The Evolutionary Robustness of Forgiveness and Cooperation

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Abstract

We study the evolutionary robustness of strategies in infinitely repeated prisoners' dilemma games in which players make mistakes with a small probability and are patient. The evolutionary process we consider is given by the replicator dynamics. We show that there are strategies with a uniformly large basin of attraction independently of the size of the population. Moreover, we show that those strategies forgive defections and, assuming that they are symmetric, they cooperate.

1 Introduction

The theory of infinitely repeated games has been very influential in the social sciences showing how repeated interaction can provide agents with incentives to overcome opportunistic behavior. However, a usual criticism of this theory is that there may be a multiplicity of equilibria. While cooperation can be supported in equilibrium when agents are sufficiently patient, there are also equilibria with no cooperation. Moreover, a variety of different punishment can be used to support cooperation.

To solve this multiplicity problem, we study what types of strategies will have a large basin of attraction regardless of what other strategies are considered in the evolutionary dynamic. More precisely, we study the replicator dynamic over arbitrary finite set of infinitely repeated strategies in which in every round of the game the strategy makes a mistake with a small probability $1-p$. We study which strategies have a non vanishing basin of attraction with a uniform size regardless of the set of strategies being consider in the population. We say that that a strategy has a uniformly large basin of attraction if it repeals invasions of a given size for arbitrarily patient players and small probability of errors and for any possible combination of alternative strategies (see definition 3 for details).

We find that two well known strategies, always defect and grim, do not have uniformly large basins of attraction. Moreover, any strategy that does not forgive cannot have a uniformly large basin either. The reason is that, as players become arbitrarily patient and the probability of errors becomes small, unforgiving strategies lose in payoffs relative to strategies that forgive and the size of the basins of attraction between these two strategies will favor the forgiving one. This is the case even when the inefficiencies happen off the equilibrium path (as it is the case for grim).

Moreover, we show that symmetric strategies leading to inefficient payoffs even when players are arbitrarily patient and the probability of errors is sufficiently small cannot have uniformly large basins of attractions.

However, it could be the case that inefficient and unforgiving strategies do not have uniformly large basins since actually there may be no strategies with that property! We prove that that is not the case by showing that the strategy win-stay-loose-shift has a uniformly large basin of attraction, provided a sufficiently small probability of mistakes. As this strategy is efficient (and symmetric), we show that the concept of uniformly large basins of attraction provides a (partial) solution to the long studied problem of equilibrium selection in infinitely repeated games: only efficient equilibria survive for patient players if we focus on symmetric strategies. We suspect that the efficiency result can be extended to non-symmetric strategies in which case the concept of uniformly large basin of attraction would provide a complete solution to the problem of equilibrium selection in infinitely repeated games.

Note that we not only provide equilibrium selection at the level of payoffs but also at the level of the type of strategies used to support those payoffs: the payoffs from mutual cooperation can only be supported by strategies that do not involve asymptotically inefficient punishments. This provides theoretical support to Axelrod's claims ([Ax]) that successful strategies should be cooperative and forgiving.

In addition, in our study of the replicator dynamics we develop technologies that can be used to analyze the basins of attractions outside of the particular case of infinitely repeated games. In fact the results are based in a series of theorems about general replicator dynamics which can be used to study the robustness of steady states for games in general. In addition, we prove that our results are robust to perturbation of the replicator dynamic provided that it is still the case that the only growing strategies are those that perform better than the average.

An extensive previous literature has addressed the multiplicity problem in infinitely repeated games. Part of this literature focuses on strategies of finite complexity with costs of complexity to select a subset of equilibria (see Rubinstein [R], Abreu and Rubinstein [AR], Binmore and Samuelson [BiS], Cooper [C] and Volij [V]). This literature finds that the selection varies with the equilibrium concept being used and the type of cost of complexity. Another literature appealed to ideas of evolutionary stability as a way to select equilibria and found that no strategy is evolutionary stable in the infinitely repeated prisoners' dilemma (Boyd and Lorberbaum [BL]). The reason is that for any strategy there exist another strategy that differs only after events that are not reached by this pair of strategies. As such, the payoff from both strategies is equal when playing with each other and the original strategy cannot be an attractor of an evolutionary dynamic. Bendor and Swistak ([BeS]) circumvent the problem of ties by weakening the stability concept and show that cooperative and retaliatory strategies are the most robust to invasions.

In a different approach to deal with the problem of ties, Boyd ([B]) introduced the idea of errors in decision making. If there is a small probability of errors in every round, then all events in a game occur with positive probability destroying the certainty of ties allowing for some strategies to be evolutionary stable. However, as shown by Boyd ([B]) and Kim ([Ki]), many strategies that are sub-game perfect for a given level of patience and errors can also be evolutionary stable.

Fudenberg and Maskin ([FM2]) (see also Fudenberg and Maskin [FM]) show that evolutionary stability can have equilibrium selection implications if we ask that the size of invasions that the strategy can repel to be uniformly large with respect to any alternative strategy and for large discount factors and small probabilities of mistakes. They show that the only strategies with characteristic must be cooperative. There are two main differences with our contributions. First, Fudenberg and Maskin ([FM2]) focus on strategies of finite complexity while our efficiency result does not have that restriction, it applies only to symmetric strategies. Second, our robustness concept not only consider the robustness to invasion by a single alternative strategy but also robustness to invasion by any arbitrary combination of alternative strategies. In other words, we also look at the size of the basin of attraction inside the simplex.

Finally, Johnson, Levine and Pesendorfer ([JLP]), Volij ([V]) and Levine and Pesendorfer ([LP]) use the idea of stochastic stability (Kandori, Mailath and Rob [KMR] and Young [YP]) to select equilibria in infinitely repeated games.

We wonder if the present result could be useful to formulate experiment that could help to understand if individuals, when playing the repeated prisoner's dilemma, behave in the way that replicator equation assumes. In particular, if win-stay-loose-shift is highly present in a designed experiment, is it going to become prevalent?

The paper is organized as follows: In section 2 we introduce the infinite repeated prisoner's dilemma with trembles. In section 3 we start recalling the definition of replicator dynamics in any dimension and in theorem 1 we give sufficient conditions to be satisfied by a payoff matrix for a vertex to have a large local basin of attraction independent of the dimension of the matrix. Moreover, in subsection 3.4 we show that the conditions of theorem 1 are also necessary. In section 4 we recast the replicator dynamics in the context of infinite repeated prisoner's dilemma with trembles. In this section we define the notion of strategy having a uniformly large basin of attraction (see definition 3). In section 5 we show that grim does not have a uniformly large basin. In section 6 we prove that for any history, the frequency of cooperation converges to one for symmetric strategies that have a uniform large basin of attraction. In section 7 we show how to adapt theorem 1 to the context of the set of all the strategies. In particular, in subsection 7.1 it is provided sufficient condition to guarantee that a strategy has a uniform large basin of attraction. These conditions basically consist in analyzing all the possible set of three strategies; moreover, in subsection 7.2 we show that weaker conditions that consists in comparing sets of two strategies is not enough to have a uniformly large basin of attraction. In section 8 we develop a technique to calculate the payoff with trembles for certain type of strategies (see definition 10) provided certain restriction on the probability of mistakes (see lemma 15). In section 9 we apply this techniques for the particular case of win-stay-loose-shift, proving that it has a uniformly large basin of attraction. We also consider in subsection 9.1 a generalization of win-stay-loose-shift. In subsection 10 we show that theorem 1 can be reproved for a general type of equation that resembles the replicator dynamics.

2 Infinitely repeated prisoner's dilemma with trembles

In the present section, we state the definitions of the game first without trembles and later with trembles. We also explain and how the payoff is calculated whit and without trembles.

In each period $t = 0, 1, 2, ...$ the 2 agents play a symmetric stage game with action space $A = \{C, D\}$. At each period t player one choose action $a^t \in A$ and second player choose action $b^t \in A$. We denote the vector of actions until time t as $a_t = (a^0, a^1, \dots, a^t)$ for player one and $b_t = (b^0, b^1, \dots, b^t)$ for player two. The payoff from the stage game at time t is given by utility function $u(a^t, b^t) : A \times A \to \Re$ for player one and $u(b^t, a^t) : A \times A \to \Re$ for player two such that $u(D, C) = T$, $u(C, C) = R$, $u(D, D) = P$, $u(C, D) = S$, with $T > R > P > S$ and $2R > T + S$.

Agents observe previous actions and this knowledge is summarized by histories. When the game begins we have the null history $h^0 = (a^0, b^0)$, afterwards $h_t = (a_{t-1}, b_{t-1}) = ((a^0, b^0), \dots (a^{t-1}, b^{t-1}))$ and H^t is the space of all possible t histories. Let H_∞ be the set of all possible histories. A pure strategy is a function $s : H_\infty \to A$. In other words, a pure strategy s is a functions $s : H_t \to A$ for all t.

It is important to remark, that given two strategies s_1, s_2 and a finite path $h_t = (a_{t-1}, b_{t-1})$, if s_1 encounter s_2 then

$$h^t = (s_1(h_t), s_2(\hat{h}_t)),$$

where

$$\hat{h}_t := (b_{t-1}, a_{t-1}). \tag{1}$$

Given a pair of strategies (s_1, s_2) we call the history they generate as their equilibrium path and denote it as h_{s_1,s_2}. In other words, denoting with $h_{s_1,s_2 t}$ the path up to period t then equilibrium path h_{s_1,s_2}, is the path that verifies

$$s_1(h_{s_1,s_2 t}) = a^t,\ \ s_2(\widehat{h_{s_1,s_2 t}}) = b^t.$$

Given a pair of strategies s_1, s_2 the utility of the agent s_1 is

$$U(s_1, s_2) = (1-\delta)\sum_{t=0}^{\infty}\delta^t u(s_1(h_{s_1,s_2 t}), s_2(\widehat{h_{s_1,s_2 t}})),$$

where the common and constant discount factor $\delta < 1$.

Given a finite path h_t, with $h_{s_1,s_2/h_t}$ we denote the equilibrium path between s_1 and s_2 with seed h_t Given the recursivity of the discounted utility function we can write the utility starting from history h_t as $U(s_1, s_2|h_t) = (1-\delta)\sum_{k=t}^{\infty}\delta^{t-k}u(s_1(h_{s_1,s_2/h_t k}), s_2(\widehat{h_{s_1,s_2/h_t k}}))$.

For the case of trembles, we have the probability of making a mistake, more precisely, with a positive $p < 1$ we denote the probability that a strategy perform what intends. Now, given two strategies s_1, s_2 (they can be the same strategy) we define

$$U_{\delta,p}(s_1, s_2) = (1-\delta)\sum_{t\geqslant 0, a_t, b_t}\delta^t p_{s_1,s_2}(a_t, b_t)u(a^t, b^t)$$

where $u(a^t, b^t)$ denotes the usual payoff of the pair (a^t, b^t) and $p_{s_1,s_2}(a_t, b_t)$ denote the probability that the strategies s_1 and s_2 go through the path $h_t = (a_t, b_t)$ when they are playing one to each other. To define $p_{s_1,s_2}(a_t, b_t)$ we proceed inductively:

$$p_{s_1,s_2}(a_t, b_t) = p_{s_1,s_2}(a_{t-1}, b_{t-1})p^{i_t+j_t}(1-p)^{1-i_t+1-j_t} \tag{2}$$

where

(i) $i_t = 1$ if $a^t = s_1(h_t)$, $i_t = 0$ otherwise,

(ii) $j_t = 1$ if $b^t = s_2(\hat{h}_{t-1})$, $j_t = 0$ otherwise.

Therefore,

$$p_{s_1,s_2}(a_t, b_t) = p^{m_t+n_t}(1-p)^{2t+2-m_t-n_t}$$

where

$$m_t = Cardinal\{0 \leqslant i \leqslant t : s_1(h_i) = a^i\}$$
$$n_t = Cardinal\{0 \leqslant i \leqslant t : s_2(\hat{h}_i) = b^i\}.$$

Observe that if $h_t \in h_{s_1,s_2}$ (meaning that $h_t = h_{s_1,s_2 t}$) then

$$p_{s_1,s_2}(h_t) = p^{2t+2}. \tag{3}$$

With

$$U_{\delta,p,h_{s_1,s_2}}(s_1, s_2)$$

we denote the utility only along the equilibrium path. With $U_{\delta,p,h^c_{s_1,s_2}}(s_1, s_2)$ we denote the difference, i.e., $U_{\delta,p}(s_1, s_2) - U_{\delta,p,h_{s_1,s_2}}(s_1, s_2)$. Now, given a finite string h_t with

$$U_{\delta,p}(s_1, s_2/h_t)$$

we denote the utility with seed h_t and with

$$U_{\delta,p}(h_{s_1,s_2/h_t})$$

we denote the utility only along the equilibrium path with seed h_t for the pair s_1, s_2. In the same way, with $U_{\delta,p}(h^c_{s_1,s_2/h_t})$ we denote $U_{\delta,p}(s_1, s_2/h_t) - U_{\delta,p}(h_{s_1,s_2/h_t})$. Also, with $\mathcal{NE}$ we denote the set of path which are not equilibrium paths; usually those paths are called second order paths.

Definition 1. *We say that s is a subgame perfect strategy if for any s'different than s it follows that if $s(h_t) \neq s'(h_t)$ then*

$$U_{\delta,p}(s, s/h_t) - U_{\delta,p}(s', s/h_t) > 0.$$

Let us consider two strategies s_1 and s_2 and let

$$\mathcal{R}_{s_1,s_2} := \{h \in H_0 : \exists\ k \geqslant 0,\ s_1(h_t) = s_2(h_t)\ \forall\, t < k;\ s_1(h_k) \neq s_2(h_k)\}.$$

Observe that if $s_1(h_0) \neq s_2(h_0)$ then any path $h \in H_0$ belongs to $\mathcal{R}_{s_1,s_2}$. On the other hand, if $s_1(0) = s_2(0)$ then for any $h \in \mathcal{R}_{s_1,s_2}$ there is not restriction on the values that h_0 can take. In other words, we consider all the paths where s_1 and s_2 differ at some moment, including the first move. Observe that k depends on h, and it is defined as the first time that s_1 differs with s_2 along h, i.e.

$$k_h(s_1, s_2) = \min\{t \geqslant 0 : s_1(h_t) \neq s_2(h_t)\}.$$

From now on, to avoid notation we drop the dependence on the path. Observe that for $h \in \mathcal{R}_{s_1,s_2}$, the fact that $s_1(h_t) = s_2(h_t)$ for any $t < k$ does not imply that $h_{t+1} = s_1(h_t)$. Moreover, observe also that if $s_1 \neq s_2$ then

$$\mathcal{R}_{s_1,s_2} \neq \emptyset.$$

From now on, given $h \in \mathcal{R}_{s_1,s_2}$ with h_k we denote the finite path contained in h such that $s_1(h_t) = s_2(h_t)$ for any $t < k$ and $s_1(h_k) \neq s_2(h_k)$

Lemma 1. *It follows that*

$$U_{\delta,p}(s_1, s_1) - U_{\delta,p}(s_2, s_1) = \sum_{h_k, h \in \mathcal{R}_{s_1,s_2}} \delta^k p_{s_1,s_1}(h_k)(U_{\delta,p}(s_1, s_1/h_k) - U_{\delta,p}(s_2, s_1/h_k)).$$

Proof. If $s_1(h_0) \neq s_2(h_0)$ then $\mathcal{R}_{s_1,s_2} = H_0$, $h_k = h_0$ and in this case there is nothing to prove. If $s_1(0) = s_2(0)$, the result follows from the next claim that states that given a history path h then

$$p_{s_1,s_1}(h_t) = \begin{cases} p_{s_2,s_1}(h_t) & \text{if } t \leqslant k \\ p_{s_2,s_1}(h_k) p_{s_2,s_1/h_k}(\sigma^k(h)_{t-k}) = p_{s_1,s_1}(h_k) p_{s_2,s_1/h_k}(\sigma^k(h)_{t-k}) & \text{if } t > k \end{cases}$$

(recall that $\sigma^k(h)$ is a history path that verifies $\sigma^k(h)_j = h_{j+k}$). To prove the claim in the case that $t \leqslant k$ we proceed by induction: recalling (2)follows that

$$p_{s_1,s_1}(a_t, b_t) = p_{s_1,s_1}(a_{t-1}, b_{t-1}) p^{i_t^1 + j_t^1} (1-p)^{2 - i_t^1 - j_t^1} \tag{4}$$

where

(i) $i_t^1 = 1$ if $a_t = s_1(h_{t-1}) = s_1(a_{t-1}, b_{t-1})$, $i_t^1 = 0$ otherwise,

(ii) $j_t^1 = 1$ if $b_t = s_1(\hat{h}_{t-1}) = s_1(b_{t-1}, a_{t-1})$, $j_t^1 = 0$ otherwise

and

$$p_{s_2,s_1}(a_t, b_t) = p_{s_2,s_1}(a_{t-1}, b_{t-1}) p^{i_t^2 + j_t^2} (1-p)^{2 - i_t^2 - j_t^2} \tag{5}$$

where

(i) $i_t^2 = 1$ if $a_t = s_2(h_{t-1}) = s_2(a_{t-1}, b_{t-1})$, $i_t^2 = 0$ otherwise,

(ii) $j_t^2 = 1$ if $b_t = s_1(\hat{h}_{t-1}) = s_1(b_{t-1}, a_{t-1})$, $j_t^2 = 0$ otherwise.

Now, by induction follows that $p_{s_1,s_1}(a_{t-1},b_{t-1}) = p_{s_2,s_1}(a_{t-1},b_{t-1})$ and from $s_1(h_{t-1}) = s_2(h_{t-1})$ follows that $i^1_t = i^2_t, j^1_t = j^2_t$. ☐

Remark 1. *It follows that $h \in \mathcal{R}_{s_1,s_2}$ if and only if $h \in \mathcal{R}_{s_2,s_1}$ and*

$$U_{\delta,p}(s_2,s_2) - U_{\delta,p}(s_1,s_2) = \sum_{h_k, h\in\mathcal{R}_{s_1,s_2}} \delta^k p_{s_2,s_2}(h_k)(U_{\delta,p}(s_2,s_2/h_k) - U_{\delta,p}(s_1,s_2/h_k)). \tag{6}$$

Lemma 2. *Given any pair of strategies s_1, s_2 it follows that*

$$|U_{\delta,p}(h^c_{s_2,s_1/h_t})| < \frac{1-p^2}{p^2(1-\delta)}M$$

where $M = \max\{T, |S|\}$.

