constraints and, as a result, the ability of investors to benefit from the mispriced assets is limited. If these constraints are lifted for some agents, then these agents can scale their position to an arbitrary size and the presence of mispriced assets becomes inconsistent with the existence of an equilibrium. Gromb and Vayanos (2002) focus on the impact of arbitrageurs in welfare. They examine a different form of constraints which induce segmented markets. Without the arbitrageur, there is no trade across segmented markets; in the presence of the arbitrageur, who acts as an intermediary by running riskless arbitrages, trade occurs and a Pareto improvement takes place. In contrast, the arbitrageur in our model does not alleviate the portfolio constraints, quite the contrary, his trading activity may affect negatively the welfare of both unconstrained and constrained agents.

Basak and Croitoru (2006) build a production economy version of Basak and Croitoru (2000), and introduce a risk neutral arbitrageur with position constraints and zero net wealth, to show that his trading activity reduces the mispricing in securities, bringing prices closer to their fundamental values through costless and riskless trades, as in Gromb and Vayanos (2002). Our economy features a risk averse arbitrageur that also starts from a zero wealth position, however, the arbitrageur accumulates capital because not all arbitrage profits are consumed and importantly, his arbitrage trades are risky, in the sense that they may involve temporary losses prior to closure[5]. Moreover, in contrast to Basak and Croitoru (2006) where the production technology determines the stock price dynamics exogenously, rendering a flat stock volatility, we show that the credit conditions faced by the arbitrageur have important implications on the price dynamics, which are fully endogenous.

[5]The arbitrageur may need credit to close the arbitrage, which brings the results of this paper closer to the literature of limits of arbitrage. Unlike the model of Shleifer and Vishny (1997), in our model there are no outside investors or agency conflicts, yet, we build on the idea of the limited effectiveness of arbitrageurs in bringing prices closer to fundamental values. See Gromb and Vayanos (2010) for a recent survey of this literature.

The key contributions in the literature of equilibrium asset pricing bubbles in models with continuous trading have studied primarily the case of agents facing nonnegativity constraints on wealth. Loewenstein and Willard (2000a) show that, in complete-market frictionless economies, where agents are subject to this form of constraint, bubbles may exist on zero net supply securities, such as options and futures, but not on positive net supply securities such as stocks. Hugonnier (2010), as mentioned, shows that the presence of portfolio constraints can generate equilibrium pricing bubbles also on positive net supply securities, even if the economy includes unconstrained arbitrageurs that hold strategies that keep nonnegative wealth at all times. Prieto (2010) extends Hugonnier (2010) in a setting with risk aversion and beliefs heterogeneity. In addition to Loewenstein and Willard (2000a,b), Hugonnier (2010) and Prieto (2010), other papers have studied, mostly in partial equilibrium models, the properties of asset pricing bubbles in models with continuous trading. Cox and Hobson (2005) and Heston et al. (2007) study bubbles on the price of derivatives written on the stock and show that put-call parity might not hold and American calls have no optimal exercise policy. Jarrow et al. (2010) introduce regime shifts to show that a bubble on the stock can burst and be born in models with incomplete markets. An important difference with respect to these studies lies in the fact that they assume the existence of a risk neutral probability measure, in contrast, in our economy, the presence of a bubble on the riskless asset is equivalent to the non existence of a risk neutral probability measure.

The remainder of the paper is structured as follows. Section 2 present the main assumptions about the economy, the traded assets and the agents. Section 3 solves for the unique equilibrium in the economy. Section 4 discusses the main implications and insights of the model. Section 5 concludes. Proofs are gathered in the Appendix.

2 The model

2.1 Information structure

We consider a continuous time economy on an infinite horizon and assume that the uncertainty in the economy is represented by a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ that carries a standard Brownian motion Z. All random processes are assumed to be adapted with respect to the augmentation of the filtration $\mathbb{F} = (\mathcal{F}_t)_{t\geq 0}$ generated by the Brownian motion, and all statements involving random quantities are understood to hold either almost surely or almost everywhere depending on the context.

2.2 Securities markets

Agents trade in two securities: a locally riskless savings account in zero net supply and one risky asset, or stock, in positive supply of one unit. The price of the riskless asset evolves according to

$$S_{0t} = 1 + \int_0^t S_{0u} r_u du$$

for some short rate process $r \in \mathbb{R}$ that is to be determined in equilibrium. On the other hand, the stock is a claim to a dividend process δ that evolves according to a geometric Brownian motion,

$$\delta_t = \delta_0 + \int_0^t \delta_u (\mu_\delta du + \sigma_\delta dZ_u)$$

for some constant drift μ_δ and some constant volatility $\sigma_\delta > 0$. The stock price process is denoted by S and evolves according to

$$S_t + \int_0^t \delta_u du = S_0 + \int_0^t S_u (\mu_u du + \sigma_u dZ_u)$$

for some initial value $S_0 > 0$, some drift $\mu \in \mathbb{R}$ and some volatility $\sigma \in \mathbb{R}$ processes which are to be determined in equilibrium.

2.3 Trading strategies

A trading strategy is a pair of processes $(\pi; \phi)$ where π represents the amounts invested in the stock while ϕ represents the amount invested in the riskless asset. A trading strategy is said to be self-financing given initial wealth w and consumption rate c if the corresponding wealth process

$$W_t = W_t(\pi; \phi) \equiv \phi_t + \pi_t \tag{1}$$

satisfies the dynamic budget constraint

$$W_t = w + \int_0^t (\phi_u r_u + \pi_u \mu_u - c_u) \, du + \int_0^t \pi_u \sigma_u dZ_u. \tag{2}$$

Implicit in the definition is the requirement that the trading strategy be such that the above stochastic integrals are well-defined.

2.4 Agents

The economy is populated by three agents indexed by $k = 1, 2, 3$ who have homogenous beliefs about the state of the economy. The preferences of agent k are represented by

$$U_k(c) \equiv E\left[\int_0^\infty e^{-\rho t} \log(c_t) dt\right],$$

for some nonnegative discount rate $\rho > 0$ and we let $w_k \equiv \beta_k + \alpha_k S_0$ denote the initial wealth of agent k computed at equilibrium prices.

The three agents in the economy have homogenous preferences and beliefs but differ in their trading opportunities. Agent 1 is free to choose any self-financing strategy whose wealth is nonnegative and we will refer to him as the unconstrained agent. Agent 2, to whom we will refer as the constrained agent, is subject to the same requirement as agent 1, but must in addition choose a strategy that satisfies

$$\pi_t \in \mathcal{C}_t \equiv \{\pi \in \mathbb{R} : |\sigma_t \pi| \leq (1 - \varepsilon)\sigma_\delta W_t\}, \tag{3}$$

for some fixed constant $\varepsilon \in [0, 1]$. This constraint can be thought of as a stock market participation constraint, that limits the amount of risk the agent can take while trading, e.g., if $\sigma_t \geq \sigma_\delta$, this constraint forces agents to invest in the money market account. This constraint is also special case of the general risk constraints in Cuoco et al. (2008) and is also studied in Gârleanu and Pedersen (2007) and Prieto (2010), as a constraint on conditional value-at-risk.

The third agent is also free to choose any self-financing strategy but, in contrast to the two other agents, he is not required to maintain nonnegative wealth at all times. Instead, this agent is allowed to run short term deficits provided that

$$W_{3t} \geq -\psi S_t, \tag{4}$$

for some constant $\psi \geq 0$. This agent should be thought of as an arbitrageur whose funding liquidity conditions are determined by the parameter ψ. The fact that the amount of credit available to this arbitrageur increases with the size of the market captures in a simple way the observation that liquidity improves in times of where the stock market is high and dries up in bad times.

Since the agent is able to trade in states of negative wealth, the solvency constraint in (4) allows for excess borrowing. Trades in these states may be considered uncollateralized as the agent does not have enough assets to cover his liabilities in case of instantaneous liquidation. Note however that, given the preferences in this economy, the arbitrageur will never stop servicing debt or risk a freeze of debt rollover. The arbitrageur is thus balance sheet insolvent but not cash flow insolvent in these states.

To emphasize the interpretation of agent 3 as an arbitrageur we will from now on assume that $\alpha_3 = \beta_3 = 0$ so that his initial wealth is zero. This in turn implies that the initial endowments of the other agents can be summarized by the pair $(\alpha, \beta) = (\alpha_2, \beta_2)$ that describes the initial portfolio of the constrained agent. In what follows, we assume $\alpha \in [0, 1)$, so that both agents 1 and 2 start with a long position in the stock.

2.5 Equilibrium

The concept of equilibrium that we use is similar to that of equilibrium of plans, prices and expectations which was introduced by Radner (1972):

Definition 1. *An equilibrium is a pair of security price processes* (S, S_0) *and a set* $\{c_k, (\pi_k, \phi_k)\}_{k=1}^{3}$ *of consumption plans and trading strategies such that:*

1. *Given* (S, S_0) *the consumption plan* c_k *maximizes* U_k *over the feasible set of agent* k *and is financed by the trading strategy* (π_k, ϕ_k)*.*

2. *Markets clear:* $\phi_1 + \phi_2 + \phi_3 = 0$*,* $\pi_1 + \pi_2 + \pi_3 = S$ *and* $c_1 + c_2 + c_3 = \delta$*.*

An equilibrium is said to have arbitrage activity if the consumption plan of the arbitrageur is not identically zero.

