CONVENTIONAL AND UNCONVENTIONAL MONETARY POLICY WITH ENDOGENOUS COLLATERAL CONSTRAINTS

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Abstract. In this paper we consider a finite horizon model with default and monetary policy. In our model, each asset promises delivery of either one unit of money in the next period or of a real collateral payoff in case of default. Moreover, we include the possibility of purchases of the collateralizable durable good by the central bank. This is done in order to add an additional dimension of monetary policy, which may be of particular relevance when the interest rate reaches the zero lower bound (as has happened in the recent financial crisis). In order to implement such a policy, we must additionally include a rental market for the durable good (so that durables purchased by the central bank can still be used by agents).

We examine through numerical examples how the purchases of the central bank affect the equilibrium allocations of resources, and under which conditions such a policy can lead to a Pareto-improvement.

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1. Introduction

In response to the recent financial crisis, central banks cut interest rates sharply, and in some cases (as with the U.S. Federal Reserve) essentially to the lower bound of a zero nominal rate. The inability to use conventional interest-rate policy more aggressively, under circumstances of low real activity and declining inflation, led many central banks to resort to "unconventional" policies, involving purchases of assets by the central bank, or extensions of credit to private institutions, beyond those required to implement the central bank's target for the short-term nominal interest rate [CWo].

In this paper, we extend a general equilibrium model with collateral (GEIC) of the kind developed by [GZ] to include money and purchases of a durable good by the central bank. In our model, each asset promises delivery of one unit of money in the next period and a real collateral payoff in case of default. Each agent begins with an initial endowment of money, and additional money can be issued by the central bank to finance asset purchases. Conventional monetary policy (interest-rate policy) is introduced by allowing the central bank to specify the nominal interest rate on its liabilities: one unit of money held today becomes a claim to a certain number of units of money tomorrow, regardless of the states of nature.[1] Because we work for simplicity with a finite-horizon model, the value of money in the final period is determined by redemption of the outstanding money stock for goods

[1] In actual economies, the central bank's policy rate is distinct from the interest rate paid on reserve balances at the central bank (the "money" in our model), and the central bank influences the value of that policy rate *both* by varying the interest rate paid on reserves and by varying the differential between the two interest rates through variation in the supply of reserves. The latter aspect of policy (typically emphasized in textbook discussions of monetary policy) depends on a reason for reserves to be valued more than other assets paying the same pecuniary return in each state; for simplicity, we abstract from such non-pecuniary returns to money here, and allow interest-rate policy to be implemented purely through variations in the rate of interest paid on money, as in the "cashless" model proposed in [Wo] chapter 2. For an example of a model in which the policy rate can also be influenced by varying the supply of reserves, see for example [CWo].

by the central bank; the revenue required to redeem the money supply is raised through lump-sum taxation.

In this model, we also consider the possibility of purchases of the durable good by central bank, paid for by issuance of money. In the recent crisis, the market value of the most important collateralizable durable good (real estate) fell significantly, resulting in increased defaults by debtors and significant losses for financial institutions. Declines in the market value of other types of collateral (financial claims used as collateral for new borrowing by financial institutions) also played an important role in the crisis, and one possible goal of "unconventional" policy might be to raise the market value of such collateral. (In practice, such policies were addressed more to the value of collateralizable financial claims, but in our model the only form of collateral is a real durable good, and so we consider the possibility of affecting the equilibrium value of that good.) In fact, it is not obvious that central-bank purchases must change equilibrium asset prices; [Wa] presents a general-equilibrium analysis in which open-market purchases by the central bank have no effect on equilibrium asset prices or on the allocation of resources. As [CWo] discuss further, in a GEI model without collateral constraints, both the size and composition of the central-bank balance sheet are irrelevant, though the existence of additional financial constraints can break the irrelevance result. In our model, the existence of the collateral constraint breaks the irrelevance result, and we find in most cases that central-bank purchases do affect the equilibrium price of the durable good.

In section 2, we first present a two-period flexible price model and show that in the "cashless limit" of this model (when money balances are negligible), variations in interest-rate policy have no effect on the equilibrium allocation of resources. With sticky prices (our second model), instead, interest-rate policy affects the equilibrium allocation, even in the cashless limit. In this second model, there is an arbitrarily given predetermined price level for the non-durable good in the initial period; but it may not be realistic to suppose that we can choose any monetary policy we like without the anticipation of that policy having had an effect on the way that prices were set. Hence we consider a third model, in which the non-durable good price and supply commitments are endogenized (modeled as being chosen before agents learn the period 0 state of the world). Here we consider different possible states in the first period, but assume that the values of non-durable price and supply commitments are chosen prior to the realization of the state, and so are the same for all states in the first period. We examine through numerical examples how the welfare-maximizing choice of interest-rate policy (in each state in the first period) is different depending on the severity of collateral constraints. We present an example in which in the bad state (in which collateral constraints are binding) the optimal nominal interest rate is zero, so that the interest-rate lower bound is a binding constraint. This makes the question of the efficacy of alternative dimensions of policy particularly interesting.

In section 3, we consider the effects of purchases of the collateralizable durable good by the central bank in the two-period flexible-price model. We show that when markets are endogenously incomplete and collateral constraints are binding, this additional dimension of monetary policy affects equilibrium asset prices. We then consider the welfare effects of the use of this kind of policy, in some simple numerical examples involving two types of agents (rich and poor). In the numerical examples presented in section 4, we generally find that the poor agent gains from purchases of durable good by the central bank, while the rich agent loses; thus the policy is not irrelevant, but cannot achieve a Pareto improvement. We also find that the distribution of taxes and transfers (and hence the distribution of net central-bank earnings from trading), and in particular the way that this depends on the states of nature, is quite important for the equilibrium effects of the central-bank purchases. When the distribution of taxes is the same across states, central-bank purchases of the durable good can lead to Pareto-improvements, given that the utility of the rich agent is little affected, while it is possible to improve the utility of the poor agent to a significant extent through purchases of the durable in the bad state where the zero lower bound constrains interest-rate policy.

