DISCOUNTED REPEATED GAMES WITH GENERAL INCOMPLETE INFORMATION

MARCIN PĘSKI

Abstract. We analyze discounted repeated games with incomplete information, and such that the payoffs of the players depend only on their own type (*known-own payoff case*). We describe an algorithm to find all equilibrium payoffs in games for which there exists an open set of belief-free equilibria of Horner and Lovo (2009). This includes generic games with one-sided incomplete information and a large and important class of games with multi-sided incomplete information. When players become sufficiently patient, all Nash equilibrium payoffs can be approximated by payoffs in sequential equilibria in which information is revealed finitely many times. The set of equilibrium payoffs is typically larger than the set of equilibrium payoffs in repeated games without discounting, and it is also larger than the set of payoffs obtained in belief-free equilibria. The results are illustrated on bargaining and oligopoly examples.

1. Introduction

The goal of this paper is to describe all equilibrium payoffs in repeated games in which the players have private information about their own payoff types (*known-own payoffs case)* as players become increasingly patient. The original model was introduced in Aumann et al. (1967) (without discounting), and it has many important applications, like oligopoly with privately known costs, bargaining with uncertain preferences, nuclear disarmament, etc.

The major problem in analysis of repeated games with incomplete information is that there is no natural candidate for the payoff set. To see the difficulty, notice that this problem is not present in games with complete information, where it is immediate

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to show that all equilibrium payoffs must be feasible and individually rational, and the main difficulty is to find conditions under which all feasible and individually rational payoffs can be attained in subgame perfect equilibria. With incomplete information, the set of (naturally defined) feasible and individually rational payoffs is typically too large, since not all such payoffs can be attained (or even approximated) by equilibrium payoffs. Another candidate, the equilibrium payoff set obtained in the literature without discounting (Hart (1985), Shalev (1994), and Koren (1992)), is typically smaller than the set of payoffs that can be obtained in games with discounting.

Another difficulty is that the set of equilibrium payoffs typically depends on initial beliefs. Because any equilibrium play in which information is revealed has continuation play in a game with posterior beliefs that may differ from the prior, the payoff sets for different beliefs are related to each other. Thus, the characterization must simultaneously describe the *entire* equilibrium correspondence for all initial priors.

Our result solves the problem by first, providing a construction of a candidate equilibrium correspondence, and then, showing that no payoff outside of the correspondence can be attained in equilibrium. The idea is to consider payoffs in strategy profiles in which (i) there are finitely many periods in which players reveal information (by taking partially or fully separating actions), (ii) these periods are separated by long stretches of time during which players' types pool their actions, and, (iii) at each period, the continuation payoffs are individually rational. We begin with a set of individually rational payoffs in profiles in which no information is ever revealed. Next, we go through a sequence of steps. In each step, we construct a profile with continuation payoffs that belong to one of the earlier steps. We alternate between two kinds of constructions: (A) during the initial periods, players' types pool their actions or, (B) in the first period, the types reveal some substantial information. Each step has a simple geometric characterization. For sufficiently patient players, all the payoff vectors in the candidate equilibrium correspondence can be attained by payoffs in *finitely revealing equilibria*: sequential equilibria in which players information is revealed at most finitely many times.

For the second part, we assume that there exists an open set of payoffs in sequential equilibria in which during the first period of the game, all players fully reveal their information (i.e., they take fully separating actions), and such that the players are *ex post* indifferent between revealing their type truthfully or reporting any other type (i.e., they are indifferent conditional on any type of the opponent). The payoffs in such equilibria form a multi-linear cross-section (i.e., a *thread*) of payoffs across games with all possible initial beliefs. The assumption is relatively mild. It is generically satisfied in games with one-sided incomplete information, and in many important examples of games with multi-sided incomplete information (like oligopoly models). It is equivalent to the existence of belief-free equilibria of Horner and Lovo (2009), i.e., equilibria in which players' strategies form a Nash equilibrium in the complete information game with the true realized types. It is also required for the existence of equilibria in repeated games without discounting (Koren (1992)). However, there are robust examples of games with two players and incomplete information on both sides that do not satisfy the assumption.

Given the assumption, we show that *all* payoffs attained in Nash equilibria of the repeated game can be approximated by the payoffs obtained in the first part, i.e., by payoffs in finitely revealing equilibria. The proof constructs a finitely revealing profile that approximates a payoff in some Nash equilibrium. The idea is to modify the Nash continuation payoffs in order to pull them towards the thread of payoffs in the belief-free equilibria. We show that the continuation payoffs reach the neighborhood of the belief-free equilibrium payoff in finitely many periods. Once this happens, we finish the construction by immediate and full revelation of information. We believe that the argument is of independent interest, as it is very simple and possibly can be applied in other, related settings (like games with types that may slowly change over time).

The characterization of all equilibrium payoffs through finitely revealing profiles may seem intuitive at first sight. The beliefs of the players converge (as martingales), and, in any equilibrium, with high probability, no substantial information is revealed after finite number of periods. Nevertheless, this intuition does not lead to the proof as it leaves open the possibility that, after low probability histories, the continuation game requires large amount of information revelation. In fact, the intuition fails in repeated games with no discounting in which there are examples of games with equilibrium payoffs that cannot be approximated by finitely revealing profiles (see Forges (1984), Forges (1990) and Aumann and Hart (2003)).

The previous literature was not able to describe the equilibrium set except for very special cases. The closest to the current paper is Peski (2008) who characterized the equilibrium payoffs in games with incomplete information only on one side, and with the informed player having only two types. The current paper generalizes Peski (2008) to multiple types and multi-sided incomplete information. The characterization of payoffs in finitely revealing equilibria is a relatively straightforward extension of Peski (2008). The key step of the current paper, i.e., the argument that no other payoffs can be attained in equilibrium, is entirely novel. [1]

The main advantage of our characterization is that the payoffs in finitely revealing profiles are relatively easy to describe. We illustrate this claim with three examples. First, we discuss a model from Aumann and Maschler (1995) of bargaining over a pie with a cherry, where there is uncertainty about players' fondness for cherry. This model belongs to a class of games in which all feasible payoffs are individually rational. We show that in such games, all equilibrium payoffs can be approximated by payoffs in equilibria in which all players immediately and fully reveal their information.

Next, we discuss a class of oligopoly games. That class includes, as a knife-edge case, a Bertrand oligopoly with privately known production costs from Athey and Bagwell (2008). In that paper, the authors propose mechanism design methods for analyzing repeated games with incomplete information. They describe the equilibrium that maximizes the sum of ex ante payoffs among all symmetric equilibria, and they show that no information is revealed in such an equilibrium. Here, we explain that there is a relation between the mechanism design approach and equilibria in which all players fully and immediately reveal their information. We show that in oligopoly games all equilibrium payoffs can be attained by such equilibria, and can

[1] To compare, notice that Peski (2008) uses a much more complicated differential technique that, despite our best efforts, could not be extended beyond the two-type, one-sided case. Also, notice that Athey and Bagwell (2008) (see below) solve the second part of the argument using a sophisticated approach from the mechanism design literature and by making assumptions like log-concavity of the cost distribution. Their approach does not seem to generalize well beyond the particular example they analyze.

be derived as solutions to a simple mechanism design problem. We use the explicit description of payoffs to show that some (and, in some cases, complete) productive efficiency can typically be obtained in the Pareto-dominant equilibrium. In particular, we argue that the "pooling" result is not robust to alternative demand specifications.

In the third example, we discuss a bargaining game with two players, one-sided incomplete information, and two types (normal and "strong") of the informed player. We assume that the game between the normal type and the uninformed player has strictly conflicting interests (Schmidt (1993)). The strong type's payoffs are parametrized as a convex combination between the payoffs of the normal type and the payoffs of a player who is committed to play a single action (i.e., for whom repetition of the single action is a dominant strategy in the repeated game). We describe the Pareto frontier of the equilibrium set as a solution to a system of differential equations. We show that there are efficient equilibria that require any arbitrarily large number of periods with information revelation. When the payoffs of the strong type converge to the payoffs of the committed player, all equilibrium payoffs converge to the Stackelberg outcome of the informed player.

We compare our characterization to the literature on repeated games without discounting (see Aumann and Maschler (1995)). That literature typically considers the *general payoff case*, in which players' payoffs may depend on their opponents' type.[2] Hart (1985) considers one-sided uncertainty with general payoffs and characterizes the equilibrium payoffs as values of bi-martingales. Step B from our characterization corresponds directly to one of the defining properties of the bi-martingales. Because the initial periods do not matter in the no-discounting case, step A differs from its analog in the no-discounting case. The characterization in the general-payoff no-discounting case is not constructive, as there is no known algorithm that allows to find all equilibrium payoffs. In particular, one cannot construct the equilibrium set by repeated applications of (analogs of) operations A and B. This fact is related to the

[2] In order to avoid players learning about the other players' types from their own payoffs, the literature assumes that the payoffs are not observed until the end of the (infinite) repeated game. This assumption is not needed in the known-own payoffs case.

existence of games and equilibria that cannot be approximated by payoffs in finitely revealing equilibria that we mention above.

With known-own payoffs, one-sided uncertainty, and no discounting, all equilibrium payoffs can be obtained by strategies in which players reveal all their information in the first period (Shalev (1994) and Koren (1992)). We explain the similarities and the differences between the no-discounting and discounted cases in more detail in section 7.

Kreps and Wilson (1982a) and Milgrom and Roberts (1982) introduced a model of reputation with one-sided incomplete information about the type of the long-run informed player: strategic or commitment ("reputational") types. This literature was extended to equal discounting and patient players in Cripps and Thomas (1997), Chan (2000), and Cripps et al. (2005). Because, in the reputational model, the highest payoff of the commitment type is equal to his minmax payoffs, this model does not have an open set of payoffs. On the other hand, a small perturbation of the reputational types' payoffs may create an open thread and restore the assumption. We can use the "nearby" models to test the predictions of the reputational literature. Our third example illustrates the robustness of the result of Cripps et al. (2005). In the same vein, Horner and Lovo (2009) argue that Chan (2000) result is not robust.

Cripps and Thomas (2003) are the first to study repeated games with one-sided incomplete information and equal discounting. They look at the limit correspondence of payoffs when the probability of one of the types is close to 1.[3] They show that the set of payoffs of the uninformed player and the high probability type is close to the folk theorem payoff set from a complete information game. Cripps and Thomas (1997) and Chan (2000) ask similar questions within the framework of reputation games. All these results are proved by the construction of finitely revealing equilibria.

Horner and Lovo (2009) study the general payoff case with multi-sided incomplete information and they characterize the set of payoffs obtained in *belief-free* equilibria.

[3]Cripps and Thomas (2003) also discuss the limit of payoff sets when the two players become infinitely patient, but player I becomes patient much more quickly than player U. Their characterization is closely related to Shalev and Koren's results for the no-discounting case.

Horner et al. (2011) describe detailed conditions for information structures in N-player games under which the belief-free equilibria exist for *all* payoff functions. Our main result is limited to games in which the belief-free equilibria exist. However, our characterization of equilibrium payoffs is *not* limited to such equilibria. In particular, even if the belief-free equilibria exist, they may not capture all equilibrium payoffs, or even, not all efficient equilibrium payoffs. (See example at the end of section 6.2 and in section 6.3).

There are other related papers on repeated games with discounting but with different kind of incomplete information. Wiseman (2005) considers the situation in which the payoffs are initially unknown by all players (i.e., there is no asymmetric incomplete information), and the players learn the payoff function from observing the realization of their payoffs over time. Fudenberg and Yamamoto (2010) and Fudenberg and Yamamoto (2011) study the case where the payoffs and the monitoring structure are initially unknown, and the players may start the game with private information about the state of the world. The players learn over time by observing signals. The authors find conditions on the informativeness of the signals that ensure that in equilibrium players can learn the state very quickly and the set of equilibrium payoffs obtained in each state is equal to the folk theorem set in the complete information game *as if* the players knew the state from the beginning. In their setting, the set of payoffs is not affected by initially incomplete information.

The next section describes the model and preliminary results. Section 3 describes the geometric construction of the candidate payoff set. Section 4 shows that each element of the payoff set can be attained in finitely revealing equilibria. Section 5 shows that given the existence of an open set of payoffs in belief-free equilibria, each Nash equilibrium payoff can be approximated by a payoff in a finitely revealing equilibrium. We illustrate the result with examples in Section 6. Section 7 discusses the relation to the no-discounting literature. Section 8 concludes. Most of the proofs are postponed to the Appendix.

2. Model

2.1. **Repeated game.** For each set $X \subseteq R^d$, we write int$X$, cl$X$, and con$X$ to denote the interior, closure, and convexification of X. For each $u \in R^d$, each $\varepsilon > 0$, let $B(u, \varepsilon) = \{u' : \sup_i |u_i - u'_i| < \varepsilon\}$ be an open ball in the "sup" metric.

There are I players, $i = 1, 2, ..., I$. In each period $t \geq 0$, each player i takes an action a_i from finite set A_i and receives payoffs $g_i(a_i, a_{-i}, \theta_i)$. The payoffs depend on the actions of all players and on the privately known type θ_i of player i (*known-payoff case*). We assume that $|A_i| \geq |\Theta_i|$ for each player i. Let $M = \max_{i,a,\theta_i} |g_i(a, \theta_i)| < \infty$ be an upper bound on the absolute value of payoffs.

The type of player i is chosen by Nature from finite set Θ_i and revealed to player i before the first period of the repeated game. We write $\Theta_{-i} = \times_{j \neq i} \Theta_j$ to denote the set of type tuples of all players but i, and $\Theta = \times_i \Theta_i$ to denote the set of type profiles. We also write $\Theta^* = \Theta_1 \cup ... \cup \Theta_I$ to denote the disjoint union of the sets of types for each player.

We encode the payoffs of different types of different players as a tuple $v = (v_i(\theta_i))_{i,\theta_i} \in R^{\Theta^*}$ with an interpretation that $v_i(\theta_i)$ is the (expected) payoff of type θ_i of player i. We write $v_i \in R^{\Theta_i}$ to refer to the component of v that consists of payoffs of player i's types.[4]

Each type θ_i of player i starts the game with beliefs $\pi^{\theta_i} \in \Delta\Theta_{-i}$ about the distribution of the other players' types. The beliefs may differ across types and we do not assume that they are derived from a common prior. However, we assume that all types of all players agree on which types have positive or zero probability. Precisely, from now on, we assume that each belief system $\pi = \left(\pi^{\theta_i}\right)_{i,\theta_i \in \Theta_i}$ satisfies *common rectangular support* property: for each player j, there exists set $\Theta_j^\pi \subseteq \Theta_j$ such that for each type θ_i of each player i, $\pi^{\theta_i}(\theta_{-i}) > 0$ if and only if $\theta_{-i} \in \times_{j \neq i} \Theta_j^\pi$. We refer to Θ_j^π as the π-support of player j. We say that type θ_j has π-positive probability if

[4]The convention of encoding payoffs *given one's own type* follows Peski (2008) and differs from some other papers in the literature. For example, Horner and Lovo (2009) write $v \in R^{I \times \Theta}$ to denote the vector of payoffs of players *given the realization of the entire type profile*, and not only player's own type. Our convention is simpler and more natural in the known-own payoff case.

$\theta_j \in \Theta_j^{\pi}$, and π-zero probability otherwise. Let Π denote the space of belief systems with common rectangular support.

Players discount the future with common discount factor $\delta < 1$. We refer to the game with discount factor δ and initial beliefs π as $\Gamma(\pi, \delta)$.

For simplicity, we assume that players have access to public randomization. As it is standard practice in the literature, we omit the reference to public randomization in the formal definition of a history.

