The Polynomial First-Order Approach to Principal-Agent Problems

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Analysis of Principal-Agent Problems

Standard solution method: first-order approach (FOA)

FOA replaces the agent's utility maximization problem by the necessary first-order optimality condition

Mirrlees (1975, 1999): FOA is generally not correct without additional technical assumptions

Subsequent literature identified different sets of assumptions on the probability distribution of outcomes

"The current state-of-the-art conditions are highly restrictive." Kadan, Reny, Swinkels (2011)

Alternative Approach

Key assumption: agent's expected utility is a rational function

Modern methods of polynomial optimization become applicable

New reformulation of the principal-agent problem

Agent's problem characterized by a system of equations and inequalities

With add'l assumptions: global optimality conditions

Advantages of Our Solution Approach

Our approach dispenses with several assumptions

Utility functions

- separability of agent's utility function
- monotonicity
- curvature

Probability functions

- monotonicity properties
- curvature

Effort may be multi-dimensional

Outline of Presentation

Basic Framework

Agent chooses action ("effort level") $\mathbf{a} \in A \subset \mathbb{R}^L$
unobservable to the principal

Finitely many possible outcomes ("output values"), $y_i \in \mathbb{R}$
$y_1 < y_2 < \ldots < y_N$

Probability $p_i(\mathbf{a})$ of outcome y_i if agent chooses action $\mathbf{a}$

Principal pays wage $w_i \in \mathcal{W} \subset \mathbb{R}$ to the agent if outcome y_i occurs

Contract ("compensation scheme") $\mathbf{w} = (w_1, \ldots, w_N) \in W \equiv \mathcal{W}^N$

Utility Functions

Principal has a Bernoulli utility over income,
$u : I \to \mathbb{R}$, with $I = (\underline{I}, \infty) \subset \mathbb{R}$

Outcome y_i, wage w_i, then the principal's utility is $u(y_i - w_i)$

Agent has a Bernoulli utility over income and actions,
$v : J \times A \to \mathbb{R}$, with $J = (\underline{J}, \infty) \subset \mathbb{R}$

Principal and agent have both von Neumann-Morgenstern utilities

$$\begin{aligned} U(\mathbf{w}, \mathbf{a}) &= \sum_{i=1}^{N} u(y_i - w_i) p_i(\mathbf{a}) \\ V(\mathbf{w}, \mathbf{a}) &= \sum_{i=1}^{N} v(w_i, \mathbf{a}) p_i(\mathbf{a}) \end{aligned}$$

Principal-Agent Problem

Optimization problem

$$\max_{\mathbf{w}\in W, \mathbf{a}\in A} \quad U(\mathbf{w}, \mathbf{a})$$

$$\text{s.t.} \quad \mathbf{a} \in \arg\max_{\mathbf{b}\in A} V(\mathbf{w}, \mathbf{b})$$

$$V(\mathbf{w}, \mathbf{a}) \geq \underline{V}$$

In general very difficult to find globally optimal solution

Popular technique for models with one-dimensional action set,
$A = [\underline{a}, \bar{a}]$ with $\bar{a} \in \mathbb{R} \cup \{\infty\}$,

First-Order Approach

First-Order Approach

Replace the incentive-compatibility constraint

$$a \in \arg\max_{b \in A} V(\mathbf{w}, b)$$

by the necessary first-order stationarity condition

$$\frac{\partial}{\partial a} V(\mathbf{w}, a) = 0$$

(assuming an interior solution)

Application of the first-order approach requires some assumptions

Assumptions for First-Order Approach

Part I: standard monotonicity, curvature, and differentiability assumptions (Rogerson, 1985)

(1) The probability functions p_i are twice continuously differentiable on A for $i \in \{1, 2, \ldots, N\}$.

(2) The principal's Bernoulli utility function $u : I \to \mathbb{R}$ is strictly increasing, concave, and twice continuously differentiable on I.

(3) The agent's Bernoulli utility function $v : J \times A \to \mathbb{R}$ satisfies $v(w, a) = \psi(w) - a$. The function $\psi : J \to \mathbb{R}$ is strictly increasing, concave and twice continuously differentiable.

Additional assumptions on the probabilities $p_i(a)$ are needed for the concavity of

$$V(\mathbf{w}, a) = \sum_{i=1}^{N} v(w_i, a) p_i(a)$$

Assumptions on Probabilities

Cumulative distribution of outcomes, $F_j(a) = \sum_{j=1}^{i} p_i(a)$

Part II: assumptions on probabilities (with $p_i(a) > 0$ for all $a \in A$)
Mirrlees (1979) and Rogerson (1985)

(MLRC) (monotone likelihood-ratio condition) The functions p_i have the property that for $a_1 \leq a_2$ the ratio $\frac{p_i(a_1)}{p_i(a_2)}$ is decreasing in i.