2. A two period model with one dimension of monetary policy

We first consider a two period model, which includes money in a general equilibrium model with collateral. In this model, we examine the effect of the monetary policy, through interest rate. We show that in the cashless limit (money supply goes to zero), variations in interest rate policy have no effect on the equilibrium. If we consider that prices are sticky, interest-rate policy effects the equilibrium allocation of resources, even in the cashless limit.

2.1. **Basic monetary model: flexible prices.** We consider a pure exchange economy over two time periods $t = 0, 1$ with uncertainty over the state of nature in period 1 denoted by the subscript $s \in \mathcal{S} = \{1, \ldots, S\}$.

The economy consists of a finite number H of agents denoted by the superscript $h \in \mathcal{H} = \{1, \ldots H\}$ and $L = 2$ goods or commodities, denoted by the subscript $l \in \mathcal{L} = \{1, 2\}$. Throughout the analysis we assume that good 1 is perishable and good 2 is durable, i.e. there is a possibly risky (and possibly productive) storage technologies, represented by $Y_s \in \mathbb{R}^2_+$. Using one unit of the durable good in the first period yields Y_{sl} units of good l in state s. The most natural assumption is of course that $Y_s = (0, 1)$ for each state s, i.e. it is possible to store the durable good without depreciation. Each agent has an initial endowment of the goods in each state, $e^h \in \mathbb{R}^{2(S+1)}_+$. The preference ordering of agent h is represented by a utility function $u^h : \mathbb{R}^{2(S+1)}_+ \to \mathbb{R}$, defined over consumption $x^h = (x^h, x^h_1, \ldots, x^h_S) \in \mathbb{R}^{2(S+1)}_+$.

Money: each agent h has an endowment of money in the amount m^h in period 0, where $\sum_h m^h = M > 0$ is the money supply. Monetary policy also specifies the nominal interest rate i: one unit of money hold in period 0 becomes a claim to $(1 + i)$ units of money in period 1, regardless of the states s. Finally, monetary policy specifies the redemption value of money in each state s in period 1. Each unit of money is redeemed for a specified (positive) number of units of good 1 in state s; then for each state s, the price p_{s1} of good 1 in units of money is fixed by monetary policy. The revenues required to redeem the money supply are raised through lump-sum taxation. The share of taxes raised from each household h in state s is $\theta_s^h \geq 0$, where $\sum_h \theta_s^h = 1$ for each state s. Hence the tax obligation of household h in state s (in units of good 1) is $\theta_s^h M(1+i)/p_{s1}$; or $\theta_s^h M(1+i)$ in units of money.

The fact that the money is redeemed for goods in the terminal period is a way of representing the fact that in an actual economy (with a terminal period), the value of money each period is determined by monetary policy (in that period and later). Since there is no interest rate in the terminal period, and no value of end-of-period money balances, the only way that monetary policy can determine the price level in this period is by specifying the redemption value of money, as under a commodity money regime.

Additional assets: each asset j promises delivery of one unit of money in period 1, regardless of the state s. The collateral requirements for asset j is $C_j \geq 0$; any agent has to hold C_j units of good 2 in period 0 in order to sell 1 unit of asset j. Given the possibility of default the actual payoff of asset j in state s is $\min(1, p_{s2}C_j)$ in units of money, here p_{s2} is the spot price of good 2 (in units of money) in state s, period 1.

Given $p \in \mathbb{R}_{++}^{2(S+1)}$, and $q \in \mathbb{R}_+^J$ the agent h chooses consumption, portfolios and money $(x^h, \psi^h, \varphi^h, \mu^h)$, to maximize utility subject to the budget constraints.

$$\begin{aligned}
&\max_{x\geq 0, \psi\geq 0, \varphi \geq 0, \mu\geq 0} u^h(x^h) \quad \text{s.t.} \\
&p \cdot (x^h - e^h) + q \cdot (\psi - \varphi) + \mu^h - m^h \leq 0; \\
&p_s \cdot (x_s^h - e_s^h) - p_{s2}x_2^h - \sum_{j\in\mathcal{J}} (\psi_j^h - \varphi_j^h) \min\{1, p_{s2}C_j\} + (1+i)(\theta_s^h M - \mu^h) \leq 0; \quad \forall s \in \mathcal{S} \\
&x_2^h - \sum_{j\in\mathcal{J}} \varphi_j^h C_j \geq 0.
\end{aligned}$$

(2.1)

A competitive equilibrium is defined as usual by agents' optimality and market clearing.

Definition 1. An equilibrium for the economy E is a vector $[(\overline{x}, \overline{\psi}, \overline{\varphi}, \overline{\mu}); (\overline{p}, \overline{q})]$, such that:

(i) $(\overline{x}^h, \overline{\psi}^h, \overline{\varphi}^h, \overline{\mu}^h)$ solves problem 2.1.