Let $H_t = A^t$ be the set of t-period histories $h_t = (a_s)_{s=0}^{t-1}$. A (repeated game) strategy of player i is a mapping $\sigma_i : \Theta_i \times \bigcup_t H_t \to \Delta A_i$. For any profile $\sigma = (\sigma_i)_i$ of such strategies, let

$$v^{\pi,\delta}(\sigma) = (1-\delta) \sum\nolimits_{\theta_{-i} \in \Theta_{-i}} \pi^{\theta_i}(\theta_{-i}) E_{\sigma(\theta_i, \theta_{-i})} \sum\nolimits_t \delta^t g(a_t; \theta_i) \in R^{\Theta^*}$$

denote the (normalized) expected payoff of player i type θ_i, where the expectation is computed with respect to distribution over histories induced by strategies σ and given types (θ_i, θ_{-i}). Let $v^{\pi,\delta}(\sigma) \in R^{\Theta^*}$ denote the (normalized) expected payoff vector.

2.2. **Feasible, non-revealing payoffs.** Two sets play an important role in our characterization. The first set consists of stage-game payoffs obtained when all types of each player pool their actions. For each action profile $a = (a_i) \in A \equiv \times_i A_i$, let $g(a) = (g_i(a, \theta_i))_{i,\theta_i} \in R^{\Theta^*}$ be the payoff vector obtained when each type of player i plays the same action a_i. Let

$$V = \text{con}\{g(a) : a \in A\} \subseteq R^{\Theta^*}.$$

be the convex hull of payoff vectors $g(a)$. We refer to V as *feasible, non-revealing* (i.e., pooling) *payoffs*.

It is important to note that V is not the set of all feasible payoffs in game $\Gamma(\pi, \delta)$. The latter can be defined as the convex hull of payoff vectors $v^{\pi,\delta}(\sigma)$ for all strategy profiles σ, including profiles in which players types do not pool their actions.

2.3. **Individual rationality.** The second set consists of individually rational payoffs. We follow Blackwell (1956) who solved the problem of extending individual rationality to the incomplete information case (see also Peski (2008) or Horner and Lovo (2009) for games with discounting). Define weighted minmax of player i: for each for each $\phi \in R_{+}^{\Theta_i}$

$$m_i(\phi) := \min_{\alpha_{-i} \in \times_{j \neq i} \Delta A_j} \max_{\alpha_i \in \Delta A_i} \sum\nolimits_{\theta_i} \phi_{\theta_i} g(\alpha_i, \alpha_{-i} | \theta_i). \tag{2.1}$$

Define the set of *individually rational* payoffs as

$$IR = \left\{ v \in \mathtt{R}^{\Theta^*} : \forall i \forall \phi \in R_{+}^{\Theta_i},\ \phi \cdot v_i \geq m_i(\phi) \right\}.$$

2.4. **Equilibrium.** A strategy profile σ is a *(Bayesian) Nash equilibrium* in game $\Gamma(\pi, \delta)$ for some $\pi \in \Pi$ if for each player i type θ_i, strategy $\sigma_i(\theta_i)$ is the best response of type θ_i. One shows that any payoff vector in a Nash equilibrium must belong to set IR.

A strategy profile σ is *totally mixed* if for each player i, type θ_i, history h_t, action a^i, $\sigma_i(a_i | h_t, \theta_i) > 0$. Each totally mixed strategy profile σ together with the initial belief system $\pi \in \Pi$ induces through the Bayes formula well-defined belief mapping $p^{(\sigma,\pi)} : \bigcup_t H_t \to \Pi$. (Notice that if the initial beliefs have common rectangular support, then the posterior beliefs also have common rectangular support.) For any strategy profile σ, say that belief mapping $p : \bigcup_t H_t \to \Pi$ is $(\sigma, \pi)-$*consistent*, if there exists a sequence of totally mixed strategy profiles $\sigma_n \to \sigma$ such that $p^{(\sigma_n,\pi)} \to p$.[5] If history h_t *has a positive probability*, i.e., if for each player i,

$$\prod\nolimits_{s<t} \sigma_i\left(a_s^i | h_s, \theta_i\right) > 0,$$

then $p(h_t)$ does not depend on the choice of sequence σ_n. We use this observation without any further reminder.

A strategy profile σ is a *sequential equilibrium* in game $\Gamma(\pi, \delta)$ if there exists (σ, π)-consistent belief mapping p such that for each player i type θ_i, history h_t, continuation strategy $\sigma_i(h_t, .)$ is the best response to continuation strategy $\sigma_{-i}(h_t, .)$ given beliefs $p_i(h_t)$. A sequential equilibrium is *n-revealing* if for any positive probability history

[5] In both cases, we use the "pointwise" notion of convergence, i.e., $\sigma_n \to \sigma$ if and only if $\sigma_n(\theta_i, h) \to \sigma(\theta_i, h)$ for each type θ_i and history h. Our analysis would not be affected if instead we used the "uniform" convergence across infinitely many histories. This is, despite the fact that in general, different notions of convergence lead to different definitions of sequential equilibrium. (Notice that the original definition of sequential equilibrium from Kreps and Wilson (1982b) applies only to finite games.)

h, there exists at most n periods t such that $p(h_t) \neq p(h_{t-1})$. A *finitely revealing equilibrium* is a sequential equilibrium profile σ that is n-revealing for some n.

Let $NE^{\delta}(\pi)$, $FR_n^{\delta}(\pi) \subseteq R^{\Theta^*}$ be the sets of expected payoff vectors $v^{\pi,\delta}(\sigma)$ in, respectively, Nash, and n-revealing equilibria σ.

We are going to simplify our description of the equilibrium correspondences by focusing on the payoffs of the positive probability types (see, among others, Hart (1985) and Aumann and Hart (2003) for analogous approach). For any belief system $\pi \in \Pi$, any two payoff vectors $v, v' \in R^{\Theta^*}$, write $v \preceq_{\pi} v'$ if $v(\theta_i) \leq v'(\theta_i)$ for all player i type θ_i and $v(\theta_i) = v'(\theta_i)$ for all π-positive probability types θ_i. In other words, vector v' contains exactly the same payoffs for π-positive probability types and possibly higher payoffs for the zero-probability types. For each set $A \subseteq R^{\Theta^*}$, define

$$A^{\pi+} = \{v' : \exists v \in A \text{ st. } v \preceq_{\pi} v'\}.$$

For any payoff correspondence $E(\pi)$, define the enhancement of E as a payoff correspondence E^+ such that $E^+(\pi) = (E(\pi))^{\pi+}$ for each π. If $E = E^+$, we say that correspondence E is *enhanced*.

2.5. **Comments.** Our analysis of repeated games of incomplete information is restricted to the known-own payoffs case. There are two reasons for this assumption. The first reason is technical. In the known-own payoffs case, the set of i's individually rational payoffs, as well as $-i$'s strategy that minmaxes player i do not depend on the beliefs of player i or the type of player $-i$ (see section 2.3 above or Lemma 4 from appendix B). This fact allows us to construct equilibria in which minmax strategies are used without any information revelation; we only need to make sure to choose continuation payoffs so that all types of the minmaxing players have incentives to randomize with the same probabilities between all pure strategies in the support of (possibly, mixed) minmax strategy.

On the other hand, f player i payoffs depend on the information of player $-i$, the value of player i minmax may depend on his beliefs about the type of player $-i$. It follows that in order to punish player i, player $-i$ punishing strategy may depend on $-i$'s type. This complicates using the punishment in sequential equilibria.

The second reason is substantive. If players' payoffs depend on the type of the other player, they may learn information about the type of the other players simply from observing their own payoffs. In order to avoid such a learning, the literature typically assumes that the players do not observe their own payoffs before the end of the repeated game. Although one can imagine scenarios under which waiting infinitely many periods for the revelation of payoffs is not unrealistic in the no-discounting case (for example, see Aumann and Hart (2003)), we are not aware of a credible story that would justify such an assumption in the discounted case.

We restrict our analysis to belief systems that have common rectangular support. The restriction ensures that all players can agree with each other (and with an outside observer) on which types have positive or zero probability. Moreover, because Bayesian updating respects the restriction, it is inherited along the equilibrium path. The distinction between the zero and positive probability types is important because their payoffs are analyzed differently (see the above definition of enhanced payoffs).

3. Payoff correspondence

In this section, we define a candidate payoff correspondence $F^* : \Pi \rightrightarrows R^{\Theta^*}$ as the smallest correspondence that satisfies three conditions.

Payoffs in non-revealing equilibria. For each belief system π, let

$$F_0(\pi) = \operatorname{int}\left(IR \cap V^{\pi+}\right). \tag{3.1}$$

It is well known that correspondence F_0 is equal to the payoffs in equilibria in which no information is revealed (see Hart (1985), Koren (1992), and Shalev (1994) for Nash equilibrium and no-discounting, and Peski (2008), and Horner and Lovo (2009) for the sequential equilibrium in the discounted case). To see an intuition for this observation, suppose that π is a full support belief system. Then, $F_0(\pi) = \operatorname{int}(IR \cap V)$. If such a set is non-empty, there exists a (possibly, correlated) action profile a such that $g(a) \in F_0(\pi)$. One can construct equilibria, in which a is played on the equilibrium path and deviations by single player are punished with Blackwell's minmaxing.

Initially pooling actions. For each correspondence F, define correspondence $\mathcal{A}F$: for each belief π, let

$$\mathcal{A}F(\pi) = \operatorname{int}(IR \cap \operatorname{con}(F(\pi) \cup V)).$$

Correspondence $\mathcal{A}F$ contains all individually rational payoffs $u = \alpha v + (1-\alpha)u'$ that can be obtained as convex combination of a (possibly, not individually rational) non-revealing payoff v and payoff vector $u' \in F(\pi)$. If u' is an expected payoff in some strategy profile, then u is a payoff in a profile in which, initially, players types pool their behavior on profile a, and after t periods, where $\delta^t = \alpha$, players continue with the original profile with payoffs u'.

Revelation of information. Information is revealed (possibly, only partially) whenever different types of a player play different (possibly, mixed) actions.

Let $\pi \in \Pi$ be the initial belief system. We represent the revelation of information in the form of *continuation lottery* $l = (\alpha, u)$, where $\alpha = (\alpha_i)$ is a profile of the first-period strategies $\alpha_i : \Theta_i \to \Delta A_i$, and $u : A \to R^{\Theta^*}$ is an assignment of continuation payoffs following the realization of the first-period actions. We consider only strategies α in which all actions a_i that are played with positive probability by some type (i.e., there exists θ_i such that $\alpha_i(a_i|\theta_i) > 0$), are played with positive probability by some π-positive probability type. This allows us to use the Bayes formula to compute thee posterior belief system define $p^{\pi,l}(a) = \left(p^{\pi,l,\theta_i}(a_{-i})\right)_{i,\theta_i}$ following positive probability action profile a. Notice that the beliefs of player i depend only on the actions chosen by other players. Additionally, we require that lottery l ensures that all types of all players are indifferent between all positive probability actions:

$$E_{\pi^{\theta_i}} u_i(a_i, \alpha_{-i}(\theta_{-i}), \theta_i) = E_{\pi^{\theta_i}} u_i(\alpha_i(\theta_i), \alpha_{-i}(\theta_{-i}), \theta_i). \tag{3.2}$$

(We ignore the payoff consequences of playing actions for one period.) Define the value of the lottery l as a payoff vector $v^{\pi,l} \in R^{\Theta^*}$ such that for each player i and type θ_i, $v_i^{\pi,l}(\theta_i)$ is equal to (3.2). Let $L(\pi)$ denote the set of lotteries that satisfies the above conditions.

The incentive condition (3.2) requires that all types of all players are indifferent between all (positive probability) actions, including actions a_i that type θ_i is not supposed to play with positive probability, $\alpha_i(a_i|\theta_i) = 0$. This is stronger than a typical incentive condition, which only requires weak inequality. However, this is without loss of generality: due to the enhancement property, we can always increase the continuation payoffs of type θ_i after action a_i so to replace weak inequality by equality.

For each correspondence F, define correspondence $\mathcal{B}F$: for each belief π, let

$$\mathcal{B}F\left(\pi\right)=\left\{v^{\pi,l}:l\in L\left(\pi\right)\text{ and }u\left(a\right)\in F\left(p^{\pi,l}\left(a\right)\right)\text{ for each pos. prob. }a\right\}.$$

Set $\mathcal{B}F\left(\pi\right)$ contains the values of all continuation lotteries payoffs with prior belief π and with posteriors and continuation payoffs vectors that belong to correspondence F.

For any two payoff correspondences $F,G\rightrightarrows R^{\Theta^*}$, write $F\subseteq G$ if $F\left(\pi\right)\subseteq G(\pi)$ for any belief system. The next result follows immediately from the fact that operations $\mathcal{A}$ and $\mathcal{B}$ are monotonic: for any two correspondences, if $F\subseteq G$, then $\mathcal{A}F\subseteq\mathcal{A}G$ and $\mathcal{B}F\subseteq\mathcal{B}G$.

Theorem 1. *There exists the smallest correspondence F^* such that $F_0\subseteq F^*$, $\mathcal{A}F^*\subseteq F^*$, and $\mathcal{B}F^*\subseteq F^*$. Moreover, $F^*=\bigcup_n F_n^B$, where $F_1^A=\mathcal{A}F_0$, and, for each $n\geq 1$, $F_n^B=\mathcal{B}F_n^A$ and $F_{n+1}^A=\mathcal{A}F_n^B$.*

The Theorem defines correspondence F^* as the smallest correspondence that contains F and that is closed with respect to operations $\mathcal{A}$ and $\mathcal{B}$. Additionally, the Theorem provides a method of constructing set F by alternating application of the two operations to initial correspondence F_0. Each step has a simple geometric characterization. In general, it is not possible to simplify the description by eliminating any of the steps (section 6.3 contains an example of a game and constructions for which all steps are required).

For future reference, notice that correspondences F, F_n^A, and F_n^B are enhanced. This follows from the fact that correspondence F_0 is enhanced, and that operations $\mathcal{A}$ and $\mathcal{B}$ preserve the enhancement property.

4. Finitely Revealing payoffs

In this section, we show that correspondence F^* is a lower bound on the set of payoffs obtained in finitely revealing equilibria. For each n, define the limit payoff correspondences[6]

$$FR_n(\pi) = \liminf_{\delta \to 1} FR_n^{\delta}(\pi).$$

Let $FR(\pi) = \bigcup_n FR_n(\pi)$ be the limit set of payoffs in finitely revealing equilibria.

Theorem 2. *For each $\pi \in \Pi$, $F^*(\pi) \subseteq FR^+(\pi)$ and for each n, $F_n^B(\pi) \subseteq F_{n+1}^A(\pi) \subseteq FR_n^+(\pi)$.*

The proof goes by induction on n and it is contained in Appendix B. In each step, we construct finitely revealing equilibria with the required payoffs. The constructions are relatively standard and rely on techniques originated in Fudenberg and Maskin (1986) adapted to games with incomplete information. (See also constructions used in Cripps and Thomas (2003) and Peski (2008)).

5. Equilibrium payoffs

In this section, we state our main assumption and show that under this assumption, all Nash equilibrium payoffs are contained in the closure of correspondence F^*.

5.1. **Open thread assumption.** For each type profile $\theta = (\theta_i)_i \in \Theta$, let $\pi^\theta \in \Pi$ be the belief system in which all types of player i assign full probability to the opponents' profile θ_{-i}.