Proof. Observe that fixed t then

$$\sum_{h_t\in H_t} p_{s_1,s_2}(h_t) = 1,$$

since in the equilibrium path at time t the probability is p^{2t+2} it follows that

$$\sum_{h_t\notin H_t\cap\mathcal{NE}} p_{s_1,s_2}(h_t) = 1 - p^{2t+2}.$$

Therefore, and recalling that $u(h^t) \leqslant M$,

$$\begin{aligned}
|U_{\delta,p}(h^c_{s_2,s_1/h_t})| &= |\frac{1-p^2\delta}{p^2} \sum_{t\geqslant 0, h_t\notin\mathcal{NE}} \delta^t p_{s_1,s_2}(h_t)u(h^t)| \\
&\leqslant (1-\delta)\sum_{t\geqslant 0}\delta^t \sum_{h_t\notin\mathcal{NE}} p_{s_1,s_2}(h_t)|u(h^t)| \\
&\leqslant (1-\delta)M\sum_{t\geqslant 0}\delta^t(1-p^{2t+2}) \\
&= M[(1-\delta)\sum_{t\geqslant 0}\delta^t - (1-\delta)\sum_{t\geqslant 0}\delta^t p^{2t+2}] \\
&= M[1-p^2\frac{1-\delta}{1-p^2\delta}] \\
&= \frac{1-p^2}{(1-p^2\delta)}M.
\end{aligned}$$

☐

From previous lemma, we can conclude the next two lemmas:

Lemma 3. *Given two strategies s_1 and s_2*

$$\lim_{p\to 1} \sum_{h_t\in\mathcal{NE}} U_{\delta,p}(s_1,s_2/h_t) = 0.$$

Lemma 4. *given $s_1 s_2$ then*

$$lim_{p\to 1} U_{\delta,p}(s_2,s_2) - U_{\delta,p}(s_1,s_2) = \sum_{h_k, h\in\mathcal{R}_{s_1,s_2}} \delta^k [U_\delta(h_{s_2,s_2/h_k}) - U_\delta(h_{s_1,s_2/h_k})].$$

Now, we are going to rewrite the equation (6) considering at the same time the paths h and $\hat{h}$. The reason to do that it will become more clear in subsection 7.1.

Remark 2. *Observe that given a strategy s if $\hat{h}_t \neq h_t$ it could hold that $s(\hat{h}_t) \neq s(h_t)$. Also, given two strategies s_1, s_2 it also could hold that $k_h(s_1, s_2) \neq k_{\hat{h}}(s_1, s_2)$. However, it follows that if $k_h(s_1, s_2) \leqslant k_{\hat{h}}(s_1, s_2)$ then*

$$p_{s_1,s_1}(h_k) = p_{s_1,s_1}(\hat{h}_k) = p_{s_1,s_2}(h_k) = p_{s_1,s_2}(\hat{h}_k) =$$
$$p_{s_2,s_1}(h_k) = p_{s_2,s_1}(\hat{h}_k) = p_{s_2,s_2}(h_k) = p_{s_2,s_2}(\hat{h}_k)$$

Using previous remark, we define the set $\mathcal{R}^*_{s_1,s_2}$ as the set

$$\mathcal{R}^*_{s_1,s_2} = \{h \in \mathcal{R}_{s_1,s_2} : k_h(s_1, s_2) \leqslant k_{\hat{h}}(s_1, s_2)\}$$

and therefore the differences $U_{\delta,p}(s_2, s_2) - U_{\delta,p}(s_1, s_2)$ can be written in the following way (denoting k as $k_h(s_1, s_2)$)

$$U_{\delta,p}(s_2, s_2) - U_{\delta,p}(s_1, s_2) =$$
$$\sum_{h_k, h \in \mathcal{R}^*_{s_1,s_2}} \delta^k p_{s_1,s_1}(h_k)[U_{\delta,p}(s_1, s_1/h_k) - U_{\delta,p}(s_2, s_1/h_k) + U_{\delta,p}(s_1, s_1/\hat{h}_k) - U_{\delta,p}(s_2, s_1/\hat{h}_k)].$$

Now we are going to give a series of lemmas that relates equilibrium paths with seeds h_t and $\hat{h}_t$; later, we also relate the payoff along those paths. The proofs of the first two next lemmas are obvious and left to the reader.

Lemma 5. *Given two strategies s, s^* and a path h_t follows that*

$$\widehat{h_{s*,s/h_t}} = h_{s,s*/\hat{h}_t} \tag{7}$$

Now, we try to relates the payoffs. Given two strategies s, s^* and a path h_k, we take

$$b_1 = (1-\delta) \sum_{j:u^j(s^*,s/h_k)=R} \delta^j, \quad b_2 = (1-\delta) \sum_{j:u^j(s^*,s/h_k)=S} \delta^j,$$

$$b_3 = (1-\delta) \sum_{j:u^j(s^*,s/h_k)=T} \delta^j, \quad b_4 = (1-\delta) \sum_{j:u^j(s^*,s/h_k)=P} \delta^j.$$

Observe that $b_1 + b_2 + b_3 + b_4 = 1$ and

$$U(s^*, s) = b_1 R + b_2 S + b_3 T + b_4 P.$$

In the same way, for $\hat{h}_k$ we define $\hat{b}_1, \hat{b}_2, \hat{b}_3, \hat{b}_4$

$$b_1 = (1-\delta) \sum_{j:u^j(s^*,s/\hat{h}_k)=R} \delta^j, \quad b_2 = (1-\delta) \sum_{j:u^j(s^*,s/\hat{h}_k)=S} \delta^j,$$

$$b_3 = (1-\delta) \sum_{j:u^j(s^*,s/\hat{h}_k)=T} \delta^j, \quad b_4 = (1-\delta) \sum_{j:u^j(s^*,s/\hat{h}_k)=P} \delta^j.$$

Observe that $\hat{b}_1 + \hat{b}_2 + \hat{b}_3 + \hat{b}_4 = 1$. Now we define

$$B_1 = b_1 + \hat{b}_1, \ B_2 = b_2 + \hat{b}_2, \ B_3 = b_3 + \hat{b}_3, \ B_4 = b_4 + \hat{b}_4.$$

Remark 3. *The above numbers b_j depend on δ and the infinite sums converge fixed δ. However, they could not converge as δ goes to 1.*

Lemma 6. *Given two strategies s, s^* and a path h_k, if*

$$U_\delta(h_{s^*,s/h_k}) = b_1R + b_2S + b_3T + b_4PS$$

then

$$U_\delta(h_{s,s^*/\hat{h}_k}) = b_1R + b_2T + b_3S + b_4P.$$

Moreover, if

$$U_\delta(h_{s^*,s/h_k}) + U_\delta(h_{s^*,s/\hat{h}_k}) = B_1R + B_2T + B_3S + B_4P,$$

then

$$U_\delta(h_{s,s^*/h_k}) + U_\delta(h_{s,s^*/\hat{h}_k}) = B_1R + B_2S + B_3T + B_4P.$$

Lemma 7. *Given two strategies s, s^* and a path h_k, follows that*

$$U_\delta(h_{s,s^*/h_k}) + U_\delta(h_{s^*,s/\hat{h}_k}) \leqslant 2R.$$

Lemma 8. *For any $\lambda_0 < 1$ follows that there exists $\hat{\lambda}_0 < 1$ such that if $U_\delta(h_{s,s/h_t}) = \lambda_0 R$ then*

$$U_\delta(h_{s,s/h_t}) + U_\delta(h_{s,s/\hat{h}_t}) \leqslant 2\hat{\lambda}_0 R.$$

Moreover, if $\lambda_0' < \lambda_0$ then $\hat{\lambda}_0' < \hat{\lambda}_0$. In particular,

$$U_\delta(h_{s,s/h_t}) + U_\delta(h_{s,s/\hat{h}_t}) < 2R.$$

Proof. If $U_\delta(h_{s,s/h_t}) = b_1R + b_2S + b_3T + b_4P = \lambda_0 R$ then it follows that

$$\max\{b_2, b_3, b_4\} > \frac{1-\lambda_0}{3}. \tag{8}$$

In fact, if it is not the case,

$$b_1R + b_2S + b_3T + b_4P \geqslant b_1R = (1 - (b_2 + b_3 + b_4)) \geqslant [1 - (1 - \lambda_0)]R = \lambda_0 R,$$

a contradiction. From equality (7) follows $U(h_{s,s/\hat{h}_t}) = b_1R + b_2T + b_3S + b_4P$ so

$$U_\delta(h_{s,s/h_t}) + U_\delta(h_{s,s/\hat{h}_t}) = 2b_1R + (b_2 + b_3)(T + S) + 2b_4P$$

and from the fact that $b_1 + b_2 + b_3 + b_4 = 1$ follows that is equal to

$$2R - [2b_2(R - P) + (b_3 + b_4)(2R - (T + S))]$$

So taking

$$\hat{R} = \min\{R - P, \frac{R - (T + S)}{2}\}$$

which is positive, follows from inequality (8) that

$$U_\delta(h_{s,s/h_t}) + U_\delta(h_{s,s/\hat{h}_t}) < 2R - 2\frac{1-\lambda_0}{3}\hat{R},$$

and taking

$$\hat{\lambda}_0 = 1 - \frac{1-\lambda_0}{3}\frac{\hat{R}}{R}$$

the result follows. □

3 Replicator dynamics

In this section we introduce the notion of replicator dynamics and we analyze the attractors.

Given the payoff matrix

$$A = \begin{pmatrix} a_{11} & \dots & a_{1i} & \dots & a_{1n} \\ \dots & \dots & \dots & \dots & \dots \\ a_{i1} & \dots & a_{ii} & \dots & a_{in} \\ \dots & \dots & \dots & \dots & \dots \\ a_{n1} & \dots & a_{ni} & \dots & a_{nn} \end{pmatrix}$$

Let Δ be the $n-$dimensional simplex

$$\Delta = \{(x_1 \dots x_n) \in \mathbb{R}^n : x_1 + \dots + x_n = 1, \ x_j \geqslant 0, \forall j\}.$$

We consider the replicator dynamics X associated to the payoff matrix A on the n dimensional simplex given by the equations:

$$\dot{x}_j = X_j(x) := x_j F_j(x) = x_j(f_j - \bar{f})(x) \tag{9}$$

where

$$f_j(x) = (Ax)_j, \ \ \bar{f}(x) = \sum_{l=1}^{n} x_l f_l(x),$$

where $(AX)_j$ denotes the $j-$th coordinate of the vector Ax. In other words, provided a payoff matrix A, the replicator equation is given by

$$\dot{x}_j = x_j[(Ax)_j - x^t Ax], \quad j = 1, \dots, n$$

where x^t denotes the transpose vector.

Using that $1 = x_1 + x_2 + \dots + x_n$ we can write

$$F_j = f_j(x)(x_1 + x_2 + \dots + x_n) - \bar{f}(x) = f_j(x)(x_1 + x_2 + \dots + x_n) - \sum x_l f_l(x) = \sum_{l \neq j} x_l(f_j - f_l)(x).$$

We denote with φ the associated flow:

$$\varphi : \mathbb{R} \times \Delta \to \Delta.$$

Giving $t \in \mathbb{R}$ with $\varphi_t : \Delta \to \Delta$ we denote the $t - time$ diffeomorphism. Observe that any vertex is a singularity of the replicator equation, therefore, any vertex is a fixed point of the flow.

3.1 Affine coordinates for the replicator equation

We consider an affine change of coordinates to define the dynamics in the positive quadrant of $\mathbb{R}^{n-1}$ instead of the simplex Δ. The affine change of coordinates is given by

$$\bar{x}_1 = 1 - \sum_{j \geqslant 2} x_j, \ \ \bar{x}_j = x_j \ \ \forall \ j \geqslant 2$$

and so, the replicator equation is defined as

$$\dot{x}_j = F_j(\bar{x}) x_j, \ \ j = 2, \dots, n$$

where $\bar{x} = (\bar{x}_1, \bar{x}_2 \ldots, \bar{x}_n)$ with $x_i \geqslant 0, x_2 + \cdots + x_n \leqslant 1$ and

$$F_j(\bar{x}) = (f_j - \bar{f})(1 - \sum_{i \geqslant 2} x_i, x_2, \ldots, x_n).$$

Observe that in these coordinates the point $e_1 = (1, 0, \ldots, 0)$ corresponds to $(0, \ldots, 0)$ and in the new coordinates the simplex Δ is replaced by $\{(x_2, \ldots, x_n) : x_i \geqslant 0, \sum_{i=2}^n x_i \leqslant 1\}$.

We also can rewrite F_j in the following way:

$$\begin{aligned}
F_j(\bar{x}) &= \sum_{l \neq j, l \geqslant 1} (f_j - f_l)(\bar{x})\bar{x}_l \\
&= (f_j - f_1)(\bar{x})(1 - \sum_{l \geqslant 2} x_l) + \sum_{l \neq j, l \geqslant 2} (f_j - f_l)(\bar{x})\bar{x}_l \\
&= (f_j - f_1)(\bar{x})(1 - \sum_{l \geqslant 2} x_l) + \sum_{l \neq j, l \geqslant 2} (f_j - f_l)(\bar{x})x_l \\
&= (f_j - f_1)(\bar{x}) - \sum_{l \geqslant 2} (f_j - f_1)(\bar{x})x_l + \sum_{l \neq j, l \geqslant 2} (f_j - f_l)(\bar{x})x_l \\
&= (f_j - f_1)(\bar{x}) - (f_j - f_1)(\bar{x})x_j + \sum_{l \neq j, l \geqslant 2} [(f_j - f_l)(\bar{x}) - (f_j - f_1)(\bar{x})]x_l \\
&= (f_j - f_1)(\bar{x}) - (f_j - f_1)(\bar{x})x_j + \sum_{l \neq j, l \geqslant 2} (f_1 - f_l)(\bar{x})x_l \\
&= (f_j - f_1)(\bar{x}) + (f_1 - f_j)(\bar{x})x_j + \sum_{l \neq j, l \geqslant 2} (f_1 - f_l)(\bar{x})x_l \\
&= (f_j - f_1)(\bar{x}) + \sum_{l \geqslant 2} (f_1 - f_l)(\bar{x})x_l.
\end{aligned}$$

Denoting

$$R(\bar{x}) := \sum_{l \geqslant 2} (f_1 - f_l)(\bar{x})x_l, \tag{10}$$

it follows that

$$F_j(\bar{x}) = (f_j - f_1)(\bar{x}) + R(\bar{x}) \tag{11}$$

where

$$\begin{aligned}
(f_j - f_l)(\bar{x}) &= \sum_{k \geqslant 1} (a_{jk} - a_{lk})\bar{x}_k = (a_{j1} - a_{l1})\bar{x}_1 + \sum_{k \geqslant 2} (a_{jk} - a_{lk})\bar{x}_k \\
&= (a_{j1} - a_{l1})(1 - \sum_{l \geqslant 2} x_l) + \sum_{k \geqslant 2} (a_{jk} - a_{lk})x_k \\
&= a_{j1} - a_{l1} + \sum_{k \geqslant 2} (a_{jk} - a_{lk} - a_{j1} + a_{l1})x_k.
\end{aligned}$$

Observe that if we take the matrix $M \in \mathbb{R}^{(n-1)\times(n-1)}$ and the vector $N \in \mathbb{R}^(n-1)$ such that

$$M_{jk} = a_{jk} - a_{1k} + a_{11} - a_{j1}$$

and

$$N_j = a_{j1} - a_{11}$$

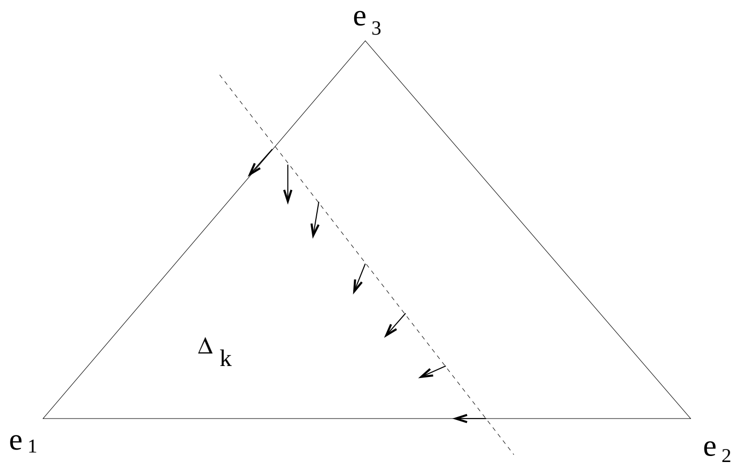

Figure 1: Attracting fixed point. Basin of attraction.

then the replicator equation o affine coordintaes is given by

$$\dot{x}_j = x_j[(v + Mx)_j - x^t(v + Mx)], \quad j = 2, \dots, n; \tag{12}$$

where $(v + Mx)_j$ is the $j - th$ coordinate of $v + Mx$.

3.2 Attracting fixed points

Given a point e and a positive constant ϵ, $B_\epsilon(e)$ denotes the ball of radius ϵ and center e.

Definition 2. Attracting fixed point and local basin of attraction. *Let e be a singular point of X (i.e.: $X(e) = 0$). It is said that e is an attractor if there exists an open neighborhood U of e such that for any $x \in U$ follows that $\varphi_t(x) \to e$. The global basin of attraction $B^s(e)$ is the set of points that its forward trajectories converges to e. Moreover, given $\epsilon > 0$ we say that $B_\epsilon(e)$ is contained in the local basin of attraction of e if $B_\epsilon(e)$ is contained in global basin of attraction and any forward trajectory starting in $B_\epsilon(e)$ remains inside $B_\epsilon(e)$. This is denoted with $B_\epsilon(e) \subset B^s_{loc}(e)$.*

For the sake of completeness, we give a folklore's sufficient condition for the vertex e_1 to be an attractor. Before that, we need to calculate the derivative DX of the function $X = (X_1 \dots X_n)$ given by the replicator equation (see equation 9). For that, for any l, we compute $DX_l = (\frac{\partial X_l}{\partial x_1} \dots \frac{\partial X_l}{\partial x_n})$ and observe that for $k \neq l$ then $\frac{\partial X_l}{\partial x_k} = (\partial_{x_k} f_l - \partial_{x_k} \bar{f})x_k$, and for $k = l$ follows that $\frac{\partial X_l}{\partial x_l} = (\partial_{x_l} f_l - \partial_{x_l} \bar{f})x_l + f_l - \bar{f}$.

