The equilibrium dynamics of liquidity and illiquid asset prices

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Adrian Buss
Goethe Universität Frankfurt
Joining INSEAD

Bernard Dumas
INSEAD
NBER, CEPR

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Transactions fees

- Access to a financial market is a service that investors make *available to each other*.
- We bypass brokers and let the investors serve as brokers for, and pay the fees to each other. We assume that the trading fee is proportional to the *value of the shares traded*.
  - fees are received as a lump sum
- Given the presence of those fees, an investor may decide not to trade, thereby preventing other investors from trading with him/her, *which is an additional indirect, stochastic and perhaps quantitatively more important consequence of the fees*.

Literature: partial equilibrium

- Seminal papers by Constantinides (1976a, 1976b, 1986), demonstrating the existence of the no-trade region
- Davis and Norman (1990), Dumas and Luciano (1991), Gennotte and Jung (1994) among others

Literature: general equilibrium

- Other authors have postulated a physical cost of transacting and exhibited the resulting equilibrium behavior.
- In Vayanos (1998) and Vayanos and Vila (1999), an investor's only motive to trade is the fact that he has a finite lifetime.
  - Transactions costs induce him to trade twice in his life: when young, he buys some securities that he can resell when old
  - Here, we intend to introduce a higher-frequency motive to trade
- In the paper of Lo, Mamaysky and Wang (2004), costs of trading are fixed costs, all traders have the same negative exponential utility function, individual endowment provide the motive to trade (as in our paper) but aggregate endowment is not stochastic.
- In our current paper, fees are proportional, utility is a power utility that differs across traders and endowments are free to follow an arbitrary stochastic process, so that we are able to generate higher-frequency dynamics. To our knowledge, ours is the first paper to reach that goal.

Literature: empirical

- Amihud and Mendelson (1986a, 1986b)
- Amihud (1992): LIQ variable
- Pastor and Stambaugh (2003): liquidity risk factor for asset pricing and CAPM tested
- Acharya and Pedersen (2005): expected liquidity and liquidity risk; CAPM tested

The problem of each agent (1/3)

- population of two investors $l = 1, 2$ and
- a set of exogenous time sequences of individual endowments $\{e_{l,t} \in \mathbb{R}_+; l = 1, 2; t = 0, \ldots T\}$ on a tree or lattice.
  - For simplicity, we consider a binomial tree so that a given node at time t is followed by two nodes at time $t + 1$ at which the endowments are denoted $\{e_{l,t+1,u}, e_{l,t+1,d}\}$.
  - The transition probabilities are denoted $\pi_{t,t+1,j}$ (with $\sum_{j=u,d} \pi_{t,t+1,j} = 1$).
  - Notice that the tree only accomodates the *exogenous* state variables.

The problem of each agent (2/3)

- In the financial market, there are two long- or short-lived securities in zero net supply, defined by their payoffs $\{\delta_{t,i}; i = 1, 2; t = 0, ..T\}$. The subscript $i = 1$ refers to a short-lived riskless security and the subscript $i = 2$ refers to a long-lived risky security.
  - The "ticker" prices of the securities, which are not always transactions prices, are denoted: $\{S_{t,i}; i = 1, 2; t = 0, ... T\}$.
  - The ticker price is an effective transaction price if and only if a transaction takes place but it is posted all the time by the Walrasian auctioneering computer (which works at no cost).
- Financial-market transactions entail transactions fees. When an investor sells one unit of security i, turning it into consumption good,
  - he receives the price and pays $\varepsilon_{i,t}$ units of consumption goods as transaction fee and
  - the buyer of the securities must pay $\lambda_{i,t}$ units.