(ii) $\sum_{h=1}^{H}(\overline{x}^h - e^h) = 0$

(iii) $\sum_{h=1}^{H}(\overline{x}^h_{s1} - e^h_{s1}) = 0$ and $\sum_{h=1}^{H}(\overline{x}^h_{s2} - e^h_{s2} - \overline{x}^h_2) = 0$, $s = 1, \ldots, S$

(iv) $\sum_{h=1}^{H}(\overline{\psi}^h - \overline{\varphi}^h) = 0$

(v) $\sum_{h=1}^{H}\overline{\mu}^h - M = 0$

In this model changes in the quantities of money, or interest rate or taxation will have effects on equilibrium, but if we consider the limiting case in which $M \to 0$ we show that money is neutral.

"Cashless limit": The analysis is simplified if we consider the limiting case in which $M \to 0$, so that we can abstract from redistributive effects of changes in the nominal price level. In this case, monetary policy is simply specified by $1 + i > 0$ and the $\{p_{s1}\}$. The initial endowments m^h are also set to zero, and the specification of the tax shares $\{\theta^h_s\}$ no longer matters. In this case, the economy is completely characterized by the agents' utility functions $\{u^h\}$, their endowments $\{e^h\}$, the asset structure $\{C_j\}$, and the monetary policy $(i, \{p_{s1}\})$. The following proposition shows that monetary policy is neutral in the cashless limit.

Proposition 1. ***Neutrality of money:*** *In the flexible-price model, with cashless limit [$M \to 0$, and $m^h \to 0 \quad \forall h$ as a convergence], variations in interest-rate policy have no effect on the equilibrium allocation of resources.*

Proof. the household by problem, and hence all equilibrium conditions, can be written entirely in terms of quantities $\frac{\mu}{p_1}, \frac{p_2}{p_1}, \frac{q}{p_1}$, and$(1+i)p_1$, with no reference to p_1 or i individually. Hence the real allocation $\overline{x}$, the portfolios, and relative prices are all determined by data that do not involve i. It follows that changing i changes the value of p_1, so as to keep $(1+i)p_1$, the same, but has no effect in the real allocation or on relative prices. □

Central-bank balance sheet: It may be more realistic to suppose that the money supply M represents liabilities of a central bank, which has corresponding assets on its balance sheet. [This is of interest if one wishes to consider the effects of alternative kinds of asset purchases by the central bank.] The above model can be equivalent desenhed as one in which the central bank holds riskless government debt as an asset, of the same value as its liabilities M. In equilibrium, the riskless government debt must earn the same interest rate i as the liabilities of the central bank, so the central bank has no income or losses in period 1. The taxes in period 1 can now be thought of as being levied in order to pay off the government debt in period 1. Later, we generalize the model to allow the central bank to hold other assets as well. In this case, the central bank can have income or losses on its portfolio, which are distributed to the Treasury, and so affect the state-contingent taxes that must be collected from the agents.

2.1.1. *Example: Cashless limit in the flexible price model.* We consider a simple example with two states in period 1 $S = 2$, two agents $H = 2$ and two assets $J = 2$. Each individual $h =$ has a utility function of the form:

$$u^h(x) = \log(x_1) + \log(x_2) + \frac{1}{2}\sum_{s=1}^{2}(\log(x_{s1}) + \log(x_{s2}))$$

Endowments are:

$$e^1 = (e_1^1, e_2^1, e_{11}^1, e_{12}^1, e_{21}^1, e_{22}^1) = (4, 1, 4, 0, 4, 0);$$

$$e^2 = (e_1^2, e_2^2, e_{11}^2, e_{12}^2, e_{21}^2, e_{22}^2) = (2, 1, 6, 0, 2, 0).$$

To illustrate the cashless limit in the flexible price model, we consider several quantity of money supply in which $M \to 0$ $M = [1; 0.8; 0.5; 0.1; 0.001]$.

Money m^h in the first period are: $m^1 = 0.7 * M$; $m^2 = 0.3 * M$.

Taxes θ_s^h are: $\theta^1 = (\theta_1^1, \theta_2^1) = (0.5, 0.5)$; $\theta^2 = (\theta_1^2, \theta_2^2) = (0.5, 0.5)$.

In the second period the price of non-durable good p_{s1} is fixed: $p_{11} = 1$; $p_{21} = 1$.

The collateral requirements are: $C_1 = 0.2$; $C_2 = 0.333$.

In the Figures 1 and 2, we show a few different points in utility space with random choice of the interest rate between 0 and 1.