A *thread* is an assignment $u^* : \Theta \to R^{\Theta^*}$ of payoff vectors to type profiles such that (a) for each type profile $\theta \in \Theta$, $u^*(\theta)$ is an (enhanced) payoff vector in a non-revealing equilibrium in game with initial beliefs π^θ,

$$u^*(\theta) \in \mathrm{cl} F_0\left(\pi^\theta\right),$$

and (b) for each player i, all types $\theta_i, \theta_i' \in \Theta_i$, and all type profiles $\theta_{-i} \in \Theta_{-i}$,

$$u_i^*(\theta_i, \theta_{-i}) = u_i^*(\theta_i', \theta_{-i}).$$

The second condition ensures *ex post* incentive compatibility for player i. We say that there exists an *open thread* if u^* can be chosen so that $u_i^*(\theta) \in F_0\left(\pi^\theta\right)$.

[6] The infimum limit $\liminf_{\delta\to 1} FR_n^\delta(\pi)$ is defined as the set of payoff vectors v such that for each sequence $\delta_n \to 1$, there exists sequence $v_n \to v$ and such that $v_n \in FR_n^{\delta_n}(\pi)$. It is the greatest lower bound on the set of accumulation points.

For each open thread u^*, each $\pi \in \Pi$, define $u^*(\pi) \in R^{\Theta^*}$ so that for each player i type θ_i,

$$u_i^*(\theta_i|\pi) = \sum_{\theta_i,\theta_{-i}} \pi^{\theta_i}(\theta_{-i})\, u^*(\theta_i|.,\theta_{-i})\,.$$

($u^*(\theta_i|.,\theta_{-i})$ is equal to θ_i-coordinate of the payoff vector $u^*(\theta_i',\theta_{-i})$ for some θ_i'; by the assumption, this value does not depend on the choice of θ_i'.). By Theorem 2, $u^*(\pi)$ is a payoff vector in a fully and immediately revealing equilibrium of the game with initial beliefs π and sufficiently high δ. We say that $u^*(\pi)$ forms a multi-linear thread of equilibrium payoffs that passes through games $\Gamma(\pi,\delta)$ for each $\pi \in \Pi$.

All games with two players and one-sided incomplete information have a thread. This follows from the analysis of the non-discounted games in Shalev (1994) (see also Peski (2008)). Additionally, in the case of two players, the threads are essentially equivalent to payoffs in belief-free equilibria of Horner and Lovo (2009), and the existence of a thread is a necessary and of an open thread is a sufficient condition for the existence of such equilibria (see Appendix A).

5.2. **Main result.** Our main result provides a characterization of the set of equilibrium payoffs. Define the limit payoff correspondence[7]

$$NE(\pi) = \limsup_{\delta\to 1} NE^{\delta}(\pi)\,.$$

Theorem 3. *If there exists an open thread, then,*

$$clNE^{+}(\pi) = clF^{*}(\pi)\,.$$

The Theorem provides a characterization of the limit set of payoffs in Nash equilibria of the repeated games with incomplete information. It shows that all Nash equilibrium payoffs for a sufficiently high discount factor can be approximated by payoffs in finitely revealing equilibria that were constructed in Theorem 2. Because finitely revealing equilibria are sequential, the Theorem shows that the repeated games with incomplete information preserve a folk-theorem-like feature of games with complete

[7]The supremum limit $\limsup_{\delta\to 1} NE^{\delta}(\pi)$ is defined as the set of payoff vectors v such that there exists sequences $v_n \to v$ and $\delta_n \to 1$, such that $v_n \in NE_n^{\delta_n}(\pi)$. It is the smallest upper bound on the set of accumulation points.

information in which all payoffs in Nash equilibria can be approximated by payoffs in subgame perfect equilibria.

Together with Theorem 1, Theorem 3 provides a method to find an explicit description of the set of equilibria payoffs. We illustrate the method on examples in section 6.

We explain below that the open thread assumption plays an important role in the proof. We do not know whether the result holds in games that do not satisfy the assumption.

The proof shows that any Nash equilibrium profile in game $\Gamma(\pi, \delta)$ with expected payoffs v can be modified into a profile with expected payoff that belongs to $F^*(\pi)$ and that is arbitrarily close to v. The idea is to modify the original Nash profile so to pull the continuation payoffs towards the multi-linear thread u^*. Once the continuation payoffs get sufficiently close to the thread, we conclude the modified profile with one period of full revelation of information followed by an equilibrium of the "complete" information game.

To see how it works in an example, suppose that v is a payoff in a Nash profile σ in which during the first period the players choose non-revealing action profile a (i.e., all types of each player i play the same action a_i). Let $v(a)$ be the equilibrium continuation payoffs (we can always choose strategies in such a way so that the continuation payoff after positive probability history is a payoff in a Nash equilibrium). Then, v is a convex combination of instantaneous payoffs $g(a)$ and the equilibrium continuation payoffs $v(a)$, $v = (1-\delta) g(a) + \delta v(a)$. See Figure 5.1.

Suppose that v' is a payoff vector that is a convex combination between v and the value of the thread $u^*(\pi)$, $v' = \gamma v + (1-\gamma) u^*(\pi)$. We can find a vector $v'(a)$ such that

- v' is a convex combination between $v'(a)$ and $g(a)$, $v' = (1-\delta') g(a) + \delta' v'(a)$. Thus, we can interpret v' as a payoff in a profile that starts with action a followed by continuation payoffs $v'(a)$, in a game with discount factor $\delta' > \delta$.
- $v'(a)$ is a convex combination between $v(a)$ and the thread $u^*(\pi)$, $v'(a) = \gamma' v(a) + (1-\gamma') u^*(\pi)$.

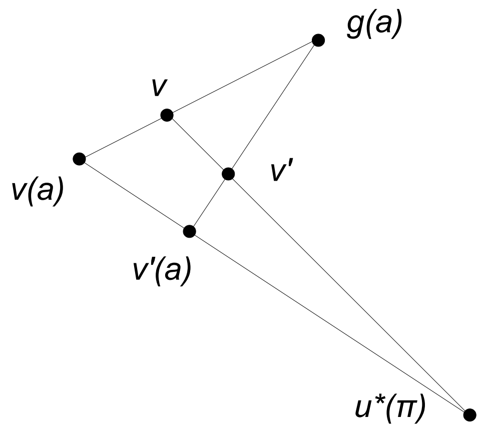

FIGURE 5.1.

Simple algebra shows that

$$\gamma = \frac{\gamma'}{\gamma'(1-\delta)+\delta}$$

which implies that $\gamma' < \gamma$. Thus, the relative distance between $v'(a)$ and the thread $u^*(\pi)$ is smaller than the relative distance between v' and the thread.

The above procedure can be applied as long as the players choose non-revealing action profile a, leads to continuation payoffs getting closer and closer to the thread. If some information is revealed, we show that the continuation payoffs have the same relative distance to the value of the thread in games with new posterior beliefs, as the relative distance of the original payoffs to the thread in the game with prior beliefs. The argument relies on the fact that the expected payoff in the continuation lottery is a convex combination of the continuation payoffs and that the prior belief is a convex combination of the posterior beliefs. The multi-linearity of thread $u^*$is essential for the argument.

Formally, Theorem 3 follows from two inclusions

$$F^*(\pi) \subseteq FR^+(\pi) \subseteq NE^+(\pi) \tag{5.1}$$

and

$$NE^{+}(\pi) \subseteq \text{cl}F^{*}(\pi). \tag{5.2}$$

The first inclusion is a consequence of Theorem 2. We need to show the other inclusion.

Suppose that $u^{*}(\pi)$ is an open thread. Let $r > 0$ be such that for all type profiles θ,

$$B\left(u^{*}\left(\pi^{\theta}\right), r\right) \subseteq F_{0}\left(\pi^{\theta}\right).$$

For each $\delta < 1$, define $\gamma_{1}^{\delta} = \frac{r}{2M}$. For each $n > 1$, inductively define

$$\gamma_{n}^{\delta} = \frac{\gamma_{n-1}^{\delta}}{\gamma_{n-1}^{\delta}(1-\delta)+\delta} \in \left(\gamma_{n-1}^{\delta}, 1\right). \tag{5.3}$$

Notice that $\gamma_{n}^{\delta} > \gamma_{n-1}^{\delta}$ and $\lim_{n\to\infty} \gamma_{n}^{\delta} = 1$. Inclusion (5.2) follows from the following result.

Lemma 1. *For each n such that $\left(1-\gamma_{n}^{\delta}\right) r > (1-\delta) M$, for each $\pi \in \Pi$, each $v \in NE^{\delta}(\pi)$,*

$$\gamma_{n}^{\delta} v + \left(1-\gamma_{n}^{\delta}\right) u^{*}(\pi) \subseteq intF_{n}^{B}(\pi)$$

5.3. **Proof of Lemma 1.** The proof of Lemma 1 goes by induction on n. First, we show the inductive claim for $n = 1$. Because $\|v\| \leq M$ for each $v \in NE^{\delta}(\pi)$, we have

$$\frac{r}{2M} v + \left(1 - \frac{r}{2M}\right) u^{*}(\pi) \in B\left(u^{*}(\pi), r\right) \subseteq F_{1}^{B}(\pi).$$

The inclusion comes from Theorem 2 and the definition of an open thread.

Next, suppose that the inductive claim holds for $n-1$. Take any prior beliefs π and Nash payoff vector $v \in NE^{\delta}(\pi)$. Find an equilibrium profile σ that supports v. Say that action a_i is played with positive probability by player i in the first period if there exists π-positive probability type θ_i such that $\sigma_i(a_i|\varnothing, \theta_i) > 0$. Let A_i^0 denote the set of actions played with positive probability by some .

We assume without loss of generality that the continuation strategies are the best responses for all players and all types after all positive probability histories. (If Nash profile σ does not have such a property, it can be easily modified without affecting the initial payoffs and equilibrium conditions.)

Non-revealing payoffs. For each positive probability action profile $a \in A^0 := \times_i A_i^0$, each type θ_i, let

$$v(a) = \left(v_i^{p(a)}(\sigma(a,.);\theta_i)\right)_{i,\theta_i} \in R^{\Theta^*}$$

be the vector of continuation payoffs after a. Because a occurs with positive probability, $v(a)$ is a Nash equilibrium payoff in game $\Gamma(p(a),\delta)$. By the inductive assumption,

$$\gamma_{n-1}^{\delta} v(a) + \left(1-\gamma_{n-1}^{\delta}\right) u^*(p(a)) \in \mathrm{int} F_{n-1}^{B}(p(a)).$$

Define

$$u(a) = (1-\delta) g(a) + \delta v(a).$$

Using (5.3), we get

$$\begin{aligned}
&\gamma_n^{\delta} u(a) + \left(1-\gamma_n^{\delta}\right) u^*(p(a)) \\
=&\gamma_n^{\delta}\left[\delta v(a) + (1-\delta) g(a)\right] + \left(1-\gamma_n^{\delta}\right) u^*(p(a)) \qquad (5.4)\\
=&\left(1-(1-\delta)\gamma_n^{\delta}\right)\left[\gamma_{n-1}^{\delta} v(a) + \left(1-\gamma_{n-1}^{\delta}\right) u^*(p(a))\right] + (1-\delta)\gamma_n^{\delta} g(a) \qquad (5.5)\\
\in&\mathrm{intcon}\left(F_{n-1}^{B}(p(a)) \cup V\right).
\end{aligned}$$

Because $v(a)$ is a payoff in a Nash equilibrium, $v(a) \in IR$. Because $(1-\delta) M \leq \left(1-\gamma_{n-1}^{\delta}\right) r$, it must be that

$$\gamma_n^{\delta}\left[\delta v(a) + (1-\delta) g(a)\right] + \left(1-\gamma_n^{\delta}\right) u^*(p(a)) \in \mathrm{int} IR. \qquad (5.6)$$

Then, (5.5) and (5.6) imply that for each positive probability a,

$$\gamma_n^{\delta} u(a) + \left(1-\gamma_n^{\delta}\right) u^*(p(a)) \in F_n^{A}(p(a)) \qquad (5.7)$$

Revelation of information. For each π-positive probability type θ_i, let

$$\alpha_i(\theta_i) = \sigma_i(\varnothing, \theta_i) \in \Delta A_i^0.$$

For each π-zero probability type θ_i, let

$$\alpha_i(\theta_i) \in \arg\max_{a_i \in A_i^0} u(a_i, \alpha_{-i}).$$

Because profile σ is a Nash equilibrium and because of the choice of $\alpha_i(\theta_i)$, for all types θ_i, all positive probability actions a_i,

$$E_{\pi^{\theta_i}} u_i(a_i, \alpha_{-i}(\theta_{-i}), \theta_i) \leq E_{\pi^{\theta_i}} u_i(\alpha_i(\theta_i), \alpha_{-i}(\theta_{-i}), \theta_i).$$

The inequality turns into equality for all actions a_i that are played with positive probability by type θ_i. We can replace the inequality by equality for all actions a_i by enhancing the continuation payoffs $u(\theta_i|a)$ of types θ_i that do not play action a_i in strategy α_i (i.e., $\alpha_i(a_i|\theta_i)=0$). Because correspondence F_n^A is enhanced (see remark at the end of section 3), (5.7) holds for the enhanced continuation payoffs.

The above implies that the continuation lottery $l=(\alpha,u)$ satisfies (3.2) and it belongs to set $L(\pi)$ (we use the same symbol u to denote the enhanced continuation payoffs). Consider lottery $l'=\left(\alpha,\gamma_n^\delta u(.)+\left(1-\gamma_n^\delta\right)u^*(p(.))\right)$. The properties of the thread u^* imply that for each positive probability $a_i\in A_i$, all types θ_i, θ_{-i},

$$\begin{aligned}
&E_{\pi^{\theta_i}}E_{\alpha_{-i}(\theta_{-i})}u^*(\theta_i|p(a_i,a_{-i}))\\
&=\sum\nolimits_{\theta_{-i},a_{-i},\theta'_{-i}}\pi^{\theta_i}(\theta_{-i})\,\alpha_{-i}(a_{-i};\theta_{-i})\,p^{\theta_i}\left(\theta'_{-i}|a_i,a_{-i}\right)u_i^*\left(\theta_i|.,\theta'_{-i}\right)\\
&=\sum\nolimits_{\theta'_{-i}}\pi^{\theta_i}\left(\theta'_{-i}\right)u_i^*\left(\theta_i|.,\theta'_{-i}\right)\\
&=u_i^*(\theta_i|\pi).
\end{aligned}\tag{5.8}$$

In particular, the first line of (5.8) does not depend on positive probability action a_i. Together with the fact that lottery $l\in L(\pi)$, the above implies that lottery l' satisfies (3.2) for each type θ_i.

The value of lottery l' is equal to

$$v^{\pi,l'}=\gamma_n^\delta v^{\pi,l}+\left(1-\gamma_n^\delta\right)u^*(\pi),$$

where $v_i^{\pi,l}$ is the value of lottery l. Then, (5.7) implies that

$$\gamma_n^\delta v^{\pi,l}+\left(1-\gamma_n^\delta\right)u^*(\pi)=v^{\pi,l'}\in\operatorname{int}F_n^B(\pi).$$

Notice that

$$\begin{aligned}
&v_i^{\pi,l}(\theta_i)=v_i(\theta_i)\ \text{ for } \pi\text{-positive probability } \theta_i,\\
&v_i^{\pi,l}(\theta_i)\le v_i(\theta_i)=v_i^{\pi,\delta}(\sigma;\theta_i)\ \text{ for } \pi\text{-zero probability } \theta_i.
\end{aligned}$$

The latter follows from the fact that action $\alpha_i(\theta_i)$ is not necessarily the best response action of zero probability type θ_i. Because correspondence F_n^B is enhanced,

$$\gamma_n^\delta v+\left(1-\gamma_n^\delta\right)u^*(\pi)\in F_n^B(\pi).$$

This ends the proof.