Lemma 9. *If e_1 is a strict Nash equilibrium (i.e. $a_{11} - a_{j1} > 0$ for any $j \neq 1$) then e_1 is an attractor. Moreover, the eigenvalues of DX at e_1 are given by $\{a_{11} - a_{j1}\}_{j>1}$.*

Proof. To prove the result, observe first that $\bar{0}$ (the point e_1 in the simplex) is a fixed point. To finish, observe that D_0X is a diagonal matrix with $\{a_{j1} - a_{11}\}_{j \neq 1}$ in the diagonal. Therefore, $\{a_{j1} - a_{11}\}_{j \neq 1}$ are the eigenvalues which by hypothesis they are all negative.

□

3.3 Large Basin of attractions for fixed points

The goal of the following theorem is to give sufficient conditions for a vertex to have a "large local basin of attraction", independent of the dimension of the space. In other words, provided a vertex e and a positive number K, the goal is to find sufficient condition for any payoff matrix A, independently of the dimension, the neighborhood $B_R(e)$ is contained in the local basin of attraction of e.

A natural condition is to assume that the eigenvalues are "uniformly negative". But this criterion is not appropriate for the context of games, since the quantities $a_{j1} - a_{11}$ even when negative could be arbitrary close to zero. However, we take advantage that the replicator equations are given by a special type of cubic polynomials, and we provide a sufficient condition for "large local basin of attraction" even for the case that the eigenvalues are close to zero. To do that, we need to introduce some other quantities. From now on we use the L^1-norm

$$||x|| = \sum_{i\geqslant 1} |x_i|.$$

Now, let us go back to the replicator equations and let us assume from now on that e is a strict Nash equilibrium, i.e.

$$a_{11} - a_{j1} > 0$$

for any $j \neq 1$. Recall as we define in the previous subsection the matrix M and N given by

$$N_{j1} = a_{j1} - a_{11} \quad (13)$$
$$M_{ij} = a_{ji} - a_{1i} + a_{11} - a_{j1} \quad (14)$$
$$. \qquad M_{ji} = a_{ij} - a_{1j} + a_{11} - a_{i1}. \quad (15)$$

Moreover, we assume that the vertex $\{e_2 \dots e_n\}$ are ordered in such a way that

$$a_{11} - a_{i1} \geqslant a_{11} - a_{j1},\ \forall\, 2 \leqslant i < j.$$

Theorem 1. *Let $A \in \mathbb{R}^{n\times n}$ (n arbitrary) such that $a_{j1} < a_{11}$. Let*

$$M_0 \;=\; \max_{i,j\geqslant i}\{\frac{M_{ij}+M_{ji}}{-N_i},\ 0\}. \quad (16)$$

Then,

$$\Delta_{\frac{1}{M_0}} = \{\bar{x} : \sum_{i\geqslant 2} x_i \leqslant \frac{1}{M_0}\} \subset B^s_{loc}(e_1).$$

The proof of the theorem is based on a crucial lemma about quadratic polynomials (see lemma 10). So, first we recall a series of definitions and results involving quadric, we state the lemma, provide its proof and latter we prove theorem 1.

First recall that a *quadratic polynomial* Q is a function from $\mathbb{R}^n$ to $\mathbb{R}$ of the form $Q(x) = Nx + x^tMx$ (where N is a vector, M is a square matrix and x^t means the transpose of x). It is said that Q is *positive defined* if $x^tMx \geqslant 0$ for any x. It is said that Q is negative-definite if $x^tMx \leqslant 0$ for any x. Now, associated to a quadratic polynomial Q we consider the set

$$\{x \in \mathbb{R}^n : Q(x) = 0\}$$

which is smooth submanifold of codimension one. Observe that $Q(0) = 0$. If Q is either positive-definite or negative-definite then $\{Q(x) = 0\}$ is an ellipsoid, in particular, it is a connected compact

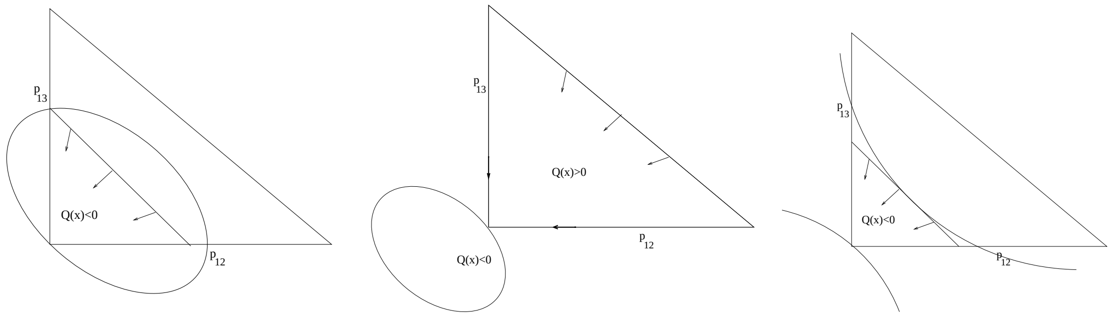

Figure 2: Q positive-definite, negative-definite and neither.

set and $\{x \in \mathbb{R}^n : Q(x) \leqslant 0\}$ is a convex set (see first two cases in figure 2). If Q is neither positive-definite nor negative-definite then $\{Q(x) = 0\}$ is a hyperboloid, and in particular, it is not a bounded set. However, it could be connected or not (see third case of figure 2).

Lemma 10. *Let $Q : \mathbb{R}^n \to \mathbb{R}$ given by*

$$Q(x) = Nx + x^t Mx$$

with $x \in \mathbb{R}^n$, $N \in \mathbb{R}^n$ and $M \in \mathbb{R}^{n\times n}$. Let us assume that $N_i < 0$ for any i and for any $j > i$, $|N_i| \geqslant |N_j|$. Let

$$M_0 = \max_{i,j>i}\{\frac{M_{ij} + M_{ji}}{-N_i},\ 0\}.$$

Then, the set $\Delta_{\frac{1}{M_0}} = \{x \in \mathbb{R}^n : x_i \geqslant 0, \sum_{i=1}^n x_i < \frac{1}{M_0}\}$ is contained in $\{x : Q(x) < 0\}$. In particular, if $M_0 = 0$ then $\frac{1}{M_0}$ is treated as ∞ and this means that $\{x \in \mathbb{R}^n : x_i \geqslant 0\} \subset \{x : Q(x) \leqslant 0\}$.

Proof. For any $v \in \mathbb{R}^n$ such that $v_i \geqslant 0$ and $\sum_i v_i = 1$, we consider the following one dimensional quadratic polynomial, $Q^v : \mathbb{R} \to \mathbb{R}$ given by

$$Q^v(s) := Q(sv) = sNv + s^2 v^t Mv.$$

To prove the thesis of the lemma, we claim that is enough to show that

$$\text{“for any positive vector } v \text{ with norm equal to 1, if } 0 < s < \frac{1}{M_0} \text{ then } Q^v(s) < 0\text{”;} \tag{17}$$

in fact, to prove that claim, we can argue by contradiction: if there is a point $x_0 \in \Delta_{\frac{1}{M_0}}$ different than zero (i.e.: $0 < |x_0| < \frac{1}{M_0}$) such that $Q(x_0) = 0$, then taking $v = \frac{x_0}{|x_0|}$ and $s = |x_0|$ follows that $Q^v(s) = Nx_0 + x_0^t Mx_0 = 0$, but $|v| = 1, s < \frac{1}{M_0}$, a contradiction.

Now we proceed to show (17). Observe that the roots of $Q^v(s)$ are given by $s = 0$ and

$$s = \frac{-Nv}{v^t Mv}.$$

Observe that

$$-Nv = \sum(-N_i)v_i > 0.$$

If $v^t Mv < 0$ then it follows that Q^v is a one dimensional quadratic polynomial with negative quadratic term and two non-positive roots, so for any $s > 0$ holds that $Q^v(s) < 0$ and therefore proving the claim in this case. So, it remains to consider the case that $v^tMv > 0$. In this case, since Q^v is a one dimensional quadratic polynomial with positive quadratic term (v^tMv), therefore for any s between both roots $(0, \frac{-Nv}{v^tMv})$ follows that $Q < 0$ so to finish we have to prove that

$$\frac{-Nv}{v^tMv} \geqslant \frac{1}{M_0}. \tag{18}$$

Using that $\sum_{j\geqslant i} v_j \leqslant 1$ observe

$$\begin{aligned}
v^tMv &= \sum_{ij} v_i v_j M_{ij} \\
&= \sum_i [v_i^2 M_{ii} + \sum_{j>i} v_i v_j (M_{ij} + M_{ji})] \\
&\leqslant \sum_i [v_i^2 (-N_i) M_0 + \sum_{j>i} v_i v_j (-N_i) M_0] = \\
&= M_0 \sum_i (-N_i) v_i [\sum_{j\geqslant i} v_j] \\
&\leqslant M_0 \sum_i (-N_i) v_i \\
&= M_0(-Nv).
\end{aligned}$$

Therefore, (18) holds and so proving (17). □

Now we provide the proof of theorem 1.

Proof of theorem 1: We consider the affine change of coordinates: $\bar{x}_1 = 1 - \sum_{j\geqslant 2} x_j, \bar{x}_j = x_j, j = 2, \ldots, n$ introduced before. Let $X = (X_2, \ldots, X_n)$ the vector field in these coordinates, where $X_j = \bar{x}_j F_j(\bar{x})$. For any $k < 1$ we denote

$$\Delta_k := \{\bar{x} : \sum_{i\geqslant 2} x_i \leqslant k\}, \ \partial\Delta_k = \{\bar{x} : \sum_{i\geqslant 2} x_i = k\}.$$

We want to show that for any initial condition $\bar{x}$ in the region $\Delta_{\frac{1}{M_0}}$ follows that the map

$$t \to \bar{x}(t) = \sum_{i\geqslant 2} \bar{x}_k(t)$$

is a strict decreasing function and so the trajectories remains inside $\Delta_{\frac{1}{M_0}}$ and since it can not escape Δ it follows that $\bar{x}(t) \to 0$ and therefore the trajectory converge to $(0, \ldots, 0)$. To do that, we prove

$$\dot{\bar{x}} < 0.$$

Therefore, we have to show

$$Q(\bar{x}) := \dot{\bar{x}} = \sum_{j\geqslant 2} X_j = \sum_{j\geqslant 2} x_j F_j(\bar{x}) < 0. \tag{19}$$

Recall that $F_j = (f_j - f_1)(\bar{x}) + R(\bar{x})$ where $R(\bar{x}) = \sum_{l\geqslant 2} (f_1 - f_l)(\bar{x}) x_l$ (see equations (10) and (11)). Therefore,

$$\begin{aligned}
Q(\bar{x}) &= \sum_{j\geqslant 2} (f_j - f_1)(\bar{x}) x_j + \sum_{j\geqslant 2} R(\bar{x}) x_j \\
&= \sum_{j\geqslant 2} (f_j - f_1)(\bar{x}) x_j + R(\bar{x}) \sum_{j\geqslant 2} x_j.
\end{aligned}$$

Since $\sum_{j\geqslant 2} x_j = k$ (with $k<1$) follows that

$$Q(\bar{x}) \;=\; \sum_{j\geqslant 2}(f_j - f_1)(\bar{x})x_j + R(\bar{x})k.$$

Recalling the expression of R we get that

$$Q(\bar{x}) = (1-k)\sum_{j\geqslant 2}(f_j - f_1)(\bar{x})x_j.$$

So, to prove inequality (19) is enough to show that

$$Q(\bar{x}) = (1-k)\sum_j x_j(f_j - f_1)(\bar{x}) < 0 \quad \forall\, \bar{x} \in \Delta_k,\ k < \frac{1}{M_0}.$$

First we rewrite Q. Observe that

$$\begin{aligned}(f_j - f_1)(\bar{x}) &= \sum_i (a_{ji} - a_{1i})\bar{x}_i = \\ &= a_{j1} - a_{11} + \sum_{i\geqslant 2}(a_{ji} - a_{1i} + a_{11} - a_{j1})\, x_i.\end{aligned}$$

If we note the vector

$$N := (a_{j1} - a_{11})_j$$

and the matrix

$$M := (M_{ij}) = a_{ji} - a_{1i} + a_{11} - a_{j1}.$$

Therefore,

$$Q(\bar{x}) = N\bar{x} + \bar{x}^t M \bar{x}.$$

So we have to find the region given by $\{\bar{x} : Q(\bar{x}) = 0\}$. To deal with it, we apply lemma 10 and we use equation (16) and the theorem is concluded.

□

Remark 4. *Observe that in the theorem 10 it only matters to compare $a_{11} - a_{i1}$ with the entries $M_{ij} + M_{ji}$ that are positive.*

Remark 5. *If we apply the proof of lemma 10 to the particular case that $v = e_j$, we are considering the map*

$$Q^v(s) = s[a_{j1} - a_{11} + (a_{jj} - a_{1j} + a_{11} - a_{j1})\, s]$$

and $Q(s) = 0$ if and only if $s = 0$ or

$$s = \frac{a_{11} - a_{j1}}{a_{11} - a_{j1} + a_{jj} - a_{1j}} = \frac{1}{1 + \frac{a_{jj} - a_{1j}}{a_{11} - a_{j1}}} = p_{1j} \tag{20}$$

and so

$$Q(s) < 0,\ \ \forall\, 0 < s < p_{1j}.$$

In particular, if we apply this to theorem 10, it follows that the whole segment $[0, p_{1j})$ is in the basin of attraction of e_1. In particular, observe that $\hat{p}_{1j} = (1 - p_{1j}, \ldots, p_{1j} \ldots)$, is the fixed point of the replicator dynamics different than e_1, e_j inside the one dimensional simplex that contains e_1, e_j.

Remark 6. *Observe that the basin of attraction could be much larger than the region given by the previous theorem. It may be the case that better linear upper bounds for the quadratics map F_j could provide better estimates for the size of the basin of attraction.*

3.4 Comparing strategies by pairs is not enough

It is natural to wonder if conditions of theorem 10 are necessary? More precisely, is it true that if M_0 is small then the basin of attraction is small? Related to this question, we provide the following theorem that shows that is not enough to bound by below the basin of attraction only considering populatios of two strategies. In other words, it is possible to show examples of strategies such that the basin of attraction of e_1 restricted to the axis are large but the whole basin is not large.

We consider a replicator dynamics in dimension two and we write the equation in affine coordinates. Given $\lambda > 0$ and close to zero, we consider the almost horizontal and vertical lines given by

$$H_\lambda(x_1) = (x_1, \lambda(1-x_1)), \quad V_\lambda(x_2) = (\lambda(1-x_2), x_2).$$

Theorem 2. *Given $\lambda > 0$ close to zero, $a > 0$ there exist $A \in \mathbb{R}^{3\times3}$ such that $0 < a_{ij} < a$, satisfying that*

(i) *$(0,0)$ is an attractor and the horizontal line $(x_1, 0), 0 \leqslant x_1 < 1$ and vertical line $(0, x_2), 0 \leqslant x_2 < 1$ are contained in the basin of attraction of $(0,0)$;*

(ii) *$(1,0)$ and $(0,1)$ are repellers;*

(iii) *there is a point $p = (p_1, p_2)$ with $p_1 + p_2 = 1$ which is an attractor;*

(iv) *the region bounded by H_λ, V_λ and $x_1 + x_2 = 1$ is contained in the basin of attraction of p.*

Proof. To prove the result, we are going to choose $A \in \mathbb{R}^{3\times3}$ in a proper way such that for any $(x_2, x_3) \in H_\lambda$ and $(x_2, x_3) \in V_\lambda$ follows that $X(x_2, x_3)$ points towards the region bounded by H_λ, V_λ and $x_1 + x_2 = 1$. For that, it is enough to show that

$$\frac{\lambda(1-x_2)F_3(H(x_2))}{|x_2F_2(H(x_2))|} > \frac{1}{4}, \quad F_3(H(x_2)) > 0 \ \text{ for} \frac{\lambda}{1-\lambda} < x_2 < 1, \tag{21}$$

and

$$\frac{\lambda(1-x_3)F_2(V(x_3))}{|x_3F_3(V(x_3))|} > \frac{1}{4}, \quad F_2(V(x_3)) > 0 \ \text{ for} \frac{\lambda}{1-\lambda} < x_3 < 1, \tag{22}$$

where $(\frac{\lambda}{1-\lambda}, \frac{\lambda}{1-\lambda})$ is the intersection point of H_λ and V_λ. Recall the definition of $N \in \mathbb{R}^2, M \in \mathbb{R}^{2\times2}$ that induce the replicator dynamics in affine coordinates. Given λ we assume that

(i) $N_2 = N_3$,

(ii) $\frac{m_{32}}{N_3} = \frac{m_{23}}{N_3} = \frac{1}{\lambda}$,

(iii) $\frac{m_{22}}{N_2} = \frac{m_{33}}{N_2} = 2$.

To get that, and recalling the relation between the coordinates of M and A, we choose the matrix A such that

(i) $\frac{a_{33}-a_{13}}{N_3} = -1, \frac{a_{22}-a_{12}}{N_2} = -1$;

(ii) $a_{32} > a_{22}$, $a_{23} > a_{33}$ and $\frac{a_{32}-a_{22}}{N_2} = \frac{a_{23}-a_{33}}{N_2} = \frac{1}{\lambda} - 2$.