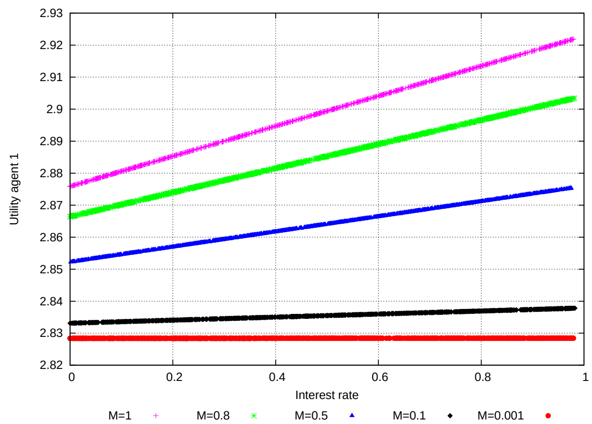


Figure 1. Interest rate effect when $M \to 0$ on agent 1

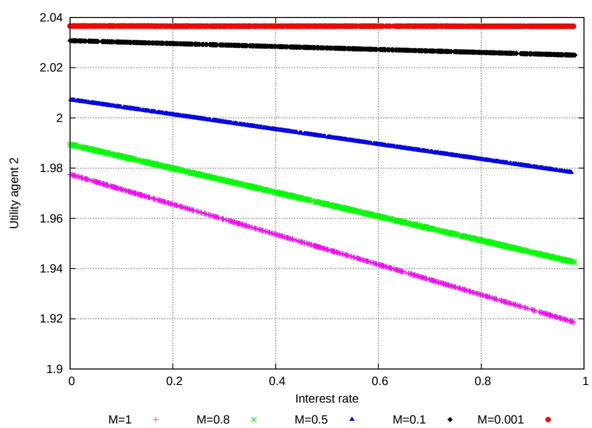


Figure 2. Interest rate effect when $M \to 0$ on agent 2

In the flexible-price model cashless limit ($M \to 0$, and $m^h \to 0\,\forall h$ as a convergence), variations in interest-rate policy have no effect on the equilibrium allocation of resources. Real allocation $\overline{x}$, the portfolios, and relative prices are all determined by data that do not involve i. It follows that changing i changes the value of p_1, so as to keep $(1+i)p_1$, the same, but has no effect in the real allocation or on relative prices. We show in the next section that with sticky prices, interest-rate policy affects the equilibrium allocation even in the cashless limit.

2.2. **Sticky price model.** We now suppose instead that the price p_1 of the non-durable good in period 0 is given [set some time prior to date $t = 0$], and suppliers of the good must supply whatever quantity is demanded at that price, rather than having a fixed endowment of the good to sell, regardless of the market price. Households choose the quantity that they wish to buy of the good at this price; we suppose they are able to buy any amount that they choose, consistent with their budget constraints. A rationing rule specifies how much of the total demand for the non-durable good must be supplied by each supplier. [In the model of endogenous price setting proposed below, buyers commit themselves in advance to supply a given share of the market, whatever that quantity may prove to be. In another formulation, the goods supplied by different suppliers are imperfect substitutes, and consumers distribute their purchases over suppliers accordingly, given the prices fixed in advance by each of the suppliers. The formulation proposed here economizes on notation, by not having to allow for different prices in the case of suppliers with different marginal disutilities of supplying goods or different needs for income.]

The prices of non-durable goods in period 1 are instead assumed to be flexible [determined to clear competitive markets]. The idea is that period 1 represents "the long run", in which monetary policy affects only the price of goods in terms of money, but not real aggregate demand. [This can however be easily generalized, if one wishes to consider real effects of monetary policy decisions in period 1 as well.] The prices of durable goods are flexible in both periods, as are the prices of all assets.

Notation: let σ^h be the fraction of the demand for non-durable goods in period 0 that must be supplied by household h, where $\sigma^h \geq 0$ for each h, and $\sum_h \sigma^h = 1$. [At present, both the predetermined price p_1 and the supply commitments $\{\sigma^h\}$ are part of the specification of the model. Below, we consider a more elaborate model in which they are endogenized.] Each household has an endowment (non-negative) of each of the other $2S+1$ goods, as before. Preferences of household h are defined by a utility function $u^h(x^h) - v^h(y^h)$, where $y^h \geq 0$ is the quantity of the non-durable good that is supplied by household h in period 0. [Additive separability is not essential, but allows us to write the consumer demand problem in terms of the same preferences as before.]

Given $p \in \mathbb{R}^{2(S+1)}_{++}$, $q \in \mathbb{R}^J_+$ and aggregate demand $y \in \mathbb{R}_+$ for the non-durable good in period 0 the agent h chooses consumption, portfolios and money $(x^h, \psi^h, \varphi^h, \mu^h)$, to maximize utility subject to the budget constraints.

$$\max_{x\geq 0, \psi \geq 0, \varphi \geq 0, \mu \geq 0} u^h(x^h) \quad \text{s.t.}$$

$$p_1(x_1^h - \sigma^h y) + p_2(x_2^h - e_2^h) + q_j \cdot (\psi - \varphi) + \mu^h - m^h \leq 0;$$

$$p_s \cdot (x_s^h - e_s^h) - p_{s2}x_2^h - \sum_{j\in\mathcal{J}} (\psi_j^h - \varphi_j^h) \min\{1, p_{s2}C_j\} + (1+i)(\theta_s^h M - \mu^h) \leq 0; \quad \forall s \in \mathcal{S}$$

$$x_2^h - \sum_{j\in\mathcal{J}} \varphi_j^h C_j \geq 0.$$

(2.2)

A competitive equilibrium is defined as usual by agents' optimality and market clearing.

Definition 2. An equilibrium for the economy E is a vector $[(\overline{x}, \overline{\psi}, \overline{\varphi}, \overline{\mu}); (\overline{p}, \overline{q}); \overline{y}]$ where $\overline{p}_1$ is the given (predetermined) price, that is part of the data delivery the economy, such that:

(i) $(\overline{x}^h, \overline{\psi}^h, \overline{\varphi}^h, \overline{\mu}^h)$ solves problem 2.2.

(ii) $\sum_{h=1}^H \overline{x}_1^h - \overline{y} = 0$

(iii) $\sum_{h=1}^H (\overline{x}_2^h - e_2^h) = 0$

(iv) $\sum_{h=1}^H (\overline{x}_{s1}^h - e_{s1}^h) = 0$ and $\sum_{h=1}^H (\overline{x}_{s2}^h - e_{s2}^h - \overline{x}_2^h) = 0$, $s = 1, \ldots, S$

(v) $\sum_{h=1}^H (\overline{\psi}^h - \overline{\varphi}^h) = 0$

(vi) $\sum_{h=1}^{H} \overline{\mu}^h - M = 0$

Since $\overline{p}_1$ is predetermined, and $\overline{p}_{s1}$ is specified for each s by monetary policy, only $\overline{p}_2$ is endogenously determined to satisfy the above requirements.