(CDFC) (convexity of the distribution function condition) The cumulative distributions are convex, $F_i''(a) \geq 0$ for all $i = 1, 2, \ldots, N$ and $a \in A$.

A Brief and Incomplete History

Mirrlees (1975, 1999), Mirrlees (1979)

Rogerson (1985)

Jewitt (1988)

Sinclair-Desgagné (1994)

Alvi (1997)

Araujo and Monteiro (2001)

Jewitt, Kadan, Swinkels (2008)

Conlon (2009)

Key Assumption

The agent's expected utility function is a rational function of the form

$$V(\mathbf{w}, a) = \sum_{j=1}^{N} v(w_j, a)p_j(a) = -\frac{\sum_{i=0}^{d} c_i(\mathbf{w})a^i}{\sum_{i=0}^{d} f_i(\mathbf{w})a^i}$$

for functions $c_i, f_i : W \to \mathbb{R}$ with $\sum_{i=0}^{d} f_i(\mathbf{w})a^i > 0$ for all $(\mathbf{w}, a) \in W \times A$.

The two polynomials in the variable a, $\sum_{i=0}^{d} c_i(\mathbf{w})a^i$ and $\sum_{i=0}^{d} f_i(\mathbf{w})a^i$, have no common factors and $d \in \mathbb{N}$ is maximal such that $c_d(\mathbf{w}) \neq 0$ or $f_d(\mathbf{w}) \neq 0$.

New Solution Approach

Agent has rational utility; set of actions, $A = [-1, 1]$ (w.l.o.g.)

Then $(\mathbf{w}^*, a^*)$ solves the principal-agent problem if and only if there exist $\rho^* \in \mathbb{R}$ as well as matrices $Q^{(0)*} \in \mathbb{R}^{(D+1)\times(D+1)}$ and $Q^{(1)*} \in \mathbb{R}^{D\times D}$ such that $(\mathbf{w}^*, a^*, \rho^*, Q^{(0)*}, Q^{(1)*})$ solves the following optimization problem.

$$\max_{\mathbf{w}, a, \rho, Q^{(0)}, Q^{(1)}} \quad U(\mathbf{w}, a)$$
$$\text{s.t.} \quad \text{a set of constraints}$$

Constraints

$$
\begin{aligned}
V(\mathbf{w}, a) &\geq \underline{V} \\
-a^2 + 1 &\geq 0 \\
\mathbf{w} \in W &
\end{aligned}
$$

Constraints

$$\begin{aligned} V(\mathbf{w}, a) &\geq \underline{V} \\ -a^2 + 1 &\geq 0 \\ \mathbf{w} &\in W \end{aligned}$$

$$\begin{aligned} \rho &= -V(\mathbf{w}, a) \\ c_0(\mathbf{w}) - \rho f_0(\mathbf{w}) &= Q^{(0)}_{0,0} + Q^{(1)}_{0,0} \\ \text{for} \quad l = 1, \ldots, d & \\ c_l(\mathbf{w}) - \rho f_l(\mathbf{w}) &= \sum_{i+j=l} Q^{(0)}_{ij} + \sum_{i+j=l} Q^{(1)}_{ij} - \sum_{i+j=l-2} Q^{(1)}_{ij} \\ Q^{(0)}, Q^{(1)} &\succcurlyeq 0 \end{aligned}$$

Mathematical Framework

Constrained optimization problem of minimizing a univariate rational function on an interval of $\mathbb{R}$ is a convex problem and reduces to a semi-definite program (SDP)

Necessary tools

- polynomials
- sum of squares
- representation of sums of squares
- polynomial optimization

Polynomials

Ring of polynomials in n variables over the reals, $\mathbb{R}[x_1, \ldots, x_n]$

Monomial $\mathbf{x}^{\boldsymbol{\alpha}} \equiv x_1^{\alpha_1} \cdots x_n^{\alpha_n}$

Degree of monomial $|\boldsymbol{\alpha}| = \sum_{i=1}^{n} \alpha_i$

Polynomial $p \in \mathbb{R}[\mathbf{x}]$, $p = \sum_{\boldsymbol{\alpha}} a_{\boldsymbol{\alpha}} \mathbf{x}^{\boldsymbol{\alpha}}$ with finitely many nonzero $a_{\boldsymbol{\alpha}} \in \mathbb{R}$