With this assumption, now we prove that inequality (21) is satisfied: Let us denote $x := x_2$ and we first calculate $F_3(x, \lambda(x-1))$ and $F_2(x, \lambda(x-1))$,

$$\begin{aligned} F_3(x,\lambda(1-x)) \quad &= N_3 + m_{32}x + m_{33}\lambda(1-x) - \\ &[x(N_2 + m_{22}x + m_{23}\lambda(1-x)) + \lambda(1-x)(N_3 + m_{32}x + m_{33}\lambda(1-x))] \end{aligned}$$

so,

$$\begin{aligned} \frac{F_3(x,\lambda(1-x))}{N_3} \quad &= 1 + \frac{m_{32}}{N_3}x + \frac{m_{33}}{N_3}\lambda(1-x) - \\ &[x(\frac{N_2}{N_3} + \frac{m_{22}}{N_3}x + \frac{m_{23}}{N_3}\lambda(1-x)) + \lambda(1-x)(1 + \frac{m_{32}}{N_3}x + \frac{m_{33}}{N_3}\lambda(1-x))] \\ &= 1 + \frac{1}{\lambda}x + 2\lambda(1-x) - \\ &[x(1 + 2x + \frac{1}{\lambda}\lambda(1-x)) + \lambda(1-x)(1 + \frac{1}{\lambda}x + 2\lambda(1-x))] \\ &= 1 + 2\lambda + (\frac{1}{\lambda} - 2\lambda)x - [2\lambda^2 + \lambda + (3 - \lambda - 4\lambda^2)x + 2\lambda^2x^2] \\ &= 1 + \lambda - 2\lambda^2 + (\frac{1}{\lambda} - \lambda + 4\lambda^2 - 3)x - 2\lambda^2x^2, \end{aligned}$$

$$\begin{aligned} F_2(x,\lambda(1-x)) \quad &= N_2 + m_{22}x + m_{23}\lambda(1-x) - \\ &[x(N_2 + m_{22}x + m_{23}\lambda(1-x)) + \lambda(1-x)(N_3 + m_{32}x + m_{33}\lambda(1-x))] \end{aligned}$$

so,

$$\begin{aligned} \frac{F_2(x,\lambda(1-x))}{N_2} \quad &= 1 + \frac{m_{22}}{N_2}x + \frac{m_{23}}{N_2}\lambda(1-x) - \\ &[x(1 + \frac{m_{22}}{N_2}x + \frac{m_{23}}{N_2}\lambda(1-x)) + \lambda(1-x)(1 + \frac{m_{32}}{N_2}x + \frac{m_{33}}{N_2}\lambda(1-x))] \\ &= 1 + 2x + \frac{1}{\lambda}\lambda(1-x) - \\ &[x(1 + 2x + \frac{1}{\lambda}\lambda(1-x)) + \lambda(1-x)(1 + \frac{1}{\lambda}x + 2\lambda(1-x))] \\ &= 2 + x - [x + 2x^2 + (1-x)[1 + \lambda + 2\lambda^2 + (1 - 2\lambda^2)x]] \\ &= (1-x)[2(1+x) - [1 + \lambda + 2\lambda^2 + (1 - 2\lambda^2)x]] \\ &= (1-x)[1 - \lambda - 2\lambda^2 + (1 + 2\lambda^2)x]. \end{aligned}$$

Therefore, on one hand observe that $1 + \lambda - 2\lambda^2 + (\frac{1}{\lambda} - \lambda + 4\lambda^2 - 3)x - 2\lambda^2x^2$ is a quadratic polynomial with negative leading term that is positive at 1 and $\frac{\lambda}{1-\lambda}$ (provided that $|\lambda|$ is small) so is positive for $\frac{\lambda}{\lambda-1} < x < 1$, on the other hand $(1-x)[1 - \lambda - 2\lambda^2 + (1 + 2\lambda^2)x]$ is positive in the same range, so

$$\frac{\lambda(x-1)F_3(x,\lambda(x-1))}{|xF_2(x,\lambda(x-1))|} = \frac{\lambda[1 + \lambda - 2\lambda^2 + (\frac{1}{\lambda} - \lambda + 4\lambda^2 - 3)x - 2\lambda^2x^2]}{x[1 - \lambda - 2\lambda^2 + (1 + 2\lambda^2)x]};$$

since the minimum of the numerator is attained at $\frac{\lambda}{1-\lambda}$ getting a value close to 1 and the maximum of the denominator is attained at 1 getting a value close to 2, follows that in the range $\frac{\lambda}{\lambda-1} < x < 1$ holds

$$\frac{\lambda(x-1)F_3(x,\lambda(x-1))}{|xF_2(x,\lambda(x-1))|} \geqslant \frac{1}{3},$$

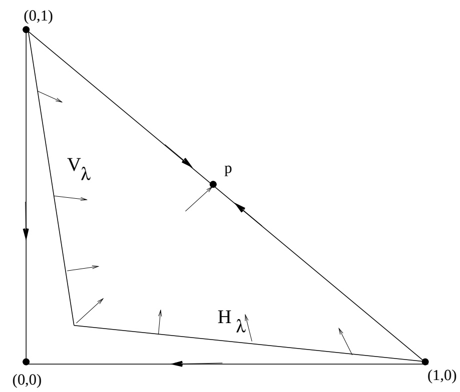

Figure 3: Comparing strategies by pairs is not enough.

and therefore the inequality (21) is proved. The proof of inequality (22) is similar and left for the reader. □

4 Replicator dynamics and Infinitely Repeated Prisoner's dilemma with trembles. Strategies having a uniformly large basin of attraction

In the rest of the paper we study the replicator dynamics when the matrix of payoffs is given by a finite set of strategies $\mathcal{S} = \{s_1, \dots, s_n\}$ from an infinitely repeated prisoners' dilemma game with discount factor δ and error probability $1 - p$. It is well known, that any strict sgp is an attractor in any population containing it. In this case, with $B_{loc}(s, \delta, p, \mathcal{S})$ we denote the local basin of attraction of s in any set of strategies $\mathcal{S}$ and identifying s with s_1. Related to that we give the following definition:

Definition 3. *We say that a strategy s has a uniformly large basin if there is K_0 verifying that for any finite set of strategies $\mathcal{S}$ containing s and any δ and p close to one, it holds that*

$$\{(x_1, \dots, x_n) : x_2 + \dots + x_n \leqslant K_0\} \subset B_{loc}(s, p, \delta, \mathcal{S})$$

where $n = cardinal(\mathcal{S})$.

One particular case of previous definition is when $\mathcal{S}$ has only one strategy different than s. In this case, and based on remark 5 we can obtain the following remark:

Lemma 11. *If s has a uniformly large basin then there exists C_0 such that for any strategy s^* and for any p, δ large (independently of s^*) follows that*

$$\frac{U_{\delta,p}(s^*, s^*) - U_{\delta,p}(s, s^*)}{U_{\delta,p}(s, s) - U_{\delta,p}(s^*, s)} < C_0.$$

In particular,

$$\lim_{\delta \to 1} \lim_{p \to 1} \frac{U_{\delta,p}(s^*, s^*) - U_{\delta,p}(s, s^*)}{U_{\delta,p}(s, s) - U_{\delta,p}(s^*, s)} < C_0.$$

The goal of this paper is to understand which characteristics of strategies lead them to have uniformly large basin of attraction. We show first that a strategy that is commonly used in the literature, grim, does not have a uniformly large basin of attraction. Then, we show that is due to the fact that grim never forgives a defection. As a positive results we show that another well known strategy, win-stay-loose-shift, does have a uniformly large basin of attraction under certain conditions.

5 Grim does not have a uniformly large basin of attraction

In this section, we prove that the strategy Grim (g from now on), which cooperates in the first period and then cooperates if there has been no defection before, does not have a uniformly large basin. To prove it, we are going to find a strategy s such that the basin of attraction of g when it is considered the population formed by s and g is arbitrary small provided that δ and p are close to 1. In fact, we use the equation (20) to determine the boundary point $p_{g,s} = \frac{1}{1+\frac{U_{\delta,p}(s,s)-U_{\delta,p}(g,s)}{U_{\delta,p}(g,g)-U_{\delta,p}(s,g)}}$ of the basin of attraction of g (the smaller $p_{g,s}$ is, the smaller the basin of attraction of g is).

Theorem 3. *Grim does not have a uniformly large basin of attraction. More precisely, there exists a strategy s such that for any population $\mathcal{S} = \{s, g\}$ and $\epsilon > 0$ small, there exist p_0, δ_0 such that for any $p > p_0, \delta > \delta_0$, the size of the basin of attraction of grim is smaller than ϵ.*

Proof. We consider the strategy s that behaves like g but forgives defections in the first period ($t = 0$). We need to show that for any $\epsilon > 0$ small, there exist p_0, δ_0 such that for any $p > p_0, \delta > \delta_0$, follows that

$$\frac{1}{1+\frac{U_{\delta,p}(s,s)-U_{\delta,p}(g,s)}{U_{\delta,p}(g,g)-U_{\delta,p}(s,g)}} < \epsilon.$$

From the definition of s, for any h verifying that $h^0 \neq (D, D)$ and any t it follows that

$$p_{g,g}(h_t) = p_{s,g}(h_t) = p_{g,s}(h_t) = p_{s,s}(h_t).$$

Therefore,

$$U_{\delta,p}(s, s/(C, C)) = U_{\delta,p}(s, g/(C, C)) = U_{\delta,p}(g, g/(C, C)) = U_{\delta,p}(g, s/(C, C)),$$

$$U_{\delta,p}(s, s/(D, C)) = U_{\delta,p}(s, g/(D, C)) = U_{\delta,p}(g, g/(D, C)) = U_{\delta,p}(g, s/(D, C)),$$

$$U_{\delta,p}(s, s/(C, D)) = U_{\delta,p}(s, g/(C, D)) = U_{\delta,p}(g, g/(C, D)) = U_{\delta,p}(g, s/(C, D)),$$

so

$$U_{\delta,p}(s, s) - U_{\delta,p}(g, s) = U_{\delta,p}(s, s/(D, D))p_{s,s}(D, D) - U_{\delta,p}(g, s/(D, D))p_{g,s}(D, D),$$

$$U_{\delta,p}(g, g) - U_{\delta,p}(s, g) = U_{\delta,p}(g, g/(D, D))p_{g,g}(D, D) - U_{\delta,p}(s, g/(D, D))p_{s,g}(D, D).$$

Recalling that s after (D, D) behaves as g and g after (D, D) behaves as the strategy always defect (denoted as a) and $p_{s,s}(D, D) = p_{s,g}(D, D) = p_{g,s}(D, D) = p_{g,g}(D, D) = (1 - p)^2$, then

$$U_{\delta,p}(s, s) - U_{\delta,p}(g, s) = (1 - p)^2\delta[U_{\delta,p}(g, g) - U_{\delta,p}(a, g)],$$

$$U_{\delta,p}(g, g) - U_{\delta,p}(s, g) = (1 - p)^2\delta[U_{\delta,p}(a, a) - U_{\delta,p}(g, a)].$$

Therefore, it remains to calculate the payoffs involving a and g. Also observe that for any path h if we take k as the first non-negative integer such that $h_k \neq (C, C)$ then for any $t > k$ $p_{s_1,s_2}(h_t) =$

$p_{s_1,s_2}(h_k)p_{a,a}(\sigma^k(h)_{t-k})$ where s_1 and s_2 is either g or a and $\sigma^k(h)$ is a history path that verifies $\sigma^k(h)_j = h_{j+k}$.

Therefore

$$U_{\delta,p}(g,g/h_k) = U_{\delta,p/h_k}(a,g) = U_{\delta,p}(g,a/h_k) = U_{\delta,p/h_k}(a,a).$$

So, noting with $(C,C)^t$ a path of t consecutive simultaneous cooperation and

$$L = \sum_{t\geqslant 0,h_t} \delta^t p_{a,a}(h_t)u(h_t) = \frac{1}{1-\delta}[(1-p)^2R + (S+T)(1-p)p + p^2P],$$

follows that

$$\begin{aligned}
&U_{\delta,p}(g,g) - U_{\delta,p}(a,g) = \\
&(1-\delta)\{\sum_{t\geqslant 0} \delta^t u(C,C)[p_{g,g}((C,C)^t) - p_{a,g}((C,C)^t)] + \\
&\sum_{t\geqslant 0} \delta^t [u(C,D)+\delta L][p_{g,g}((C,C)^t(C,D) - p_{a,g}((C,C)^t(C,D))] + \\
&\sum_{t\geqslant 0} \delta^t [u(D,C)+\delta L][p_{g,g}((C,C)^t(D,C)) - p_{a,g}((C,C)^t(D,C))] + \\
&\sum_{t\geqslant 0} \delta^t [u(D,D)+\delta L][p_{g,g}((C,C)^t(D,D)) - p_{a,g}((C,C)^t(D,D))]\} = \\
&(1-\delta)\{\sum_{t\geqslant 1} \delta^{t-1} R[p^{2t} - p^t(1-p)^t] + \sum_{t\geqslant 0} \delta^t [S+\delta L][p^{2t}p(1-p) - p^t(1-p)^t(1-p)^2] + \\
&\sum_{t\geqslant 0} \delta^t [T+\delta L][p^{2t}(1-p)p - p^t(1-p)^t p^2] + \sum_{t\geqslant 0} \delta^t [P+\delta L][p^{2t}(1-p)^2 - p^t(1-p)^t(1-p)p]\}.
\end{aligned}$$

Therefore

$$U_{\delta,p}(g,g) - U_{\delta,p}(a,g) = (1-\delta)GA(\delta,p)$$

where

$$\begin{aligned}
GA(\delta,p) \quad = \quad & R[\frac{p^2}{1-p^2\delta} - \frac{p(1-p)}{1-p(1-p)\delta}] + [S+\delta L][\frac{p(1-p)}{1-p^2\delta} - \frac{(1-p)^2}{1-p(1-p)\delta}] + \\
& [T+\delta L][\frac{(1-p)p}{1-p^2\delta} - \frac{p^2}{1-p(1-p)\delta}] + [P+\delta L][\frac{(1-p)^2}{1-p^2\delta} - \frac{(1-p)p}{1-p(1-p)\delta}].
\end{aligned}$$

and we write

$$GA(\delta,p) = GA^0(\delta,p) + GA^1(\delta,p)$$

where

$$\begin{aligned}
GA^0(\delta,p) \quad = \quad & R[\frac{p^2}{1-p^2\delta} - \frac{p(1-p)}{1-p(1-p)\delta}] + S[\frac{p(1-p)}{1-p^2\delta} - \frac{(1-p)^2}{1-p(1-p)\delta}] + \\
& T[\frac{(1-p)p}{1-p^2\delta} - \frac{p^2}{1-p(1-p)\delta}] + P[\frac{(1-p)^2}{1-p^2\delta} - \frac{(1-p)p}{1-p(1-p)\delta}] = \\
& [Rp^2 + (S+T)p(1-p) + P(1-p)^2][\frac{1}{1-p^2\delta} - \frac{1}{1-p(1-p)\delta}],
\end{aligned}$$

$$
\begin{aligned}
GA^1(\delta,p) &= \delta L[\frac{p(1-p)}{1-p^2\delta} - \frac{(1-p)^2}{1-p(1-p)\delta}] + \\
&\delta L[\frac{(1-p)p}{1-p^2\delta} - \frac{p^2}{1-p(1-p)\delta}] + \delta L[\frac{(1-p)^2}{1-p^2\delta} - \frac{(1-p)p}{1-p(1-p)\delta}] = \\
&\delta L[\frac{1-p^2}{1-p^2\delta} - \frac{1-(1-p)p}{1-p(1-p)\delta}].
\end{aligned}
$$

Observe that when $p, \delta \to 1$ then

$$
Rp^2 + (S+T)p(1-p) + P(1-p)^2 \to R, \quad \frac{1}{1-p(1-p)\delta} \to 1, \quad \frac{1-(1-p)p}{1-p(1-p)\delta} \to 1
$$

and recalling that $(1-\delta)L = \hat{P} = (1-p)^2R + (S+T)(1-p)p + p^2P$ then for δ, p large follows that

$$
(1-\delta)GA^0(\delta,p) \geqslant \frac{R}{2}\frac{1-\delta}{(1-p^2\delta)} \tag{23}
$$

$$
(1-\delta)GA^1(\delta,p) \geqslant \frac{\hat{P}}{2}\frac{1-p^2}{(1-p^2\delta)}. \tag{24}
$$

In the same way

$$
\begin{aligned}
&U_{\delta,p}(a,a) - U_{\delta,p}(g,a) = \\
&(1-\delta)\{\sum_{t\geqslant 0} \delta^t u(C,C)[p_{a,a}((C,C)^t) - p_{g,a}((C,C)^t)] + \\
&\sum_{t\geqslant 0} \delta^t[u(C,D)+\delta L][p_{a,a}((C,C)^t(C,D) - p_{g,a}((C,C)^t(C,D))] + \\
&\sum_{t\geqslant 0} \delta^t[u(D,C)+\delta L][p_{a,a}((C,C)^t(D,C)) - p_{g,a}((C,C)^t(D,C))] + \\
&\sum_{t\geqslant 0} \delta^t[u(D,D)+\delta L][p_{a,a}((C,C)^t(D,D)) - p_{g,a}((C,C)^t(D,D))]\} = \\
&(1-\delta)\{\sum_{t\geqslant 1} \delta^{t-1}R[(1-p)^{2t} - p^t(1-p)^t] + \sum_{t\geqslant 0}\delta^t[S+\delta L][(1-p)^{2t}p(1-p) - p^t(1-p)^tp^2] + \\
&\qquad \sum_{t\geqslant 0}\delta^t[T+\delta L][(1-p)^{2t}(1-p)p - p^t(1-p)^t(1-p)^2] + \\
&\qquad \sum_{t\geqslant 0}\delta^t[P+\delta L][(1-p)^{2t}p^2 - p^t(1-p)^t(1-p)p]\}.
\end{aligned}
$$

Therefore

$$
U_{\delta,p}(a,a) - U_{\delta,p}(g,a) = (1-\delta)AG(\delta,p)
$$

where

$$
\begin{aligned}
AG(\delta,p) &= R[\frac{(1-p)^2}{1-(1-p)^2\delta} - \frac{p(1-p)}{1-p(1-p)\delta}] + [S+\delta L][\frac{p(1-p)}{1-(1-p)^2\delta} - \frac{p^2}{1-p(1-p)\delta}] + \\
&[T+\delta L][\frac{(1-p)p}{1-(1-p)^2\delta} - \frac{(1-p)^2}{1-p(1-p)\delta}] + [P+\delta L][\frac{p^2}{1-(1-p)^2\delta} - \frac{(1-p)p}{1-p(1-p)\delta}]
\end{aligned}
$$

and we write

$$AG(\delta,p) = AG^0(\delta,p) + AG^1(\delta,p)$$

where

$$\begin{aligned} AG^0(\delta,p) &= R[\frac{(1-p)^2}{1-(1-p)^2\delta} - \frac{p(1-p)}{1-p(1-p)\delta}] + S[\frac{p(1-p)}{1-(1-p)^2\delta} - \frac{p^2}{1-p(1-p)\delta}] + \\ & T[\frac{(1-p)p}{1-(1-p)^2\delta} - \frac{(1-p)^2}{1-p(1-p)\delta}] + P[\frac{p^2}{1-(1-p)^2\delta} - \frac{(1-p)p}{1-p(1-p)\delta}] \end{aligned}$$

$$\begin{aligned} AG^1(\delta,p) &= \delta L[\frac{p(1-p)}{1-(1-p)^2\delta} - \frac{p^2}{1-p(1-p)\delta}] + \\ & \delta L[\frac{(1-p)p}{1-(1-p)^2\delta} - \frac{(1-p)^2}{1-p(1-p)\delta}] + \delta L[\frac{p^2}{1-(1-p)^2\delta} - \frac{(1-p)p}{1-p(1-p)\delta}] = \\ & \delta L[\frac{2p(1-p)}{1-(1-p)^2\delta} - \frac{1-p}{1-p(1-p)\delta}] + \delta L[\frac{p^2}{1-(1-p)^2\delta} - \frac{p^2}{1-p(1-p)\delta}] = \\ & \delta L[\frac{2p(1-p)}{1-(1-p)^2\delta} - \frac{1-p}{1-p(1-p)\delta}] + \delta L[\frac{p^2(1-p)}{(1-(1-p)^2\delta)(1-p(1-p)\delta)}] = \\ & \delta L(1-p)[\frac{2p}{1-(1-p)^2\delta} - \frac{1-p}{1-p(1-p)\delta} + \frac{p^2\delta}{(1-(1-p)^2\delta)(1-p(1-p)\delta)}] \end{aligned}$$