Under the assumption of additive separability made here, the functions $v^h(y^h)$ are irrelevant to equilibrium determination. However, they matter for welfare evaluation of alternative monetary policies. [They also matter for the endogenization of the predetermined price level, treated below.]

As shown in Proposition 1, interest-rate policy has real effects in the flexible-price model only to the extent that it changes the distribution across households of real cash balances $\frac{m^h}{p_1(0)}$ or of tax obligations $\theta^h(s)M(1+i)$. The latter effects are not obviously of quantitative relevance in actual economies; hence the usefulness of considering the cashless limit. With sticky prices, instead, interest-rate policy effects the equilibrium allocation of resources even in the cashless limit.

2.2.1. *Example: Cashless limit of the sticky price model.* We consider a similar economy as in the example of section (2.1.1), but with the following modifications. Each individual h has a utility function of the form:

$$\Gamma^h = u^h(x) - v^h(\sigma^h y)$$

where $u^h(x)$ has the same logarithmic form as to Example 1; and $v^h(\sigma^h y) = (\sigma^h y)^\eta$, with $\eta = 1$. Endowments are:

$$e^1 = (e_2^1, e_{11}^1, e_{12}^1, e_{21}^1, e_{22}^1) = (1, 4, 0, 4, 0);$$

$$e^2 = (e_2^2, e_{11}^2, e_{12}^2, e_{21}^2, e_{22}^2) = (1, 6, 0, 2, 0).$$

Rationing rule for the supply of non-durable goods in period 0: $\sigma^1 = 0.67$ and $\sigma^2 = 0.33$. Finally, the price of non-durable good in period 0 is given by $p_1 = 1$.

Unlike what we showed for the flexible-price model, with sticky prices the interest rate effects the equilibrium allocation of resources even in the cashless limit (see Figures 3 and 4).

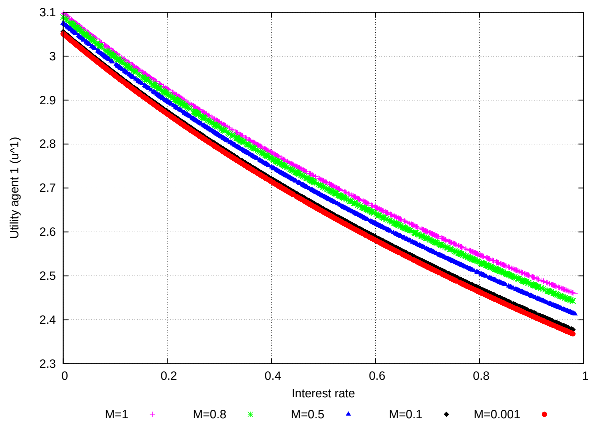


FIGURE 3. Interest rate effect when $M \to 0$ on agent 1

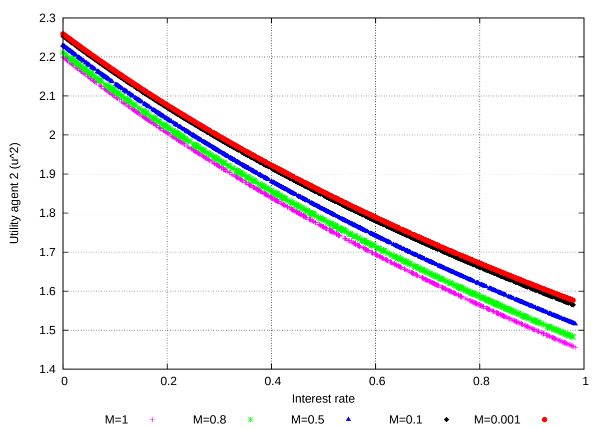


FIGURE 4. Interest rate effect when $M \to 0$ on agent 2

It may be considered undesirable to specify the predetermined price level p_1 and the supply commitments $\{\sigma^h\}$ arbitrarily. These can in fact be endogenized, if we imagine them to have been chosen before agents learn the period 0 state of the world.

2.2.2. *Extended model.* Now, consider that there exist N different possible states of the world in period 0, $n = 1, ...N$. These states have ex-ante probabilities $\pi_n > 0$ such that $\sum_n \pi_n = 1$. For each state n, there is a two period economy specified by (u^h, v^h, e^h, C) and a monetary-fiscal policy specified by $(\{m^h, \theta^h\}, i, \{p_{s1}\})$. Neither the economy nor the monetary-fiscal policy need be the same for different realizations of the state n. However, the values of p_1 and the $\{\sigma^h\}$ are chosen prior to the realization of the state n, so these are the same for all n. The fact that these are the same for all n is the only assumed link between the economies corresponding to different states n. [Of course, we could also introduce markets for risk-sharing across the states n: this would mean moving to a full-fledged 3-period model.] Once the state n is realized, the equilibrium concept is the same as the one defined above. The share of taxes raised from each household h in state n and s is $\theta^h_{ns} \geq 0$, where $\sum_h \theta^h_{ns} = 1$ for each state n and s.

We may suppose that p_1 and the $\{\sigma^h\}$ are chosen ex ante as follows. An auctioneer proposes different prices p_1, and each of the households h chooses a share σ^h of demand that it is willing to supply in each of the states n. There is assumed to be no possibility of offering to supply different shares in different states; this is because a single auction is determining a single "market-clearing" price to apply regardless of the state n.

Given a predetermined price p_1 [independent of n]; an interest rate i_n for each state n; and goods prices $(p_{n2}, \{p_{nsl}\})$, asset prices q_n and aggregate demand y_n for each of the states n; agent h chooses a value of σ^h [independent of n] and a plan $(\mu_n, x_n, \psi_n, \varphi_n)$ for each of the states n, to maximize:

$$\sum_n \pi_n [u^h(x^h_n) - v^h(\sigma^h y_n)]$$

subject to the same set of constraints as above (Problem 2.2) for each of the states n.