Degree of p is $\deg(p) = \max_{\{\boldsymbol{\alpha} | a_{\boldsymbol{\alpha}} \neq 0\}} |\boldsymbol{\alpha}|$

Basic closed semi-algebraic set

$$K = \{\mathbf{x} \in \mathbb{R}^n \mid g_i(\mathbf{x}) \geq 0, \ \forall i = 1, \ldots, m\}$$

with $g_1, \ldots, g_m \in \mathbb{R}[\mathbf{x}]$

Sum of Squares

A polynomial $\sigma \in \mathbb{R}[\mathbf{x}]$ is called a *sum of squares* if there exists finitely many polynomials $p_1, \ldots, p_m \in \mathbb{R}[\mathbf{x}]$ such that $\sigma = \sum_{i=1}^{m} p_i^2$.

Link to positive semi-definite matrices

Consider the vector

$$\begin{aligned}\mathbf{v}_d(\mathbf{x}) &= (\mathbf{x}^{\boldsymbol{\alpha}})_{|\boldsymbol{\alpha}| \leq d} \\ &= \left(1, x_1, \ldots, x_n, x_1^2, x_1 x_2, \ldots, x_{n-1} x_n, x_n^2, \ldots, x_n^d\right)^T\end{aligned}$$

of all monomials $\mathbf{x}^{\boldsymbol{\alpha}}$ of degree at most d

Representation of Sums of Squares

A polynomial $\sigma \in \mathbb{R}[\mathbf{x}]$ of degree $2d$ is a sum of squares if and only if there exists a positive semidefinite $\binom{n+d}{d} \times \binom{n+d}{d}$ matrix Q such that $\sigma = \mathbf{v}_d(\mathbf{x})^T Q \mathbf{v}_d(\mathbf{x})$, where $\mathbf{v}_d(\mathbf{x})$ is the vector of monomials in $\mathbf{x}$ of degree at most d.

Representation of Sums of Squares

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Illustration for $n = 1$, so $\mathbf{v}_d(x) = \left(1, x, x^2, \ldots, x^d\right)^T$

Polynomial $p_i(x) = \sum_{j=0}^{d} a_{ij} x^j = \mathbf{a}_i \mathbf{v}_d(x)$,
vector of coefficients $\mathbf{a}_i = (a_{i0}, a_{i1}, \ldots, a_{id})$

Illustration in $\mathbb{R}$

Aggregate m such polynomials in a matrix-vector product

$$\begin{bmatrix} p_1(x) \\ p_2(x) \\ \vdots \\ p_m(x) \end{bmatrix} = \begin{bmatrix} a_{10} & a_{11} & \dots & a_{1d} \\ a_{20} & a_{21} & \dots & a_{2d} \\ \vdots & \vdots & \vdots & \vdots \\ a_{m0} & a_{m1} & \dots & a_{md} \end{bmatrix} \begin{bmatrix} 1 \\ x \\ \vdots \\ x^d \end{bmatrix}.$$

Denoting the $(m \times (d+1))$ coefficient matrix on the right-hand side by V, we can write a sum of squares as

$$\sigma(x) = \sum_{i=1}^{m} p_i^2(x) = (V\mathbf{v}_d(x))^T (V\mathbf{v}_d(x)) = \mathbf{v}_d(x)^T Q \mathbf{v}_d(x)$$

for $Q = V^T V$. By construction the matrix Q is symmetric, positive semi-definite and has at most rank m.

Sum of Squares and SDP

Method to determine whether a polynomial is a sum of squares

Equality of two polynomials: coefficients must be identical

Conditions on coefficients: linear equations

Set of positive semi-definite matrices is closed and convex

Linear constraints combined with PSD constraints yields the feasible region of a semidefinite program (SDP)

Sum of Square Representation in $\mathbb{R}$

Any nonnegative univariate polynomial is a sum of (at most) two squares.

Nonnegative univariate polynomials on closed intervals

Two cases: $[-1, 1]$ and $[0, \infty)$

$[-1, 1] = \{x \in \mathbb{R} \mid 1 - x^2 \geq 0\}$

$[0, \infty) = \{x \in \mathbb{R} \mid x \geq 0\}$

Univariate Positivstellensatz

Let $p \in \mathbb{R}[x]$ be of degree d.