Observe that when $p, \delta \to 1$ then

$$AG^0(\delta,p) \to AG^0(1,1) = P - S,$$

$$\frac{2p}{1-(1-p)^2\delta} - \frac{1-p}{1-p(1-p)\delta} + \frac{p^2\delta}{(1-(1-p)^2\delta)(1-p(1-p)\delta)} \to 3$$

and recalling that $(1-\delta)L = \hat{P} = (1-p)^2R + (S+T)(1-p)p + p^2P$ then for δ, p large follows that

$$(1-\delta)AG^0(\delta,p) \leqslant 2(1-\delta)(P-S) \tag{25}$$

$$(1-\delta)AG^1(\delta,p) \leqslant 4(1-p)\hat{P}. \tag{26}$$

Recall now that the size of the basin of attraction of a is given by

$$E(\delta,p) := \frac{1}{1+\frac{(1-\delta)GA(\delta,p)}{(1-\delta)AG(\delta,p)}}.$$

Observe that for any $\epsilon > 0$ for p, δ large then from inequalities (23) and (25)

$$(1-\delta)AG^0(\delta,p) \leqslant \epsilon(1-\delta)GA^0(\delta,p)$$

and from inequalities (24) and (26)

$$(1-\delta)AG^1(\delta,p) \leqslant \epsilon(1-\delta)GA^1(\delta,p),$$

therefore, for p, δ large

$$E(\delta,p) \leqslant \frac{1}{1+\frac{1}{\epsilon}} = \frac{\epsilon}{1+\epsilon}$$

and so the theorem is concluded. □

Theorem 3 shows that the well known strategy grim does not have a uniformly large basin of attraction given that after a defection it behaves like always defect. In an world with trembles unforgivingness is evolutionary costly. We formalize next the idea of unforgivingness and and provide a general results regarding the basin of attraction of unforgiving strategies.

Definition 4. *We say that a strategy s is unforgiving if there exists a history h_t such that for all $h_{t+\tau}$ with $\tau = 0, 1, 2...$ follows $s(h_{t+\tau}/h_t) = D$.*

Theorem 4. *Unforgiving strategies do not have a uniformly large basin of attraction.*

The proof is similar to the proof of theorem 3 with the difference that the first point of divergence may not be $t = 1$.

It remains to be shown that there exists strategies with uniformly large basins of attraction. To do that we must first develop some simple way of calculating payoffs under the presence of trembles. This calculations will help us prove that there exist strategies with uniformly large basins of attractions.

6 Efficiency and size of basin of attraction; the symmetric case

In the present section we study the relationship between efficiency of a strategy and the size of its basin of attraction. Roughly speaking, full efficiency means that strategies cooperate with itself. We prove, that this is the case for any strategy that hs a uniformly large basin of attraction.

Given a finite path h_t, and a pair of strategies s, s^* it is defined

$$U(s, s/h_t) = \lim_{\delta \to 1} \lim_{p \to 1} U_{\delta,p}(s, s/h_t).$$

Definition 5. *It is said that a strategy s is efficient if for any finite path h_t follows that*

$$U(s, s/h_t) = R.$$

Question 1. *Which is the relation between efficiency and being a uniform large basin strategy?*

We provide a positive answer to previous question for symmetric strategues:

Definition 6. *we say that a strategy s is symmetric if for any finite path h_t it follows that*

$$s(h_t) = s(\hat{h}_t).$$

Theorem 5. *If s has a uniform large basin of attraction and is symmetric, then is efficient.*

Previous result establish efficiency if the probability of mistake is much smaller than $1 - \delta$. An easy corollary is the following:

Corollary 1. *If s has uniform large basin of attraction and is symmetric, then for any $R_0 < R$ there exists $\delta_0 := \delta_0(s)$ such that for any $\delta > \delta_0$ there exists $p_0(\delta)$ verifying that if $\delta > \delta_0, p > p_0(\delta)$ then for any path h_t follows that*

$$U_{\delta,p}(s, s/h_t) > R_0.$$

Here it is important to compare the statement of theorem 3 wit theorem 5 and corollary 1. On one hand, observe that the conclusion of theorem 3 is obtained for any $\delta > \delta_0$ and any $p > p_0$; instead, in corollary 1 is for $d > \delta_0$ but $p > p(\delta)$with $p(\delta)$ strongly depending of δ. On the other hand, a weake version of theorem 3 can be concluded from corollary 1.

Lemma 12. *If s has a uniformly large basin of attraction, then there exists C_0 such that for any s^* and h_t follows that*

$$\lim_{\delta\to 1}\lim_{p\to 1}\frac{U_{\delta,p}(s^*,s^*/h_t)-U_{\delta,p}(s,s^*/h_t)+U_{\delta,p}(s^*,s^*/\hat{h}_t)-U_{\delta,p}(s,s^*/\hat{h}_t)}{U_{\delta,p}(s,s/h_t)-U_{\delta,p}(s^*,s/h_t)+U_{\delta,p}(s,s/\hat{h}_t)-U_{\delta,p}(s^*,s/\hat{h}_t)}<C_0.$$

Proof. It follows immediately from lemma 11 considering a strategy s^* such that the first deviation from s occurs at h_t (and obviously at also at $\hat{h}_t$).

□

Proof of theorem 5: Let us assume that there exists a path h_t and $\lambda_0 < 1$ such that

$$U(s,s/h_t)=\lambda_0 R$$

and s is a Sub Game Perfect. We start assuming that h is no symmetric. Then we show how to deal with the symmetric case using the asymmetric one.

From the fact that s is symmetric, then follows that

$$U(s,s/h_t)=U(s,s/\hat{h}_t)$$

and therefore

$$U(s,s/h_t)+U(s,s/\hat{h}_t)=2\lambda_0 R.$$

Moreover, we can assume that $s(h_t)=D$. We are going to get a strategy s^* such that

(i) $U(s^*,s^*/h_t)=U(s^*,s^*/\hat{h}_t)=R$,

(ii) s^* acts like s after the sequel of h_t and $\hat{h}_t$.

To build that strategy s^*, first we take s^* such that $s^*(h_t)=s^*(\hat{h}_t)=C$ and then we consider all the equilibriums that follows after $h_t,\hat{h}_t$ for the pairs $s,s;s^*,s;s,s^*;s^*,s^*$:

(i) $h_t(D,D)$, $\hat{h}_t(D,D)$ for s,s

(ii) $h_t(C,D)$, $\hat{h}_t(C,D)$ for s^*,s

(iii) $h_t(D,C)$, $\hat{h}_t(D,C)$ for s,s^*

(iv) $h_t(C,C)$, $\hat{h}_t(C,C)$ for s^*,s^*.

Observe that the paths involving h_t are all differents and the same holds for the paths involving $\hat{h}_t$.

Now we request that s^* after $h_t(C,C)$ and $\hat{h}_t(C,C)$ plays C so

$$h_{s^*,s^*/h_t}=(C,C)\dots(C,C)\dots,\quad h_{s^*,s^*/\hat{h}_t}=(C,C)\dots(C,C)\dots,$$

and so

$$U(s^*,s^*/h_t)=U(s^*,s^*/\hat{h}_t)=R.$$

We also request that

$$s^*(h_t(C,D))=s(h_t(C,D)),\quad s^*(\hat{h}_t(C,D))=s(\hat{h}_t(C,D)),$$

and observe that both requirement can be satisfied simultaneously and inductively we get that

$$h_{s^*,s/h_t(C,D)}=h_{s,s/h_t(C,D)},\quad h_{s^*,s/\hat{h}_t(C,D)}=h_{s,s/\hat{h}_t(C,D)}.$$

From the fact that s is symmetric, it follows that each entry of $h_{s^*,s/h_t(C,D)} = h_{s,s/h_t(C,D)}$ and $h_{s^*,s/\hat{h}_t(C,D)} = h_{s,s/\hat{h}_t(C,D)}$ is (C,C) or (D,D) and recalling equality 7 follows that

$$U(s^*, s/h) + U(s^*, s/\hat{h}) = U(s, s^*/h) + U(s, s^*/\hat{h}).$$

Since, s is a Sub Game Perfect then $U(s^*, s/h_t) + U(s^*, s/\hat{h}_t) < 2\lambda_0 R$ and therefore $U(s, s^*/h_t) + U(s, s^*/h_t) < 2\lambda_0 R$; by remark (12) follows that if we denote $U(s^*, s/h_t) + U(s^*, s/\hat{h}_t) = 2\lambda_1 R$, then

$$\frac{1-\lambda_1}{\lambda_0 - \lambda_1} < C_0, \tag{27}$$

and taking a positive constant $C_1 < 1 - \lambda_0 < 1 - \lambda_1$ it follows that λ_1 satisfies inequality

$$\frac{C_1}{\lambda_0 - \lambda_1} < C_0. \tag{28}$$

Therefore, it follows that there exists $\gamma > 0$ such that

$$\lambda_1 < \lambda_0 - \gamma.$$

Now, we consider the path $h_t(C,D)$ and we denote it as h_{t_2} and as before we construct a new strategy s_2^* that satisfies the same type of properties as the one satisfied by s^* respect to s but on the path h_{t_2} instead on the path h_t. Inductively, we construct a sequences of paths h_{t_i}, strategies s_i^* and constants λ_i such that

$$U(s_i^*, s/h_{t_i}) = \lambda_i R \tag{29}$$

and they satisfy the following equation equivalent to (27)

$$\frac{1-\lambda_{i+1}}{\lambda_i - \lambda_{i+1}} < C_0, \tag{30}$$

and since $\lambda_{i+1} < \lambda_i$ then also satisfy an equation equivalent to (28)

$$\frac{C_1}{\lambda_i - \lambda_{i+1}} < C_0. \tag{31}$$

and therefore

$$\lambda_{i+1} < \lambda_i - i\gamma$$

but this implies that $\lambda_i \to -\infty$ and so $U(s^*, s/h_{t_i}) \to -\infty$, a contradiction because utilities along equilibrium are bounded by P.

To finish, we have to deal with the case that h_t is symmetric. For that, let us consider the sequel path $h_t(C,D)$. We claim that if $U(s, s/h_t) < R$ then

$$U(s, s/h_t(C,D)) < R.$$

In fact, we can consider the strategy s^* such that only differs on h_t and after that plays the same as s plays. Since s is a Sub Game Perfect, it follows that $U_{\delta,p}(s, s/h_t) > U_{\delta,p}(s^*, s/h_t)$ therefore, $U(s, s/h_t) = \lim_{\delta\to 1}\lim_{p\to 1} U_{\delta,p}(s, s/h_t) \geqslant \lim_{\delta\to 1}\lim_{p\to 1} U_{\delta,p}(s^*, s/h_t)$, but since

$$\lim_{\delta\to 1}\lim_{p\to 1} U_{\delta,p}(s^*, s/h_t) = \lim_{\delta\to 1}\lim_{p\to 1} U_{\delta,p}(s, s/h_t(C,D)) = U(s, s/h_t(C,D))$$

the result follows.

7 Revisiting the sufficient conditions to have a uniformly large basin

In the present section we provide general sufficient conditions to guarantee that a strategy has a uniformly large basin (see definition 3), i.e., conditions that implies that a strategy has a uniform large basin of attraction independent of the initial population, for large discount factor and small trembles. This is based in theorem 1. In subsection 7.1 we introduce another type of condition easier to calculate than the previous one, which also implies that a given strategy satisfying it is a uniform large basin strategy. From now on, we are going to take $p > p(\delta)$ where $p(\delta)$ is the one given by remark 46.

Given two strategies s_1, s_2 to avoid notation, we write

$$N_{\delta,p}(s_1, s_2) := U_{\delta,p}(s_1, s_1) - U_{\delta,p}(s_2, s_1).$$

Let s be a subgame perfect. Given s' and s^* with $N_{\delta,p}(s, s^*) \geqslant N_{\delta,p}(s, s')$ we consider the following number

$$M_{\delta,p}(s, s^*, s') := \frac{N_{\delta,p}(s, s^*) + N_{\delta,p}(s, s') + U_{\delta,p}(s', s^*) - U_{\delta,p}(s, s^*) + U_{\delta,p}(s^*, s') - U_{\delta,p}(s, s'),}{N_{\delta,p}(s, s^*)}.$$

$$M_{\delta,p}(s) := \sup_{N_{\delta,p}(s,s^*) \geqslant N_{\delta,p}(s,s')} \{M_{\delta,p}(s, s^*, s'), \ \ 0\}.$$

Remark 7. *If we take the payoff matrix associated to a set of strategies that includes s, s^*, s' and $s = e_1, s^* = e_i, s' = e_j$ it follows that $M_{\delta,p}(s, s^*, s') = \frac{M_{ij}+M_{ji}}{-N_i}$ as in lemma 1 and theorem 10.*

Remark 8. *Observe that in the case that $s^* = s'$, the quantity $M_{\delta,p}(s, s^*, s')$ is equal to*

$$\frac{2[N_{\delta,p}(s, s^*) + N_{\delta,p}(s, s^*)]}{N_{\delta,p}(s^*, s)} = 2M_{\delta,p}(s, s^*).$$

So, for the purpose of bounding $M_{\delta,p}(s)$ from $+\infty$ it is enough to take the supreme over $M_{\delta,p}(s, s^*, s')$. Observe also that if we only considere the population $\{s, s^*\}$ then the segment $[0, \frac{1}{M_\delta(s,s^*)})$ is in the basin of attraction of s (provided that s is identified with e_1).

Definition 7. *We say that a strategy s satisfies the "Large Basin strategy condition" if it is a subgame perfect strategy and if there exist δ_0 and M_0 such that for any $\delta > \delta_0$ and $p > p(\delta)$ there exists $M_0(\delta)$ verifying*

$$M_{\delta,p}(s) < M_0(\delta) < \infty.$$

We can also define

$$M(s) := \limsup_{\delta,p\to 1} M_{\delta,p(\delta)}(s)$$

and observe that in this case, if $M(s) < \infty$ then s has a large basin of attraction (but the size could depend on δ and p).

Remark 9. *It is important to remark that it could hold that $\limsup_{\delta\to 1} \sup_{s^*} \{M_{\delta,p(\delta)}(s, s^*)\} < +\infty$ but $M(s) = +\infty$. This means that to guarantee a uniform L^1-size basin in any population, it is not enough that a strategy has uniform size of basin against any other strategy.*

Definition 8. *We say that a strategy s satisfies the " uniformly Large Basin condition" if it is a strict subgame perfect strategy and*

$$M(s) < \infty.$$

Theorem 6. *If s satisfies the "uniformly Large Basin condition", then s has a uniformly large basin. More precisely, let β be small. Then, there exists δ_0 such that for any $\delta > \delta_0$ ($p > p(\delta)$) and any finite set of strategies $\mathcal{S}$ containing s, follows that s is an attracting point such that*

$$B(s) \subset B^s_{loc}(s)$$

where

$$B(s) = \{(x_1, \dots, x_n) : x_2 + \cdots + x_n \leqslant \frac{1}{M(s)} - \beta\}$$

and $n = cardinal(\mathcal{S})$.

Proof. The proof follows immediately from theorem 1 and the definition of $M(s)$. In fact, ordering the strategies in such a way that s corresponds to the first one and $N(s, s_i) \geqslant N(s, s_j)$ if $j > i$ then it follows that for δ large, then the constant $M_0 = \sup\{\frac{M_{ij}+M_{ji}}{-N_{ii}}, 0\} < M(s) - \beta$ and therefore $B(s)$ is contained in the basin of attraction of e_1. □

Remark 10. *Observe that to guarantee a uniform size of the basin of the attraction independent of the population, it is enough to bound a condition that only involves another two strategies.*

Remark 11. *Given a subgame perfect strategy s and a population $\mathcal{S}$, the lower bound of the size of the basin of attraction of s can be improved taking*

$$M_{\delta,p}(s, \mathcal{S}) := \sup_{N_{\delta,p}(s,s^*) \geqslant N_{\delta,p}(s,s'), s', s^* \in \mathcal{S}} \{M_{\delta,p}(s, s^*, s'),\ \ 0\}.$$

To check that $M_{\delta,p}(s) < +\infty$ observe that

$$\begin{aligned} & M_{\delta,p}(s, s^*, s') \\ = \ & \frac{N_{\delta,p}(s, s^*) + N_{\delta,p}(s, s') + U_{\delta,p}(s', s^*) - U_{\delta,p}(s, s^*) + U_{\delta,p}(s^*, s') - U_{\delta,p}(s, s')}{N_{\delta,p}(s, s^*)} \\ = \ & 1 + \frac{N_{\delta,p}(s, s')}{N_{\delta,p}(s, s^*)} + \frac{U_{\delta,p}(s', s^*) - U_{\delta,p}(s, s^*) + U_{\delta,p}(s^*, s') - U_{\delta,p}(s, s')}{N_{\delta,p}(s, s^*)}. \end{aligned}$$

Then if

$$Z_{\delta,p}(s, s^*, s') := \frac{U_{\delta,p}(s', s^*) - U_{\delta,p}(s, s^*) + U_{\delta,p}(s^*, s') - U_{\delta,p}(s, s')}{N_{\delta,p}(s, s^*)},$$

defining

$$Z_{\delta,p}(s) := \sup_{N_{\delta,p}(s,s^*) \geqslant N_{\delta,p}(s,s')} \{Z_{\delta,p}(s, s^*, s')\}$$

and using that $\frac{N_{\delta,p}(s,s')}{N_{\delta,p}(s,s^*)} \leqslant 1$ then follows that $M_{\delta,p}(s) < +\infty$ if and only if $Z_{\delta,p}(s) < +\infty$.