The problem of each agent (3/3)

Investor l solves the following problem:

$$J_l\left(\left\{\theta_{l,t-1,i}\right\},\left\{S_{t,i}\right\},e_{l,t},t\right)\triangleq\sup_{\left\{c_l,\theta_l\right\}}\mathbb{E}_0\sum_{t=0}^{T}u_l\left(\widetilde{c}_{l,t},t\right)$$

subject to a sequence of flow budget constraints:

$$\begin{aligned}c_{l,t}&+\sum_{i=1,2}\left[\theta_{l,t,i}-\theta_{l,t-1,i}\right]^{+}\times S_{t,i}\times\left(1+\lambda_{i,t}\right)\\&+\sum_{i=1,2}\left[\theta_{l,t,i}-\theta_{l,t-1,i}\right]^{-}\times S_{t,i}\times\left(1-\varepsilon_{i,t}\right)\\=e_{l,t}&+\sum_{i=1,2}\theta_{l,t-1,i}\delta_{t,i}+\sum_{i=1,2}\left[\theta_{l',t,i}-\theta_{l',t-1,i}\right]^{+}S_{t,i}\lambda_{i,t}\\&-\sum_{i=1,2}\left[\theta_{l',t,i}-\theta_{l',t-1,i}\right]^{-}S_{t,i}\varepsilon_{i,t};\forall t;l'\neq l\end{aligned}$$

$\theta_{l,t,i}$ being the number of units of Security i in the hands of Investor l *after* all transactions of time t.

Numerical solution technique

- we reformulate as problem with non negativity constraints
- solved by the "Interior-Point Algorithm"
  - amounts to relaxing the complementary-slackness condition
  - implemented in the Armand et al. (2008) version.
- we utilize the method of a "time shift" of equations proposed by Dumas-Lyasoff (2011): unknowns are time-t portfolios but time-$t+1$ consumptions
  - this allows us to solve recursively
  - endogenous state variables have closed domain

Final equation system at time t (1/4)

- First-order conditions for time $t+1$ consumption:

$$u_l'\left(c_{l,t+1,j}, t+1\right) = \phi_{l,t+1,j}$$

- The set of time-$t+1$ flow budget constraints for all investors and all states of nature of that time:

$$\begin{aligned} & e_{l,t+1,j} + \sum_{i=1,2} \theta_{l,t,i}\delta_{t+1,i,j} - c_{l,t+1,j} \\ & - \sum_{i=1,2} \left(\theta_{l,t+1,i,j} - \theta_{l,t,i}\right) \times R_{l,t+1,i,j} \times S_{t+1,i,j} \\ & = \sum_{i=1,2} \left(\widehat{\theta}_{l',t+1,i,j} - \theta_{l',t,i}\right) \times S_{t+1,i,j} \times \lambda_{i,t+1,j} \\ & - \sum_{i=1,2} \left(\widehat{\widehat{\theta}}_{l',t+1,i,j} - \theta_{l',t,i}\right) \times S_{t+1,i,j} \times \varepsilon_{i,t+1,j} \end{aligned}$$

Final equation system at time t (2/4)

- The third subset of equations says that, when they trade them, all investors must agree on the prices of traded securities and, more generally, they must agree on the posted "ticker prices" inclusive of the shadow prices R that make units of paper securities less valuable than units of consumption. We call them the "kernel conditions":

$$\frac{1}{R_{1,t,i} \times \phi_{1,t}} \sum_{j=u,d} \pi_{t,t+1,j} \times \phi_{1,t+1,j} \times \left(\delta_{t+1,i,j} + R_{1,t+1,i,j} \times S_{t+1,i,j}\right)$$

$$= \frac{1}{R_{2,t,i} \times \phi_{2,t}} \sum_{j=u,d} \pi_{t,t+1,j} \times \phi_{2,t+1,j} \times \left(\delta_{t+1,i,j} + R_{2,t+1,i,j} \times S_{t+1,i,j}\right)$$

Final equation system at time t (3/4)

- Definitions:

$$\theta_{l,t+1,i,j} = \widehat{\theta}_{l,t+1,i,j} + \widehat{\widehat{\theta}}_{l,t+1,i,j} - \theta_{l,t,i}$$