Since the arbitrageur starts from zero wealth it might be that the set of consumption plans that he can finance is empty. In such cases, his consumption and optimal portfolio are identically equal to zero and the equilibrium only involves the two other agents. To determine conditions under which the arbitrageur participates it is necessary to characterize his feasible set. This is the issue to which we now turn.

2.6 Feasible sets and bubbles

Let (S, S_0) denote the securities prices in a given equilibrium and assume that there are no trivial arbitrage opportunities for otherwise the market could not be in equilibrium. As is well-known (see e.g., Duffie (2001)), this assumption implies that

$$\mu_t = r_t + \sigma_t \theta_t.$$

for some process θ such that $\int_0^T \theta_u^2 du < \infty$, $T \geq 0$. The process θ is referred to as the market price of risk and is uniquely defined on the set where the stock volatility is non zero. Now consider the state price density defined by

$$\xi_t = \frac{1}{S_{0t}} \exp\left(-\int_0^t \theta_u dZ_u - \frac{1}{2}\int_0^t |\theta_u|^2 du\right). \tag{5}$$

The following proposition shows that ξ can be used as a pricing kernel in order to characterize the feasible sets of agents 1 and 3 and allows to determine the conditions under which the arbitrageur participates in the market:

Proposition 1. *A consumption plan c is feasible for agent 1 if and only if*

$$E\left[\int_0^\infty \xi_t c_t dt\right] \leq w_1, \tag{6}$$

and feasible for agent 3 if and only if

$$E\left[\int_0^\infty \xi_t c_t dt\right] \leq \psi(S_0 - F_0(\delta)), \tag{7}$$

where

$$F_t(\delta) \equiv E_t\left[\int_t^\infty \frac{\xi_u}{\xi_t}\delta_u du\right]$$

gives the minimal amount that agent 1 needs to hold at time t in order to replicate the dividends of the stock while maintaining nonnegative wealth. In particular, the feasible set for agent 3 is empty unless $\psi > 0$ and $S_0 > F_0(\delta)$.

The quantity $F_t(\delta)$ corresponds to the fundamental value of the stock, and hence

$$B_t \equiv S_t - F_t(\delta) \tag{8}$$

is referred to as the stock's bubble[6]. The result in the above proposition says that the feasible set of the arbitrageur is empty unless there is a bubble on the stock and he has access to credit ($\psi > 0$). We will see that ψB_0 corresponds to the value of arbitrage profits generated by the arbitrageur through dynamic trading.

[6]The definitions of fundamental value and bubble have been used previously by Santos and Woodford (1997), Loewenstein and Willard (2000a, 2010) and Jarrow et al. (2010), among others.

In this model, bubbles may be consistent with optimal choice and with the existence of an equilibrium, because agents may not have enough collaterall[7] to implement arbitrage strategies fully. For example, assume the stock has a bubble and consider the strategy which sells short $n > 0$ units of the stock, buys the (minimal cost) portfolio that replicates the corresponding dividends and invests the remaining strictly positive amount in the riskless asset for any time $\tau > 0$. The wealth process associated with this arbitrage strategy is given by

$$W_{t,\tau}^{\text{arb}}(n) = n\left(F_{t,\tau}(\delta, S_\tau) - S_t + B_0(\tau)S_{0t}\right) = -nB_t(\tau) + nB_0(\tau)S_{0t}, \tag{9}$$

where

$$F_{t,\tau}(\delta, S_\tau) \equiv E_t\left[\int_t^\tau \frac{\xi_u}{\xi_t}\delta_u du + \frac{\xi_\tau}{\xi_t}S_\tau\right]$$

denotes the fundamental value of the stock's cash flows in $[t, \tau]$ and

$$B_t(\tau) \equiv S_t - F_{t,\tau}(\delta, S_\tau) \tag{10}$$

denotes the bubble in $[t, \tau]$.

The strategy in (9) requires no initial investment and has terminal value $nB_0(\tau)S_{0\tau} > 0$. This strategy could be implemented by the arbitrageur, if $W_{t,\tau}^{\text{arb}}(n)$ satisfies the solvency constraint in (4). However, it cannot be implemented on a stand alone basis by the unconstrained agent because it can take negative values with strictly positive probability, since the stock and its fundamental value may diverge further before they eventually converge at time τ. With a sufficiently large collateral position, on the other hand, the unconstrained agent can implement the above arbitrage trade. For example, assume that the unconstrained agent already holds one unit of the stock at the initial date and $n = 1$. The corresponding wealth process $W_{t,\tau}^{\text{arb}}(1) + S_t$ is then nonnegative at all times.

Bubbles may be defined on any security, including the riskless asset. Indeed, for any time $\tau > t$ the riskless asset can be viewed as a derivative security that pays a single lump dividend equal to $S_{0\tau}$ at time τ. The fundamental value of this derivative security is

$$F_{0t}(\tau) \equiv E_t\left[\frac{\xi_\tau S_{0\tau}}{\xi_t}\right],$$

[7]As in Detemple and Murthy (1997) and Detemple and Serrat (2003), we use the term collateral to describe the position in cash or in securities that allows to borrow or shortsell a security while satisfying a solvency constraint.

whereas the market value of this security is simply S_{0t}. This naturally leads to defining the riskless asset bubble as

$$B_{0t}(\tau) \equiv S_{0t} - F_{0t}(\tau). \tag{11}$$

As can be seen from equation (11), the riskless asset has a bubble over the horizon $[0, \tau]$ if and only if the nonnegative process

$$M_t \equiv \xi_t S_{0t}$$

satisfies $E[M_\tau] < 1$, for $\tau < \infty$. This process is the unique candidate for the density of the risk neutral probability measure and it follows that the existence of a bubble on the riskless asset is equivalent to the non existence of the risk neutral probability measure. See Loewenstein and Willard (2000a,b) and Heston et al. (2007).

3 Equilibrium

3.1 Individual optimality

The next proposition summarizes the optimal consumption and portfolio policies for all agents.

Proposition 2. *Assume that equilibrium prices are such that $B \neq 0$. The optimal consumption and trading strategies of the three agents are given by*

$$c_{kt} = \rho \left(W_{kt} + 1_{\{k=3\}} \psi B_t \right) \tag{12}$$

and

$$\pi_{1t} = (\theta_t/\sigma_t) W_{1t}, \tag{13}$$

$$\pi_{2t} = (\theta_t/\sigma_t) \kappa_t W_{2t}, \tag{14}$$

$$\pi_{3t} = (\theta_t/\sigma_t)(W_{3t} + \psi B_t) - \psi(\Sigma_t^B/\sigma_t), \tag{15}$$

where

$$\kappa_t = \min \left(1; \frac{(1-\varepsilon)\sigma_\delta}{|\theta_t|} \right),$$

and the process Σ^B denotes the diffusion coefficient of the process B_t.

Note that agents 1 and 2 consume a fraction ρ of their wealth, whereas agent 3 follows a similar consumption policy, but over what we refer to as his effective total wealth $W_{3t}+\psi B_t$. The fact that they all consume the same fraction over (effective) wealth follows from the assumption of homogenous time discount rates and logarithmic preferences.

As for portfolio policies, the solution for agent 1 is standard, as he chooses the mean-variance efficient portfolio. The solution for agent 2 follows from the results of Cvitanić and Karatzas (1992) and shows that the constraint binds in those states where the market price of risk is high. Note that the process $\kappa \in [0, 1)$ encapsulates the impact of the portfolio constraint in (3), as it scales down[8] his position in the mean variance efficient portfolio.

Finally, the solution for agent 3 is novel, and shows that his position on the stock is composed of two parts[9]: a mean-variance efficient position over his effective wealth, and a position which is proportional to the credit conditions (as ψ controls the size of the arbitrage profits) and whose sign depends on the correlation of the stock price with B_t. Notice that agent 3 never exhausts the borrowing capacity described in (4), since $W_{3t} + \psi B_t > 0$ and $S_t > B_t$.

3.2 Equilibrium price system

The equilibrium is characterized using a representative agent with stochastic weights that allows to easily account for the market clearing conditions, despite the imperfect risk sharing induced by the presence of the constrained agent (see Cuoco and He (1994)). The utility function of this representative agent is defined by

$$u(c, \gamma, \lambda_t) \equiv \max_{c_1+c_2+c_3=c} \{\log(c_1) + \lambda_t \log(c_2) + \gamma \log(c_3)\},$$

where $\lambda_t > 0$ is an endogenously determined weighting process that encapsulates the differences across the agents' investment opportunity sets and $\gamma \geq 0$ is a nonnegative constant that determines the relative weight of arbitrageurs in the economy.

[8] See Prieto (2010) for details.

[9] The position of the arbitrageur bears resemblance to the optimal portfolio policy of an agent with logarithmic preferences endowed with marketable income process e in a complete market setting. This agent's optimal wealth process is determined by $W_t + F_t(e) = c_t/\rho$, where $F_t(e) \equiv E_t\left[\int_t^\infty \frac{\xi_u}{\xi_t} e_u du\right]$, and hence, his stock position is $\pi_t = (\theta_t/\sigma_t)(W_t + F_t(e)) - (\Sigma_t^{F_e}/\sigma_t)$, where Σ^{F_e} denotes the diffusion coefficient of the process $F_t(e)$.