Definition 3. An equilibrium for the economy E is a vector (p_1, σ) and vectors $[(\overline{x}_n, \overline{\psi}_n, \overline{\varphi}_n, \overline{\mu}_n); (\overline{p}_n, \overline{q}_n); \overline{y}_n]$ for each of the states n, such that:

(i) for each h, σ^h and $(\overline{x}^h_n, \overline{\psi}^h_n, \overline{\varphi}^h_n, \overline{\mu}^h_n)$ solves problem of individual h;
(ii) market-clearing conditions (ii)-(vi) above (Definition 2) are satisfied for each n;
(iii) $\sum_{h=1}^{H} \sigma^h - 1 = 0$

Note that in the special case that $N = 1$, this concept of equilibrium reduces to the flexible-price monetary equilibrium. It differs from the equilibrium definition above only in that there is now an endogenous supply of the non-durable good by each household in period 0, rather than a fixed endowment. [The case of a fixed endowment would of course be recovered as a limiting case of this model.] Thus when $N = 1$, variations in interest-rate policy have no effect in the equilibrium allocation of resources, as shown above.

What can be asked in this model: Suppose that the degree of collateral available is different in different states n. One can then consider the consequences of specifying monetary policy (for example, the value of i_n) to be different in different states n as well. In this analysis, the state-contingent character of monetary policy will be anticipated by households in choosing their supply commitments, and so in the determination of the predetermined price p_1. Because monetary policy is anticipated when p_1 is determined if $N = 1$ the specification of i has no effect on the real equilibrium allocation (only on the value of p_1). But if there are multiple possibilities ($N > 1$), then in general the specification of state-contingent interest-rate policy does matter. We wish to address the following question: How is the welfare-maximizing choice of $\{i_n\}$ different depending what one assumes about the severity of the collateral constraints? One can fix preferences $\{u^h, v^h\}$ and the endowments $\{e^h\}$ of the $2S + 1$ goods other than the non-durable good in period 0 [for each state n], but consider the effects of varying the collateral requirements $\{C_j\}$.

2.2.3. *Example: Interest rate policy in the extended model.* We consider an example with two states in period 0, $N = 2$; two states in period 1, $S = 2$; and two agents, $\mathcal{H} = \{1, 2\}$.

Each individual h has a utility function of the form as example above in the section (2.2.1) for each of the states n.

Ex ante probabilities are: $\pi^1 = 0.6$; and $\pi^2 = 0.4$.

Endowments are:

$e_1^1 = (0.1)$, $e_2^1 = (1.8)$, for l=2 $\quad e_{11l}^1 = (2, 0)$, $\quad e_{12l}^1 = (5, 0)$, $\quad e_{21l}^1 = (3, 0)$, $\quad e_{22l}^1 = (6, 0)$;

$e_1^2 = (0.9)$, $e_2^2 = (0.2)$, for l=2 $\quad e_{11l}^2 = (5, 0)$, $\quad e_{12l}^2 = (3, 0)$, $\quad e_{21l}^2 = (7, 0)$, $\quad e_{22l}^2 = (3, 0)$.

Money m_n^h in the first period are:

$$m_1^1 = 0.5, \quad m_2^1 = 0.6;$$

$$m_1^2 = 0.5, \quad m_2^2 = 0.4.$$

Money supply: $M_1 = 1$ and $M_2 = 1$.

Tax θ_{ns}^h is:

$$\theta^1 = (\theta_{11}^1, \theta_{12}^1, \theta_{21}^1, \theta_{22}^1) = (0.52, 0.50, 0.70, 0.45);$$

$$\theta^2 = (\theta_{11}^2, \theta_{12}^2, \theta_{21}^2, \theta_{22}^2) = (0.48, 0.50, 0.30, 0.55).$$

The price is fixed $p_{ns1} = 1 \quad \forall n, \quad \forall s$.

There are four assets, $J = 4$.

In this example we consider that all agents have identical homothetic utility, then prices in the period 1 do not change. According to [AKS] (Proposition 2) the level of collateral requirements defined by $C_j = 1/p(s)_2$ the markets chooses the asset structure efficiently. In our case the level de collateral requirements is defined by $C_j = 1/p_n(s)_2$, where $C_1 = 0.143$, $C_2 = 0.125$, $C_3 = 0.2$ and $C_4 = 0.222$ in this example.

We solved several samples with random choice of the nominal interest rate (i_1 and i_2) between 0 and 1.

The interest-rate policy that maximizes utility for agent 1 is $i_1 = 0$ and $i_2 = 0.23$ (see Figure 10), while the policy best for agent 2 is $i_1 = 0$ and $i_2 = 0.16$ (see Figure 11). The plot with the contours for both households in the same plane is the Figure 7. In this example, for the first state the optimal policy is a zero interest rate. It is especially interesting in this case to consider an alternative instrument of monetary policy.