- Complementary-slackness conditions:

$$\begin{aligned} (-R_{l,t+1,i,j} + 1 + \lambda_{i,t+1,j}) \times \left(\widehat{\theta}_{l,t+1,i,j} - \theta_{l,t,i}\right) &= 0 \\ (R_{l,t+1,i,j} - (1 - \varepsilon_{i,t+1,j})) \times \left(\theta_{l,t,i} - \widehat{\widehat{\theta}}_{l,t+1,i,j}\right) &= 0 \end{aligned}$$

- Market-clearing restrictions:

$$\sum_{l=1,2} \theta_{l,t,i} = 0 or 1$$

- Inequalities:

$$\widehat{\widehat{\theta}}_{l,t+1,i,j} \leq \theta_{l,t,i} \leq \widehat{\theta}_{l,t+1,i,j}; 1 - \varepsilon_{i,t+1,j} \leq R_{l,t+1,i,j} \leq 1 + \lambda_{i,t+1,j};$$

Final equation system at time t (4/4)

- The unknowns are
  - $\left\{c_{l,t+1,j}, \phi_{l,t+1,j}; l = 1, 2; j = u, d\right\}$,
  - $\left\{R_{l,t+1,i,j}; l = 1, 2; j = u, d; i = 1, 2\right\}$ and
  - $\left\{\theta_{l,t,i}, \widehat{\theta}_{l,t+1,i,j}, \widehat{\widehat{\theta}}_{l,t+1,i,j}; l = 1, 2;\ j = u, d; i = 1, 2\right\}$.
- Besides the exogenous endowments $e_{l,t+1,j}$, the "givens" are
  - the investor-specific shadow prices of consumption $\left\{\phi_{l,t}; l = 1, 2\right\}$ and paper securities $\left\{R_{l,t,i}; l = 1, 2; i = 1, 2\right\}$, which must henceforth be treated as state variables and which we refer to as "endogenous state variables".
    - Actually, the variables can be reduced to two state variables because only their ratios appear.

Backward induction

- the future securities' prices $S_{t+1,i,j}$ are obtained by backward induction (see the third equation in the system):

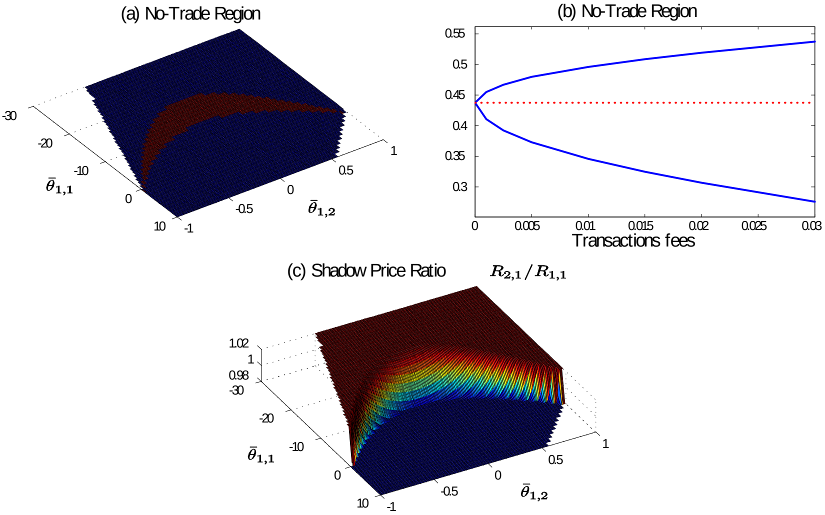
$$
\begin{aligned}
S_{t,i} &= \frac{1}{R_{l,t,i}\phi_{l,t}} \sum_{j=u,d} \pi_{t,t+1,j}\phi_{l,t+1,j} \times \left(\delta_{t+1,i,j} + R_{l,t+1,i,j} \times S_{t+1,i,j}\right) \\
S_{T,i} &= 0
\end{aligned}
$$

- the future positions $\theta_{l,t+1,i,j}$ are also obtained by an obvious backward induction of $\theta_{l,t,i}$, the previous solution of the above system, with terminal conditions $\theta_{l,T,i} = 0$.