Relying on the result of Proposition 2, we have that the first order conditions of optimality for agents 1 and 3 are given by

$$e^{-\rho t}\frac{c_{k0}}{c_{kt}} = \xi_t, \qquad k = 1, 3.$$

Comparing these first order conditions to those of the representative agent's problem shows that state price density and the equilibrium allocation are given by

$$\xi_t = e^{-\rho t}\frac{u_c(\delta_t, \gamma, \lambda_t)}{u_c(\delta_0, \gamma, \lambda_0)} = e^{-\rho t}\frac{\delta_0(1+\gamma+\lambda_t)}{\delta_t(1+\gamma+\lambda_0)},$$

and

$$\begin{aligned} c_{2t} &= s_t\delta_t, \\ c_{1t} &= \frac{1}{1+\gamma}(1-s_t)\delta_t, \\ c_{3t} &= \delta_t - c_{1t} - c_{2t} = \frac{\gamma}{1+\gamma}(1-s_t)\delta_t, \end{aligned}$$

where the process

$$s_t \equiv \frac{c_{2t}}{\delta_t} = \frac{\lambda_t}{1+\gamma+\lambda_t} \in (0,1) \tag{16}$$

represents the equilibrium consumption share of the constrained agent. In order to determine the dynamics of this process, let us assume that

$$ds_t = m_t dt + v_t dZ_t$$

for some adapted processes m and v. Applying Itô's lemma to the definition of the state price density and comparing the result to (5) shows that the market price of risk and the interest rate are given by

$$\theta_t = \sigma_\delta - \frac{v_t}{1-s_t}, \tag{17}$$

and

$$r_t = \rho + \mu_\delta - \sigma_\delta^2 + \frac{\sigma_\delta v_t - m_t}{1-s_t} - \left(\frac{v_t}{1-s_t}\right)^2. \tag{18}$$

On other hand, Proposition 2 shows that along the optimal path the wealth of agent 2 is explicitly given by $W_{2t} = \frac{\delta_t}{\rho} s_t$. Applying Itô's lemma to this expression, and comparing the result with the dynamic budget constraint (2), shows that the drift and volatility of the consumption share are related by

$$m_t + \frac{v_t^2}{1 - s_t} = 0,$$

and that the optimal portfolio of the constrained agent is given by

$$\sigma_t \pi_{2t} = W_{2t} \left(\sigma_\delta + \frac{v_t}{s_t} \right).$$

Plugging the expression for the market price of risk into equation (14) and comparing the result with the above expression shows that

$$\left(\sigma_\delta + \frac{v_t}{s_t} \right) \max \left\{ 1; \frac{|\sigma_\delta (1 - s_t) - v_t|}{(1 - \varepsilon)(1 - s_t)\sigma_\delta} \right\} = \sigma_\delta - \frac{v_t}{1 - s_t}.$$

Solving that nonlinear equation gives an explicit expression for the volatility of the consumption share process, and plugging this explicit solution back into equations (17) and (18), delivers the following characterization of equilibrium.

Proposition 3. *In equilibrium, the riskless rate of interest and the market price of risk are explicitly given by*

$$\theta_t = \sigma_\delta \left(1 + \frac{\varepsilon s_t}{1 - s_t} \right), \tag{19}$$

$$r_t = \rho + \mu_\delta - \sigma_\delta \theta_t = \rho + \mu_\delta - \sigma_\delta^2 \left(1 + \frac{\varepsilon s_t}{1 - s_t} \right), \tag{20}$$

and the consumption share of the constrained agent evolves according to

$$ds_t = -s_t \varepsilon \sigma_\delta \left(dZ_t + \frac{s_t}{1 - s_t} \varepsilon \sigma_\delta dt \right) \tag{21}$$

with initial condition $s_0 = \rho w_2 / \delta_0$.

The equilibrium is notable for two reasons. First, the participation constraint modifies the unconstrained equilibrium ($\varepsilon = 0$) and always implies a higher market price of risk and a lower interest rate. These shifts will induce agent 1 and 3 to increase their demands in the risky asset through borrowing from the constrained agent. Second, the share of the constrained agent in (21) is negatively correlated with the dividend process, a result that follows from the portfolio constraint. Since the constraint limits agent 2's position in the stock, during good times (a sequence of positive shocks) his consumption share tends to go down, whereas during bad times (a sequence of negative shocks), his consumption share tends to go up. These dynamics generate a countercyclical market price of risk since this quantity increases with s.

To compute the equilibrium price of the stock, we rely on the financial market clearing conditions which require

$$S_t = \sum_{k=1}^{3} W_{kt}.$$

This expression in conjunction with (12) and the clearing of the consumption good market imply

$$S_t = \sum_{k=1}^{3} W_{kt} = \frac{\delta_t}{\rho} - \psi(S_t - F_t(\delta)).$$

Combining this expression with the definition of the stock bubble finally shows that the price of the stock and its fundamental value are given by

$$S_t = \frac{1}{1+\psi}\left(\frac{\delta_t}{\rho} + \psi F_t(\delta)\right) \tag{22}$$

and

$$F_t(\delta) = E_t\left[\int_t^\infty \frac{\xi_u}{\xi_t}\delta_u du\right] = \delta_t(1-s_t)E_t\left[\int_t^\infty e^{-\rho(u-t)}\frac{du}{1-s_u}\right].$$

In order to complete the description of the equilibrium, it remains to determine whether the stock price includes a bubble component or not. Using the definition of the bubble together with the above expression, and the relation between the consumption share and the weighting process in (16), we obtain

$$B_t = S_t - F_t(\delta) = \frac{1}{1+\psi}\left(\frac{\delta_t}{\rho} - F_t(\delta)\right) = \frac{\delta_t}{1+\psi}E_t\left[\int_t^\infty e^{-\rho(u-t)}\left(\frac{\lambda_t - \lambda_u}{1+\gamma+\lambda_t}\right)du\right]$$

and it follows that the stock price is bubble free if and only if the weighting process is a martingale. Applying Itô's lemma to the definition of the weighting process gives

$$d\lambda_t = (1+\gamma)d\left(\frac{s_t}{1-s_t}\right) = -\lambda_t(1+\gamma+\lambda_t)\frac{\varepsilon\sigma_\delta}{1+\gamma}dZ_t,$$

so that the weighting process is a local martingale. However, the following proposition shows that this local martingale fails to be a true martingale and thereby proves that any equilibrium has arbitrage activity.

Proposition 4. *The weighting process is a strict local martingale and, a result, the price of the stock includes a strictly positive bubble component in any equilibrium.*

The following theorem provides closed form expressions for the stock price and the bubble on the stock, as well as parametric conditions for existence and uniqueness of equilibrium.

Theorem 1. *Let*

$$g(s) \equiv \beta + \frac{\delta_0}{\rho}\alpha \left(1 - \frac{\psi}{1+\psi} s^{\eta}\right) - \frac{s\delta_0}{\rho}, \tag{23}$$

where

$$\eta \equiv \frac{1}{2}\left(1 + \sqrt{1 + 8\rho/(\varepsilon\sigma_\delta)^2}\right),$$

and assume that the parameters of the model are such that

$$-\frac{\alpha\delta_0}{\rho} < \beta < \frac{\delta_0(1-\alpha)}{\rho(1+\psi)}. \tag{24}$$

There exists a unique equilibrium with arbitrage activity in which the price, the fundamental value and the bubble of the stock are given by

$$S_t = \frac{\delta_t}{\rho}\left(1 - \frac{\psi}{1+\psi} s_t^{\eta}\right) \tag{25}$$

$$F_t(\delta) = \frac{\delta_t}{\rho}\left(1 - s_t^{\eta}\right) \tag{26}$$

$$B_t = \frac{\delta_t s_t^{\eta}}{\rho(1+\psi)} \tag{27}$$

where the consumption share process s_t evolves according to equation (21) with initial condition $s_0 = s^*$, where $s^* \in (0,1)$ solves $g(s^*) = 0$.

The stock price satisfies $S_t = \delta_t/\rho - \psi B_t$, which shows that its equilibrium level is lower than the price level in an economy with no bubbles or in an economy where only agents 1 and 2 trade. We will back to this issue in the analysis, but notice that the fundamental value is not independent of the stock bubble since state prices depend on the initial distribution of wealth across agents (unless $\alpha = 0$)[10]. Furthermore, not only the stock contains a bubble, but also the riskless asset is mispriced, as we see next.

Proposition 5. *The finite horizon bubbles on the stock and on the riskless asset are given by*

$$B_t(\tau) = \frac{\delta_t}{\rho(1+\psi)} H\left(\tau - t, s_t; 2\eta - 1\right), \tag{28}$$

$$B_{0t}(\tau) = S_{0t} s_t^{-1/\varepsilon} H\left(\tau - t, s_t; 2/\varepsilon - 1\right), \tag{29}$$

where

$$H(\tau, s; \alpha) \equiv s^{\frac{1+\alpha}{2}} N(d_+(\tau, s; \alpha)) + s^{\frac{1-\alpha}{2}} N(d_-(\tau, s; \alpha)),$$

$$d_\pm(\tau, s; \alpha) \equiv \frac{1}{\varepsilon\sigma_\delta\sqrt{\tau}} \log(s) \pm \frac{\alpha}{2}\varepsilon\sigma_\delta\sqrt{\tau},$$

and $N(\cdot)$ denotes the standard normal cumulative distribution function.