3. A finite horizon model with two dimensions of monetary policy

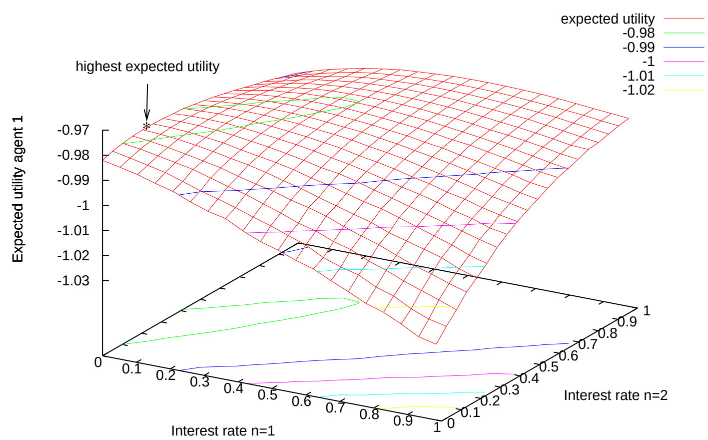
We now include the possibility of purchases of the collateralizable durable good by the central bank, which could be particularly interesting to implement when the interest rate reaches the zero lower bound (as we show in the example above in the section 2.2.3). First, we show when this alternative monetary policy has effect on the equilibrium allocations of resources, in a two period model with flexible prices. Moreover, we examine a flexible prices three period model with purchases of the collateralizable durable good by the central bank in different states in the second period.

3.1. **A simple case.** We modify the model presented in section 2.1 to include a rental market for durable services and purchases of the collateralizable durable good by the central bank. In this version of the model, there are three goods $l \in \mathcal{L} = \{1, 2, 3\}$ where good 1 is perishable, good 2 is [rental of] durable services and good 3 is [purchases of] the durable good. We assume that it is possible to store only the durable good and that there is no depreciation.

The central bank can purchase durable good in the first period, in an amount given by $\omega \sum_h e_3^h$ where $\omega \in (0, 1]$ represents the fraction of aggregate durable good supply purchased by the central bank. The central bank also issues money in the amount needed to pay for these purchases, so that $M = (p_3 - p_2)\omega \sum_h e_3^h$. The net tax obligation of household h in state s in the second period is

$$\theta_s^h[(1 + i)(p_3 - p_2)\omega \sum_h e_3^h - p_{s3}\omega \sum_h e_3^h],$$

where the quantity in square brackets is the amount by which the amount owed by the central bank on its liabilities exceeds the value of its assets.



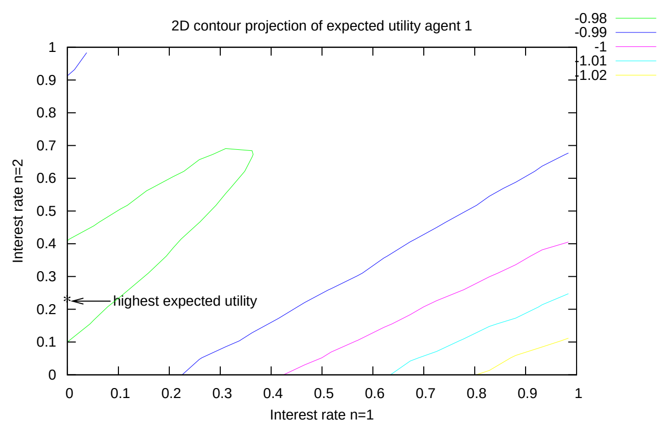


FIGURE 5. Interest rate effect on Expected utility of agent 1 (Γ^1)

Each agent has an endowment of the goods (only non-zero for the perishable and the durable goods) in each period and state, $e^h \in \mathbb{R}_+^{2(S+1)}$. The preference ordering of agent h

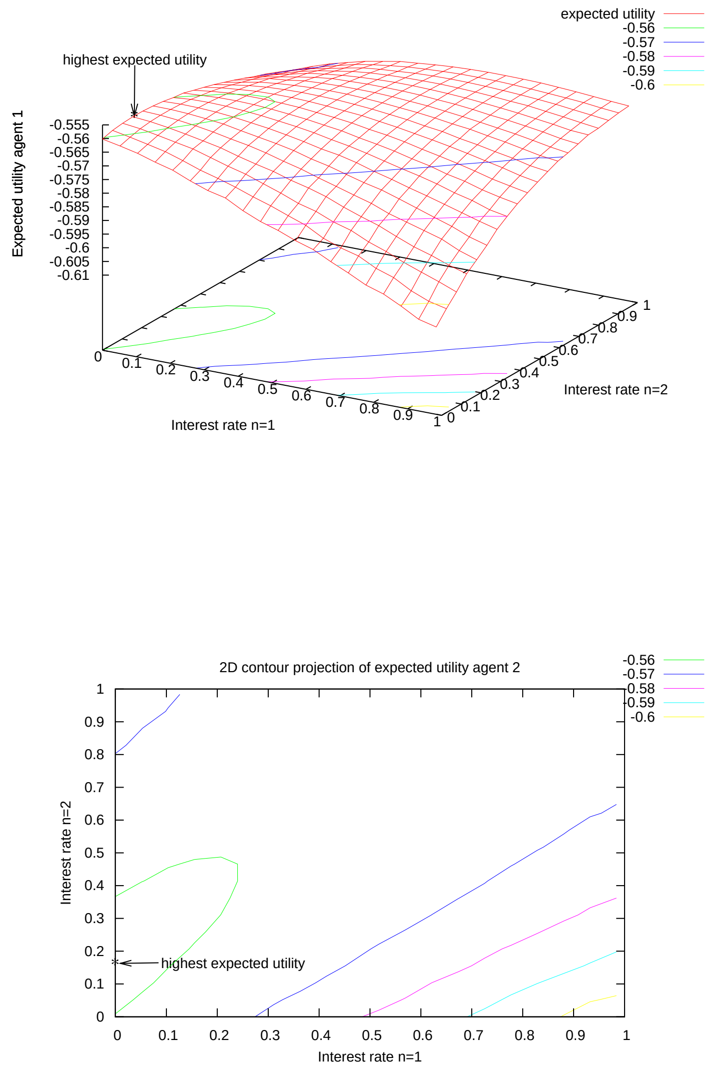


FIGURE 6. Interest rate effect on Expected utility of agent 2 (Γ^2)

is represented by a utility function $u^h : \mathbb{R}_+^{2(S+1)} \to \mathbb{R}$, defined over consumption $x_1^h \in \mathbb{R}_+^{S+1}$ and $x_2^h \in \mathbb{R}_+^{S+1}$, i.e, the utility function depends only on consumption of the perishable and