Set up and parameter values

- 50 period economy. Of the results, those that are simulations are based on first 25 periods
- Heterogeneous agents:
  - Agent 1: Risk-aversion 2, time discount 0.975
  - Agent 2: Risk-aversion 4, time discount 0.975
- Only agent 1 receives endowment. Endowment parameters are calibrated to match: 1% risk free rate and 8% expected rate of return at 20% volatility
- Second investor has an initial claim on the first investor
- Two assets:
  - one period riskless asset
  - equity defined as a claim on the endowment
- Fees, charged on equity trades only, are in the range of $[0\%, 3\%]$ per dollar traded,
  - thanks to which there is scale invariance and we only need solve the system at one node for each time t

Equilibrium: no-trade region.



Asset Prices: Stock

- Recall that the securities' prices $S_{t,i}$ are:

$$S_{t,i} = \mathbb{E}_t \left[\frac{\phi_{l,t+1}}{R_{l,t,i}\phi_{l,t}} \times (\delta_{t+1,i} + R_{l,t+1,i} \times S_{t+1,i}) \right];$$
$$S_{T,i} = 0$$

- Definition:

$$\hat{S}_{t,i,l} \triangleq \frac{1}{\phi_{l,t}} \sum_{j=u,d} \pi_{t,t+1,j}\phi_{l,t+1,j} \times \left(\delta_{t+1,i,j} + \hat{S}_{t+1,i,j}\right); \hat{S}_{T,i} = 0$$

- We show that:

$$R_{l,t,i} \times S_{t,i} = \widehat{S}_{l,t,i}$$

which means that the ticker prices of securities can at most differ from the present value of their dividends by the value of the transactions fees paid by one person at the current trading date only.

Asset Prices: Stock

$$\Delta\phi_{l,t} \triangleq \frac{\phi_{l,t}}{\phi_{l,t-1}} - \frac{\phi^*_{l,t}}{\phi^*_{l,t-1}}$$

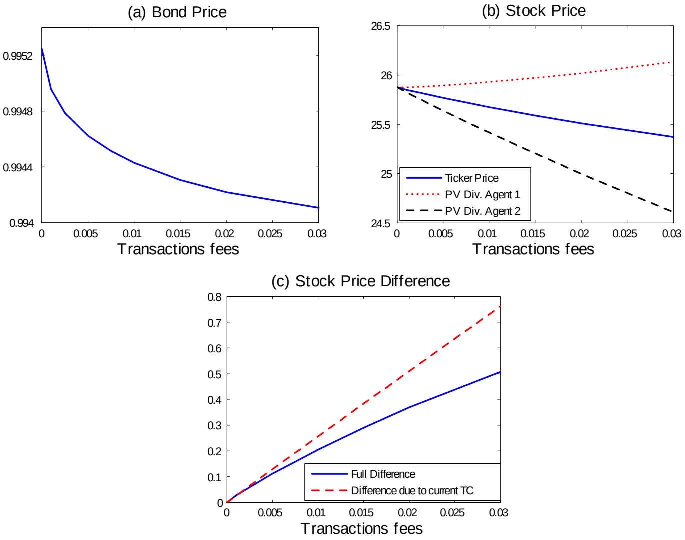
we show that:

$$R_{l,t,i} \times S_{t,i} = S^*_{t,i} + \mathbb{E}_t \left[\sum_{\tau=t+1}^{T} \frac{\phi_{l,\tau-1}}{\phi_{l,t}} \cdot \Delta\phi_{l,\tau} \cdot (\delta_\tau + S^*_\tau) \right]$$