5.4. **Quality of approximation.** The proof of Theorem 3 leads to the following bounds on the quality of the approximation of the Nash equilibrium set by n-revealing sets F_n^B. Recall that M is an upper bound on the absolute value of the payoffs and $r > 0$ is the size of the open thread.

Corollary 1. *Let* $A = \max\left\{\frac{2M}{r}, 2\right\}$. *For each* $v \in NE^{\delta}(\pi)$, *each* $\epsilon > (1-\delta) A$, *and either* $n \geq \left\lceil \frac{2\log 2A}{\epsilon(1-\delta)} \right\rceil$, *or* $n \geq \frac{1}{(1-\delta)^2}$,

$$(1-\epsilon) v + \epsilon u^{*}(\pi) \in F_n^B(\pi).$$

Proof. We show first that for each $\delta \geq \frac{1}{2}$ and each $\epsilon > 0$, if $n \geq \left\lceil \frac{\log 2A}{\epsilon(1-\delta)} \right\rceil + 1$, then $\gamma_n^{\delta} \geq 1-\epsilon$. If not, then $\gamma_1^{\delta} \leq .... \leq \gamma_n^{\delta} \leq 1-\epsilon$, and

$$\gamma_n^{\delta} \geq \frac{1}{\delta + (1-\delta)(1-\epsilon)} \gamma_{n-1}^{\delta} = \frac{1}{1-(1-\delta)\epsilon} \gamma_n^{\delta} \geq \left(\frac{1}{1-(1-\delta)\epsilon}\right)^{n-1} \frac{1}{2A},$$

where the last inequality follows from the definition of $\gamma_1^{\delta} = \frac{r}{2M}$. Because

$$-\log(1-\epsilon(1-\delta)) \geq \epsilon(1-\delta),$$

we have a contradiction:

$$\gamma_n^{\delta} \geq e^{(n-1)\epsilon(1-\delta)} \frac{1}{2A} \geq 1 > 1-\epsilon.$$

Fix $v \in NE^{\delta}(\pi)$. Take any $\epsilon > (1-\delta) A$. By Lemma 1 and the convexity of set $F_n^B(\pi)$, $\gamma v + (1-\gamma) u * (\pi) \in F_n^B(\pi)$ for each $n \geq \left\lceil \frac{\log 2A}{\epsilon(1-\delta)} \right\rceil + 1$ and any $\gamma \leq \gamma_n^{\delta}$ such that $1-\gamma \leq 1-(1-\delta) A$. Letting $\gamma = 1-\epsilon$ establishes the first result.

For the second result, take $\epsilon = (1-\delta) A$, and observe that for $A \geq 2$, $\frac{\log 2A}{A} \leq 1$. The result follows from the first part. □

6. Examples

Theorem 1 describes an algorithm for finding all the finitely revealing payoffs. In this section, we illustrate the algorithm with three examples. In the first two examples, bargaining over a pie with a cherry and a class of oligopoly games with privately known costs, all equilibrium payoffs can be approximated by the payoffs in fully and immediately revealing equilibria. In the third example, a bargaining game with one-sided incomplete information, the set of equilibrium payoffs is substantially larger than 1-revealing payoffs. In fact, the equilibria that yield the maximal payoff for the uninformed party typically involve a large number of revelation periods.

6.1. **A pie with a cherry.** In the first pages of their book, Aumann and Maschler (1995) describe repeated bargaining over a pie with a cherry. A version of this model goes like this. In each period, two players must divide a pie. The pie has two parts: with and without a cherry. In each stage, player 1 proposes a division of the two parts (x, y), where $x, y \in (0,1)$. Player 2 accepts or rejects the offer. If the offer is rejected, neither one gets anything, and the players' stage payoffs are equal to 0. If the offer is accepted, player 1 receives payoff $u_1(x) + \theta_1 u_1(y)$, and player 2 receives payoff $u_2(1-x) + \theta_2 u_2(1-y)$, where θ_i is privately known taste for the cherry and u_i are strictly increasing utility functions.

It turns out that all equilibrium payoffs for positive probability types of patient players can be obtained by the full and immediate revelation of all private information. This result follows from a slightly more general observation. Notice that in the bargaining with a cherry, the individually rational set is equal to the set of all non-negative payoffs. Because all feasible payoffs are non-negative, they are also individually rational. Moreover, if the uncertainty over tastes for cherry is nontrivial, the feasible non-revealing set has nonempty interior.

Theorem 4. *Suppose that set V has a nonempty interior and that $V \subseteq IR$. Then, for each belief system $\pi \in \Pi$,*

$$clNE^{+}(\pi) = clF_1^B(\pi).$$

Proof. The proof is an application of the characterization of the set of equilibrium payoffs from Theorem 3. We sketch the argument. It is enough to show that $F_2^A(\pi) = F_1^B(\pi)$ and $F_2^B(\pi) = F_1^B(\pi)$. In order to show the first claim, notice first that set $F_1^B(\pi)$ is convex. (It can be obtained by using only fully revealing and incentive compatible lotteries and the values of such lotteries form a convex set.) Because of the assumption, $\text{int}V \subseteq F_1^B(\pi)$. Together with the fact that $F_1^B(\pi) \subseteq IR$, this implies that first claim. The second claim follows from the first and the fact that the composition of an incentive compatible lottery with a incentive compatible and fully revealing lottery can be replaced by a single incentive compatible lottery with the same value and outcomes that are convex combinations of the outcomes in the original lotteries. □

6.2. **Oligopoly.** There are I firms on the same market. We consider an abstract model of competition that encompasses, as special cases, textbook examples of Bertrand and Cournot competitions with undifferentiated products and incomplete information about the costs.

We keep the notation from the general model of a repeated game. Additionally, we make two assumptions. The first assumption says that each payoff vector can be replicated by a scheme in which each firm spends a fraction of the period selling to the market as a single firm on the market while the other firm is inactive. In order to state it formally, let $M_i \subseteq R^{\Theta_i}$ be the convex hull of the set of payoff vectors attainable by firm i if firm i was the only firm on the market. We refer to M_i as the set of monopoly payoffs.. We assume that M_i is compact, that it contains the zero-payoff vector $\mathbf{0}_i \in M_i$, and that the intersection of M_i with the set of strictly positive payoff vectors has a nonempty interior. We assume that each payoff vector in the game between the firms is a convex combination of monopoly-inactive payoffs: for each $v \in V$, there exist monopoly payoff $m_i \in M_i$ and market share $\beta_i \geq 0$ for each player i, such that $\sum_i \beta_i \leq 1$, and the vector of payoffs of player i is equal to $v_i = \beta_i m_i$.

Second, we assume that the set of individually rational payoffs is equal to the set of vectors with non-negative coordinates, IR=$\{v : v_i(\theta_i) \geq 0$ for each $i$and $\theta_i\}$. Any game with such a structure is called a *oligopoly game*.

If we interpret θ_i as the cost parameter, actions as quantities or prices, then the above assumptions are satisfied in various oligopoly models.

Example 1. The firms play a Cournot oligopoly. The firms choose quantities q_i. The payoff of firm i with cost type θ_i is equal to $q_i(P(\sum q_j) - \theta_i)$ where $P(.)$ is an inverse demand function. The payoff is equal to the fraction $\frac{q_i}{\sum q_j}$ of the monopoly payoff obtained from producing quantity $\sum q_j$. By choosing quantity 0, each firm can ensure that its payoff is not smaller than 0. Moreover, if $\lim_{q \to \infty} P(q) < \inf \Theta_i$, then, by choosing a sufficiently large quantity, firm $-i$ can ensure that the profits of firm i are not higher than 0.

Example 2. Another model is a Bertrand oligopoly with demand $D(.)$. The firms choose prices p_i. The payoff of each firm i is equal to $D(p_i)(p_i - \theta_i)$ if the firm i's price is strictly lower than the price of its competitors, $\frac{1}{k}D(p_i)(p_i - \theta_i)$, if $k-1$ other firms choose the lowest price, and and 0 otherwise.

Theorem 5. *For each oligopoly game, each belief system $\pi \in \Pi$,*

$$clNE^+(\pi) = clF_1^B(\pi).$$

The proof of Theorem 5 follows the same structure as the proof of Theorem 4. The details can be found in Appendix C.

The proof of Theorem 5 implies a characterization of the set of equilibrium payoffs (see Lemma 8 for details) for full support belief system π: For each payoff vector v, $v \in \mathrm{cl}F_1^B(\pi)$ if and only if for each type profile $\theta = (\theta_i, \theta_{-i})$, each firm i, there exist monopoly payoffs $m_i^{\theta_i} \in M_i$ and market shares $\beta_i^\theta \geq 0$, such that $\sum_i \beta_i^\theta \leq 1$ and the following conditions hold:

(1) Individual rationality: $m_i^{\theta_i}(\theta_i) \geq 0$ for each player i and type θ_i,
(2) Incentive compatibility: for all θ_i, θ_i',

$$v(\theta_i) \geq \left(\sum\nolimits_{\theta_{-i}} \pi^{\theta_i}(\theta_{-i})\, \beta_i^{(\theta_i', \theta_{-i})}\right) m_i^{\theta_i'}(\theta_i),$$

with equality if $\theta_i' = \theta_i$.

In particular, any equilibrium payoff v can be approximated by a payoff in a profile in which firms immediately reveal their costs and if θ is the true type profile, then player i's payoff is equal to $\beta_i^\theta m_i^{\theta_i}(\theta_i)$. The first condition ensures that individual rationality is satisfied *ex post*, and the second condition ensures that firms have incentives to reveal their types truthfully (although the incentives are not necessarily *ex post*).

"Pooling" result of Athey and Bagwell (2008). As an application, we perform a quick test of the robustness of the "pooling" result from Athey and Bagwell (2008). Athey and Bagwell (2008) analyze a Bertrand model with a demand that is constant and equal to one unit below some reservation price $r > 0$ and the demand disappears

at prices higher than r. They show that for a sufficiently large discount factor, and given some assumptions on the distribution of cost types, in the (ex ante) optimal symmetric equilibrium, all players choose the same price and receive the same market share regardless of their (privately known) costs. In other words, one can sustain the best payoff in equilibrium in which no player ever reveal any information. There is no contradiction between Athey and Bagwell (2008)'s result and Theorem 5.[8] First, their characterization of optimal equilibrium is tight for all sufficiently high $\delta < 1$, whereas ours simply says that any equilibrium payoff can be approximated by fully revealing payoffs. In fact, one can construct equilibria in which players fully reveal their costs in the first period and then they proceed to ignore the revealed information. Because revelation of information is costly for discount factors strictly smaller than 1, it should be avoided in optimal equilibrium of Athey and Bagwell (2008).

Nevertheless, the pooling" claim is not robust to modifications of the demand. Define the monopoly payoff vector that maximizes the payoffs of type θ_i among all monopoly payoffs of player i:

$$m^*_{\theta_i} = \arg\max_{m \in M_i} m\left(\theta_i\right).$$

In Athey and Bagwell (2008), the optimal monopoly price is equal to r and does not depend on the player's type. In general, in both Cournot or Bertrand models, if the demand function is differentiable, then the optimal monopoly action depends on the cost.

Corollary 2. *Suppose that the monopoly actions $m^*_{\theta_i}$ are not the identical for all types of player i. Then, for any π that assigns positive probability to all types, for all sufficiently high $\delta < 1$, there is no Pareto-optimal equilibrium in which players' behavior does not depend on type.*

[8]There are other differences between Athey and Bagwell (2008)'s and our model. For example, the demand specification does not lead to nonempty interior and our result does not apply. However, it applies to "nearby" models in which the demand below price r is not completely inelastic. In addition, Athey and Bagwell (2008) work with the continuum type model, whereas this paper assumes that there are only finitely many types. These differences do not seem to be important for this discussion.

Proof. Suppose that v is an efficient payoff in a profile in which, on the equilibrium path, the players' behavior does not depend on the type. Then, there exists $\beta_i \geq 0$ and $m_i \in M_i$ such that $\sum_i \beta_i \leq 1$, and $v_i = \beta_i m_i$.

For each player, construct a payoff vector m_i^* such that for each type θ_i, $m_i^*(\theta_i) = \max\left\{m_{\theta_i}^*(\theta_i), m_i(\theta_i)\right\} \geq m_i(\theta_i)$ with some inequalities strict. Define payoff vector v^* such that player i payoffs are equal to $v_i^* = \beta_i m_i^*$. The mechanism-design characterization implies that $v^* \in F_1^B(\pi)$. Because $v^*(\theta_i) \geq v(\theta_i)$ with some inequalities strict, this contradicts the fact that vector v is efficient. □

Belief-free vs. fully and immediately revealing equilibria. In the above characterization of equilibrium payoffs, firms have incentives to reveal their private information *ex ante*, before they learn the true types of the other player. Next, we show on an example that we cannot improve the incentives to hold *ex post* (i.e., conditionally on each of the type of the other player). In particular, we show that there exist efficient repeated equilibria that are fully and immediately revealing, but that are not belief-free.

Consider a symmetric Cournot model with two players and two cost types for each player, $\Theta = \{h, l\}$, where $h > l > 0$. Let $m^q \in R^\Theta$ be the monopoly payoff vector from quantity q and let $q_\theta = \arg\max_q m^q(\theta)$ be the optimal monopoly quantity of type θ. The monopoly profits are maximized by the firm with low costs and quantity q_l. We assume that the optimum is strict:

$$m^{q_l}(l) > m^{q_h}(l), m^{q_l}(h). \tag{6.1}$$

Additionally, we assume that the payoff of the high cost type from quantity q_l is strictly positive, but much smaller than the maximum payoff attainable by this type:

$$m^{q_h}(h) > 6m^{q_l}(h) > 0. \tag{6.2}$$

We are interested in strategies that maximize the ex ante expected sum of payoffs of both firms. Because of (6.1), the first best for interior beliefs is attained if and only if:

(u, f_1, f_2)	$Weak$	$Tough$
$Weak$	$2, 2, 2x$	$0, 4, 1 + 3x$
$Tough$	$4, 0, 0$	$-2, -2, 1 - 3x$

TABLE 1. Payoffs in bargaining game.

- if both firm types are equal to θ, one of the firms is inactive, and the other one produces quantity q^θ. In symmetric equilibrium, the two allocations are chosen with equal probability,
- if firm i has low costs and firm $-i$ has high costs, firm i is active and it produces quantity q^l, and firm $-i$ is inactive (and produces 0).

We claim that the first best allocation cannot be attained in belief-free equilibrium. Indeed, notice that at least one firm i must expect strictly positive profits in state in which both firms report l (in symmetric allocation, both firms must receive strictly positive profiles). Because firm i receives zero profits if it reports l and the other firm reports h, type h of firm i does not have ex post incentives to reveal its true type if the other firm has low costs.

On the other hand, if $\pi_i(h) = \frac{1}{2}$ for both players i, then (6.2) implies that the first best profile satisfies ex ante incentive compatibility. In particular, the first best expected payoff can be attained in repeated game equilibrium in which both types are revealed immediately and then the play approximates the efficient allocation.

6.3. **Labor union - firm bargaining.** Consider the following class of games parametrized with $x \in [0, 1]$. There are two players, a labor union (U) and a firm (F). The firm can be either a normal type, $\theta_F = 1$, or a strong type, $\theta_F = 2$. Each player chooses between two actions, $Weak$ and $Tough$. The payoffs are given in Table 1.