(a) $p \geq 0$ on $[-1, 1]$ if and only if

$$p = \sigma_0 + \sigma_1(1 - x^2) \qquad \sigma_0, \sigma_1 \text{ sums of squares}$$

with $\deg(\sigma_0), \deg(\sigma_1(1 - x^2)) \leq d$ if d is even and $\deg(\sigma_0), \deg(\sigma_1(1 - x^2)) \leq d + 1$ if d is odd.

(b) $p \geq 0$ on $[0, \infty)$ if and only if

$$p = \sigma_0 + \sigma_1 x \qquad \sigma_0, \sigma_1 \text{ sums of squares}$$

with $\deg(\sigma_0), \deg(x\sigma_1) \leq d$.

Polynomial Optimization in $\mathbb{R}$

Polynomial $p \in \mathbb{R}[x]$, nonempty semi-algebraic set $K \subset \mathbb{R}$

Constrained polynomial optimization problem

$$p_{\min} = \inf_{x \in K} p(x)$$

Equivalently,

$$\begin{aligned} &\sup_{\rho} \ \rho \\ &\text{s.t. } p(x) - \rho \geq 0 \quad \forall x \in K \end{aligned}$$

Equivalent Constraints

For $K = [-1, 1] = \{x | g(x) = 1 - x^2 \geq 0\}$,

$$\begin{aligned} \sup_{\rho, \sigma_0, \sigma_1} \quad & \rho \\ \text{s.t.} \quad & p - \rho = \sigma_0 + \sigma_1 g \\ & \sigma_0, \sigma_0 \quad \text{sums of squares} \end{aligned}$$

Let $d_p = \left\lceil \frac{\deg(p)}{2} \right\rceil$, then

$$\begin{aligned} \sup_{\rho, Q^{(0)}, Q^{(1)}} \quad & \rho \\ \text{s.t.} \quad & p - \rho = v_{d_p}^T Q^{(0)} v_{d_p} + g v_{d_p - 1}^T Q^{(1)} v_{d_p - 1} \\ & Q^{(0)}, Q^{(1)} \succcurlyeq 0 \\ & Q^{(0)} \in \mathbb{R}^{(d_p+1)\times(d_p+1)}, \; Q^{(1)} \in \mathbb{R}^{d_p \times d_p} \\ & v_{d_p} = (1, x, \ldots, x^{d_p})^T, \; v_{d_p - 1} = (1, x, \ldots, x^{d_p - 1})^T \end{aligned}$$

Final Optimization Problem

$$
\begin{aligned}
\sup_{\rho, Q^{(0)}, Q^{(1)}} \quad & \rho \\
\text{s.t.} \quad & c_0 - \rho = Q^{(0)}_{0,0} + Q^{(1)}_{0,0}, \\
& c_l = \sum_{i+j=l} Q^{(0)}_{ij} + \sum_{i+j=l} Q^{(1)}_{ij} - \sum_{i+j=l-2} Q^{(1)}_{ij} \quad l = 1, \ldots, d \\
& Q^{(0)}, Q^{(1)} \succcurlyeq 0 \\
& Q^{(0)} \in \mathbb{R}^{(d_p+1)\times(d_p+1)}, \;\; Q^{(1)} \in \mathbb{R}^{d_p \times d_p}
\end{aligned}
$$

Constrained optimization problem of minimizing a univariate polynomial on an interval of $\mathbb{R}$ is a convex optimization problem and reduces to an SDP

Rational Objective Function

Described approach has been extended to rational functions

$$p_{\min} = \inf_{\mathbf{x}\in K} \frac{p(\mathbf{x})}{q(\mathbf{x})}$$

If p and q have no common factor and K is an open connected set or a (partial) closure of such a set then

(a) If q changes sign on K, then $p_{\min} = -\infty$.

(b) If q is nonnegative on K, then the problem is equivalent to

$$p_{\min} = \sup\{\rho \mid p(\mathbf{x}) - \rho q(\mathbf{x}) \geq 0,\ \forall \mathbf{x} \in K\}.$$

Literature

Laurent (2009)

Lasserre (2010)

Lasserre (2001)

Parrillo (2003)

Jibetean and de Klerk (2006)

Agent's Problem

Agent with rational utility function and $A = [-1, 1]$

$$\max_{a \in A} V(\mathbf{w}, \mathbf{a}) = -\inf_{a \in A} -V(\mathbf{w}, \mathbf{a}) = -\inf_{a \in [-1,1]} \frac{\sum_{i=0}^{d} c_i(\mathbf{w}) a^i}{\sum_{i=0}^{d} f_i(\mathbf{w}) a^i}$$