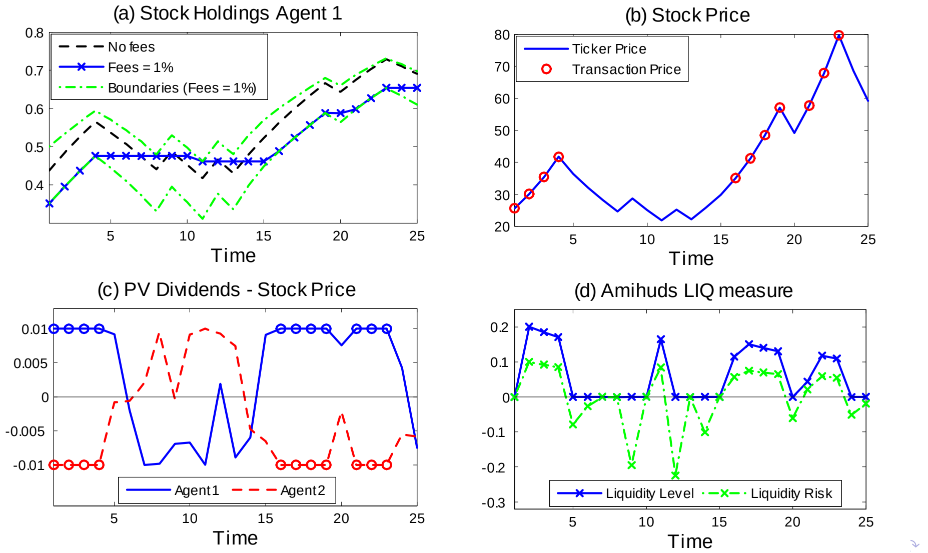
The two asset prices differ by two components:

- the current shadow price $R_{l,t,i}$, acting as a factor, of which we know that it is at most as big as the one-way transactions fees,
- the present value of all future price differences arising from the change in state prices and consumption induced by the presence of transactions fees.

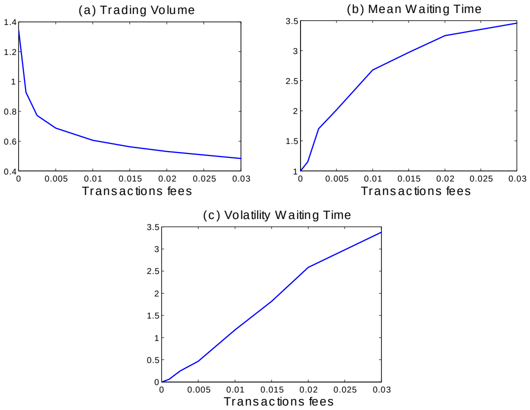
Asset Prices



Holdings and Prices: Time Path

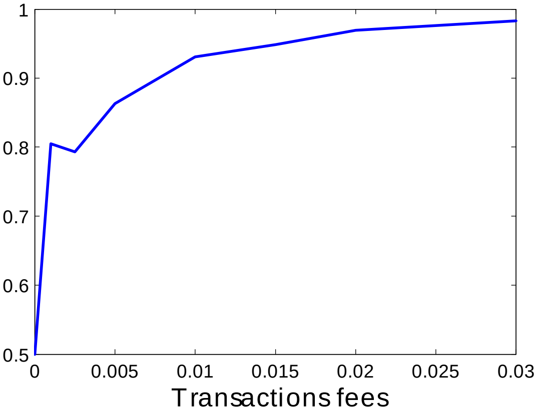


Trading activity

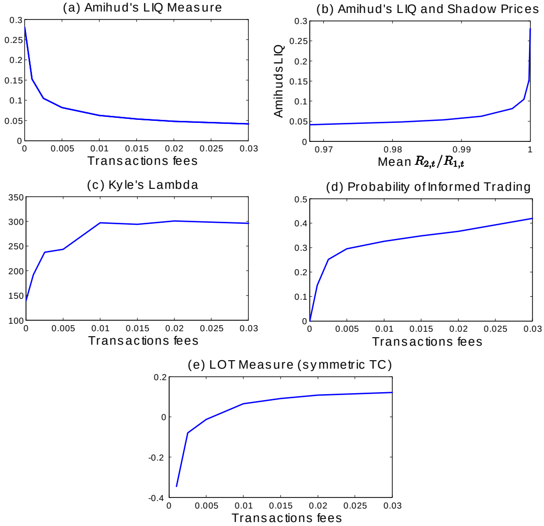


- trading volume in the stock decreases drastically
- as a result, also the bond trading decreases
- investors trade less often, liquidity vanishes in the stock market
- liquidity is stochastic as investors cannot be sure to update their positions next period