- When $x = 1$, the payoffs of the normal and strong types of the firm are equal, and the firm and the union play a multi-period bargaining model with complete information.
- When $x = 0$, the union U and the normal type have payoffs as in the complete information game. The strong type has a (repeated-game) dominant action to play T in every period. This an example of a model of reputation with equal

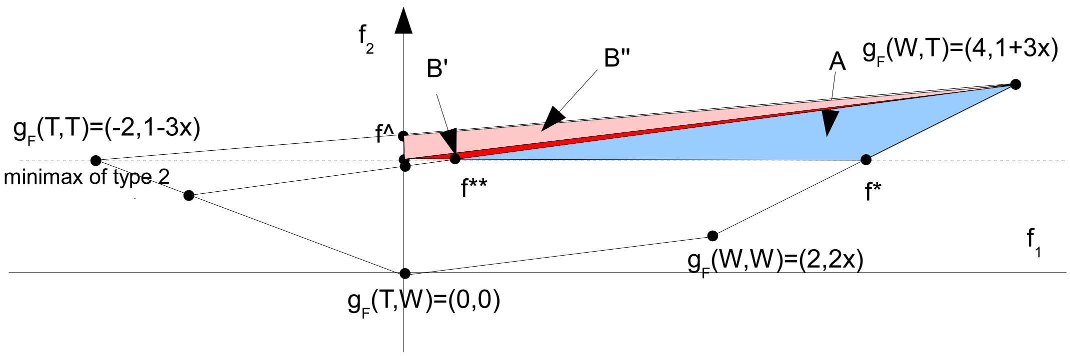

FIGURE 6.1. Payoffs of the firm's types

discount factors for two players. The complete information game has strictly conflicting interest (Schmidt (1993)): the normal type has a commitment action T such that the union's best reply gives the union its minmax payoff of 0. Cripps et al. (2005) show a reputational result for this class of games: for any $p < 1$, and for δ high enough, all Nash equilibrium payoffs of the union and the normal type are close to $(4, 0)$.

- For intermediate x, the payoff of the strong type is a convex combination between the normal type and the completely strong type of the reputation case $x = 0$. The techniques of Cripps et al. (2005) do not apply. (In fact, as we show, the reputational result does not hold). On the other hand, the game has an open thread assumption, and we can use Theorem 3 to compute the set of equilibrium payoffs.

The goal of this section is to describe an "upper," Pareto-optimal, part of the equilibrium set (the "lower" part can be described in an analogous way). To simplify the exposition, we assume that $x < \frac{1}{5}$.

Notation. Notice that the minmax strategy of each player is T, which implies that the set of individually rational payoffs is equal to

$$IR = \{(u, f_1, f_2) : u \geq 0, f_1 \geq 0, f_2 \geq 1 - 3x\}.$$

In order to describe the payoff sets, we need some notation. We use $\pi \in [0,1]$ to denote the probability of the normal type. We write $f = (f_1, f_2) \in \mathtt{R}^2$ to denote the payoffs of the two types of player F, and $v = (u, f) \in \mathtt{R}^3$ to denote the vector of payoffs of both players. For any $f^a \neq f^b$, let $I\left(f^a, f^b\right)$ be the interval on a two-dimensional plane that connects f^a and f^b. For any not co-linear $v^a, v^b, v^c \in \mathtt{R}^3$, for each $f \in \mathtt{R}^2$, let $H^{v^a,v^b,v^c}(f)$ be the unique value such that $\left(H^{v^a,v^b,v^c}(f), f\right)$ belongs to the unique affine hyperplane that passes through points $v^x, x = a, b, c$.

Figure 6.1 illustrates the payoffs of the firm's types. We find $f^* = (f_1^*, 1-3x)$ such that $f^* \in I\left(g_F(W,T), g_F(W,W)\right)$. Find $\hat{f} = \left(0, \hat{f}_2\right)$ such that $f \in I\left(g_F(W,T), g_F(T,T)\right)$. Finally, we find $f^{**} = (f_1^{**}, 1-3x)$ so that $H^{g(T,W),g(T,W),g(T,T)}(f^{**}) = 0$.

Define sets $A, B', B'' \subseteq R^2$,

$$
\begin{aligned}
A =& \text{con}\left\{f^{**}, f^*, g_F(W,T)\right\},\\
B' =& \text{con}\left\{f^{**}, (0, 1-3x), g_F(W,T)\right\},\\
B'' =& \text{con}\left\{\hat{f}, (0, 1-3x), g_F(W,T)\right\}.
\end{aligned}
$$

Sets A, B',and B'' are illustrated on Figure 6.1.

For each $f \in B'$, choose $j'(f) \in [0, f^{**}]$ so that f belongs to the interval $I\left(g_F(W,T), (j'(f), 1-3x)\right)$. Similarly, for each $f \in B''$, choose $j''(f) \in \left[1-3x, \hat{f}\right]$ so that f belongs to the interval $I\left(g_F(W,T), (0, j''(f))\right)$.

We say that function u^π *describes the upper surface of equilibria* if for each f,

$$u^\pi(f) = \sup\left\{u : (u, f) \in F^*(\pi)\right\}$$

(we take $u^\pi(f) = -\infty$, if the right-hand side set is empty).

Complete information payoffs. Using Theorem 5, we can describe the "upper" surface of the payoffs in the complete information case $\pi \in \{0,1\}$. Let

$$u^1(f) = \begin{cases} 4 - f_1, & \exists f' \in A \cup B' \cup B'' \text{ st. } f' \preceq_1 f, \\ -\infty, & \text{otherwise.} \end{cases}$$

$$u^0(f) = \begin{cases} \min\left\{H^{g(T,W),g(T,T),g(W,T)}(f), 4 - \frac{4}{1+3x} f_2\right\}, & \exists f' \in A \text{ st. } f' \preceq_0 f, \\ -\infty, & \text{otherwise.} \end{cases}$$

Incomplete information payoffs. We use the characterization from Section 4 to construct the upper surfaces of equilibria. First, we construct a sequence of payoff vectors v_n that belong to a finitely revealing set in game with initial belief $p_n = \frac{n}{N}$, where $N < \infty$. Next, we take $N \to \infty$ and show that the constructed path of equilibria converges to the solution of a certain differential equation.

First, consider the game with initial belief $p_0 = 0$. Let $j_0 = f^{**}$. Due to the above description of the upper surfaces in the complete information case, $v_0 = (0, j_0, 1-3x) \in F^*(0)$.

Next, consider the game with initial beliefs p_1. Vector

$$v' = \frac{1-p_1}{1-p_0}(0, j_0, 1-3x) + \frac{p_1-p_0}{1-p_0}\left(u^1(j_0, 1-3x), j_0, 1-3x\right).$$

is equal to the value of the p_1-incentive compatible lottery in which the firm's normal type gets revealed with probability $\frac{p_1-p_0}{1-p_0}$, upon which the players' continuation payoffs are equal to $(u^1(j_0, 1-3x), j_0, 1-3x)$. If the normal type is not revealed, the labor union updates its belief to p_0, and the play continues with payoffs $(0, j_0, 1-3x)$. Because of stage B of the construction of the finitely revealing set (Lemma 6), $v' \in F^*(p_1)$.

Further, construct a profile in which players play actions (T, T) for some fraction α of time, and then continue with a profile that leads to payoffs v'. The payoffs in such a profile are equal to

$$v_1 = \alpha g(T, T) + (1-\alpha) v'.$$

We choose α so that the payoff of the labor union in vector v is equal to 0. Then, by stage A (Lemma 7), $v_1 = (0, j_1, 1-3x) \in F^*(p_1)$, where

$$j_1 = \frac{(p_1-p_0) u^1(j_0, 1-3x)}{1-p_0}\left(2 + \frac{p_1-p_0}{1-p_0}u^1(j_0, 1-3x)\right)^{-1}(2+j_0).$$

Using the same argument, we show that if $v_n = (0, j_n, 1-3x) \in F^*(p_n)$, and j_n is not too close to 0, then $v_{n+1} = (0, j_{n+1}, 1-3x) \in F^*(p_{n+1})$,where

$$j_{n+1} = j_n + \frac{(p_{n+1}-p_n) u^1(j_n, 1-3x)}{1-p_n}\left(2 + \frac{p_{n+1}-p_n}{1-p_n}u^1(j_n, 1-3x)\right)^{-1}(2+j_n).$$

After some algebraic transformations, we obtain

$$\frac{p_{n+1}-p_n}{j_{n+1}-j_n}=\frac{2+\frac{p_{n+1}-p_n}{1-p_n}u^1\left(j_n,1-3x\right)}{2+j_n}\frac{1-p_n}{u^1\left(j_n,1-3x\right)}.$$

By taking limit $N\to\infty$, the above equation converges to the differential equation

$$\frac{dp}{dj}=-\frac{2}{2+j}\frac{1-p\left(j\right)}{u^1\left(j,1-3x\right)}. \tag{6.3}$$

(The minus comes from the fact that $p_{n+1}-p_n=-\frac{1}{N}$.)

Suppose that $p':[0,f^{**}]\to[0,1]$ is a solution to (6.3) such that $p'\left(f^{**}\right)=0$. Choose π^* so that $p'\left(0\right)=\pi^*$. The above analysis implies that for each $\pi\leq\pi^*$, each $j\in[0,f^{**}]$,

$$\left(0,j,1-3x\right)\in F^*\left(p'\left(j\right)\right).$$

Because set $F^*\left(p'\left(j\right)\right)$ is convex and it contains vector $g\left(W,T\right)$, it must be that $\left(0,f\right)\in F^*\left(p'\left(j'\left(f\right)\right)\right)$ for each $f\in B'$.

Similar equations can be derived for the elements of set B''. Let $p'':\left[1-3x,\hat{f}\right]\to[0,1]$ be a solution to the following differential equation: $p''\left(1-3x\right)=\pi^*$, and

$$\frac{dp''}{dj}=-\frac{4/3}{\hat{f}-j}\frac{1-p''\left(j\right)}{u^1\left(0,j\right)}. \tag{6.4}$$

Then, for each $f\in B''$, we have $\left(0,f\right)\in F^*\left(p''\left(j''\left(f\right)\right)\right)$.

Proposition 1. *The following functions describe the upper surfaces of equilibria:*

- *if* $\pi\leq\pi^*$, *let*

$$u^\pi\left(f\right)=\begin{cases}\pi u^1\left(f\right)+\left(1-\pi\right)u^0\left(f\right), & f\in A,\\ \frac{\pi-p'(j'(f))}{1-p'(j'(f))}u^1\left(f\right), & f\in B' \text{ and } \pi\geq p'\left(j'\left(f\right)\right),\\ -\infty, & \text{otherwise.}\end{cases}$$

- *if* $\pi>\pi^*$, *let*

$$u^\pi\left(f\right)=\begin{cases}\pi u^1\left(f\right)+\left(1-\pi\right)u^0\left(f\right), & f\in A,\\ \frac{\pi-p'(j'(f))}{1-p'(j'(f))}u^1\left(f\right), & f\in B' \text{ and } \pi\geq p'\left(j'\left(f\right)\right)\\ \frac{\pi-p''(j''(f))}{1-p''(j''(f))}u^1\left(f\right), & f\in B'' \text{ and } \pi\geq p''\left(j''\left(f\right)\right)\\ -\infty, & \text{otherwise.}\end{cases}$$

Proof. The above discussion shows that $(u^{\pi}(f), f) \in F^{*}(\pi)$ for each $f \in R^{2}$ such that $u^{\pi}(f) > -\infty$. We are left with showing that for each $u > u^{\pi}(f)$, $(u, f) \notin F^{*}(\pi)$.

Define correspondence $F(\pi) \supseteq \{(u, f) : u \leq u^{\pi}(f)\}$ for each π. We will show that none of the operations described in Section 4 adds any payoffs to correspondence F.

First, notice that $F(\pi) = IR \cap \text{con}\{V \cup F(\pi)\}$.

Second, we are going to show each π-incentive compatible lottery such that the continuation payoffs belong to correspondence $F(.)$ has its value in set $F(\pi)$. Indeed, suppose that $l = (\alpha, \psi)$ is such a lottery with value $v = (u, f)$ and continuation payoffs $\psi(a) = (u(a), f(a))$ after positive probability actions a of the firm. Then, $f(a) \leq f$ with equality if action a is played with positive probability by the two types of the firm. Moreover, if action a is played with positive probability by only one type, we can use the description of the upper surfaces in the "complete information" games to show that $u^{p(a)}(f) \geq u^{p(a)}(f(a))$.

Consider lottery $l' = (\alpha, \psi')$, where $\psi'(a) = \left(u^{p(a)}, f\right)$ for all actions a. Then, the description of the upper surface u^{π} implies that

$$u \leq \sum_{a} p(a)\, u^{p(a)} \leq u^{\pi}(f),$$

which, in turn, implies that $(u, f) \in F(\pi)$.

Finally, notice that for $\pi \in (0,1)$, both types of the firm have positive probability, and stage C does not add any payoffs. These three observations imply that $F^{*}(\pi) \subseteq F(\pi)$ and that u^{π} is the upper surface of equilibrium payoffs. □

Equilibrium behavior. One can use the above analysis to (approximately) predict the dynamics along the equilibria that support payoffs on the upper surfaces. As an example, we describe the equilibrium behavior that induces (approximately) payoff vector $(0, f_1, 1 - 3x)$ in the game with initial beliefs $p'(f_1)$ for some $f_1 \in [0, f^{**}]$. Such a profile can be described by, roughly, three phases.

- In the *revelation phase*, the labor union and the strong type of player F play Tough. The normal type of F plays Tough almost all the time. Infrequently, the normal type plays Weak fully revealing himself. The phase ends either because the normal type plays W, or because the posterior probability of the normal type becomes equal to 0. In the former case, the players continue with

(u_1, u_2, f_1, f_2)	$Weak$	$Tough$
$Weak$	$2, 2x, 2, 2x$	$0, 0, 4, 1+3x$
$Tough$	$4, 1+3x, 0, 0$	$-2, 1-3x, -2, 1-3x$

Table 2. Payoffs in bargaining game.

the "normal type" phase; in the latter, the players continue with the "strong type" phase. The continuation payoff of the normal type f_1 throughout the revelation phase gradually increases with the decreasing posterior probability $p'(f_1)$ of the normal type. The rate with which the normal type chooses $Weak$ is chosen so that the continuation payoff of the labor union is equal to 0 at each moment of the revelation phase.

- In the *"normal type" phase*, players play the "complete information" game equilibrium with payoffs equal to $(u^1(p'(f_1)), f_1, 1-3x)$, where f_1 is the expected continuation payoff of the normal type at the moment of revelation.
- In the *"strong type" phase*, players play the equilibrium of the "complete information" game with payoffs $(0, f^{**}, 1-3x)$.

In a similar way, we can describe strategy profiles that induce any other payoff on the upper surface.

6.4. **Labor union- firm bargaining with Two-sided incomplete information.** We show that if $x \leq \frac{3}{100}$, then a version of the above model with symmetric, two-sided incomplete information does not have any threads. Indeed, suppose that there are two types of each player, and the payoffs are given in Table 2.

On the contrary, suppose that $u^*(\pi)$ is the thread. Let $u^{ns} = u^*\left(\pi^{(\text{normal}_1,\ \text{strong}_2)}\right)$ be the thread Nash equilibrium payoff vector given that the first player is revealed to be normal and the second player is revealed to be strong. Because the equilibrium payoffs must be individually rational, it must be that

$$u_1^{ns}(\text{normal}_1) \geq 0 \text{ and } u_2^{ns}(\text{strong}_2) \geq 1-3x.$$

By Theorem 5, there exists $\alpha \in \Delta A$ such that

$$u_1^{ns}(\text{normal}_1) = 2\alpha_{WW} + 4\alpha_{TW} - 2\alpha_{TT} \geq 0, \quad (6.5)$$
$$u_2^{ns}(\text{strong}_2) = 2x\alpha_{WW} + (1+3x)\,\alpha_{WT} + (1-3x)\,\alpha_{TT} \geq 1-3x,$$

and

$$u_2^{ns}(\text{normal}_2) \geq 2\alpha_{WW} + 4\alpha_{WT} - 2\alpha_{TT}.$$

The next result shows that $u_2^{ns}(\text{normal}_2) > 2$.