We stress the fact that the emergence of the bubble component in the price process responds to a clear equilibrium mechanism: shifts in the interest rate and market price of risk go on the right direction but are not sufficient to reach an equilibrium, so bubbles arise to incite agents 1 and 3 to hold positions that are compatible with market clearing.

4 Analysis

In this section, we show that the mispriced assets are brought closer to their fundamental values as the availability of credit grows, and that the arbitrageur's trading activity generates excess volatility and a 'leverage effect'. We also explore welfare implications.

4.1 Portfolio strategies

Agent 2 is forced by the participation constraint to scale down his position in the risky asset and to hold a positive position in the riskless asset, becoming thus a net lender in equilibrium, as we see next.

[10] See also Weil (1990) for an economy in which the fundamental value depends on the size of the bubble and hence, the price level may be lower than the price level with no bubbles.

Corollary 1. *Agent 1 and 2's positions in the stock are given by*

$$\pi_{1t}/W_{1t} = \frac{1+\varepsilon\frac{s_{2t}}{1-s_{2t}}}{1+\frac{\psi\eta\varepsilon s_{2t}^{\eta}}{1+\psi(1-s_{2t}^{\eta})}} > 1,$$
$$\pi_{2t}/W_{2t} = \frac{1-\varepsilon}{1+\frac{\psi\eta\varepsilon s_{t}^{\eta}}{1+\psi(1-s_{t}^{\eta})}} \in [0,1].$$

Figure 1 depicts how agent 2's lending position is offset by the borrowing positions of agents 1 and 3, who lever up their positions in the stock.

Insert Figure 1 here

Agent 1's portfolio can be understood as a dynamic portfolio that uses the stock as collateral and exploits the bubble on the riskless asset.

Proposition 6. *The wealth of the unconstrained agent expressed as positions in the stock and in the riskless asset's bubble is given by*

$$W_{1t} = \phi_{1t}^{S}(\tau) + \phi_{1t}^{B_0}(\tau), \tag{30}$$

where

$$\phi_{1t}^{S}(\tau) = \frac{\delta_t(1+\psi)\left(\left(\sigma_\delta - \Sigma_{t,\tau}^{0}\right)(1-s_t) + \varepsilon\sigma_\delta s_t\right)}{\rho(\sigma_t - \Sigma_{t,\tau}^{0})(1+\gamma)} > 0 \tag{31}$$
$$\phi_{1t}^{B_0}(\tau) = -\frac{\delta_t\left(\left(\sigma_\delta - \sigma_t\right)(1-s_t) + \varepsilon\sigma_\delta s_t\right)}{\rho(\sigma_t - \Sigma_{t,\tau}^{0})(1+\gamma)} < 0 \tag{32}$$

and the process

$$\Sigma_{t,\tau}^{0} = -\frac{\varepsilon\sigma_\delta s_t \partial h_{0t}(\tau)/\partial s}{h_{0t}(\tau)} < 0,$$

with $h_{0t}(\tau) \equiv s_t^{-1/\varepsilon} H\left(\tau - t, s_t; 2/\varepsilon - 1\right)$*, corresponds to the diffusion coefficient of the process* $\log B_{0t}(\tau)$*.*

The strategy in (30) is indexed by the duration of the strategy, $[t,\tau]$. This representation converges to the standard representation in (1) as $\tau \to \infty$[11]. It is notable that even though the stock also contains a bubble component, the unconstrained agent holds a

[11] The fundamental value of the riskless asset converges to zero as $\tau \to \infty$, which implies that the money market account in this economy is akin to fiat money, $\lim_{\tau\to\infty} B_{0t}(\tau) = S_{0t}$.

long position in the stock, $\phi^S > 0$, and a short position in the riskless asset's bubble, $\phi^{B_0} < 0$. To understand this, notice that he exploits the bubble on the riskless asset because it requires less collateral per unit of initial profit.

Corollary 2. *Bubbles are such that*

$$\frac{B_{0t}(\tau)}{S_{0t}} \geq \frac{B_t(\tau)}{S_t}, \qquad \text{for } \tau \geq t.$$

On the other hand, the arbitrageur's strategy,

$$W_{3t} = \gamma W_{1t} - \psi B_t,$$

mimics agent 1's strategy while financing it by shorting the bubble in the stock, up to the scale of his credit limit. As shown in Figure 1, panels (b) and (d), credit availability allows the arbitrageur to be more aggressive than agent 1 in his arbitrage strategy, particularly in bad times, where the bubble sizes are larger.

4.2 Price level and bubble sizes

The stock price in (22) is a convex combination of the fundamental value of the stock and the stock price in the absence of the arbitrageur,

$$S_t = \frac{1}{1+\psi}\frac{\delta_t}{\rho} + \frac{\psi}{1+\psi}F_t(\delta).$$

As $\psi \to 0$, the stock price converges to δ_t/ρ, which is the price in an economy where only agents 1 and 2 trade. As $\psi \to \infty$, the stock price converges to its fundamental value and the bubble on the stock in (27) vanishes.

This result is due to a decreased value of the collateral services provided by the stock as credit conditions improve. Recall that $W_{3t} + \psi B_t = c_{3t}/\rho$ and $S_t = \delta_t/\rho - \psi B_t$, and note that $\frac{\partial}{\partial \psi}\psi B_t > 0$, which means that the arbitrageur may hold a much larger short position in the riskless asset relative to his position in the stock as $\psi \uparrow$, which coupled with the positions of agents 1 and 2, amount to a lower level required to clear the stock market. Overall, credit improvements for the arbitrageur ($\psi \uparrow$) translate into a smaller stock bubble and may also imply a larger fundamental value.

As seen in Figure 2, level effects are not only restricted to the stock's bubble. The bubble on the riskless asset also decreases with a credit improvement (note that is straightforward to show $\partial h_{0t}(\tau)/\partial \psi < 0$.)

Insert Figure 2 here

The downward move comes through a shift in the consumption share process, as the starting point of the consumption share of the constrained agent depends negatively on the credit conditions and the path of s depends monotonically on its starting point.

4.3 Volatility and the leverage effect

In contrast to other models with frictions and homogenous agents with logarithmic utility[12], the price dividend ratio in (25) is a decreasing function of the consumption share of the constrained agent (as a result of the arbitrageur's trading activity), and since the consumption share of the constrained agent is negatively correlated with dividends, the stock volatility is higher than the volatility of dividends in all states.

Corollary 3. *The volatility of the stock is given by*

$$\sigma_t = \sigma_\delta + \sigma_\delta \frac{\psi \eta \varepsilon s_t^\eta}{1 + \psi(1 - s_t^\eta)}. \tag{33}$$

The second term in (33) corresponds to what is commonly referred to as the excess volatility component. This term is a positive and increasing function of the consumption share, an observation which coupled with the fact that the price dividend ratio is a decreasing function of the consumption share, implies that the model generates the so-called 'leverage effect', i.e., the well-established stylized fact according to which volatility increases when the stock price falls[13].

4.4 Welfare

We analyze the impact of an improvement on credit conditions on the agents' welfare. The result is summarized in the following proposition.

Proposition 7. *The arbitrageur benefits from an improvement of the credit conditions ($\psi \uparrow$) at the expense of possibly both agents 1 and 2.*

[12]See for example Detemple and Murthy (1997), Basak and Cuoco (1998), Basak and Croitoru (2000) and Hugonnier (2010).

[13]See Schwert (1989) and Mele (2007) for evidence on the asymmetric nature of volatility and the cyclical behavior of stock prices.

The expected utility of the arbitrageur

$$U_3(\psi) \equiv E\left[\int_0^\infty e^{-\rho t}\log(c_{3t})dt\right]$$
$$= U_0 + \rho^{-1}\log\left(\frac{\gamma}{1+\gamma}\right) + E\left[\int_0^\infty e^{-\rho t}\log(1-s_t)dt\right], \tag{34}$$

where $U_0 \equiv E\left[\int_0^\infty e^{-\rho t}\log(\delta_t)dt\right]$, reveals that the result in Proposition 7 follows from the fact that changes in ψ impact the consumption sharing rules (γ, s) through price shifts. In particular, the second term in (34) is increasing in ψ, since the relative weight of the arbitrageur,

$$\gamma = \frac{\delta_0 - \rho S_0}{\rho(S_0 - w_2)}$$

increases as the trading activity of the arbitrageur reduces the stock price. The third term in (34) is also increasing in ψ because the starting point of the consumption share is decreasing in the credit conditions and the path of s depends monotonically on its starting point. The latter means that agent 2's welfare is decreasing in ψ.