In other words, s is a "Large Basin strategy" if and only if $Z_{\delta,p}(s) < +\infty$. Similarly, defining

$$Z(s) := \limsup_{\delta, p(\delta) \to 1} Z_{\delta,p}(s),$$

s is a "uniform Large Basin strategy" if and only if

$$Z(s) < +\infty.$$

Question 2. *Is the uniformly large basin condition (recall definition 8) a necessary condition for a strategy to have a uniformly large basin strategy? In other words, does s satisfy the large basin condition?*

7.1 Asymptotic bounded condition

We provide now a condition that implies that s is has a uniformly Large Basin of attraction. This new conditions are based on the conditions defined before but are easier to calculate. Moreover, if a strategy satisfies them it follows that has a uniformly large basin of attraction.

Definition 9. *We say that a subgame perfect strategy s satisfies the asymptotic bounded condition if*

- *there exists R_0 such that for any s^* holds*

$$\limsup_{\delta\to 1, p\to 1, p>p(\delta)} \sup_{s^*: N_{\delta,p}(s,s^*)>0} \frac{U_{\delta,p}(s,s) - U_\delta(s,s^*)}{N_{\delta,p}(s,s^*)} < R_0, \tag{32}$$

- *there exists R_1 such that for any s^*, s' that $N_{\delta,p}(s,s^*) \geqslant N_{\delta,p}(s,s^*)$ holds*

$$\limsup_{\delta\to 1, p\to 1, p>p(\delta)} \sup_{s^*: N_\delta(s,s^*)>0} \frac{U_{\delta,p}(s',s^*) + U_{\delta,p}(s^*,s') - 2U_{\delta,p}(s,s)}{N_{\delta,p}(s,s^*)} < R_1. \tag{33}$$

Theorem 7. *Let s be subgame perfect strategy satisfying the asymptotic bounded condition. Then, s has a uniformly large basin of attraction.*

Proof. Recalling that $N_{\delta,p}(s,s') \leqslant N_{\delta,p}(s,s^*)$ we need to bound by above the following expression

$$\frac{U_{\delta,p}(s',s^*) - U_{\delta,p}(s,s^*) + U_{\delta,p}(s^*,s') - U_{\delta,p}(s,s')}{N_{\delta,p}(s,s^*)}.$$

So,

$$\begin{aligned}
& \frac{U_{\delta,p}(s',s^*) - U_{\delta,p}(s,s^*) + U_{\delta,p}(s^*,s') - U_{\delta,p}(s,s')}{N_{\delta,p}(s,s^*)} = \\
= \quad & \frac{U_{\delta,p}(s',s^*) + U_{\delta,p}(s^*,s') - 2U_{\delta,p}(s,s)}{N_{\delta,p}(s,s^*)} + \\
+ \quad & \frac{U_{\delta,p}(s,s) - U_{\delta,p}(s,s^*)}{N_{\delta,p}(s,s^*)} + \frac{U_{\delta,p}(s,s) - U_{\delta,p}(s,s')}{N_{\delta,p}(s,s^*)} \leqslant \\
& \leqslant R_1 + \frac{U_{\delta,p}(s,s) - U_{\delta,p}(s,s^*)}{N_{\delta,p}(s,s^*)} + \frac{U_{\delta,p}(s,s) - U_{\delta,p}(s,s')}{N_{\delta,p}(s,s')} \frac{N_{\delta,p}(s,s')}{N_{\delta,p}(s,s^*)} \\
& \leqslant R_1 + 2R_0.
\end{aligned}$$

□

From now on, we denoted

$$N_{\delta,p}(s,s^*) := U_{\delta,p}(s,s) - U_{\delta,p}(s^*,s) \tag{34}$$

$$\bar{N}_{\delta,p}(s,s^*) := U_{\delta,p}(s,s) - U_{\delta,p}(s,s^*) \tag{35}$$

$$B_{\delta,p}(s,s^*,s') := U_{\delta,p}(s',s^*) + U_{\delta,p}(s^*,s') - 2U_{\delta,p}(s,s) \tag{36}$$

Remark 12. *From the proof of theorem 7 follows that $M(s) \leqslant 2 + 2R_0 + R_1$.*

Remark 13. *Observe that if it is assumed that (32) holds, then s satisfies the asymptotic bounded condition if and only if s satisfies the uniformly large basin condition.*

7.2 Having uniform large basin for population of two strategies is not enough

In this section we give an example that shows that when a population of three strategies are considered it can happen that one of them has a uniform large basin when it is taken the subset of two strategies but it has not a large basin when the three strategies are considered simultaneously. In other words, next theorem shows that the example given in theorem 2 can be obtained as the replicator equation associated to three strategies. In what follows, given a population of three strategies $\mathcal{S} = \{s, s^*, s'\}$ and its replicator equation (in affine coordinates), the first strategy is identified with the point $(0, 0)$. In the theorem below it is considered the repeated prisoner's dilemma without tremble and the proof in trivially adapted for the cae of trembles provided small erros of mistake.

Theorem 8. *For any λ small, there exists a population of three strategies $\mathcal{S} = \{s, s^*, s'\}$ such that*

(i) *s is an attractor in $\mathcal{S}$;*

(ii) *s always cooperate with itself;*

(iii) *in the population $\{s, s^*\}$, s is a global attractor (in the terminology of the replicator equation, the interior of the simplex associated to $\{s, s^*\}$ is in the basin of attraction of s);*

(iv) *in the population $\{s, s'\}$ s is a global attractor;*

(v) *the region bounded by $H_\lambda, V + \lambda$ and $x_2 + x_3 = 1$ does not intersect the basin of attraction of s.*

Proof. Given any small $\lambda > 0$, we build three strategies such that identifying s with $(0, 0)$, s^* with $(1, 0)$ and s' with $(0, 1)$ satisfy the hypothesis of theorem 2. We also assume that the strategies s' and s^* deviate from s at the $0-$history, s plays always cooperate with itself and so $s'(0) = s^*(0) = D$. We fixed $\gamma > 0$ and, and we take ϵ small. Observe that provided any $\epsilon > 0$ small, taking δ large, follows that there exist different b'_1, b'_2, b'_3, b'_4 and $b^*_1, b^*_2, b^*_3, b^*_4$ such that

$$0 < R - (b'_1 R + b'_2 T + b'_3 S + b'_4 P) = R - (b^*_1 R + b^*_2 T + b^*_3 S + b^*_4 P) = \epsilon$$

but

$$R - (b'_1 R + b'_2 S + b'_3 T + b'_4 P) = R - (b^*_1 R + b^*_2 S + b^*_3 T + b^*_4 P) > \gamma.$$

Now, from (C, D) we choose s, s', s^* such that

$$U_\delta(s, s^*) = U_\delta(s, s') = b'_1 R + b'_2 T + b'_3 S + b'_4 P$$

but in such a way that $s' \neq s^*$. To show that it is possible to choose s' independently of s^* against s is enough to take $s'(C, D) \neq s^*(C, D)$. Now, we take s^* and s' from (D, D) such that

$$s^*(D, D) \neq s'(D, D)$$

and

$$U_\delta(s^*, s^*) - U_\delta(s, s^*) = U_\delta(s^*, s^*) - (b^*_1 R + b^*_2 S + b^*_3 T + b^*_4 P) = -\epsilon,$$

$$U_\delta(s', s') - U_\delta(s, s') = U_\delta(s', s') - (b'_1 R + b'_2 S + b'_3 T + b'_4 P) = -\epsilon.$$

Moreover, we can take s', s^* such that

$$U_\delta(s', s^*) = U_\delta(s', s^*) = R$$

therefore,

$$U_\delta(s', s^*) - U_\delta(s^*, s^*) = U_\delta(s', s^*) - U_\delta(s', s') > \gamma.$$

So,

$$\frac{U_\delta(s', s^*) - U_\delta(s^*, s^*)}{U_\delta(s, s) - U_\delta(s^*, s)} > \frac{\gamma}{\epsilon}$$

and so choosing ϵ properly we can assume that the quotient is equal to $\frac{1}{\lambda}$. □

8 Recalculating payoff with trembles

Now, we are developing a criterion to calculate the payoff for certain strategies which roughly speaking consists in approximating the payoff using equilibrium paths, provided that the probability of mistake is small. This first order approximation allows to prove the asymptotic bounded condition (see inequalities (32), (33), (39), (41) and lemma 16) for certain types of stratgies (namely strict subgame perfect strategies, see definition 10). In few words, the difference in utility between two strategies can be estimated in the following way (provided that p is sufficiently close to 1):

- first, we consider all the paths (on and off equilibrium) up to its first node of divergence between the two strategies, namely $h_k, \hat{h}_k$ (see equalities (37, 38, 40)),
- from the node of divergence we only consider equilibrium payoffs (see lemma 15).

In particular, if $s(h_0) \neq s^*(h_0)$ then $U_{\delta,p}(s, s) - U_{\delta,p}(s^*, s)$ is approximated by $U_{\delta,p,h_{s,s}}(s, s) - U_{\delta,p,h_{s^*,s}}(s^*, s)$.

More precissely, recalling that

$$N_{\delta,p}(s, s^*) = \sum_{h_k, h \in \mathcal{R}^*_{s,s^*}} \delta^k p_{s,s}(h_k)[U_{\delta,p}(s, s/h_k\hat{h}_k) - U_{\delta,p}(s^*, s/h_k\hat{h}_k)]. \tag{37}$$

$$\bar{N}_{\delta,p}(s, s^*) = \sum_{h_k, h \in \mathcal{R}^*_{s,s^*}} \delta^k p_{s,s}(h_k)[U_{\delta,p}(s, s/h_k\hat{h}_k) - U_{\delta,p}(s, s^*/h_k\hat{h}_k)]. \tag{38}$$

we define

$$N^e_{\delta,p}(s, s^*) := \sum_{h_k, h \in \mathcal{R}^*_{s,s^*}} \delta^k p_{s,s}(h_k)[U_{\delta,p}(h_{s,s/h_k\hat{h}_k}) - U_{\delta,p}(h_{s^*,s/h_k\hat{h}_k})].$$

$$\bar{N}^e_{\delta,p}(s, s^*) := \sum_{h_k, h \in \mathcal{R}^*_{s,s^*}} \delta^k p_{s,s}(h_k)[U_{\delta,p}(h_{s,s/h_k\hat{h}_k}) - U_{\delta,p}(h_{s,s^*/h_k\hat{h}_k})]$$

where given strategies s_1, s_2

$$U_{\delta,p}(h_{s_1,s_2/h_k\hat{h}_k}) := U_{\delta,p}(h_{s_1,s_2/h_k}) + U_{\delta,p}(h_{s_1,s_2/h_k}).$$

We look for conditions such that there exists a uniform constant C

$$\frac{\bar{N}_{\delta,p}(s, s^*)}{N_{\delta,p}(s, s^*)} \leqslant \frac{\bar{N}^e_{\delta,p}(s, s^*)}{N^e_{\delta,p}(s, s^*)} + C. \tag{39}$$

A similar approach we develop for $B_{\delta,p}(s,s',s^*)$ that consists in comparing different paths for three strategies s,s^*,s'. Given any pair of paths $h,\hat{h}$ where s,s',s^* differ (meaning that at least two of the strategies differ at some finite paths contained either in h or $\hat{h}$), there exist $k'=k(s,s',h), \hat{k}'=\hat{k}(s,s',\hat{}),\ k^*=k(s,s^*,h), \hat{k}^*=\hat{k}(s,s^*,\hat{h})$, such that $s(h_{k'})\neq s'(h_{k'}), s(\hat{h}_{k'})\neq s'(\hat{h}_{k'})$ and $s(\hat{h}_{k^*})\neq s^*(\hat{h}_{k^*})$. Observe that some of them could be infinity.

We take

$$k(s,s',s^*):=\min\{k',\hat{k}',\ k^*,\hat{k}^*\}$$

which is finite and observe that

$$\begin{aligned}p_{ss}(h_k)=p_{s's^*}(h_k)=p_{s^*s'}(h_k)=p_{s^*s}(h_k)=p_{s's}(h_k)\\ p_{ss}(\hat{h}_k)=p_{s's^*}(\hat{h}_k)=p_{s^*s'}(\hat{h}_k)=p_{s^*s}(\hat{h}_k)=p_{s's}(\hat{h}_k).\end{aligned}$$

so

$$B_{\delta,p}(s,s^*,s')=\sum_{h:k(s,s',s^*)}\delta^k p_{ss}(h_k)[U_{\delta,p}(s',s^*/h_k\hat{h}_k)+U_{\delta,p}(s^*,s'/h_k\hat{h}_k)-2U_{\delta,p}(s,s/h_k\hat{h}_k)]. \quad (40)$$

Now we define

$$B^e_{\delta,p}(s,s^*,s')=\sum_{h:k(s,s',s^*)}\delta^k p_{ss}(h_k)[U_{\delta,p}(h_{s',s^*/h_k\hat{h}_k})+U_{\delta,p}(h_{s^*,s'/h_k\hat{h}_k})-2U_{\delta,p}(h_{s,s/h_k\hat{h}_k})].$$

So, in a similar way we look for conditions such that there exists a uniform constant C

$$\frac{B_{\delta,p}(s,s^*,s')}{N_{\delta,p}(s,s^*)}\leqslant\frac{B^e_{\delta,p}(s,s^*,s')}{N^e_{\delta,p}(s,s^*)}+C. \quad (41)$$

We are going to restrict a relation between p and δ. From now on we assume that

$$p\geqslant\sqrt{\delta}. \quad (42)$$

Moreover, and to simplify calculations we change the usual renormalization factor $1-\delta$ by $\frac{1-p^2\delta}{p^2}$ and so we calculate the payoff as following:

$$U_{\delta,p}(s_1,s_2)=\frac{1-p^2\delta}{p^2}\sum_{t\geqslant0,a_t,b_t}\delta^t p_{s_1,s_2}(a_t,b_t)u(a^t,b^t).$$

Both ways calculating the payoff (either with renormalization $1-\delta$ or $\frac{1-p^2\delta}{p^2}$) are equivalent as they rank histories in the same way.

In addition it holds that:

$$\frac{1}{2}<\frac{1-\delta}{1-\delta p^2}<1.$$

Observe that if $s_1=s_2$ along the equilibrium it follows that

$$U_{\delta,p}(h_{s,s})=\frac{1-\delta p^2}{p^2}\sum_{t\geqslant0}p^{2t+2}\delta^t u(a^t,a^t)\leqslant R.$$

Lemma 13. *Given any pair of strategies s_1,s_2 it follows that*

$$|U_{\delta,p}(h^c_{s_2,s_1/h_t})|<\frac{1-p^2}{p^2(1-\delta)}M$$

where $M=\max\{T,|S|\}$.

Proof. Observe that fixed t then

$$\sum_{h_t} p_{s_1,s_2}(h_t) = 1,$$

since in the equilibrium path at time t the probability is p^{2t+2} it follows that

$$\sum_{h_t \notin \mathcal{NE}} p_{s_1,s_2}(h_t) = 1 - p^{2t+2}.$$

Therefore, and recalling that $u(h^t) \leqslant M$,

$$\begin{aligned}
|U_{\delta,p}(h^c_{s_2,s_1/h_t})| &= |\frac{1-p^2\delta}{p^2} \sum_{t\geqslant 0, h_t\notin\mathcal{NE}} \delta^t p_{s_1,s_2}(h_t) u(h^t)| \\
&\leqslant \frac{1-p^2\delta}{p^2} \sum_{t\geqslant 0} \delta^t \sum_{h_t\notin\mathcal{NE}} p_{s_1,s_2}(h_t)|u(h^t)| \\
&\leqslant \frac{1-p^2\delta}{p^2} M \sum_{t\geqslant 0} \delta^t (1-p^{2t+2}) \\
&= M[\frac{1-p^2\delta}{p^2} \sum_{t\geqslant 0} \delta^t - \frac{1-p^2\delta}{p^2} \sum_{t\geqslant 0} \delta^t p^{2t+2}] \\
&= M[\frac{1-p^2\delta}{p^2(1-\delta)} - 1] \\
&= \frac{1-p^2}{p^2(1-\delta)} M.
\end{aligned}$$

□

Lemma 14. *It follows that*

$$N_{\delta,p}(s,s^*) \leqslant N^e_{\delta,p}(s,s^*) + 2\frac{1-p^2}{p^2(1-\delta)} M;$$

$$\bar{N}_{\delta,p}(s,s^*) \leqslant \bar{N}^e_{\delta,p}(s,s^*) + 2\frac{1-p^2}{p^2(1-\delta)} M;$$

$$B_{\delta,p}(s,s^*,s') \leqslant B^e_{\delta,p}(s,s^*,s') + 3\frac{1-p^2}{p^2(1-\delta)} M.$$

The next definition is an extension of the definition of subgame perfect strategies.

Definition 10. *We say that s is a uniformly strict sub game perfect if for any s^* follows that given $h \in \mathcal{R}_{s,s^*}$ then*

$$(1-p^2\delta)C_0 < U_{\delta,p}(h_{s,s/h_k}) - U_{\delta,p}(h_{s^*,s/h_k}), \tag{43}$$

for $p > p_0, \delta > \delta_0$ where C_0, δ_0, p_0 are positive constants that only depend on T, R, P, S.