Lemma 2. *Suppose that $x \leq \frac{3}{100}$. Then, $2\alpha_{WW} + 4\alpha_{WT} - 2\alpha_{TT} > 2$ for each $\alpha \in \Delta A$ that satisfies inequalities (6.5) .*

The proof of Lemma 2 can be found in Appendix D.

A symmetric argument shows that $u_1^{sn}(\text{normal}_1) > 2$, where u^{sn} is the thread equilibrium payoff vector if the first player is strong, and the second player is normal. Because players must be ex post indifferent about revealing their type truthfully, we have

$$u_2^{nn}(\text{normal}_2) = u_2^{ns}(\text{normal}_2) > 2,$$
$$u_1^{nn}(\text{normal}_1) = u_1^{sn}(\text{normal}_1) > 2,$$

where u^{nn} is the thread payoff vector if both players are revealed to be normal.

On the other hand, the sum of the payoffs of the normal types given any action profile is never higher than 4. This implies that for any equilibrium payoff vector $u \in NE(\text{normal}_1, \text{normal}_2)$, $u_1(\text{normal}_1) + u_2(\text{normal}_2) \leq 4$. The contradiction shows that u^* cannot be a thread.

7. Comparison with no-discounting case

We compare our characterization of payoffs with Hart (1985) characterization in the case of no-discounting. Hart (1985) considers the general payoff-case and he assumes that there are two players, uninformed U (with one type) and informed I. Let Θ_I be the finite set of the types of the informed player, and let Δ_{Θ_I} be the simplex of beliefs of the uninformed player. Then, the correspondence of Nash equilibrium payoffs can be characterized as the set of initial values $(v_{U,0}, v_{I,0}, p_0) \in R \times R^{\Theta_I} \times \Delta_{\Theta_I}$ of a class of bimartingales, ie., stochastic processes that satisfy the following three properties:

- for all odd t, $p_t = p_{t+1}$ and $E\left(v_{U,t+1}, v_{I,t+1} | \mathcal{F}_t\right) = \left(v_{U,t}, v_{I,t}\right)$,
- for all even t, $v_{I,t} = v_{I,t+1}$ and $E\left(v_{U,t+1}, p_{t+1} | \mathcal{F}_t\right) = \left(v_{U,t}, p_t\right)$,
- the limit payoff $\left(v_{U,\infty}, v_{I,\infty}\right) = \lim_{t\to\infty}\left(v_{U,t}, v_{I,t}\right)$ is a payoff in a repeated game with initial prior $p_\infty = \lim_{t\to\infty} p_t$ and in which no further substantial information is revealed. In the known-own payoff case, the set of such a payoffs is equal to $F_0(p)$ defined in (3.1).

The second and the third property are equivalent to, respectively, the revelation of information (operation $\mathcal{B}$) and the no-revealing payoffs F_0, from our characterization.

The first property convexifies the set of payoffs obtained in the previous steps (Hart does not assume public randomization and, instead, uses Aumann-Maschler's jointly controlled lotteries) and it corresponds to operation $\mathcal{A}$ from our characterization with a *key difference*: In the discounted case, the payoffs are additionally convexified with the set of feasible and non-revealing payoffs V. To compare the first property of bi-martingales and operation $\mathcal{A}$ side by side, let $E_N : \Delta_{\Theta_I} \rightrightarrows R \times R^{\Theta_I}$ denote the equilibrium payoff correspondence in the and discounted cases. Then, Hart's characterization implies that for each $p \in \Delta_{\Theta_I}$,

$$E_N(p) = \text{con}\left(E_N(p)\right) = \text{con}\left(E_N(p) \cup (V \cap IR)\right).$$

The second equality comes from the fact that $V \cap IR \subseteq F_0(p) \subseteq E_N(p)$. Because for any set E,

$$\text{con}\left(E \cup (V \cap IR)\right) \subseteq \text{con}\left(E \cup V\right) \cap IR,$$

and the inclusion is typically strict, the set of payoffs in the no-discounting case is included, and it is typically smaller than the set of payoffs in the discounted case.

In the known-own payoffs case, Shalev (1994) provides a much simpler characterization of no-discounting equilibrium payoffs. For each $p \in \Delta_{\Theta_I}$, $E_N(p)$ is equal to payoff vectors (v_U, v_I) such that $(v_U, v_I) \in IR$ and for each type $\theta \in \Theta_I$ of the informed player, there exist $v^\theta \in V$ so that

$$v_U = \sum_\theta p(\theta)\, v_U^\theta,$$

and for each $\theta, \theta' \in \Theta_I$,

$$v_I(\theta) = v_I^\theta(\theta) \geq v_I^{\theta'}(\theta).$$

All such payoffs can be obtained by immediate and full revelation of the informed player's type θ followed by the equilibrium play of a profile that corresponds to payoff vector v^{θ}. It is easy to show that the set of such payoffs is equal to $F_1^B(p)$ (see Cripps and Thomas (2003)), and, or the set of payoffs obtained in the belief-free equilibria. (See Horner and Lovo (2009). Note that that the latter is true because Shalev (1994) is limited to one-sided case). In particular, it is always included in the set of payoffs attained in the equilibria with discounting, and it is strictly smaller in games in which there exist non-trivial n-revealing equilibria for $n > 1$.

There is another important difference between the Hart's characterization and our result. In the general payoff case, there are games with Nash payoffs that cannot be approximated by equilibria with finite and bounded number of revelations (Forges (1984), Forges (1990); see also "four frogs" from Aumann and Hart (2003). Mathematically, the result follows from the fact that di-span of a set might be strictly larger than its di-convexification (Aumann and Hart (1986)).) More importantly, the bi-martingale characterization is not constructive and there exists no known algorithm that allows one to find all the payoffs in the general case. On the other hand, in our case, we show that all equilibrium payoffs can be approximated by payoffs in equilibria with bounded number of revelations. The characterization is constructive and there exists an algorithm that allows one to approximate the true set of payoffs in finitely many steps.

The reason for the difference is not clear. On one hand, the characterization from Shalev (1994) shows that only one revelation is necessary in the no-discounting case with known-own payoffs. This would suggest that, at least in the one-sided case, the difference is due to the known-won payoffs assumption. On other hand, we do not know whether one can find a version of the "four frogs" example with known-own payoffs and multi-sided incomplete information (as far as we know, the characterization of payoffs in such a case remains an open problem.)

8. Conclusions

This paper provides a characterization of the equilibrium payoffs in repeated games with incomplete information, with discounting, known-own payoffs, and permanent types. We assume that there exists an open multi-linear thread of payoffs in equilibria in which in the first period of the game, players fully reveal their information (i.e., all types of each players take separating actions), and such that the players are *ex post* indifferent between revealing their type truthfully or reporting any other type (i.e., they are indifferent conditionally on any type of the opponent). The assumption is generically satisfied in games with one-sided incomplete information as well as some important examples of games with multi-sided incomplete information.

Our characterization provides an algorithm to construct the equilibrium set through a sequence of geometric operations. This algorithm can be implemented numerically. In examples, we show the characterization can be used to find the exact description of the equilibrium sets analytically. Further work is required to build tools that allow for analytical description in general games. For instance, the equilibrium set in the bargaining problem from Section 6.3 is described as a solution to a certain ordinary differential equation. This method can be easily generalized to other games with one-sided uncertainty and two types. We suspect that differential equations play an important role in more general settings (with more types or with multi-sided uncertainty), but we do not know how to do it.

Other questions are left open by this paper. Most importantly, we would like to know whether a similar characterization holds for games in which an open thread assumption is not satisfied (see an example at the end of section 6.3 or Horner and Lovo (2009)). Our current methods do not allow us to form a hypothesis one way or the other. It would be interesting to check whether the current analysis extends in some way to the case of persistent types.[9] We leave these questions for future research.

References

Athey, S., Bagwell, K., Apr. 2008. Collusion with persistent cost shocks. Econometrica 76 (3), 493–540. 1, 1, 6.2, 8, 9

[9] Athey and Bagwell (2008) introduce a model of persistent types. Escobar and Toikka (2012) prove a folk theorem for limit $\delta \to 1$ and fixed rates of transitions. One can consider an alternative limit $\delta \to 1$ when the probability of transitions scales with $1 - \delta$.

Aumann, R. J., Hart, S., Jun. 1986. Bi-convexity and bi-martingales. Israel Journal of Mathematics 54 (2), 159–180. 7

Aumann, R. J., Hart, S., Oct. 2003. Long cheap talk. Econometrica 71 (6), 1619–1660. 1, 2.4, 2.5, 7

Aumann, R. J., Maschler, M., May 1995. Repeated Games with Incomplete Information. The MIT Press. 1, 6.1

Aumann, R. J., Maschler, M., Stearns, R. E., 1967. Reports to the U.S arms control and disarmament agency, tech report. ST-80 (1968), ST-116 (1967), ST-140 (1968). 1

Blackwell, D., 1956. An analog of the minimax theorem for vector payoffs. Pacific Journal of Mathematics 6 (1), 1–8. 2.3, B

Chan, J., 2000. On the Non-Existence of reputation effects in Two-Person Infinitely-Repeated games. Economics Working Paper Archive 441, The Johns Hopkins University,Department of Economics. 1

Cripps, M. W., Dekel, E., Pesendorfer, W., Apr. 2005. Reputation with equal discounting in repeated games with strictly conflicting interests. Journal of Economic Theory 121 (2), 259–272. 1, 6.3

Cripps, M. W., Thomas, J. P., 1997. Reputation and perfection in repeated common interest games. Games and Economic Behavior 18 (2), 141–158. 1

Cripps, M. W., Thomas, J. P., Aug. 2003. Some asymptotic results in discounted repeated games of One-Sided incomplete information. Mathematics of Operations Research 28 (3), 433–462. 1, 3, 4, 7

Escobar, J. F., Toikka, J., 2012. A Folk Theorem with Markovian Private Information. 9

Forges, F., 1984. Note on nash equilibria in infinitely repeated games with incomplete information. International Journal of Game Theory 13 (3), 179–187. 1, 7

Forges, F., May 1990. Equilibria with communication in a job market example. The Quarterly Journal of Economics 105 (2), 375–398. 1, 7

Fudenberg, D., Maskin, E., 1986. The folk theorem in repeated games with discounting or with incomplete information. Econometrica, 533–554. 4

Fudenberg, D., Yamamoto, Y., Oct. 2010. Repeated games where the payoffs and monitoring structure are unknown. Econometrica 78 (5), 1673–1710. 1

Fudenberg, D., Yamamoto, Y., Sep. 2011. Learning from private information in noisy repeated games. Journal of Economic Theory 146 (5), 1733–1769. 1

Hart, S., Feb. 1985. Nonzero-Sum Two-Person repeated games with incomplete information. Mathematics of Operations Research 10 (1), 117–153. 1, 2.4, 3, 7, B

Horner, J., Lovo, S., 2009. Belief-Free equilibria in games with incomplete information. Econometrica 77 (2), 453–487. (document), 1, 4, 2.3, 3, 5.1, 7, 8, A, B

Horner, J., Lovo, S., Tomala, T., Sep. 2011. Belief-free equilibria in games with incomplete information: Characterization and existence. Journal of Economic Theory 146 (5), 1770–1795. 1

Koren, G., 1992. Two-Person repeated games where players know their own payoffs. Tech. rep., Courant Institute. 1, 3, B

Kreps, D. M., Wilson, R., 1982a. Reputation and imperfect information. Journal of Economic Theory 27 (2), 253–279. 1

Kreps, D. M., Wilson, R., 1982b. Sequential equilibria. Econometrica 50 (4), 863–94. 5

Milgrom, P., Roberts, J., 1982. Predation, reputation, and entry deterrence. Journal of Economic Theory 27 (2), 280–312. 1

Peski, M., 2008. Repeated games with incomplete information on one side. Theoretical Economics 3 (1). 1, 1, 4, 2.3, 3, 4, 5.1, B

Schmidt, K. M., 1993. Reputation and equilibrium characterization in repeated games with conflicting interests. Econometrica 61 (2), 325–51. 1, 6.3

Shalev, J., 1994. Nonzero-Sum Two-Person repeated games with incomplete information and Known-Own payoffs. Games and Economic Behavior 7 (2), 246–259. 1, 3, 5.1, 7, B

Wiseman, T., Feb. 2005. A partial folk theorem for games with unknown payoff distributions. Econometrica 73 (2), 629–645. 1

Appendix A. Threads and belief-free equilibria with two players

Horner and Lovo (2009) give two necessary conditions for the existence of belief-free equilibria in the case of two players. We restate the conditions in our notation and in the known-payoff case. For each probability distribution $\alpha \in \Delta A$, let $g(\alpha) \in V$ be the expectation of payoff vectors $g(a)$ taken with respect to α. Take a pair of vectors $v_i \in R^{\Theta_1 \times \Theta_2}$ for each player $i = 1, 2$.

- Vectors v_1 and v_2 satisfy *Individual Rationality* if for each player i, each type θ_{-i}, the payoffs of player i types are individually rational: $\forall \phi \in R_+^{d_i}$, $\phi \cdot v_i^{\cdot,\theta_{-i}} \geq m_i(\phi)$, where $m_i(\phi)$ is the value of the ϕ-weighted minmax defined in (2.1).
- Vectors v_1 and v_2 satisfy *Incentive Compatibility* if for each type profile (θ_1, θ_2), there exists $\alpha_{\theta_1,\theta_2} \in \Delta A$ such that for each type profile (θ_1, θ_2), player i, type θ_i',
$$v_i^{\theta_i,\theta_{-i}} = g_i\left(\alpha_{\theta_i,\theta_{-i}}|\theta_i\right) \geq g_i\left(\alpha_{\theta_i',\theta_{-i}}|\theta_i\right).$$

The next result shows that the threads are essentially equivalent to Individually Rational and Incentive Compatible payoff vectors.

Lemma 3. *Suppose that $u^* : \Theta_1 \times \Theta_2 \to R^{\Theta^*}$ is a thread. Let v_1 and v_2 be a pair of vectors $v_i \in R^{\Theta_1 \times \Theta_2}$ such that $v_i^{\theta_i,\theta_{-i}} = u^*(\theta_i, \theta_{-i}|\theta_i)$ for each player i. Then, v_1 and v_2 satisfy Individual Rationality and Incentive Compatibility.*
Conversely, suppose that a pair of vectors v_1 and v_2 satisfies Individual Rationality and Incentive Compatibility. For each player i types $\theta_i, \theta_i' \in \Theta_i$, and $\theta_{-i} \in \Theta_{-i}$, let
$$u^*(\theta_i', \theta_{-i}|\theta_i) = v_i^{\theta_i,\theta_{-i}}.$$
Then, u^* is a thread.