A similar analysis shows that the welfare of agent 1, represented by

$$U_1(\psi) \equiv E\left[\int_0^\infty e^{-\rho t}\log(c_{1t})dt\right]$$
$$= U_0 - \rho^{-1}\log(1+\gamma) + E\left[\int_0^\infty e^{-\rho t}\log(1-s_t)dt\right]. \tag{35}$$

may be negatively affected by an improvement of the credit conditions. The second term is decreasing in ψ, since better credit conditions reduce the level of the stock and hence his initial wealth. Credit improvements also increase the stock's volatility. The latter effect is detrimental for agent 1, because the stock's 'collateral quality' worsens from the unconstrained agent's standpoint. Indeed, Figure 3 shows that the unconstrained agent drastically reduces his arbitrage position per unit of collateral as $\psi \uparrow$.

Insert Figure 3 here

The negative wealth effect caused by an improvement in credit conditions may be compensated by a decrease in the share of consumption of the constrained agent, which increases the third term in (35). Note however that if the initial endowment of agent 2 does not depend on the stock ($\alpha = 0$), credit conditions do not have welfare implications for the constrained agent, as the process s is independent of ψ. In this case, an increase in the availability of credit necessarily implies a welfare loss for agent 1, as it will impact the stock price, but not the interest rate.

4.5 The composition of the credit bound

In this section, we show that an equilibrium may fail to exist when the wealth constraint depends also on the riskless asset, whereas if the wealth constraint depends on fundamental values, the equilibrium prices correspond to those found in an economy where only agents 1 and 2 trade.

4.5.1 Failure of equilibrium

Assume that the arbitrageur faces a wealth constraint given by

$$W_{3t} \geq -\left(\psi S_t + \ell S_{0t}\right). \tag{36}$$

for some constants $(\psi, \ell) \in \mathbb{R}_+^2$. Observe that when $\psi = 0$, the bound depends on the evolution of the short rate only[14] and requires that temporary losses be bounded by ℓ in discounted terms. Given the wealth constraint in (36), feasible plans satisfy the static budget constraint

$$E\left[\int_0^\infty \xi_t c_{3t} dt\right] \leq \psi B_0 + \ell,$$

where the second term in the right hand side corresponds to $\lim_{\tau\to\infty} \ell B_{0t}(\tau)$. This term provides the value of the arbitrage profits from the bubble on the riskless asset.

The candidate stock price now depends linearly on the price of the riskless asset

$$S_t = F_t(\delta) + \frac{1}{1+\psi} E_t\left[\int_t^\infty e^{-\rho(u-t)} \frac{\lambda_t - \lambda_u}{1+\gamma+\lambda_t} du\right] - \frac{\ell}{1+\psi} S_{0t}, \tag{37}$$

and contains a negative quantity. Since a viable model requires prices to be strictly positive, the individual components in (37) must be characterized in order to assess whether the stock price is uniformly bounded from below by zero. We use the equilibrium interest rate and the dynamics of the consumption share of the constrained agent to

[14] See Loewenstein and Willard (2000a) for a similar wealth constraint.

compute the price of the riskless asset,

$$S_{0t} = e^{\left(\rho - \sigma_\delta^2/2(1-\varepsilon)\right)t} \frac{\delta_t s_t^{1/\varepsilon}}{\delta_0 s_0^{1/\varepsilon}},$$

so that the candidate price process is given by

$$S_t = \frac{\delta_t}{\rho}\left(1 - \frac{\psi}{1+\psi} s_t^\eta\right) - \frac{\ell}{1+\psi} e^{\left(\rho - \sigma_\delta^2/2(1-\varepsilon)\right)t} \frac{\delta_t s_t^{1/\varepsilon}}{\delta_0 s_0^{1/\varepsilon}}.$$

The process above is strictly negative with strictly positive probability, since $s_t \in (0, 1)$. The equilibrium fails to exist because the size of the arbitrage induced by the solvency constraint in (36), in particular, the arbitrage that exploits the bubble on the riskless asset, is too big to be supported by a viable price system.

4.5.2 Wealth constraints with fundamental values

In order for the arbitrageur to participate in the market, the wealth constraint must contain assets with bubbles. To illustrate this observation, we take a wealth constraint that moves in fixed proportion to the some cash flows' fundamental value,

$$W_{3t} \geq -\psi F_t(\nu) \tag{38}$$

for some cash flow stream $\nu \geq 0$. The expression in (38) is a valid benchmark, since the fundamental value of a stream of cash flows ν is composed by self financing strategies of the stock and the riskless asset. Simple computations reveal, however, that the arbitrageur's feasible set is empty.

5 Concluding remarks

This paper studies a pure exchange economy populated by three types of agents: constrained agents who are subject to participation constraints, unconstrained agents who are only subject to a standard nonnegativity constraint on wealth, and arbitrageurs who, in addition to being unconstrained, may incur transitory losses that are bounded by a state-dependent credit limit.

We show that this credit condition is valuable when it fluctuates with the size of the market and there are asset pricing bubbles, which arise endogenously in our model due to the presence of constrained agents. The size of the bubbles depend, primarily, on the ability of the arbitrageur to exploit the mispriced assets, and hence, on the credit conditions. The equilibrium with risky arbitrage activity is able to generate excess volatility and the 'leverage effect', as well as a countercyclical market price of risk and low interest rates.

We also explore the implications of policy interventions directed at deflating bubbles, such as those based on relaxing collateralizability standards for arbitrageurs. In particular, we show that constrained and unconstrained agents may be made worse off as the arbitrageur's increased trading activity impact prices in a way that can be unfavorable to them.

A Proofs

Proof of Proposition 1. The static budget constraint in (6) is a well known result (see e.g., Duffie (2001), Chapter 9.E.). For an arbitrary consumption and investment plan, the deflated wealth process of the arbitrageur is

$$\xi_t W_{3t} + \int_0^t \xi_u c_{3u} du = \int_0^t \xi_s \left(\pi_{3u}\sigma_u - W_{3u}\theta_u\right) dZ_u.$$

The deflated stock and riskless asset price processes satisfy

$$\xi_t S_t + \int_0^t \xi_u \delta_u du = S_0 + \int_0^t \xi_u \left(S_u \sigma_u - S_u \theta_u\right) dZ_u,$$
$$\xi_t S_{0t} = 1 - \int_0^t \xi_u S_{0s} \theta_u dZ_u.$$

Let N_t be defined by

$$\begin{aligned} N_t &= \xi_t W_{3t} + \psi \xi_t S_t + \ell \xi_t S_{0t} + \int_0^t \xi_u (c_{3u} + \psi \delta_u) du \\ &= \psi S_0 + \ell + \int_0^t \xi_u \left((\pi_{3u} + \psi S_u)\sigma_u - (\psi S_u + W_{3u} + \ell S_{0u})\theta_u\right) dZ_u \end{aligned} \tag{39}$$

with $\psi \geq 0$ and $\ell \geq 0$. N_t is a nonnegative local martingale for positive consumption plans, and hence a supermartingale, since the price system (S_{0t}, S_t) is nonnegative. This implies that

$$E\left[\int_0^T \xi_t (c_{3t} + \psi \delta_t) dt + \ell \xi_T S_{0T}\right] \leq \psi S_0 + \ell.$$

The static budget constraint in (7) follows by setting $\ell = 0$ and letting $T \to \infty$.

■

Proof of Proposition 2. The optimal policies of the unconstrained agent in (12) and (13) are a well known result (see e.g., Duffie (2001), Chapter 9.E and Karatzas and Shreve (1998), p.32).

The optimal policies of the constrained agent follow from Cvitanić and Karatzas (1992). Agent 2 faces an implicit state price representation given by

$$\xi_{2t} = e^{-\int_0^t \left(r_u + \beta_u + \frac{1}{2}\theta_{2u}^2\right) ds - \int_0^t \theta_{2u} dZ_u},$$

where $\theta_{2t} = \theta_t + \sigma_t^{-1}\nu_t$. $\beta_t(\nu)$ is the support function of the set $-\mathcal{C}_t$. Optimality conditions imply that ν is defined by the relation

$$\nu_t = \arg\min_{\nu\in\mathcal{B}_t}\left\{\frac{1}{2}\left(\theta_t + \sigma_t^{-1}\nu_t\right)^2 + \beta_t(\nu)\right\},$$

where $\mathcal{B}$ is the set of points where the support function is finite. Using the definition of the support function and Fenchel's duality theorem (see Rockafellar (1996), Theorem 31.1), the problem above is transformed into a mean-variance program given by

$$\sup_{\hat{\pi}\in\mathcal{C}_t}\left\{\hat{\pi}\sigma_t\theta_t - \frac{1}{2}\left(\sigma_t^\top\hat{\pi}\right)^2\right\}, \tag{40}$$

Since $\mathcal{C}_t$ is a closed convex subset of $\mathbb{R}$, the problem admits a unique solution given by

$$\sigma_t\hat{\pi}_t = \Pi\left[\theta_t|\,\sigma_t\mathcal{C}_t\right] = \kappa_t\theta_t, \qquad \kappa_t = \min\left(1;\frac{(1-\varepsilon)\sigma_\delta}{|\theta_t|}\right)$$

where Π denotes the projection operator. The projection operator satisfies

$$\left(\sigma_t\varpi - \sigma_t\hat{\pi}_t\right)\left(\sigma_t\hat{\pi}_t - \theta_t\right) \leq 0$$

(see Hiriart-Urruty and Lemaréchal (2001), Theorem 3.1.1). Taking the maximum on the left hand side gives $\max_{\varpi\in\mathcal{C}_t}\left\{\left(\varpi - \hat{\pi}_t\right)y_t\right\} = 0$, where $y_t = \sigma_t\left(-\theta_t + \sigma_t\hat{\pi}_t\right)$. In conjunction with the definition of the support function, this implies that the vector $y_t \in \mathcal{B}_t$. Note that the vector y_t attains the infimum on the left hand side of equation (40) and it follows that $\nu_t = y_t = -(1-\kappa_t)\sigma_t\theta_t$.