Reformulation

$$\begin{aligned}
c_0(\mathbf{w}) - \rho f_0(\mathbf{w}) &= Q^{(0)}_{0,0} + Q^{(1)}_{0,0} \\
\text{for} \quad l = 1, \ldots, d & \\
c_l(\mathbf{w}) - \rho f_l(\mathbf{w}) &= \sum_{i+j=l} Q^{(0)}_{ij} + \sum_{i+j=l} Q^{(1)}_{ij} - \sum_{i+j=l-2} Q^{(1)}_{ij} \\
Q^{(0)}, Q^{(1)} &\succcurlyeq 0
\end{aligned}$$

Additional Constraint

Constraints imply that for a given $\mathbf{w}$ the variables a, ρ satisfy

$$-V(\mathbf{w}, a) - \rho \geq 0$$

Strict inequality still possible

Additional constraint

$$V(\mathbf{w}, a) = -\rho$$
$$\iff \quad \rho \left(\sum_{i=0}^{d} f_i(\mathbf{w}) a^i \right) = \sum_{i=0}^{d} c_i(\mathbf{w}) a^i$$

For each $\mathbf{w} \in W$, there exist a, ρ such that $-V(\mathbf{w}, a) - \rho = 0$

Feasible region of the principal-agent problem is the projection of the new problem's feasible region on $W \times A$

Original principal-agent problem and the reformulation have the same objective function

New Optimization Problem

$$
\begin{aligned}
\sup_{\mathbf{w},a,\rho,Q^{(0)},Q^{(1)}} \quad & U(\mathbf{w},a) \\
\text{s.t.} \quad & c_0(\mathbf{w}) - \rho f_0(\mathbf{w}) = Q^{(0)}_{0,0} + Q^{(1)}_{0,0} \\
& c_l(\mathbf{w}) - \rho f_l(\mathbf{w}) = \sum_{i+j=l} Q^{(0)}_{ij} + \sum_{i+j=l} Q^{(1)}_{ij} - \sum_{i+j=l-2} Q^{(1)}_{ij} \\
& Q^{(0)}, Q^{(1)} \succcurlyeq 0 \\
& \rho \left(\sum_{i=0}^{d} f_i(\mathbf{w}) a^i \right) = \sum_{i=0}^{d} c_i(\mathbf{w}) a^i \\
& \sum_{i=0}^{d} c_i(\mathbf{w}) a^i \le -\underline{V} \left(\sum_{i=0}^{d} f_i(\mathbf{w}) a^i \right) \\
& -a^2 + 1 \ge 0 \\
& \mathbf{w} \in W
\end{aligned}
$$

Polynomial Optimization

Additional assumptions

1. Principal's expected utility U is polynomial
2. Agent's utility $V(\mathbf{w}, a) = -\frac{\sum_{i=0}^{d} c_i(\mathbf{w}) a^i}{\sum_{i=0}^{d} f_i(\mathbf{w}) a^i}$ with polynomials c_i, f_i
3. Semi-algebraic set W of wage contracts

Additional conclusions

1. Reformulation is a polynomial optimization problem over a basic semi-algebraic set
2. Solution methods applicable that deliver a numerical certificate of global optimality

▸ Extension to Multi-Dimensional Effort

▸ Numerical Example

Summary

New solution approach for principal-agent problems

Key assumption: agent's expected utility is a rational function

Much weaker assumptions on utility and probability functions

No MLRC, CDFC or variations thereof

Effort may be multi-dimensional

New approach offers theoretical advantages

Easily implementable

Polynomial First-Order Approach for $A \subset \mathbb{R}^L$

Classical first-order approach applies only to $A \subset \mathbb{R}$

Polynomial approach can be extended to $A \subset \mathbb{R}^L$

Necessary tools:

- Sums-of-squares representation of nonnegative multivariate polynomials on a semi-algebraic set
- Reformulation of a polynomial optimization problem

Putinar's Positivstellensatz

Let $f, g_1 \ldots, g_m \in \mathbb{R}[\mathbf{x}]$ be polynomials and $K = \{\mathbf{x} \in \mathbb{R}^n \mid g_1(\mathbf{x}) \geq 0, \ldots, g_m(\mathbf{x}) \geq 0\} \subset \mathbb{R}^n$ a basic semi-algebraic set such that for some j the set $\{\mathbf{x} \in \mathbb{R}^n \mid g_j(\mathbf{x}) \geq 0\}$ is compact. If f is strictly positive on K then

$$f = \sigma_0 + \sum_{i=1}^{m} \sigma_i g_i$$

for some sums of squares $\sigma_0, \ldots, \sigma_m$.