Given δ we take p such that it is verified,

$$3\frac{1-p^2}{p^2(1-\delta)} \frac{M}{C_0(1-p^2\delta)} < 1. \tag{44}$$

Since $p<1$ follows that $1-p^2\delta < 1-\delta$ and taking $p>\frac{1}{2}$ then to satisfies (44) we require that

$$\frac{3}{4}\frac{1-p^2}{(1-\delta)^2}\frac{M}{C_0} < 1. \tag{45}$$

Therefore, we take So, we take

$$p_1(\delta) = \sqrt{1-\frac{4}{3}\frac{C_0}{M}(1-\delta)^2}$$

and bbserve that is a function smaller than 1 for $\delta < 1$. Then, we define

$$p(\delta) = \max\{\frac{1}{2}, p_1(\delta), \sqrt{\delta}\} \tag{46}$$

Lemma 15. *If s^* is strict subgame perfect and $p > p(\delta)$ (giving by equality 46) then*

$$\frac{\bar{N}_{\delta,p}(s,s^*)}{N_{\delta,p}(s,s^*)} \leqslant \frac{\bar{N}^e_{\delta,p}(s,s^*)}{N^e_{\delta,p}(s,s^*)} + 1;$$

$$\frac{B_{\delta,p}(s,s^*,s')}{N_{\delta,p}(s,s^*)} \leqslant \frac{B^e_{\delta,p}(s,s^*,s')}{N^e_{\delta,p}(s,s^*)} + 1.$$

Proof. It follows from lemma 14, s is a subgame perfect and that inequality (44) is satisfied

$$\frac{\bar{N}_{\delta,p}(s,s^*)}{N_{\delta,p}(s,s^*)} \leqslant$$
$$\frac{\bar{N}^e_{\delta,p}(s,s^*) + 2\frac{1-p^2}{p^2(1-\delta)}M}{N^e_{\delta,p}(s,s^*)(1+2\frac{1-p^2}{p^2(1-\delta)}M\frac{1}{N^e_{\delta p}(s,s^*)})} \leqslant$$
$$\frac{\bar{N}^e_{\delta,p}(s,s^*)}{N^e_{\delta,p}(s,s^*)(1+2M\frac{1-p^2}{(1-\delta)p^2C_0(1-p^2\delta)})} + 2M\frac{1-p^2}{(1-\delta)p^2C_0(1-p^2\delta)}) \leqslant$$
$$\frac{\bar{N}^e_{\delta,p}(s,s^*)}{N^e_{\delta,p}(s,s^*)} + 1.$$

In a similar way it is done the estimate for $\frac{B_{\delta,p}(s,s^*,s')}{N_{\delta,p}(s,s^*)}$. □

Now we will try to estimate $\frac{U_{\delta,p}(s,s)-U_{\delta,p}(s,s^*)}{U_{\delta,p}(s,s)-U_{\delta,p}(s^*,s)}$ based on lemma 15.

Lemma 16. *If $p > p(\delta)$ (giving by equality 46) and s is a uniform strict and there exists D such that for any $h \in \mathcal{R}^*_{s,s^*}$ holds*

$$\frac{U_{\delta,p}(h_{s,s/h_k\hat{h}_k}) - U_{\delta,p}(h_{s,s^*/h_k\hat{h}_k})}{U_{\delta,p}(h_{s,s/h_k\hat{h}_k}) - U_{\delta,p}(h_{s^*,s/h_k\hat{h}_k})} < D$$

then

$$\frac{U_{\delta,p}(s,s) - U_{\delta,p}(s,s^*)}{U_{\delta,p}(s,s) - U_{\delta,p}(s^*,s)} < D+1.$$

Proof. It is enough to estimate $\frac{\bar{N}_{\delta,p}(s,s^*)}{N_{\delta,p}(s,s^*)}$

$$\frac{\bar{N}_{\delta,p}(s,s^*)}{N_{\delta,p}(s,s^*)} =$$
$$\frac{\sum_{h\in\mathcal{R}_{s,s^*}\delta,p} \delta^k p_{s,s}(h_k)(U_{\delta,p}(h_{s,s/h_k\hat{h}_k}) - U_{\delta,p}(h_{s,s^*/h_k\hat{h}_k}))}{\sum_{h\in\mathcal{R}_{s,s^*},\delta,p} \delta^k p_{s,s}(h_k)(U_{\delta,p}(h_{s,s/h_k\hat{h}_k}) - U_{\delta,p}(h_{s^*,s/h_k\hat{h}_k}))}$$
$$= \frac{\sum_{k,h_k} \delta^k p_{s,s}(h_k) \frac{U_{\delta,p}(h_{s,s/h_k\hat{h}_k}) - U_{\delta,p}(h_{s,s^*/h_k\hat{h}_k})}{U_{\delta,p}(s,s/h_k\hat{h}_k) - U_{\delta,p}(s^*,s/h_k\hat{h}_k)}(U_{\delta,p}(h_{s,s/h_k\hat{h}_k}) - U_{\delta,p}(h_{s,s^*/h_k\hat{h}_k})}{\sum_{k,h_k} \delta^k p_{s,s}(h_k)(U_{\delta,p}(h_{s,s/h_k\hat{h}_k}) - U_{\delta,p}(h_{s^*,s/h_k\hat{h}_k})}) \leqslant$$
$$D\frac{\sum_{h\in\mathcal{R}_{s,s^*}\delta,p} \delta^k p_{s,s}(h_k)(U_{\delta,p}(h_{s,s/h_k\hat{h}_k}) - U_{\delta,p}(h_{s^*,s/h_k\hat{h}_k}))}{\sum_{h\in\mathcal{R}_{s,s^*},\delta,p} \delta^k p_{s,s}(h_k)(U_{\delta,p}(h_{s,s/h_k\hat{h}_k}) - U_{\delta,p}(h_{s^*,s/h_k\hat{h}_k}))} = D.$$

□

9 win-stay-loose-shift has a uniformly large basin of attraction

In the present section we show that strategies like win-stay-loose-shift satisfy the conditions introduced in subsection 7.1.

Definition 11. win-stay-loose-shift *Let us define the strategy known as Win-stay lose-shift: f it gets either T or R stays, if not, shifts. To be a subgame perfect strategy it is required that $2R > T+P$. From now on, we denote win-stay lose-shift as w. See [NS] and [RC].*

The next lemma is obvious but we state it since is fundamental to do a series of calculations related to w.

Lemma 17. *Given a finite path h_t it follows that w is a symmetric strategy, meaning that*

$$w(h_t) = w(\hat{h}_t).$$

Proof. It follows from the fact that

$$w(C,D) = w(D,C) = D.$$

□

Theorem 9. *If $2R > T + P$ then w has a uniformly large basin.*

We are going to show that w has a uniformly large basin of attraction strategy. For that, first we prove that w is a uniform strict subgame perfect (this is done in subsection 9.0.1), and later we show that w satisfies the "Asymptotic bounded condition". Recall that we need to bound

$$\frac{\bar{N}_{\delta,p}(s,s^*)}{N_{\delta,p}(s,s^*)} \tag{47}$$

and

$$\frac{\bar{B}_{\delta,p}(s,s^*,s')}{N_{\delta,p}(s,s^*)} \tag{48}$$

this is done in subsection 9.0.2 and 9.0.3, respectively.

9.0.1 w is a uniformly strict subgame perfect.

Given h_k we have to estimate

$$U_{\delta,p}(h_{w,w/h_k}) - U_{\delta,p}(h_{s,w/h_k})$$

where $h_{w,w/h_k}$ is the equilibrium path for w, w starting with h_k and $h_{s,w/h_k}$ is the equilibrium path for s, w starting with h_k.

In what follows, to avoid notation, with $U(.,.)$ we denote $U_{\delta,p}(h_{.,./h_k})$. Following that, we take

$$b_1 = \frac{1-p^2\delta}{p^2} \sum_{j:u^j(s,w/h_k)=R} p^{2j+2}\delta^j, \quad b_2 = \frac{1-p^2\delta}{p^2} \sum_{j:u^j(s,w/h_k)=S} p^{2j+2}\delta^j,$$

$$b_3 = \frac{1-p^2\delta}{p^2} \sum_{j:u^j(s,w/h_k)=T} p^{2j+2}\delta^j, \quad b_4 = \frac{1-p^2\delta}{p^2} \sum_{j:u^j(s,w/h_k)=P} p^{2j+2}\delta^j.$$

Observe that

$$b_1 + b_2 + b_3 + b_4 = 1$$

and

$$U(s, w) = b_1 R + b_2 S + b_3 T + b_4 P.$$

From the property of w, for each T that s can get (s plays D and w plays C) follows that in the next move s may get either S or P because w plays D, so,

$$b_2 + b_4 \geqslant p^2\delta b_3. \tag{49}$$

To calculate $U(w, w)$ we have to consider either $s(h_k) = C$, $w(h_k) = D$ or $s(h_k) = D$, $w(h_k) = C$. So, from lemma 17

$$U(w, w) = \begin{cases} R & \text{if } w(h_k) = C \\ \frac{1-p^2\delta}{p^2} P + p^2\delta R & \text{if } w(h_k) = D \end{cases}$$

To calculate $U(s, w)$ in case that $s(h_k) = D, w(h_k) = C$, writing $R = b_1 R + b_2 R + b_3 R + b_4 R$ by inequality (49) it follows that

$$\begin{aligned} U(w, w) - U(s, w) &= b_2(R - S) + b_3(R - T) + b_4(R - P) \\ &\geqslant (b_2 + b_4)(R - P) + b_3(R - T) \\ &\geqslant \delta p^2 b_3(R - P) + b_3(R - T) \\ &\geqslant b_3[(1 + p^2\delta)R - (T + P)]. \end{aligned}$$

Observing that if $s(h_k) = D, w(h_k) = C$, then

$$b_3 \geqslant 1 - p^2\delta$$

and since $2R - (T + P) > 0$ it follows that for δ and p large (meaning that they are close to one), then $[(1 + p^2\delta)R - (T + P)] > C_0$ for a positive constant smaller than $2R - (T + P)$ and therefore (provided that δ and p large are large) follows that

$$U(w, w) - U(s, w) > (1 - p^2\delta)C_0,$$

concluding that w is a uniform strict subgame perfect in this case.

In the case that $s(h_k) = C, w(h_k) = D$, observe that $b_2 \geqslant 1-\delta$ and calculating again the quantities b_1, b_2, b_3, b_4 but starting from $j \geqslant 1$ then we get that

$$U(s,w) = (1-p^2\delta)S + p^2\delta[b_1R + b_2S + b_3T + b_4P].$$

Therefore, writing $p^2\delta R = p^2\delta[b_1R + b_2R + b_3R + b_4R]$ and arguing as before,

$$\begin{aligned} U(w,w) - U(s,w) &= (1-p^2\delta)(P-S) + \delta[b_2(R-S) + b_3(R-T) + b_4(R-P)] \\ &\geqslant (1-p^2\delta)(P-S) + \delta[(b_2+b_4)(R-P) + b_3(R-T)] \\ &\geqslant (1-p^2\delta)(P-S) + \delta[\delta b_3(R-P) + b_3(R-T)] \\ &\geqslant (1-p^2\delta)(P-S) + \delta b_3[(1+\delta)R - (T+P)] \end{aligned}$$

since $2R-(T+P) > 0$ it follows that for δ large (b_3 now can be zero)

$$U(w,w) - U(s,w) > (1-p^2\delta)(P-S),$$

proving that w is a uniform strict subgame perfect in this case.

Remark 14. *Given ϵ small follows that for δ large then C_0 can be estimated as*

$$C_0 = \min\{P-S, 2R-(T+S)-\epsilon\}. \tag{50}$$

Remark 15. *To prove that w is a uniform strict sgp, the main two properties of w used are*

(i) *it cooperates after seeing cooperation and so $U(w,w) = R$ after $w(h_k) = C$,*

(ii) *after getting P it goes back to cooperate, so $U(w,w) = (1-\delta p^2)P + \delta p^2 R$ after $w(h_k) = D$,*

(iii) *it punishes after getting S,*

(iv) $2R > T+P$.

Observe, that the previous calculation does not use that w keeps defecting after obtaining T.

9.0.2 Bounding (47).

First we estimate $U_{\delta,p}(w,w) - U_{\delta,p}(s,w)$ and $U_{\delta,p}(w,w) - U_{\delta,p}(w,s)$. Recall that from lemma 16 it follows that is enough to bound for any $h \in \mathcal{R}^*_{w,s}$.

$$\frac{U_{\delta,p}(h_{w,w/h_k\hat{h}_k}) - U_{\delta,p}(h_{w,s/h_k\hat{h}_k})}{U_{\delta,p}(h_{w,w/h_k\hat{h}_k}) - U_{\delta,p}(h_{s,w/h_k\hat{h}_k})}.$$

Therefore, we have to bound the first term.
Calculating numerator and denominator.
For the moment, to avoid notation, we denote

$$U(s,s') := U_{\delta,p}(h_{s,s'/h_k\hat{h}_h}) = U_{\delta,p}(h_{s,s'/h_k}) + U(h_{s,s'/\hat{h}_k}).$$

Observe that if $U(w,w) - U(s,w) = B_2(R-S) + B_3(R-T) + B_4(R-P)$, then

$$U(w,w) - U(w,s) = B_2(R-T) + B_3(R-S) + B_4(R-P).$$

To avoid notation, let us denote $L = U(w,w) - U(s,w) = B_2(R-S) + B_3(R-T) + B_4(R-P)$ so, $B_4(R-P) = L - [B_2(R-S) + B_3(R-T)]$ and therefore

$$\begin{aligned} U(w,w) - U(w,s) &= B_2(R-T) + B_3(R-S) + L - [B_2(R-S) + B_3(R-T)] \\ &= L + B_2(S-T) + B_3(T-S) \\ &= L + (B_3 - B_2)(T-S) \\ &\leqslant L + B_3(T-S) \end{aligned}$$

recalling that in case that $b_3 \neq 0$ then $L = U(w,w) - U(s,w) \geqslant B_3[(1+\delta)R - (T+P)]$ (if $B_3 = 0$ then $\frac{U(w,w)-U(w,s)}{U(w,w)-U(s,w)} \leqslant 1$) it follows that

$$\begin{aligned} \frac{U(w,w) - U(w,s)}{U(w,w) - U(s,w)} &\leqslant \frac{L + B_3(T-S)}{L} \\ &\leqslant 1 + \frac{B_3(T-S)}{B_3[(1+\delta)R - (T+P)]} \\ &= 1 + \frac{T-S}{(1+\delta)R - (T+P)}. \end{aligned}$$

Therefore,

$$\frac{U_{\delta,p}(h_{w,w/h_k\hat{h}_k}) - U_{\delta,p}(h_{w,s/h_k\hat{h}_k})}{U_{\delta,p}(h_{w,w/h_k\hat{h}_k}) - U_{\delta,p}(h_{s,w/h_k\hat{h}_k})} \leqslant 1 + \frac{T-S}{(1+\delta)R - (T+P)}, \tag{51}$$

so by lemma 16

$$\frac{U_{\delta,p}(w,w) - U_{\delta,p}(w,s)}{U_{\delta,p}(w,w) - U_{\delta,p}(s,w)} \leqslant 2 + \frac{T-S}{(1+\delta)R - (T+P)}.$$

Remark 16. *The main property of w used to bound (47) is that if $b_3 \neq 0$ then*

$$U(w,w) - U(s,w) \geqslant b_3[(1+\delta)R - (T+P)]$$

and this follows from the properties listed in remark 15.

9.0.3 Bounding (48)

By lemma 15 we need to bound

$$\frac{B^e_{\delta,p}(s,s^*,s')}{N^e_{\delta,p}(s,s^*)}.$$

Recall that

$$B^e_{\delta,p}(s,s^*,s') = \sum_{h:k(s,s',s^*)} \delta^k p_{ss}(h_k)[U_{\delta,p}(h_{s',s^*/h_k\hat{h}_k}) + U_{\delta,p}(h_{s^*,s'/h_k\hat{h}_k}) - 2U_{\delta,p}(h_{s,s/h_k\hat{h}_k})].$$

For the particular case of $s = w$ we divide the paths in two types: either $w(h_k) = C$ or $w(h_k) = D$. In the first case we claim that

$$U_{\delta,p}(h_{s',s^*/h_k\hat{h}_k}) + U_{\delta,p}(h_{s^*,s'/h_k\hat{h}_k}) - 2U_{\delta,p}(h_{w,w/h_k\hat{h}_k}) \leqslant 0.$$

Observe that $U_{\delta,p}(h_{w,w/h_k\hat{h}_k}) = 2R$ and by lemma 7 follows tha assertion above. Therefore,

$$B^e_{\delta,p}(s,s^*,s') \leqslant \sum_{h:k(s,s',s^*),w(h_k)=D} U_{\delta,p}(h_{s',s^*/h_k\hat{h}_k}) + U_{\delta,p}(h_{s^*,s'/h_k\hat{h}_k}) - 2U_{\delta,p}(h_{w,w/h_k\hat{h}_k}).$$

In case that $w(h_k) = D$ observe that $U(h_{w,w/h_k\hat{h}_k}) = 2\frac{1-p^2\delta}{p^2}P + 2R\delta$. To deal with this situation we consider two cases: $i)$ $s'(h_k) = C$ or $s'(\hat{h}_k) = C$, and $ii)$ $s^*(h_k) = C$ or $s^*(\hat{h}_k) = C$. So,

$$\begin{aligned} B^e_{\delta,p}(s,s^*,s') \leqslant & \sum_{h:s'(h_k)=C \text{ or} s'(\hat{h}_k)=C} U_{\delta,p}(h_{s',s^*/h_k\hat{h}_k}) + U_{\delta,p}(h_{s^*,s'/h_k\hat{h}_k}) - 2U_{h_{\delta,p}(w,w/h_k\hat{h}_k)} + \\ & \sum_{h:s^*(h_k)=C \text{ or} s^*(\hat{h}_k)=C} U_{\delta,p}(h_{s',s^*/h_k\hat{h}_k}) + U_{\delta,p}(h_{s^*,s'/h_k\hat{h}_k}) - 2U_{\delta,p}(h_{w,w/h_k\hat{h}_k}). \end{aligned}$$

Case $i)$ $s'(h_k) = C$ *or* $s'(\hat{h}_k) = C$*:* In this situation follows that $h \in \mathcal{R}^*(s',w)$. We rewrite

$$\begin{aligned} & \sum_{h:s'(h_k)=C \text{ or} s'(\hat{h}_k)=C} U_{\delta,p}(h_{s',s^*/h_k\hat{h}_k}) + U_{\delta,p}(h_{s^*,s'/h_k\hat{h}_k}) - 2U_{\delta,p}(h_{w,w/h_k\hat{h}_k}) = \\ & \sum_{h:s'(h_k)=C \text{ or} s'(\hat{h}_k)=C} U_{\delta,p}(h_{s',s^*/h_k}) + U_{\delta,p}(h_{s^*,s'/\hat{h}_k}) - U_{\delta,p}(h_{w,w/h_k\hat{h}_k}) + \\ & \sum_{h:s'(h_k)=C \text{ or} s'(\hat{h}_k)=C} U_{\delta,p}(h_{s^*,s'/h_k}) + U_{\delta,p}(h_{s',s^*/\hat{h}_k}) - U_{\delta,p}(h_{w,w/h_k\hat{h}_k}). \end{aligned}$$