The optimal policy of the arbitrageur is derived as follows. The consumption process $c_{3t} = (e^{\rho t}y_3\xi_t)^{-1}$ is optimal iff there exists a constant $y_3 > 0$ that satisfies

$$E\left[\int_0^\infty \xi_t(e^{\rho t}y_3\xi_t)^{-1}dt\right] = \psi\left(S_0 - E\left[\int_0^\infty \xi_t\delta_t dt\right]\right).$$

The wealth process implied in (12) follows from the fact that N_t in (39) evaluated at the optimal is a true martingale,

$$\xi_t W_{3t} + \psi\xi_t S_t + \int_0^t \xi_u(c_{3u} + \psi\delta_u)du = E_t\left[\int_0^\infty \xi_u(c_{3u} + \psi\delta_u)du\right],$$

so that

$$W_{3t} = \frac{c_{3t}}{\rho} - \psi B_t. \tag{41}$$

Applying Itô's lemma to (41) and matching terms with the process in (2), gives (15).

■

Proof of Proposition 3. The marginal utility of the representative agent is identified from the clearing condition

$$\frac{1}{u_c(\delta_t, \lambda_t)} + \frac{\gamma}{u_c(\delta_t, \lambda_t)} + \frac{\lambda_t}{u_c(\delta_t, \lambda_t)} = \delta_t,$$

so that $u_c(\delta_t, \lambda_t) = \frac{1+\gamma+\lambda_t}{\delta_t}$. The constants (y_3, γ, λ_0) are given by

$$\begin{aligned} y_3 &= (\delta_0 - \rho(w_1 + w_2))^{-1}, \\ \gamma &= (\delta_0 - \rho(w_1 + w_2))/\rho w_1, \\ \lambda_0 &= w_2/w_1. \end{aligned}$$

The procedure to compute the interest rate and the market price of risk in (19) and (20) is described in the text.

■

Proof of Proposition 4. Applying Itô's lemma to the weighting process λ gives

$$\frac{d\lambda_t}{\lambda_t} = [\beta_t(\nu) + \theta_{2t}(\theta_{2t} - \theta_t)]\, dt + (\theta_{2t} - \theta_t)\, dZ_t. \tag{42}$$

From the proof of Proposition 2, $\sigma_t \hat{\pi}_{2t} = \theta_{2t}, \theta_{2t} = \kappa_t \theta_t, \nu_t = -(1-\kappa_t)\sigma_t \theta_t$. and replacing in the drift of equation (42)

$$\beta_t(\nu) + \theta_{2t}(\theta_{2t} - \theta_t) = \kappa_t(1-\kappa_t)(\sigma_t^{-1}\theta_t)\sigma_t\theta_t - \kappa_t(1-\kappa_t)\theta_t\theta_t = 0$$

The process λ is a local martingale, whose dynamic follow

$$\begin{aligned} \lambda_t &= \lambda_0 + \int_0^t \lambda_u \theta_u (\kappa_u - 1)\, dZ_u \\ &= \lambda_0 + \int_0^t \lambda_u \left[(1-\varepsilon)\sigma_\delta - \sigma_\delta \frac{1-(1-\varepsilon)s_u}{1-s_u} \right] dZ_u, \end{aligned}$$

with $\varepsilon \in (0,1]$ and $\sigma_\delta > 0$. This process admits a unique and strong solution, and is a positive local martingale but fails to be a true martingale (see Prieto (2010), Lemma 2). Note that this result implies that $s_t = \frac{\lambda_t}{1+\gamma+\lambda_t} \in (0,1)$ since λ is a strictly positive supermartingale and thus, under $\mathbb{P}$, is a.s. finite.

■

Proof of Theorem 1. Let $h_{t,\tau}(\cdot) : (0,1) \to \mathbb{R}_+$ be defined by

$$h_{t,\tau}(s_t) \equiv \rho s_t E_t \left[\int_t^\tau e^{-\rho(u-t)}(1 - \lambda_u/\lambda_t) du \right].$$

From Lemma 3 in Hugonnier (2010),

$$h_{t,\tau}(s_t) = H(\tau - t, s_t; 2\eta - 1) - e^{-\rho(\tau - t)} H(\tau - t, s_t, 1), \tag{43}$$

where $H(\cdot)$ is defined by

$$\begin{aligned} H(x,y;z) &\equiv y^{\frac{1+z}{2}} N(d_+(x,y;z)) + y^{\frac{1-z}{2}} N(d_-(x,y;z)), \\ d_\pm(x,y;z) &\equiv \frac{1}{\varepsilon\sigma_\delta\sqrt{x}} \log(y) \pm \frac{z}{2}\varepsilon\sigma_\delta\sqrt{x}, \end{aligned}$$

where $N(\cdot)$ denotes the standard normal cumulative distribution function. The infinite horizon bubble on the stock in (27) is given by

$$\begin{aligned} B_t &= \frac{\delta_t}{(1+\psi)\rho} \lim_{\tau\to\infty} h_{t,\tau}(s_t) \\ &= \frac{\delta_t}{(1+\psi)\rho} s_t^\eta. \end{aligned} \tag{44}$$

The stock price and fundamental value in (25) and (26) follow from solving for $(S_t, F_t(\delta))$ in (8) and (22), using (44).

To show existence of equilibrium, we impose parametric restrictions such that the constants

$$\begin{aligned} w_1 &= (1-\alpha)\rho^{-1}\delta_0 \left(1 - \frac{\psi}{1+\psi} s_0^\eta\right) - \beta, \\ w_2 &= \alpha\rho^{-1}\delta_0 \left(1 - \frac{\psi}{1+\psi} s_0^\eta\right) + \beta, \end{aligned}$$

are strictly positive. This ensures that (y_3, γ, λ_0) are strictly positive. Necessary and sufficient conditions such that $(w_1, w_2) \in \mathbb{R}_+^2$ are provided in (24). Uniqueness follows from the fact that the function in (23), implied by the constrained agent's initial endowment, is decreasing and continuous, since $\eta > 0$. There is a unique root $s^* \in (0,1)$ which solves $g(s^*) = 0$, since

$$g(0) = \beta + \alpha\rho^{-1}\delta_0 > 0, \qquad g(1) = \beta + \rho^{-1}\delta_0 \left(\frac{\alpha}{1+\psi} - 1 \right) < 0.$$

are verified if conditions in (24) hold. ■

Proof of Proposition 5. The definition in (10) can be rewritten as

$$B_t(\tau) = B_t - E_t \left[\frac{\xi_\tau}{\xi_t} B_\tau \right].$$

Note that

$$E_t \left[\xi_\tau B_\tau \right] = \frac{\delta_0}{1+\gamma+\lambda_0} \frac{1}{1+\psi} E_t \left[\frac{e^{-\rho\tau}}{\rho} (1+\gamma+\lambda_\tau) s_\tau^\eta \right],$$

but

$$\frac{e^{-\rho\tau}}{\rho} (\lambda_\tau - (1+\gamma+\lambda_\tau) s_\tau^\eta) = E_\tau \left[\int_\tau^\infty e^{-\rho u} \lambda_u du \right],$$

which in conjunction with (43) and the fact that $E_t[\lambda_{t+\tau}] = \lambda_t(1 - s_t^{-1} H(\tau, s_t; 1))$ (see Lemma A.2. in Hugonnier (2010)), gives

$$\begin{aligned}
E_t \left[\xi_\tau B_\tau \right] &= \underbrace{\frac{\delta_0}{1+\gamma+\lambda_0} \frac{1}{1+\psi}}_{\Theta} E_t \left[\frac{e^{-\rho\tau}}{\rho} (1+\gamma+\lambda_\tau) s_\tau^\eta \right] \\
&= \Theta E_t \left[\frac{e^{-\rho\tau}}{\rho} \lambda_\tau - \int_\tau^\infty e^{-\rho u} \lambda_u du \right] \\
&= \Theta E_t \left[\frac{e^{-\rho\tau}}{\rho} \lambda_\tau - \int_t^\infty e^{-\rho u} \lambda_u du + \int_t^\tau e^{-\rho u} \lambda_u du \right] \\
&= \Theta \Bigg[\frac{e^{-\rho\tau}}{\rho} \lambda_t (1 - s_t^{-1} H(\tau - t, s_t; 1)) + \frac{e^{-\rho t}}{\rho} (\lambda_t - (1+\gamma+\lambda_t) s_t^\eta) \\
&\quad + \frac{e^{-\rho t}}{\rho} \lambda_t (1 - e^{-\rho(\tau - t)}) - \frac{e^{-\rho t}}{\rho} \frac{\lambda_t}{s_t} (H(\tau - t, s_t; 2\eta - 1) - e^{-\rho(\tau - t)} H(\tau - t, s_t; 1)) \Bigg] \\
&= \Theta \frac{e^{-\rho t}}{\rho} (1+\gamma+\lambda_t) \left[s_t^\eta - H(\tau - t, s_t; 2\eta - 1) \right]
\end{aligned}$$

and hence the expression in (28) obtains

$$
\begin{aligned}
B_t(\tau) &= B_t - E_t\left[\frac{\xi_\tau}{\xi_t}B_\tau\right] \\
&= \frac{\delta_t s_{2t}^\eta}{\rho(1+\psi)} - \frac{\delta_t}{\rho(1+\psi)}\left[s_{2t}^\eta - H(\tau-t, s_t; 2\eta-1)\right] \\
&= \frac{\delta_t}{\rho(1+\psi)}H(\tau-t, s_t; 2\eta-1).
\end{aligned}
$$