Proof. Part I. Suppose that u^* is a thread. By the definition of sets $NE(\theta_1, \theta_2)$ from Theorem 5, there exist probability distributions $\alpha_{\theta_i,\theta_{-i}}^{\theta_1^*,\theta_2^*} \in \Delta A$ such that for each type profile (θ_1^*, θ_2^*), and for each θ_i, θ_{-i},
$$u^*(\theta_i|\theta_1^*, \theta_2^*) = g_i\left(\alpha_{\theta_i,\theta_{-i}^*}^{\theta_1^*,\theta_2^*}|\theta_i\right),$$
and for each player i and all types θ_i, θ_i',
$$g_i\left(\alpha_{\theta_i,\theta_{-i}^*}^{\theta_1^*,\theta_2^*}|\theta_i\right) \geq g_i\left(\alpha_{\theta_i',\theta_{-i}^*}^{\theta_1^*,\theta_2^*}|\theta_i\right)$$

Define

$$v_i^{\theta_1,\theta_2} = u^*\left(\theta_i, \theta_{-i}|\theta_i\right).$$

Because u^* is a thread, for each player i, type θ_{-i}, each type θ_i'

$$v_i^{\theta_i,\theta_{-i}} = u^*\left(\theta_i', \theta_{-i}|\theta_i\right).$$

Because $u^*\left(\theta_i', \theta_{-i}\right) \in IR$, the payoffs of types of player i in the vector $u^*\left(\theta_i', \theta_{-i}\right)$ are individually rational. This shows that vectors (v_1, v_2) satisfy Individual Rationality.

Next, we show that (v_1, v_2) satisfies Incentive Compatibility. For each type profile (θ_1, θ_2), define

$$\alpha^*_{\theta_1,\theta_2} = \alpha^{\theta_1,\theta_2}_{\theta_1,\theta_2} \in \Delta A.$$

Then,

$$v_i^{\theta_1,\theta_2} = g\left(\alpha^*_{\theta_1,\theta_2}|\theta_i\right),$$

and

$$\begin{aligned} v_i^{\theta_1,\theta_2} &= g\left(\alpha^*_{\theta_1,\theta_2}|\theta_i\right) = g\left(\alpha^{\theta_1,\theta_2}_{\theta_1,\theta_2}|\theta_i\right) = u^*\left(\theta_i|\theta_1,\theta_2\right) = u^*\left(\theta_i|\theta_1',\theta_2\right) \\ &= g\left(\alpha^{\theta_1',\theta_2}_{\theta_1,\theta_2}|\theta_i\right) \geq g\left(\alpha^{\theta_1',\theta_2}_{\theta_1',\theta_2}|\theta_i\right) = g_i\left(\alpha^*_{\theta_i',\theta_{-i}}|\theta_i\right). \end{aligned}$$

Part II. Suppose that pair of vectors $v_i \in R^{\Theta_1\times\Theta_2}$ satisfies Individual Rationality and Incentive Compatibility. Let $\alpha_{\theta_1,\theta_2} \in \Delta A$ be as in the definition of Incentive Compatibility. For each profile (θ_1, θ_2), each player i type θ_i', define

$$u^*\left(\theta_1, \theta_2|\theta_i'\right) = v_i^{\theta_i',\theta_{-i}} = g_i\left(\alpha_{\theta_i',\theta_{-i}}|\theta_i'\right).$$

Then, for each profile (θ_1^*, θ_2^*), the vector of the payoffs of player i types, $u_i^*\left(.|\theta_1^*, \theta_2^*\right) = v_i^{.,\theta_{-i}}$, is individually rational. Thus, $u^*\left(\theta_1^*, \theta_2^*\right) \in IR$. Moreover, for each profile (θ_1^*, θ_2^*), and any two types θ_i, θ_i',

$$u^*\left(\theta_1^*, \theta_2^*|\theta_i\right) = g_i\left(\alpha_{\theta_i,\theta_{-i}^*}|\theta_i\right) \geq g_i\left(\alpha_{\theta_i',\theta_{-i}^*}|\theta_i\right).$$

This shows that $u^*\left(\theta_1^*, \theta_2^*\right) \in NE\left(\theta_1^*, \theta_2^*\right)$. □

Appendix B. Proof of Theorem 2

The proof of Theorem 2 follows from Lemmas 5, 6, and 7 below.

We begin with preliminary result.

Lemma 4. *For each $\varepsilon > 0$, there exist $\delta^{\epsilon} < 1$ and $m^{\epsilon} < \infty$ such that for each player i, each $m \geq m_{\epsilon}$, and each v such that $B(v, \varepsilon) \subseteq IR$, there exists m-period strategies of players $j \neq i$, $\mu_j^{i,v,m,\epsilon} : \bigcup_{s<m^{\epsilon}} (A_i)^{s-1} \to \Delta A_j$ such that such that for any sequence $\hat{a}^i = \left(a_0^i, ..., a_{m^{\epsilon}-1}^i\right)$ of actions of player i, each type θ_i, each $\delta \geq \delta^{\epsilon}$, the following inequality is satisfied:*

$$\begin{aligned} M_i^{v,m,\epsilon,\delta}\left(\hat{a}^i; \theta_i\right) &:= \frac{1-\delta}{1-\delta^m} \sum\nolimits_{s=0}^{m-1} \delta^s E g_i\left(a_s^i, \mu_{-i}^{i,v,m,\epsilon}\left(a_0^i, ..., a_{s-1}^i\right); \theta_i\right) \\ &\leq v_i(\theta_i). \end{aligned}$$

Here, the expectation is taken over actions induced by strategies $\mu_{-i}^{i,v,m,\epsilon}$.

Proof. The Lemma is a discounted version of the Blackwell approachability argument (Blackwell (1956)). The proof follows the same line and an observation that when $\delta \to 1$, the discounted payoff criterion in a game with finitely many periods converges to the average payoff criterion. □

Lemma 5. *For each $\pi \in \Pi$,*

$$F_0(\pi) \subseteq FR_0^+(\pi).$$

We omit the formal proof, because this result is well-known (see Hart (1985), Koren (1992), and Shalev (1994) for Nash equilibrium and no-discounting, and Peski (2008), and Horner and Lovo (2009) for the sequential equilibrium in the discounted case).

Lemma 6. *If $F_{n-1}^B(\pi) \subseteq FR_{n-1}^+(\pi)$, then $F_n^A(\pi) \subseteq FR_{n-1}^+(\pi)$.*

Proof. Take any $v^* \in F_n^A(\pi) = \text{int} IR \cap \text{con}\left\{\text{int} F_{n-1}^B(\pi) \cup V\right\}$. Find $\alpha^* \in (0,1)$, $g^* \in V$, and $u^* \in \text{conint} F_{n-1}^B(\pi)$ such that $v^* = \alpha^* g^* + (1-\alpha^*) u^*$. Assume that there exists a pure action profile a^* such that $g(a^*) = g^*$. The assumption is without loss of generality due to public correlation.

Find a sequence of t_δ such that $\delta^{t_\delta} \to 1-\alpha^*$ as $\delta \to 1$. We are going to compute the payoffs in a profile in which players play action profile a^* during the initial t^δ periods, and then receive continuation payoffs u chosen so that $v^* = \left(1-\delta^{t^\delta}\right) g^* + \delta^{t^\delta} u$. Any deviation by player i during period t triggers a punishment phase in which player i is initially minimaxed using the strategy from Lemma 4, and then the players continue with a strategy profile with payoffs $v^i(\hat{a})$ that depend on the realized actions during the minmaxing. The continuation payoffs $v^i(\hat{a})$ are chosen so that all players are indifferent among all actions during the minmaxing phase and the overall payoff from the punishment of player i phase is equal to $v^{i,t^\delta - t} = \left(1-\delta^{t^\delta - t}\right) g^* + \delta^{t^\delta - t} u^{i*}$. We choose u and u^{i*} so that they are sufficiently close to u^* and such that for sufficiently high $\delta < 1$, there exists continuation $(n-1)$-revealing equilibria $\sigma^{u,\delta}$ and $\sigma^{u^{i*},\delta}$ with payoffs, respectively, $v^{\pi,\delta}\left(\sigma^{u,\delta},\right) \preceq_\pi u$ and $v^{\pi,\delta}\left(\sigma^{u^{i*},\delta},\right) \preceq_\pi u^{i*}$. Moreover, we need to choose u^{i*}so that no player has incentives not to deviate.

Let $k^* = \frac{100}{1-\alpha^*}$ and find $\epsilon > 0$ so that $B(u^*, 2k\epsilon) \subseteq \text{conint} F^B_{n-1}(\pi)$. Using compactness, one can show that for sufficiently high δ, for each $u \in B(u^*, k\epsilon)$, there exists a strategy profile $\sigma^{u,\delta}$ that induces payoff $v^{\pi,\delta}(\sigma) \preceq_\pi u$ and such that $\sigma^{u,\delta}$ is a $(n-1)$-revealing equilibrium of game $\Gamma(\pi,\delta)$. (It might be necessary to use public randomization if $u^* \notin \text{int} F^B_{n-1}(\pi)$.)

For each player i, find $u^{i*} \in B(u^*, k\epsilon)$ so that

$$u^{i*}(\theta_i) \leq u^*(\theta_i) - \frac{2\epsilon}{1-\alpha^*} \text{ for each } \theta_i, \text{ and} \tag{B.1}$$
$$u^{i*}(\theta_j) \geq u^*(\theta_j) \text{ for each type } \theta_j \text{ of player } j \neq i.$$

For each $t \leq t^\delta$ and each player i, let $v^{i,t} = (1-\delta^t) g^* + \delta^t u^{i*}$. Because of (B.1), for sufficiently high δ, and each player $j \neq i$,

$$v^{i,t}(\theta_j) \geq v^{j,t}(\theta_j) + 2\epsilon. \tag{B.2}$$

Find m^ϵ and δ^ϵ from Lemma 4. Assume that $m \geq m^\epsilon$ and the discount factor $\delta \geq \delta^\epsilon$ are high enough so that $(1-\delta^m) M < \epsilon$, and $(1-\delta^m)\epsilon > 2(1-\delta) M$.

Let $\mu^{i,t,*}_j = \mu^{i,v^{i,t}-\epsilon,m,\epsilon}_j$ be the minmax strategies of players $j \neq i$ from Lemma 4. Let $M^{t,*}_i(\hat{a}^i)$ be the associated payoff vector of player i playing action sequence $\hat{a}^i = \left(a^i_0, ..., a^i_{m-1}\right)$. For each sequence of actions $\hat{a}^i$ of player i and $\hat{a}^{-i}$ of players $-i$, define $\hat{a} = (\hat{a}^i, \hat{a}^{-i})$ and payoff vector $v^i(a)$ so that for each type θ_i of player i,

$$(1-\delta^m) M^{t,*}_i\left(\hat{a}^i; \theta_i\right) + \delta^m v^i(\hat{a}; \theta_i) = v^{i,t}(\theta_i),$$

and for each type θ_j of player $j \neq i$,

$$(1-\delta)\sum_{s=0}^{m-1} \delta^s g_j\left(a_s^i, a_s^{-i}; \theta_j\right) + \delta^m v^i\left(\hat{a}, \theta_j\right) = v^{i,t}\left(\theta_j\right).$$

Notice that because $M_i^{t,*}\left(a^i; \theta_i\right) \leq v^{i,t}\left(\theta_i\right) - \epsilon$ for each type θ_i of player i,

$$v^i\left(\hat{a}, \theta_i\right) \geq v^{i,t}\left(\theta_i\right) + \left(1-\delta^m\right)\epsilon > v^{i,t}\left(\theta_i\right) + 2M\left(1-\delta\right).$$

Moreover, due to (B.2), for each type θ_i of player $j \neq i$,

$$v^i\left(\hat{a}, \theta_j\right) \geq v^{i,t}\left(\theta_j\right) - \left(1-\delta^m\right)M > v^{i,t}\left(\theta_j\right) - \epsilon > v^{j,t}\left(\theta_j\right) + 2M\left(1-\delta\right).$$

We are going to construct strategy profile σ. There are two types of regimes:

- *Normal*(v,t) for each $t \leq t^\delta$ and v so that (a) there exists $u \in B\left(u^*, k\epsilon\right)$ such that $v = \left(1-\delta^t\right)g^* + \delta^t u$, and (b) $v\left(\theta_i\right) \geq v^{i,t}\left(\theta_i\right) + 2M\left(1-\delta\right)$ for each player i and type θ_i. Players play action profile a^* for t periods $s = 0, 1, ..., t-1$. If there is no deviation, player continue with strategy profile $\sigma^{u,\delta}$. Simultaneous deviations of two or more players are ignored. A deviation by single player i in period s initiates regime *Punishment*$(i, t-s)$.
- *Punishment*(i,t): The regime lasts m periods. Players $-i$ play strategies $\mu_{-i}^{i,t,*}$. Player i randomizes uniformly across all action sequences $\left(a_0^i, ..., a_{m-1}^i\right)$. After m periods, regime *Normal*$\left(v^i\left(\hat{a}\right), t\right)$ is initiated, where $\hat{a}$ are the actions played during the regime.

The profile starts in regime *Normal*$\left(v^*, t^\delta\right)$.

We compute the payoffs and verify the incentives in the above profile. Initially, we make a preliminary (and, perhaps, incorrect) assumption that the payoff in profiles that ends phase *Normal*$\left(\left(1-\delta^t\right)g^* + \delta^t u, t\right)$ is equal to u (instead of $v^{\pi,\delta}\left(\sigma^{u,\delta}\right) \preceq_\pi u$). Then, the expected payoff in the beginning of regime *Normal*(v,t) is equal to v and the expected payoff in the beginning of regime *Punishment*(i,t) is equal to $v^{i,t}$. Any one-shot deviation during the *Normal*(v,t) period leads to the payoff not higher than $\left(1-\delta\right)M + \delta v^{i,t}$. If $v\left(\theta_i\right) \geq v^{i,t}\left(\theta_i\right) + 2M\left(1-\delta\right)$, the deviation is not profitable. In each period of the *Punishment*(i,t) regime, all players are indifferent between all actions. In particular, they do not have one-shot profitable deviations. Thus, the expected payoff from profile σ under the preliminary assumption is equal to v^*.

Because our preliminary assumption is possibly incorrect, the above argument may not reflect correctly the incentives faced by the players. On one hand, the preliminary assumption does not affect the payoffs of the π-positive probability types. Thus, the behavior prescribed by strategy profile σ is a best response for all such type given that all positive probability types of other players follow σ. On the other hand, the behavior prescribed by profile σ may not be a best response for the π-zero probability types. We can modify profile σ so that all the zero-probability types choose best responses given the assumption that all (the positive probability types of) other players follow σ. (Notice that such a modification does not change the incentives for the positive probability types.) Because the preliminary assumption may artificially increase the continuation payoffs of the π-zero probability type θ_i, the true expected best response payoffs of such type cannot be higher than $v^*(\theta_i)$. Thus, the true expected payoff from profile σ is equal to $v^{**} \preceq_\pi v^*$.

Finally, because the strategies prescribe the same (possibly, mixed) actions for all π-positive probability types of each player, the beliefs do not get updated before $(n-1)$-revealing profile σ^u is started. □

Lemma 7. *If* $F_n^A(\pi) \subseteq F_{n-1}^+(\pi)$, *then* $F_n^B(\pi) \subseteq F_{n-1}^+(\pi)$.

Proof. Take any $v \in F_n^B(\pi)$ and find here exists $\epsilon > 0$ and an incentive compatible lottery $l = (\alpha, u)$ such that $v = v^{\pi,l}$ and $B(u(a), 2\epsilon) \subseteq \text{int} F_n^A\left(p^{\pi,l}(a)\right)$ for each positive probability action profile a. We can assume w.l.o.g. that all actions have positive probability.

Using the compactness argument (and, possibly, public randomization), we can show that there exists δ_0 such that for all $\delta \geq \delta_0$, each a, and each $u' \in B(u(a), \epsilon)$, there exists a strategy profile that induces payoff u' and that is a $(n-1)$-revealing equilibrium of game $\Gamma\left(p^{\pi,l}(a,), \delta\right)$.

For each action profile, let $u^\delta(a) = \frac{1}{\delta} u(a) - (1-\delta) g(a) \in B(u(a), \epsilon)$. For each a, find n-revealing equilibrium profile σ^a that induces $u^\delta(a)$.