Using that $h \in \mathcal{R}^*(s',w)$, and again lemma 7 then

$$\begin{aligned} & \sum_{h:s'(h_k)=C \text{ or} s'(\hat{h}_k)=C} U_{\delta,p}(h_{s',s^*/h_k}) + U_{\delta,p}(h_{s^*,s'/\hat{h}_k}) - U_{\delta,p}(h_{w,w/h_k\hat{h}_k}) \leqslant \\ & \frac{1-p^2\delta}{p^2} \sum_{h:h\in\mathcal{R}^*(s',w)} p_{ws'}(h_k)\delta^k[S+T-2P] \end{aligned}$$

and

$$\begin{aligned} & \sum_{h:s'(h_k)=C \text{ or} s'(\hat{h}_k)=C} U_{\delta,p}(h_{s^*,s'/h_k}) + U_{\delta,p}(h_{s',s^*/\hat{h}_k}) - U_{\delta,p}(h_{w,w/h_k\hat{h}_k}) \leqslant \\ & \frac{1-p^2\delta}{p^2} \sum_{h:h\in\mathcal{R}^*(s',w)} p_{ws'}(h_k)\delta^k[S+T-2P] \end{aligned}$$

but since

$$U_{\delta,p}(w,w) - U_{\delta,p}(s',w) \geqslant \frac{1-p^2\delta}{p^2} \sum_{h:h\in\mathcal{R}^*(s',w)} p_{ws'}(h_k)\delta^k[2P-(S+P)]$$

follows that

$$\begin{aligned} & \sum_{h:s'(h_k)=C \text{ or} s'(\hat{h}_k)=C} U_{\delta,p}(h_{s',s^*/h_k}) + U_{\delta,p}(h_{s^*,s'/\hat{h}_k}) - U_{\delta,p}(h_{w,w/h_k\hat{h}_k}) \leqslant U_{\delta,p}(w,w) - U_{\delta,p}(s',w) \\ & \sum_{h:s'(h_k)=C \text{ or} s'(\hat{h}_k)=C} U_{\delta,p}(h_{s^*,s'/h_k}) + U_{\delta,p}(h_{s',s^*/\hat{h}_k}) - U_{\delta,p}(h_{w,w/h_k\hat{h}_k}) \leqslant U_{\delta,p}(w,w) - U_{\delta,p}(s',w). \end{aligned}$$

Case ii) $s^*(h_k) = C$ or $s^*(\hat{h}_k) = C$: In this situation follows that $h \in \mathcal{R}^*(s^*, w)$, and using this key statement we conclude in a similar way that

$$\sum_{h: s^*(h_k)=C \text{ or } s^*(\hat{h}_k)=C} U_{\delta,p}(h_{s^*,s'/h_k}) + U_{\delta,p}(h_{s^*,s'/\hat{h}_k}) - U_{\delta,p}(h_{w,w/h_k\hat{h}_k}) \leqslant$$
$$\leqslant U_{\delta,p}(w,w) - U_{\delta,p}(s^*,w)$$
$$\sum_{h: s^*(h_k)=C \text{ or } s^*(\hat{h}_k)=C} U_{\delta,p}(h_{s^*,s'/h_k}) + U_{\delta,p}(h_{s',s^*/\hat{h}_k}) - U_{\delta,p}(h_{w,w/h_k\hat{h}_k}) \leqslant$$
$$\leqslant U_{\delta,p}(w,w) - U_{\delta,p}(s^*,w).$$

Therefore, recalling that

$$U_{\delta,p}(w,w) - U_{\delta,p}(s^*,w) \geqslant U_{\delta,p}(w,w) - U_{\delta,p}(s',w)$$

we conclude that

$$\frac{B_{\delta,p}(s,s^*,s')}{U_{\delta,p}(w,w) - U_{\delta,p}(s^*,w)}$$

is uniformly bounded and therefore bounding (48).

9.1 Generalized w for any payoff system

Recall that w is a uniform large basin strategy, provided that $2R > S + T$. Now, we consider $w-$type strategies that are large basin strategy for any payoff system.

Definition 12. n-win-stay-loose-shift *$n-$win-stay lose-shift. If it gets either T or R stays; if it gets S, shifts to D and stays for $n-$period and then acts as w. We denote it whith w^n.*

Theorem 10. *For any payoff set there exists n such that w^n is has a uniformly large basin.*

Proof. The proof goes following the same steps that we used to prove that w is a uniform Large Basin strategy when $2R - (T + P) > 0$ but using that for any payoff matrix there exists n such that

$$nR > T + (n-1)P.$$

To show that w^n has a uniformly large basin of attraction, we calculate the quantities b_1, b_2, b_3, b_4 for $u(s, w^n)$ as it was done for w in subsection 9.0.1. In addition, observe that for w^n it follows that

$$b_2 + b_4 \geqslant \delta p^2 \frac{1-(\delta p^2)^n}{1-\delta p^2} b_3$$

and if n is large enough then $\frac{1-(\delta p^2)^n}{1-\delta p^2} > n-1$ and therefore,

$$b_2 + b_4 \geqslant (n-1)b_3.$$

Repeating the same calculation done for w, in case $w^n(h_k) = C, s(h_k) = D$ follows that

$$U(w^n, w^n) - U(s, w^n) \geqslant (n-1)b_3(R-P) + b_3(R-T) \geqslant (1-\delta p^2)[nR - T - (n-1)P].$$

In case $w^n(h_k) = D, s(h_k) = C$, the calculation is similar.

To bound uniformly the quantities (47) and (48) for w^n, we proceed in a same way that was done for w and it is only changed the upper bound $2R - (T + S)$ by $nR - T - (n-1)P$.

□

9.2 Examples of strategies with low frequency of cooperation which have large basin but they do not have uniformly large basin

In what follows, we give examples of strategies with arbitrary low frequency of cooperation which they have large basin (with size depending on δ and p), however, those strategies do not have uniformly large basin of attraction (the last assertion follows from theorem 5). In other words, the lower bounds of their basin shrinks to zero when $\delta, p \to 0$. More precisely, they can not have uniformly large basin due to theorem 5. Those strategies are built combining w with a. Moreover, we establish some relation between the frequency of cooperation and the lower bounds of the size of their local basin (but depending on δ and p).

Definition 13. *We take n large and $b_0 < 1$, we define the strategy aw^{n,b_0} as the strategy that in blocks of times $I_w^l = [l(n+m_0n), l(n+m_0n)+n-1]$ behaves as w and in the blocks of times $I_a^l = [l(n+m_0n)+n, (l+1)(n+m_0n)-1]$ behaves as a, where m_0 denotes the integer part of $\frac{1}{b_0}$ and l is a non-negative integer.*

Theorem 11. *For any n large, and any positive b_0 the strategy aw^{n,b_0} is a large basin strategy, but not a uniform large basin strategy.*

Proof. From now on, and to avoid notation, we denote aw^{n,d_0} with aw. First we are going to prove that aw is a uniform strict sgp.

The strategy aw is a uniform strict sgp: The proof is similar to the one performed for w. Let s be another strategy and given a path h let k be the first deviation ($s(h_h) \neq aw(h_k)$). Either $k \in I_w^l$ or $k \in I_a^l]$ for some non-negative l. It follows that

$$U_{\delta,p}(h_{aw,aw}/h_k) = b_0R + (1-b_0)P$$

where

$$b_0 = \frac{1-p^2\delta}{p^2} \sum_{j \geqslant 0: u^j(aw,aw/h_k)=R} = \frac{1-p^2\delta}{p^2} \sum_{j\geqslant 0, I_w^l}. \tag{52}$$

Observe that provided δ large, then b_0 is close to d_0. Now we take s and assuming that it differs in h_k and $aw(h_k) = R, s(h_k) = D$. In what follows, to avoid notation, with $U(.,.)$ we denote $U_{\delta,p,h_{.,.}}(.,./h_k)$. Following that, we take

$$b_1 = \frac{1-p^2\delta}{p^2} \sum_{j:u^j(s,aw/h_k)=R} p^{2j+2}\delta^j, \quad b_2 = \frac{1-p^2\delta}{p^2} \sum_{j:u^j(s,aw/h_k)=S} p^{2j+2}\delta^j,$$

$$b_3 = \frac{1-p^2\delta}{p^2} \sum_{j:u^j(s,aw/h_k)=T} p^{2j+2}\delta^j, \quad b_4 = \frac{1-p^2\delta}{p^2} \sum_{j:u^j(s,aw/h_k)=P} p^{2j+2}\delta^j.$$

Observe that

$$b_1 + b_2 + b_3 + b_4 = 1$$

and

$$U(s,w) = b_1R + b_2S + b_3T + b_4P.$$

Moreover, since in blocks I_a^l aw behaves as a then

$$b_4 \geqslant 1 - b_0 \tag{53}$$

From the property that aw behaves as w in blocks of the form $[l(n+m_0n),(l+1)(n+m_0n)+n]$, for each T that s can get on those blocks (s plays D and w plays C) follows that in the next move s may get either S or P because w plays D, so, noting

$$b_4^w = \frac{1-p^2\delta}{p^2} \sum_{j\in I_w^l : u^j(s,w/h_k)=P} p^{2j+2}\delta^j$$

then

$$b_4 \geqslant 1 - b_0 + b_u^w \tag{54}$$

$$b_2 + b_4^w \geqslant p^2\delta b_3. \tag{55}$$

Writing

$$U(aw,aw) = b_0R + (1-b_0)P = [b_0 - (1-b_4)]R + b_1R + b_2R + b_3R + (1-b_0)R$$

by inequalities (53, 54, 55) it follows that

$$\begin{aligned} U(aw,aw) - U(s,aw) &= [b_0 - (1-b_4)]R + b_2(R-S) + b_3(R-T) + (1-b_0-b_4)(R-P) \\ &\geqslant (b_0 + b_4 - 1 + b_2)(R-P) + b_3(R-T) \\ &\geqslant (b_4^w + b_2)(R-P) + b_3(R-T) \\ &\geqslant \delta p^2 b_3(R-P) + b_3(R-T) \\ &\geqslant b_3[(1+p^2\delta)R - (T+P)]. \end{aligned}$$

Observing that if $s(h_k)=D, aw(h_k)=C$, then

$$b_3 \geqslant 1 - p^2\delta$$

and since $2R-(T+P)>0$ it follows that for δ and p large (meaning that they are close to one), then $[(1+p^2\delta)R-(T+P)] > C_0$ for a positive constant smaller than $2R-(T+P)$ and therefore (provided that δ and p large are large) follows that

$$U(aw,aw) - U(s,aw) > (1-p^2\delta)C_0,$$

concluding that w is a uniform strict subgame perfect in the case $aw(h_k)=C, s(h_k)=D$.

In the case that $s(h_k)=C$, $aw(h_k)=D$ so we know that

$$U(aw,aw) = \frac{1-p^2\delta}{p^2}P + p^2\delta[b_0R + (1-b_0)P]$$

where b_0 is calculated as in (52), but starting from $j=1$. Calculating again the quantities b_1, b_2, b_3, b_4 but starting from $j \geqslant 1$ then we get that

$$U(s,aw) = (1-p^2\delta)S + p^2\delta[b_1R + b_2S + b_3T + b_4P].$$

Therefore, Writing

$$p^2\delta[b_0R + (1-b_0)P] = p^2\delta[b_0 - (1-b_4)]R + b_1R + b_2R + b_3R + (1-b_0)R]$$

and observing that also holds inequalities (53, 54, 55) and arguing as before it follows that

$$U(aw, aw) - U(s, aw) \geqslant (1 - p^2\delta)(P - S) + \delta b_3[(1 + \delta)R - (T + P)]$$

since $2R - (T + P) > 0$ it follows that for δ large (b_3 now can be zero)

$$U(aw, aw) - U(s, aw) > (1 - p^2\delta)(P - S),$$

proving that aw is a uniform strict subgame perfect in the case $aw(h_k) = D, s(h_k) = C$.

The strategy aw verifies the asymptotic bounded condition, but depending on δp^2: Bounding (47) and (48) for aw: To bound $U_{\delta,p}(s, s) - U_{\delta,p}(s, aw)$ we repeat the argument done for w and observe that the key point is that $U(aw, aw/h_k) - U(s, aw/h_k) > b_3((1 + \delta)R - (T + S))$ which has been proved when is proved that aw is a uniform strict sgp.

To bound (48) we perform the same approach for w, however the estimates changes depending on d_0. More precisely, given s' and s^* follows that

$$B(s', s^*, aw) \leqslant 2(1 - d_0)(R - P),$$

therefore, arguing as in the case of w follows that

$$\frac{B_{\delta,p}(s', s^*, aw)}{N_{\delta,p}(aw, s^*)} \leqslant \frac{2(1 - d_0)(R - P)}{(1 - p^2\delta)(P - S)}.$$

The strategy aw is not a uniform large basin strategy: It follows from the fact that aw is symmetic but no efficient.

□

10 Perturbed Replicator Dynamics

We consider more general equation than the replicator dynamics with the solely restrictions that *individual with low scores dies off and the ones with high ones flourish.* More precisely, given a payoff matrix A we consider equations defined in the usual $n-$dimensional simplex $\sum$, of the form

$$\dot{x}_i = x_i G_i(x)$$

such that

$$G_i(x) > 0, \quad \text{if and only if } \ (Ax)_i - x^t Ax > 0$$

$$G_i(x) < 0, \quad \text{if and only if } \ (Ax)_i - x^t Ax < 0.$$

In this case, it follows that

$$G_i(x) = [(Ax)_i - x^t Ax]H_i(x) \tag{56}$$

where $H_i : \sum \to \mathbb{R}$. Moreover, form previous assumption it holds that H_i is always positive in the simplex $\sum$. We require a slightly strong condition: $C^+ = \max\{H_i(x), x \in \sum, i = 1 \dots m\} < +\infty$, and $C^- = \min\{H_i(x), x \in \sum, i = 1 \dots m\} > 0$.

Then

$$0 < C^- \leqslant H_i < C^+. \tag{57}$$

The goal is to show that a version of theorem 1 can be obtained in the present case. More precisely, provided the hypothesis of theorem 1 and assuming equations as above, it is shown that

$$\Delta_{\frac{1}{M_0}} \cap \Delta_{\frac{C^-}{2C^+}}$$

is contained in the local basin of attraction of e_1. The proof, goes through the same strategy: we shows that for any $k \leqslant \min\{\frac{1}{M_0}, \frac{C^-}{2C^+}\}$, re-writing the equations in affine coordinates follows that

$$\sum_{i\geqslant 2} x_i G_i = \sum_{i\geqslant 2} x_i F_i H_i < 0$$

where F_i is $(Ax)_i - x^t Ax$ in affine coordinates. From inequalities (57) it follows that

$$x_i F_i(x) H_i(x) < C^+ x_i F_i(x), \text{ if } F_i(x) > 0; \;\; x_i F_i(x) H_i(x) < C^- x_i F_i(x), \text{ if } F_i(x) < 0.$$

Recalling that $F_j(x) = (f_j - f_1)(x) + R(x)$ with $R(x) = \sum_l (f_1 - f_l)(x) x_l$ (the variable x is already assumed in affine coordinates) follows that

$$\begin{aligned} \sum_i x_i F_i(x) H_i(x) &\leqslant \sum_{\{i:F_i(x)>0\}} C^+ x_i F_i(x) + \sum_{\{i:F_i(x)<0\}} C^- x_i F_i(x) \\ &= \sum_{\{i:F_i(x)>0\}} x_i C^+ (f_i - f_1)(x) + \sum_{\{i:F_i(x)<0\}} x_i C^- (f_i - f_1)(x) \\ &+ R(x)[\sum_{\{i:F_i(x)>0\}} C^+ x_i + \sum_{\{i:F_i(x)<0\}} C^- x_i]. \end{aligned}$$

If $x \in \Delta_k$ with $k < \frac{C^-}{2C^+}$ it follows that $\sum_{\{i:F_i(x)>0\}} C^+ x_i + \sum_{\{i:F_i(x)<0\}} C^- x_i] \leqslant \frac{C^-}{2}$ and recalling the definition of R_0 follows that

$$\begin{aligned} &\sum_{\{i:F_i(x)>0\}} x_i C^+ (f_i - f_1)(x) + \sum_{\{i:F_i(x)<0\}} x_i C^- (f_i - f_1) + R(x)[\sum C^+ x_i + \sum C^- x_i] \leqslant \\ &\sum_{\{i:F_i(x)>0\}} x_i \hat{C}^+ (f_i - f_1)(x) + \sum_{\{i:F_i(x)<0\}} x_i \hat{C}^- (f_i - f_1) \end{aligned}$$

where $\hat{C}^+ = C^+ - \frac{C^-}{2}, \hat{C}^- = \frac{C^-}{2}$. Therefore, rewriting the equation as it was done in the proof of theorem 1 to finish we have to prove that

$$N(cx) + x^t M(cx) < 0 \tag{58}$$

where $cx = (c_1 x_1, c_2 x_2, \ldots, c_n x_n)$ and c_i is either $\hat{C}^+$ or $\hat{C}^-$ and N, M are the vector and matrix induce by A and so. To prove (58), we need a more general version of lemma 10. The proofs are similar.

Lemma 18. *Let $c = (c_1, \ldots, c_m) \in \mathbb{R}^m$ such that each coordinate is positive. Let $Q_c : \mathbb{R}^m \to \mathbb{R}$ given by*

$$Q(x) = N(cx) + x^t M(cx)$$

with $x \in \mathbb{R}^m$, $N \in \mathbb{R}^m$, $M \in \mathbb{R}^{m\times m}$ and $cx := (c_1 x_1, \ldots, c_m x_m)$. Let us assume that $N_i < 0$ for any i and for any $j > i$, $|N_i| \geqslant |N_j|$. Let

$$M_0 = \max_{i,j>i}\{\frac{M_{ij} + M_{ji}}{-N_i}, \; \frac{M_{ii}}{-N_i}, \; 0\}.$$

Then, the set $\Delta_{\frac{1}{M_0}} = \{x \in \mathbb{R}^m : x_i \geqslant 0, \sum_{i=1}^m x_i < \frac{1}{M_0}\}$, is contained in $\{x : Q_c(x) < 0\}$. In particular, if $M_0 = 0$ then $\frac{1}{M_0}$ is treated as ∞ and this means that $\{x \in \mathbb{R}^m : x_i \geqslant 0\} \subset \{x : Q_c(x) \leqslant 0\}$.

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