Probability of buy following buy



Joint behavior of transactions prices and trades: "price impact"



Example of price-impact measurement

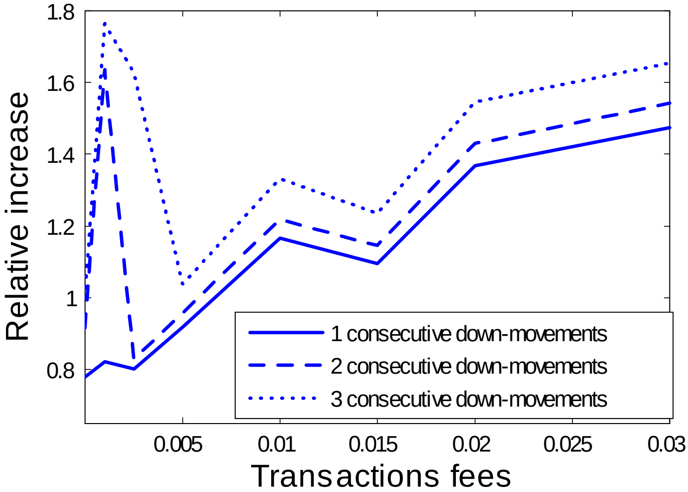
Madhavan and Smidt regression:

$$S_{t+1,2} - S_{t,2} = \text{intcpt} + \lambda \times \underbrace{\left(\theta_{1,t+1,2} - \theta_{1,t,2}\right)}_{\text{Signed trade volume}} + \text{coeff} \times \underbrace{\theta_{1,t,2}}_{\text{Stock holdings agent1}}$$

	intcpt	λ	coeff
	-8.40	291.33	2.47
t stat	(-1.56)	(3.82)	(1.36)

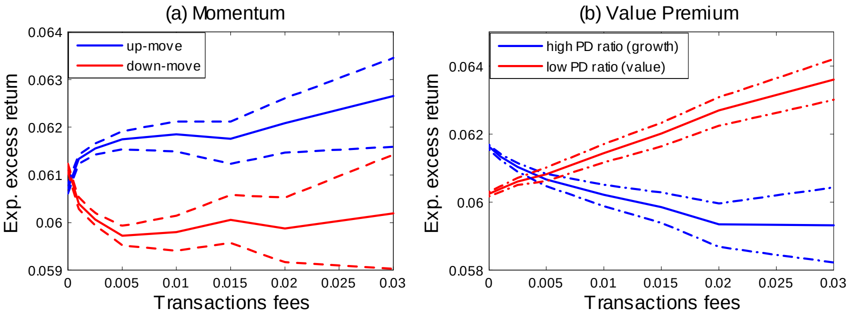
Joint behavior of transactions prices and trades: volume post price drop (pointed out by Xi Dong)

Joint behavior of transactions prices and trades: volume post price drop



Asset Prices: Anomalies

Can asset-pricing anomalies be explained by transactions fees?



Risk adjustment?

The pricing of liquidity and of liquidity risk: deviations from the classic CAPM

In our equilibrium, the expected return of the stock is given by:

$$\mathbb{E}_t\left[r_{t+1,i}\right] = r_{t+1,1} - cov_t\left(r_{t+1,i}, \frac{\phi_{l,t+1}}{\mathbb{E}_t\left[\phi_{l,t+1}\right]}\right) \underbrace{+\mathbb{E}_t\left[\tau_{l,t+1,i}\right] + cov_t\left(\tau_{l,t+1,i}, \frac{\phi_{l,t+1}}{\mathbb{E}_t\left[\phi_{l,t+1}\right]}\right)}_{\text{CAPM deviation}}$$