The bubble on the riskless asset in (29) follows from Lemma 5 and the proof of Proposition 6 in Hugonnier (2010),

$$
\begin{aligned}
B_{0t}(\tau) &= S_{0t}\left(1 - E_t\left[M_\tau/M_t\right]\right) \\
&= S_{0t}s_t^{-1/\varepsilon}H(\tau-t, s_t; 2/\varepsilon - 1)
\end{aligned}
$$

where

$$
\begin{aligned}
dM_t &= -M_t\left(1 + \varepsilon\frac{\lambda_t}{1+\gamma}\right)\sigma_\delta dZ_t \\
&= -M_t\left(1/\varepsilon + X_t\right)\varepsilon\sigma_\delta dZ_t
\end{aligned}
$$

and

$$
dX_t = d\left(\lambda_t/(1+\gamma)\right) = -X_t\left(1+X_t\right)\varepsilon\sigma_\delta dZ_t.
$$

■

Proof of Corollary 1. The result follows from using equilibrium quantities in Proposition 2. ■

Proof of Proposition 6. The portfolio decomposition follows from an application of Itô's lemma to the expression in (30) and matching the process and the diffusion terms with that of the wealth process $W_{1t} = \frac{1}{(1+\gamma)\rho}(1-s_{2t})\delta_t$. In particular, (31) and (32) solve the system

$$
\phi_{1t}^S(\tau)\sigma_t + \phi_{1t}^{B_0}(\tau)\Sigma_{t,\tau}^0 = \frac{\sigma_\delta\delta_t(1-(1-\varepsilon)s_t)}{\rho(1+\gamma)}, \qquad \phi_{1t}^S(\tau) + \phi_{1t}^{B_0}(\tau) = \frac{\delta_t(1-s_t)}{\rho(1+\gamma)}.
$$

The sign of $\phi_t^S(\tau)$ is easily determined from the fact that $\Sigma_{t,\tau}^0 < 0$. The sign of $\phi_{1t}^{B_0}(\tau)$ follows from noting that

$$\text{sign}\left[\phi_{1t}^{B_0}(\tau)\right] = -\text{sign}\left[s_t^{\eta-1}(s_t(\eta-1)-\eta)\psi + 1 + \psi\right]$$

The function in the right hand side is strictly decreasing with range $(1, 1+\psi)$ for $s_2 \in (0,1)$, ■

Proof of Corollary 2. From Proposition 5,

$$B_{0t}(\tau)/S_{0t} = s_t^{-1/\varepsilon} H(\tau - t, s_t; 2/\varepsilon - 1),$$

and

$$\begin{aligned} B_t(\tau)/S_t &= \frac{H(\tau - t, s_t; 2\eta - 1)}{1 + \psi(1 - s_t)} \\ &\leq H(\tau - t, s_t; 2\eta - 1) \quad (\text{since } \eta > 0 \text{ and } s < 1) \\ &\leq H(\tau - t, s_t; 1) \quad (\text{since } H(\tau - t, s_t; x) \text{ is} \uparrow \text{in } x > 0)\ (*) \\ &\leq s_t^{-1} H(\tau - t, s_t; 1) \quad (\text{since } s < 1) \\ &\leq s_t^{-1/\varepsilon} H(\tau - t, s_{2t}; 2/\varepsilon - 1) \quad (\text{since } s_t^{-\alpha} H(\tau - t, s_t; 2\alpha - 1) \text{ is} \downarrow \text{in } \alpha > 0)\ (**) \end{aligned}$$

(*) Note that

$$\frac{\partial H(\tau, s, x)}{\partial x} = \frac{1}{4} s^{1/2 - x/2} \log(s) G(s) \leq 0$$

which follows from

$$G(s) = s^x \text{Erf}\left(\frac{2\log(s) + x(\varepsilon\sigma_\delta)^2\tau}{2\sqrt{2}\varepsilon\sigma_\delta\sqrt{\tau}}\right) + s^x + \text{Erf}\left(\frac{-2\log(s) + x(\varepsilon\sigma_\delta)^2\tau}{2\sqrt{2}\varepsilon\sigma_\delta\sqrt{\tau}}\right) - 1$$

with

$$\begin{aligned} G(0) &= 0 \\ G(1) &= 2\text{Erf}\left(\frac{x\varepsilon\sigma_\delta\sqrt{\tau}}{2\sqrt{2}}\right) \geq 0 \\ G'(s) &= x s^{x-1}\left(1 + \text{Erf}\left(\frac{2\log(s) + x(\varepsilon\sigma_\delta)^2\tau}{2\sqrt{2}\varepsilon\sigma_\delta\sqrt{\tau}}\right)\right) \geq 0. \end{aligned}$$

(**) It follows by direct differentiation. ■

Proof of Corollary 3. The volatility of the stock follows from an application of Itô's lemma to (25). ■

Proof of Proposition 7. The comparative statics with respect to ψ in (34) can be readily assessed since:

1. The starting point of the consumption share process is decreasing in ψ,

$$\frac{\partial s_0}{\partial \psi} = -\frac{\partial g/\partial \psi}{\partial g/\partial s_0} = -\frac{s_0^{1+\eta}\alpha}{(1+\psi)\left((1+\psi)s_0 + \alpha\eta\psi s_0^{\eta}\right)} < 0.$$

2. The relative weight of the arbitrageur $\gamma = \frac{\delta_0 - \rho S_0}{\rho(S_0 - w_2)}$ is increasing in ψ,

$$\frac{\partial \gamma}{\partial \psi} = \frac{\delta_0(\delta_0(1-\alpha) - \beta\rho)(1+\psi)s_0^{1+\eta}}{\left((\delta_0(1-\alpha) - \beta\rho)(1+\psi) + \delta_0(1-\alpha)\psi s_0^{\eta}\right)^2\left((1+\psi)s_0 + \alpha\eta\psi s_0^{\eta}\right)} > 0,$$

since from (24), $\delta_0(1-\alpha) - \beta\rho > 0$.

3. The third term in (34) is increasing in ψ. We use $\frac{\partial s_0}{\partial \psi} < 0$ and show s is increasing in its starting point, s_0. To verify this, it suffices to check that the process defined by $\varpi_t = \frac{s_t}{1-s_t}$, is increasing on its starting point. Using Protter (2004) (Theorem 29), $\partial \varpi_t / \partial \varpi_0 = v_t$ is a strictly positive process given by $dv_t/v_t = -(1+\varpi_t)\varepsilon\sigma_\delta dZ_t$.

■

References

S. Basak and B. Croitoru. Equilibrium mispricing in a capital market with portfolio constraints. *Review of Financial Studies*, 13(3):715–748, 2000.

S. Basak and B. Croitoru. On the role of arbitrageurs in rational markets. *Journal of Financial Economics*, 81(1):143–173, 2006.

S. Basak and D. Cuoco. An equilibrium model with restricted stock market participation. *Review of Financial Studies*, 11(2):309–341, 1998.

R. Caballero and A. Krishnamurthy. Global imbalances and financial fragility. *American Economic Review Papers and Proceedings*, 99(2):584–588, 2009.

A. Cox and D. Hobson. Local martingales, bubbles and option prices. *Finance and Stochastics*, 9(4):477–492, 2005.

D. Cuoco and H. He. Dynamic equilibrium in infinite–dimensional economies with incomplete financial markets. *Working paper, The Wharton School, University of Pennsylvania*, 1994.

D. Cuoco, H. He, and S. Isaenko. Optimal Dynamic Trading Strategies with Risk Limits. *Operations Research*, 56(2):358–368, 2008.

J. Cvitanić and I. Karatzas. Convex duality in convex portfolio optimization. *Annals of Applied Probability*, 2(4):767–818, 1992.

J. Detemple and S. Murthy. Equilibrium asset prices and no-arbitrage under portfolio constraints. *Review of Financial Studies*, 10(4):1133–1174, 1997.

J. Detemple and A. Serrat. Dynamic equilibrium with liquidity constraints. *Review of Financial Studies*, 16(2):597–629, 2003.

D. Duffie. *Dynamic asset pricing.* Princeton University Press, third edition, 2001.

P. Dybvig and C.-F. Huang. Nonnegative wealth, absence of arbitrage and feasible consumption plans. *Review of Financial Studies*, 1(4):377–401, 1988.

E. Fahri and J. Tirole. Bubbly liquidity. *Forthcoming, Review of Economics Studies*, 2010.

N. Gârleanu and L. Pedersen. Liquidity and risk management. *American Economic Review P&P*, 97(2):193–197, 2007.

D. Gromb and D. Vayanos. Equilibrium and welfare in markets with financially constrained arbitrageurs. *Journal of Financial Economics*, 66:361–407, 2002.