Two problems:

- Strict positivity
- Unknown degrees of the sums of squares

Relaxation of the Optimization Problem

Optimization problem

$$\sup_{\rho} \ \rho$$
$$\text{s.t. } p(\mathbf{x}) - \rho > 0 \ \forall \mathbf{x} \in K$$

Relaxation by restricting the degrees of the involved sums of squares

$$\rho_d = \sup_{\rho,\sigma_0,\sigma_1,\ldots,\sigma_m} \ \rho$$
$$\text{s.t.} \quad p - \rho = \sigma_0 + \sum_{i=1}^{m} \sigma_i g_i$$
$$\sigma_0, \sigma_i \quad \text{sums of squares of degree } d, d - d_{g_i}, \text{resp.}$$

Convergence

If the assumptions of Putinar's Positivstellensatz hold, then the optimal solution ρ_d of the relaxed problem converges (from below) to the optimal value of the original problem as $d \to \infty$.

In applications, convergence often for small finite d

Multivariate First-Order Approach

Assumption: $A = \left\{\mathbf{a} \in \mathbb{R}^L \mid g_1(\mathbf{a}) \geq 0, \ldots, g_m(\mathbf{a}) \geq 0\right\}$ is a compact semi-algebraic set with a nonempty interior

Relaxation of principal-agent problem includes constraints of the type

$$\epsilon \geq -\rho - V(\mathbf{w}, \mathbf{a}) = \sigma_0 + \sum_{i=1}^{m} \sigma_i g_i$$

For $\epsilon \to 0$ and $d \to \infty$ convergence of solutions

▸ Return to Polynomial Optimization

Numerical Example

$A = [0,1]$, $\mathcal{W} = \mathbb{R}_+$, $N = 3$ possible outcomes $y_1 < y_2 < y_3$

Probabilities

$$(p_1(a), p_2(a), p_3(a)) = \left(\binom{2}{0} a^0 (1-a)^2, \binom{2}{1} a(1-a), \binom{2}{2} a^2 (1-a)^0 \right)$$

Principal is risk-neutral, $u(y-w) = y - w$

Agent's utility

$$v(w,a) = \frac{w^{1-\eta} - 1}{1-\eta} - \kappa a^2$$

Expected utility is not concave

$$\frac{\partial^2 V}{\partial a^2} = \frac{2w_1^{1-\eta}}{1-\eta} - \frac{4w_2^{1-\eta}}{1-\eta} + \frac{2w_3^{1-\eta}}{1-\eta} - 2\kappa$$

Classical first-order approach may not hold

Preparation for New Approach

Specific problem: $\eta = \frac{1}{2}$, $\underline{V} = 0$, $\kappa = 2$, $(y_1, y_2, y_3) = (0, 2, 4)$

Transform the set of actions $A = [0, 1]$ into the interval
$A = [-1, 1]$ via the bijective mapping $a \mapsto \frac{a+1}{2}$

Since V is quadratic in effort a,

$$Q^{(0)} = \begin{pmatrix} n_{00} & n_{01} \\ n_{01} & n_{11} \end{pmatrix} \quad \text{and} \quad Q^{(1)} = m$$

Reformulation of the principal-agent problem

Reformulation

$$
\begin{aligned}
\max_{w_1,w_2,w_3,a,\rho,n_{00},n_{01},n_{11},m} \quad & U(w_1,w_2,w_3,a) \\
\text{s.t.} \quad & \frac{5}{2} - \frac{\sqrt{w_1}}{2} - \sqrt{w_2} - \frac{\sqrt{w_3}}{2} - \rho = n_{00} + m \\
& 1 + \sqrt{w_1} - \sqrt{w_3} = 2n_{01} \\
& \frac{1}{2} - \frac{\sqrt{w_1}}{2} + \sqrt{w_2} - \frac{\sqrt{w_3}}{2} = n_{11} - m \\
& \rho = -V(w_1,w_2,w_3,a) \\
& n_{00} \geq 0, n_{11} \geq 0, n_{00}n_{11} - n_{01}^2 \geq 0, m \geq 0 \\
& V(w_1,w_2,w_3,a) \geq 0 \\
& -a^2 + 1 \geq 0 \\
& w_1, w_2, w_3 \geq 0
\end{aligned}
$$

$\mathbf{w}^* = (0.3417, 1.511, 3.511) \quad a^* = 0.6446$

Return