Let σ be a strategy profile in which in the first period players play according to α, and continue with $\sigma(a)$ after first period history a. Then, σ is a $(n-1)$-revealing equilibrium for sufficiently high δ with expected payoff v. □

Appendix C. Proof of Theorem 5

In this Appendix, we assume that the game has a structure described in Section 6.2. In particular,

$$\text{int}IR = \left\{v \in R^{\Theta^*} : v_i(\theta_i) > 0 \text{ for each type } \theta_i\right\},$$

and there exist sets $M_i \subseteq \mathtt{R}^{\Theta_i}$ such that $\mathbf{0}_i \in M_i$ and the set

$$\text{int}V = \text{intcon}\left\{\bigcup_i M_i \times \{\mathbf{0_{-i}}\}\right\}$$

is not empty.

We begin with a convenient characterization of set $F_1^B(\pi)$.

Lemma 8. *Let $v \in R^{\Theta^*}$ be a payoff vector. Then, $v \in F_1^B(\pi)$ if and only if for each player i, there exist mappings $\beta_i : \Theta \to [0,1]$ and $m_i : \Theta_i \to M_i$ such that $\sum_i \beta_i^\theta \leq 1$ and the following conditions hold:*

1. *Individual rationality: $v_i(\theta_i) > 0$ for each player i and type θ_i, and $m_i^{\theta_i}(\theta_i) > 0$ for each player i and π-positive probability type θ_i,*
2. *Incentive compatibility: for all θ_i, θ_i',*

$$v(\theta_i) \geq m_i^{\theta_i'}(\theta_i) \sum\nolimits_{\theta_{-i}} \pi^{\theta_i}(\theta_{-i}) \beta_i^{(\theta_i', \theta_{-i})},$$

with the equality if type θ_i has π-positive probability and $\theta_i' = \theta_i$.

In particular, set $F_1^B(\pi)$ is convex.

Proof. If v satisfies the above two conditions, then one can easily construct appropriate lottery to show that $v \in F_1^B$. We show the other direction. Take some $v \in F_1^B(\pi)$ and find π-incentive compatible lottery $l^0 = (\alpha^0, u^0)$ with value v and such that for each action profile a, either beliefs $p(a)$ are degenerate on the type tuple θ and

$$u^0(a) \in F_1^A\left(\pi^\theta\right) = F_0\left(\pi^\theta\right),$$

or the beliefs $p(a)$ are non-degenerate, and

$$u^0(a) \in F_1^A(p(a)) = \text{int}IR \cap \text{int}V.$$

Because $\text{int}IR \cap \text{int}V \subseteq F_0^B\left(\pi^\theta\right)$, we can assume that $u^0(a) \in F_0^B\left(\pi^\theta\right)$ for each a that is played with positive probability by types θ in strategy profile α^0.

For each π-positive probability type profile θ and action profile a that is played by positive probability by types in θ, we can find $u^1(a) \in \text{int}V$ such that $u^1(a) \preceq_{\pi^\theta} u^0(a)$. Because payoffs $u^0(a)$ are strictly individually rational, we have

$$\max\left\{0, u^1(\theta_i'|a)\right\} \leq u^0(\theta_i'|a) \text{ for each type } \theta_i'.$$

Define allocation $u : \Theta \to \text{int}V$ so that for each type profile θ (not necessarily positive probability),

$$u^\theta = \sum_a \left(\prod_i \alpha_i^0(a_i|\theta_i)\right) u^1(a).$$

For each type profile θ, player i, find $\beta_i^\theta \geq 0$ and $\hat{m}_i^\theta \in M_i$ so that $\sum_i \beta_i^\theta \leq 1$ and $u_i^\theta = \beta_i^\theta \hat{m}_i^\theta$. Finally, for each type θ_i, define

$$m_i^{\theta_i} = \frac{\sum_{\theta_{-i}} \pi^{\theta_i}(\theta_{-i}) u_i^{(\theta_i,\theta_{-i})}}{\sum_{\theta_{-i}} \pi^{\theta_i}(\theta_{-i}) \beta_i^{(\theta_i,\theta_{-i})}} = \frac{\sum_{\theta_{-i}} \pi^{\theta_i}(\theta_{-i}) \beta_i^{(\theta_i,\theta_{-i})} \hat{m}_i^{(\theta_i,\theta_{-i})}}{\sum_{\theta_{-i}} \pi^{\theta_i}(\theta_{-i}) \beta_i^{(\theta_i,\theta_{-i})}}.$$

Notice that $m_i^{\theta_i}$ is a convex combination of elements of M_i, hence $m_i^{\theta_i} \in M_i$.

We check that β and m satisfy the thesis of the Lemma. For each π-positive probability type profile $\theta = (\theta_i, \theta_{-i})$, each player i and each action profile $a = (a_i, a_{-i})$ such that a_i is played with positive probability by type θ_i, $u^1(\theta_i|a) = u^0(\theta_i|a) > 0$. It follows that $m_i^{\theta_i}(\theta_i) > 0$ as a convex combination of strictly positive values.

Further, because lottery l^0 is π-incentive compatible, for each action a_i,

$$v(\theta_i) \geq \sum\nolimits_{\theta_{-i}} \pi^{\theta_i}(\theta_{-i}) u_i^0\left(\theta_i|a_i, \alpha_{-i}^0(\theta_{-i})\right)$$

with equality when action a_i is played with positive probability by type θ_i, i.e., $\alpha_i^0(a_i|\theta_i) > 0$. It follows that for π-positive probability type θ_i

$$\begin{aligned}
v(\theta_i) &= \sum\nolimits_{\theta_{-i}} \pi^{\theta_i}(\theta_{-i}) u^0(\theta_i|\alpha_i(\theta_i), \alpha_{-i}(\theta_{-i})) \\
&= \sum\nolimits_{\theta_{-i}} \pi^{\theta_i}(\theta_{-i}) u^1\left(\theta_i|\alpha_i^0(\theta_i), \alpha_{-i}^0(\theta_{-i})\right) \\
&= \sum\nolimits_{\theta_{-i}} \pi^{\theta_i}(\theta_{-i}) u^\theta\left(\theta_i|\alpha_i^0(\theta_i), \alpha_{-i}^0(\theta_{-i})\right) \\
&= \sum\nolimits_{\theta_{-i}} \pi^{\theta_i}(\theta_{-i}) \beta^{(\theta_i,\theta_{-i})} \hat{m}_i^{(\theta_i,\theta_{-i})}(\theta_i) \\
&= \sum\nolimits_{\theta_{-i}} \pi^{\theta_i}(\theta_{-i}) \beta^{(\theta_i,\theta_{-i})} m_i^{\theta_i}(\theta_i)
\end{aligned}$$

and for all types θ_i, θ_i',

$$\begin{aligned} v(\theta_i) &\geq \sum\nolimits_{\theta_{-i}} \pi^{\theta_i}(\theta_{-i}) E_{\alpha^0\left(\theta_i',\theta_{-i}\right)} u_i^0(\theta_i|a) \\ &\geq \sum\nolimits_{\theta_{-i}} \pi^{\theta_i}(\theta_{-i}) E_{\alpha^0\left(\theta_i',\theta_{-i}\right)} u_i^1(\theta_i|a) \\ &\geq \sum \pi^{\theta_i}(\theta_{-i}) E_{\alpha^0\left(\theta_i',\theta_{-i}\right)} u_i^1\left(\theta_i|\alpha_i^0(\theta_i'), \alpha_{-i}^0(\theta_{-i})\right) \\ &\geq m_i^{\theta_i'}(\theta_i)\left(\sum\nolimits_{\theta_{-i}} \pi^{\theta_i}(\theta_{-i}) \beta_i^{\left(\theta_i',\theta_{-i}\right)}\right). \end{aligned}$$

The last claim follows from the characterization. □

Take any individually rational payoff vector $v^* =$ of payoffs that are individually rational for all positive π-probability types of all players and that can be obtained by a play of non-revealing actions followed by a payoff vector from stage $1B$, $v^* = \gamma g + (1-\gamma) v'$ for some $g \in V$ and $v' \in F_1^B(\pi)$. The next Lemma shows that there exist a corresponding fully revealing payoff v, with the same payoffs as v^* for the positive probability types and not smaller, and individually rational payoffs for the zero-probability types. The idea is to delay the play of non-revealing actions after the revelation. We need to be careful so that the expected payoffs and the incentives to reveal information truthfully are not affected and that continuation payoffs after the revelation are individually rational.

Lemma 9. *For each $\pi \in \Pi$, each $F_2^A(\pi) = F_1^B(\pi)$.*

Proof. Take $v^* \in \text{int}\left(IR \cap \text{con}\left(F_1^B(\pi) \cup V\right)\right)$. Find $\gamma_i \geq 0$ and $m_i^* \in M_i$, and $u^* \in F_1^B(\pi)$ so that $\sum_i \gamma_i \leq 1$ and for each player i,

$$v_i^*(\theta_i) = \gamma_i m_i^*(\theta_i) + \left(1 - \sum_i \gamma_i\right) u^*(\theta_i),$$

with equality for π-positive probability types θ_i. Using Lemma 8, find $\beta_i^\theta \geq 0$ and $m_i^{\theta_i} \in M_i$ for each player type tuple θ so that $\sum_i \beta_i^\theta \leq 1$ and $m_i^{\theta_i} \geq 0$ for each $\theta = (\theta_i, \theta_{-i})$, and

$$\begin{aligned} v_i^*(\theta_i) &= \gamma_i m_i^*(\theta_i) + \left(\sum\nolimits_{\theta_{-i}} \pi^{\theta_i}(\theta_{-i}) \beta_i^{\left(\theta_i',\theta_{-i}\right)}\right) m_i^{\theta_i'}(\theta_i) \\ &\geq \gamma_i m_i^*(\theta_i) + \left(\sum\nolimits_{\theta_{-i}} \pi^{\theta_i}(\theta_{-i}) \beta_i^{\left(\theta_i',\theta_{-i}\right)}\right) m_i^{\theta_i'}(\theta_i) \end{aligned}$$

with the equality if type θ_i has π-positive probability and $\theta_i' = \theta_i$.

For each player i and type profile $\theta = (\theta_i, \theta_{-i})$, let

$$\hat{\beta}_i^\theta = \gamma_i + \left(1 - \sum_i \gamma_i\right) \beta_i^\theta.$$

For all π-positive probability types θ_i, let

$$\hat{m}_i^{\theta_i} = \frac{\gamma_i m_i^* + \left(1 - \sum_i \gamma_i\right) \sum_{\theta'_{-i}} \pi^{\theta_i}\left(\theta_{-i}\right) \beta_i^{\left(\theta_i, \theta'_{-i}\right)} m_i^{\theta_i}}{\gamma_i + \left(1 - \sum_i \gamma_i\right) \sum_{\theta'_{-i}} \pi^{\theta_i}\left(\theta_{-i}\right) \beta_i^{\left(\theta_i, \theta'_{-i}\right)}}.$$

For all π-zero probability types θ_i, let $\hat{m}_i^{\theta_i} = \mathbf{0}_i$. For each player i type θ_i, define

$$\begin{aligned} v\left(\theta_i\right) &= \textstyle\sum_{\theta_{-i}} \pi^{\theta_i}\left(\theta_{-i}\right) \hat{\beta}_i^{(\theta_i, \theta_{-i})} \hat{m}_i^{\theta_i}\left(\theta_i\right) \text{ for } \pi\text{-positive prob. } \theta_i \\ v\left(\theta_i\right) &= v_i^*\left(\theta_i\right) \text{ for } \pi\text{-zero prob. } \theta_i. \end{aligned}$$

Simple calculations show that $v = v^*$.

We check that assignments $\hat{\beta}_i$ and $\hat{m}_i^\theta$ satisfy the conditions of Lemma 8 for v. Indeed, $v_i\left(\theta_i\right) > 0$ and $\hat{m}_i^{\theta_i} \in M_i$ because $\hat{m}_i^{\theta_i} = \mathbf{0}_i$ or $\hat{m}_i^{\theta_i}$ is a convex combination of elements of M_i. Moreover, for each tuple θ,

$$\begin{aligned} \sum_i \left(\gamma_i + \left(1 - \sum_i \gamma_i\right) \beta_i^\theta\right) &= \sum_i \gamma_i + \left(1 - \sum_i \gamma_i\right) \sum_i \beta_i^\theta \\ &\leq \sum_i \gamma_i + \left(1 - \sum_i \gamma_i\right) \leq 1. \end{aligned}$$

The individual rationality holds because, in the first case, $\hat{m}_i^{\theta_i}\left(\theta_i\right)$ is equal to $v^*\left(\theta_i\right)$ multiplied by a positive factor, and, in the second case, $\hat{m}_i^\theta\left(\theta_i\right) = 0$.

We check the incentive compatibility: For all types π-positive probability type θ_i and all types θ_i' ,

$$\begin{aligned}v(\theta_i) =& \sum\nolimits_{\theta_{-i}} \pi^{\theta_i}(\theta_{-i})\, \hat{\beta}_i^{(\theta_i,\theta_{-i})} \hat{m}_i^{\theta_i}(\theta_i) \\ =& \gamma_i m_i^* + \left(1 - \sum_i \gamma_i\right) \sum\nolimits_{\theta'_{-i}} \pi^{\theta_i}(\theta_{-i})\, \beta_i^{\left(\theta_i,\theta'_{-i}\right)} m_i^{\theta_i}(\theta_i) \\ \\ \geq& \gamma_i m_i^* + \left(1 - \sum_i \gamma_i\right) \left(\sum\nolimits_{\theta'_{-i}} \pi^{\theta_i}(\theta_{-i})\, \beta_i^{\left(\theta'_i,\theta'_{-i}\right)}\right) m_i^{\theta'_i}(\theta_i) \\ =& \left(\sum\nolimits_{\theta_{-i}} \pi^{\theta_i}(\theta_{-i})\, \hat{\beta}_i^{\left(\theta'_i,\theta_{-i}\right)}\right) \hat{m}_i^{\theta'_i}(\theta_i).\end{aligned}$$

where the first inequality follows from the choice of β. The incentive compatibility in case of π-zero probability types θ_i is trivial. It follows that $v \in F_1^B(\pi)$. □

Theorem 5 follows from the argument sketched in the proof of Theorem 4 with a key step replaced by Lemma 9.

Appendix D. Proof of lemma 2

The first inequality in (6.5) implies that

$$\alpha_{TT} \leq \frac{2}{3} - \frac{1}{3}\alpha_{WW} - \frac{2}{3}\alpha_{WT}.$$

Substituting into the second inequality, we obtain

$$2x\alpha_{WW} + (1+3x)\,\alpha_{WT} \geq (1-3x)\left(\frac{1}{3} + \frac{1}{3}\alpha_{WW} + \frac{2}{3}\alpha_{WT}\right),$$

or, after some algebra,

$$\alpha_{WT} \geq \frac{1-3x}{1+15x} + \frac{1-9x}{1+15x}\alpha_{WW}.$$

It follows that

$$\begin{aligned}& 2\alpha_{WW} + 4\alpha_{WT} - 2\alpha_{TT} \\ \geq& \frac{8}{3}\alpha_{WW} + \frac{16}{3}\alpha_{WT} - \frac{4}{3} \\ \geq& \left(\frac{8}{3} + \frac{16}{3}\frac{1-9x}{1+15x}\right)\alpha_{WW} + \frac{16}{3}\frac{1-3x}{1+15x} - \frac{4}{3} > 2,\end{aligned}$$

where the last inequality holds for all $\alpha_{WW} \geq 0$ and all $x < \frac{3}{100}$.

University of Toronto
E-mail address: mpeski@gmail.com.