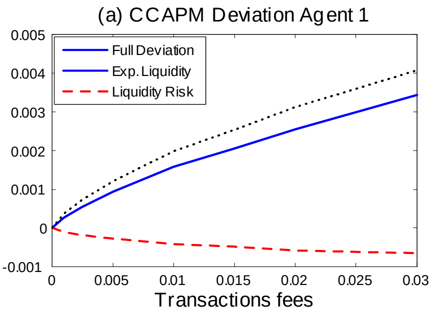
where:

$$r_{t+1,i,j} \triangleq \frac{\delta_{t+1,i,j} + S_{t+1,i,j}}{S_{t,i}}; \tau_{l,t+1,i,j} \triangleq (1 - R_{l,t+1,i,j}) \times \frac{S_{t+1,i,j}}{S_{t,i}} - (1 - R_{l,t,i})$$

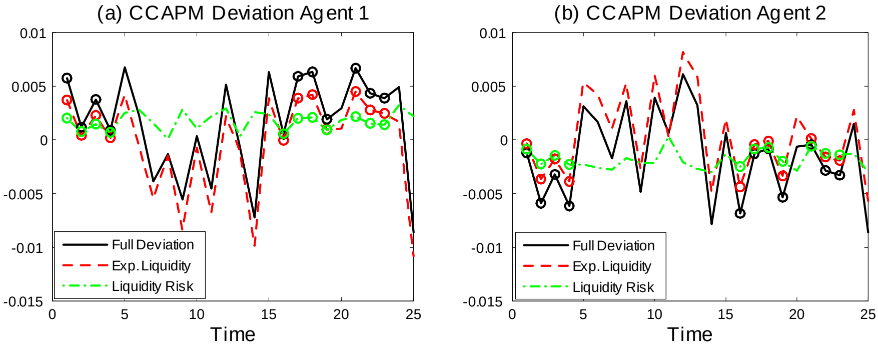
The CAPM deviation has two components:

- **expected liquidity change**: $\mathbb{E}_t\left[\tau_{l,t+1,i}\right]$
- **liquidity risk premium**: $cov_t\left(\tau_{l,t+1,i}, \frac{\phi_{l,t+1}}{\mathbb{E}_t\left[\phi_{l,t+1}\right]}\right)$

Equilibrium deviations from the classic CAPM

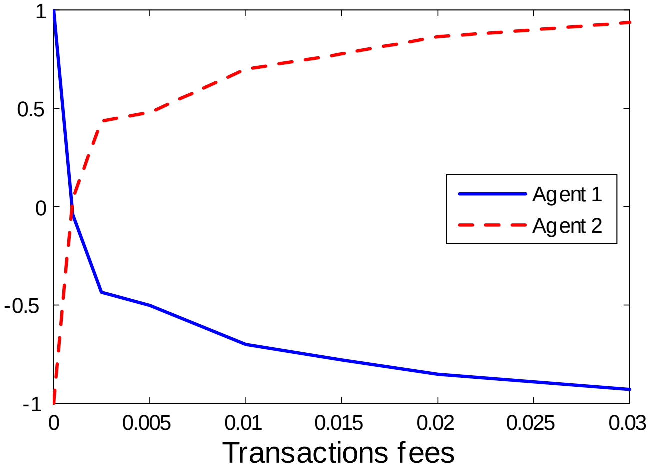


Equilibrium deviations from the classic CAPM over time



Liquidity and asset pricing

- to test the TC CAPM one needs to find proxy for τ
- Acharya-Pedersen use the LIQ (ILLIQ) measure
- how good is this measure?



Conclusion

- present a new method to compute financial-market equilibria in the presence of proportional transactions fees.
- deliver the optimal, market-clearing moves of each investor and the resulting ticker and transactions prices.
- conclude that:
  - transactions fees have a strong effect on investors' asset holdings
  - deviations in asset prices from a frictionless economy are due to:
    - current transactions fees
    - all future price differences due to state price differences
  - intertemporal optimization under transactions fees explain "price impact"
  - transactions fees may explain some parts of empirical asset pricing anomalies
- present a transactions-fees adjusted CAPM model and identify the risk factors