D. Gromb and D. Vayanos. Limits of arbitrage: The state of the theory. *Working paper, INSEAD*, 2010.

J. M. Harrison and D. Kreps. Martingales and arbitrage in multiperiod securities markets. *Journal of Economic Theory*, 20:381–408, 1979.

S. Heston, M. Loewenstein, and G. Willard. Options and bubbles. *Review of Financial Studies*, 20(2):359–390, 2007.

J.-B. Hiriart-Urruty and C. Lemaréchal. *Fundamentals of convex analysis*. Springer Verlag, 2001.

J. Hugonnier. Bubbles and multiplicity of equilibria under portfolio constraints. *Working paper, Ecole Polytechnique Fédérale de Lausanne*, 2010.

R. Jarrow, P. Protter, and K. Shimbo. Asset price bubbles in incomplete markets. *Mathematical Finance*, 20(2):145–185, 2010.

I. Karatzas and S. Shreve. *Methods of Mathematical Finance*. Springer Verlag, 1998.

M. Loewenstein and G. Willard. Rational equilibrium asset pricing bubbles in continuous trading models. *Journal of Economic Theory*, 91(1):17–58, 2000a.

M. Loewenstein and G. Willard. Local martingales, arbitrage, and viability: Free snacks and cheap thrills. *Economic Theory*, 16:135–161, 2000b.

M. Loewenstein and G. Willard. Consumption and bubbles. *Working paper, University of Maryland*, 2010.

A. Mele. Asymmetric stock market volatility and the cyclical behavior of expected returns. *Journal of Financial Economics*, 86(2):446–478, 2007.

R. Prieto. Dynamic equilibrium with heterogeneous agents and risk constraints. *working paper, Boston University*, 2010.

P. Protter. *Stochastic integration and differential equations*. Springer, second edition, 2004.

R. Radner. Existence of equilibrium of plans, prices and prices expectations in a sequence of markets. *Econometrica*, 40(2):289–303, 1972.

R. T. Rockafellar. *Convex Analysis.* Princeton University Press, second edition, 1996.

M. Santos and M. Woodford. Rational asset pricing bubbles. *Econometrica*, 65(1):19–57, 1997.

W. G. Schwert. Why does stock market volatility change over time. *Journal of Finance*, 44:1115–1153, 1989.

A. Shleifer and R. Vishny. The limits of arbitrage. *Journal of Finance*, 52:35–55, 1997.

P. Weil. On the possibility of decreasing bubbles. *Econometrica*, 58(6):1467–1474, 1990.

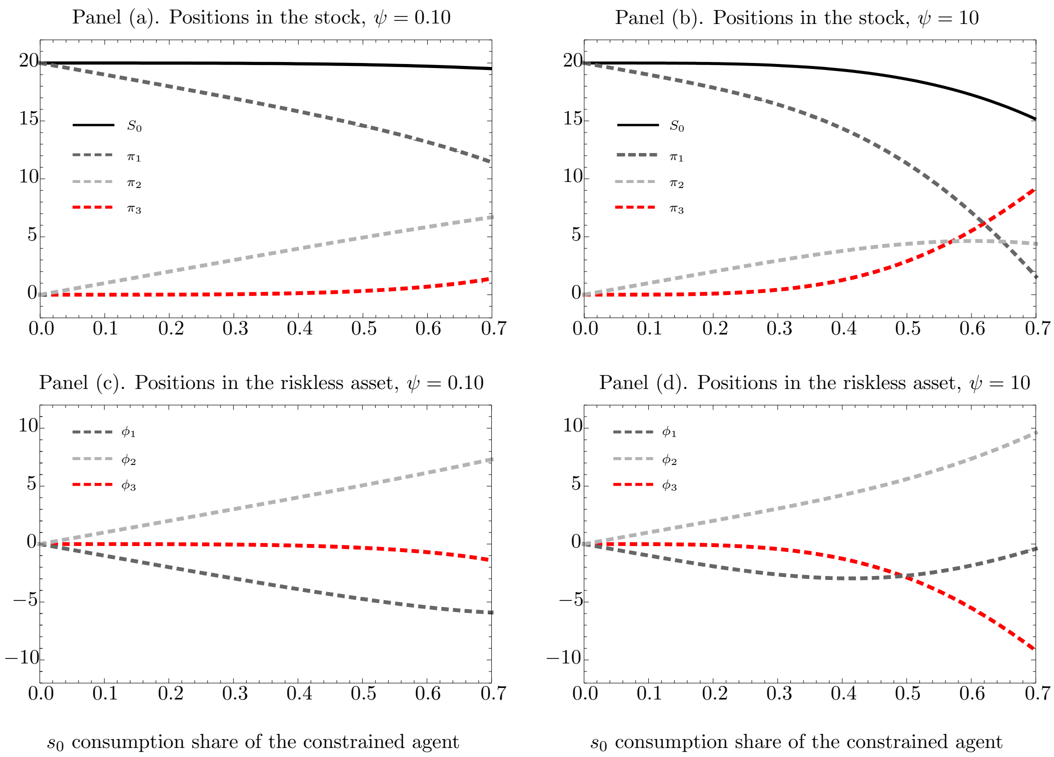

Figure 1: Panels (a) and (c) plot the portfolio holdings (scaled by the level of the dividend) with $\psi = 0.10$. Panels (b) and (d) plot the portfolio holdings (scaled by the level of the dividend) with $\psi = 10$. The constrained agent holds the offsetting position in the riskless asset. Agent 1's (the arbitrageur's) short position in the riskless asset is much less (more) aggressive as ψ increases, especially in bad times (high states of s_0) . The level of the stock decreases with ψ. Parameters are set to $\delta_0 = 1$, $\mu_\delta = 0.03$, $\sigma_\delta = 0.2$, $\rho = 0.05$ and $\varepsilon = 0.5$.

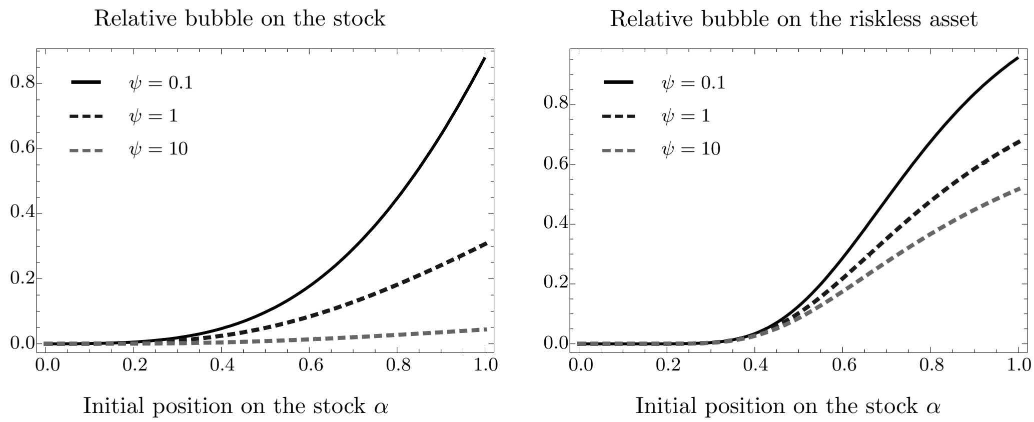

Figure 2: The left (right) panel plots the relative size of the bubble on the stock $B_t(\tau)/S_t$ (on the risk less asset $B_{0t}(\tau)/S_{0t}$) for different levels of ψ. Parameters are set to $\tau = 20$, $\delta_0 = 1$,$\mu_\delta = 0.03$, $\sigma_\delta = 0.2$, $\rho = 0.05$, $\varepsilon = 0.5$.

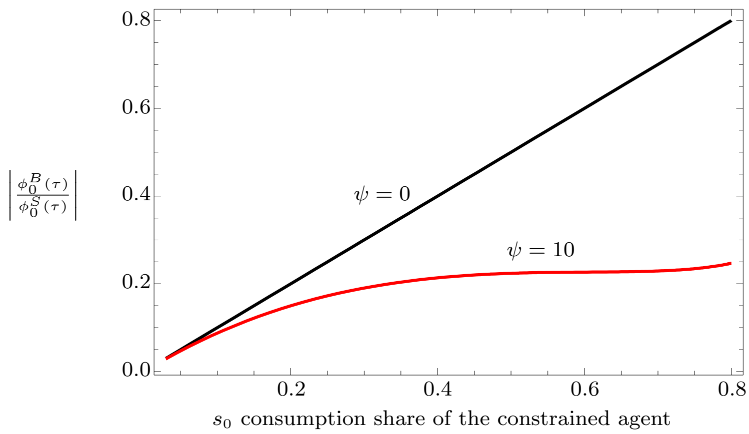

Figure 3: The figure plots the arbitrage holdings of agent 1 relative to collateral. As ψ increases, the collateral quality of the stock decreases for the unconstrained agent. Parameters are set to $\tau = 100$, $\delta_0 = 1$, $\mu_\delta = 0.03$, $\sigma_\delta = 0.2$, $\rho = 0.05$, $\varepsilon = 1$.