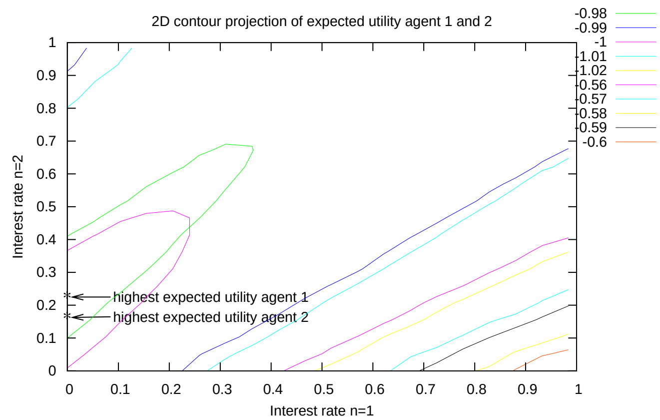


FIGURE 7. Interest rate effect on Expected utility of agent 1 (Γ^1) and agent 2 (Γ^2)

the durable services. The durable good can be held as an asset by agents and can also be purchased by the central bank.

Given $p \in \mathbb{R}^{3(S+1)}_{++}$, and $q \in \mathbb{R}^J_+$ the agent h aims to maximize his utility function under the budget constraints:

$$\max_{x_1\geq 0, x_2\geq 0, \psi\geq 0, \varphi\geq 0, \mu\geq 0} u^h(x^h) \quad \text{s.t.}$$

$$p_1(x_1^h - e_1^h) + p_2(x_2^h - x_3^h) + p_3(x_3^h - e_3^h) + q \cdot (\psi - \varphi) + \mu^h \leq 0;$$

$$p_{s1}(x_{s1}^h - e_{s1}^h) + p_{s2}(x_{s2}^h - x_{s3}^h) + p_{s3}(x_{s3}^h - e_{s3}^h - x_{s3}^h) - \sum_{j\in\mathcal{J}}(\psi_j - \varphi_j)\min\{1, p_{s3}C_j\}$$

$$+ \theta_s^h[(1+i)(p_3 - p_2)\omega \sum_h e_3^h - p_{s3}\omega \sum_h e_3^h]$$

$$-(1+i)\mu^h \leq 0; \quad \forall s \in \mathcal{S}$$

$$x_3^h - \sum_{j\in\mathcal{J}} \varphi_j C_j \geq 0. \tag{3.1}$$

An equilibrium for the economy E is a vector $[(\overline{x}_1, \overline{x}_2, \overline{\psi}, \overline{\varphi}, \overline{\mu}); (\overline{p}, \overline{q})]$, such that:

(i) $(\overline{x}_1^h, \overline{x}_2^h, \overline{\psi}^h, \overline{\varphi}^h, \overline{\mu}^h)$ solves problem 3.1.

(ii) $\sum_{h=1}^{H}(\overline{x}_1^h - e_1^h) = 0$

(iii) $\sum_{h=1}^{H}(\overline{x}_2^h - e_3^h) = 0$

(iv) $\sum_{h=1}^{H}\overline{x}_3^h - (1-\omega)\sum_{h=1}^{H} e_3^h = 0$

(v) $\sum_{h=1}^{H}(\overline{x}_{s1}^h - e_{s1}^h) = 0$

(vi) $\sum_{h=1}^{H}(\overline{x}_{s2}^h - e_{s3}^h - \overline{x}_3^h) = 0$

(vii) $\sum_{h=1}^{H}(\overline{x}_{s3}^h - e_{s3}^h - \overline{x}_3^h) = 0$

(viii) $\sum_{h=1}^{H}(\overline{\psi}^h - \overline{\varphi}^h) = 0$

(x) $\sum_{h=1}^{H}\overline{\mu}^h - (p_3 - p_2)\omega\sum_{h=1}^{H} e_3^h = 0$

In our model, central-bank purchases of collateralizable durable goods generally affect equilibrium prices and the allocation of resources; they are not irrelevant as in the result of [Wa]. We illustrate this effect with a numerical example.

3.1.1. *Example: Effect on the equilibrium with purchases of durable good by central bank.* We consider an example with a scarce collateralizable durable good, in which only a very few assets are traded and hence markets are endogenously incomplete (see [AKS]). In this economy there are two states in period 1 $S = 2$ and two assets $J = 2$, but in equilibrium only one asset is traded.

Each individual $h = 1, 2$ has a utility function in terms of perishable good ($l = 1$) and durable services good ($l = 2$) of the form:

$$u^h(x) = \log(x_1) + \log(x_2) + \frac{1}{2}\sum_{s=1}^{2}(\log(x_{s1}) + \log(x_{s2}))$$

Endowments of perishable good ($l = 1$) and durable good ($l = 3$) are:

$$e^1 = (e_1^1, e_3^1; e_{11}^1, e_{13}^1, e_{21}^1, e_{23}^1) = (4, 0.9; 1, 0, 2, 0);$$

$$e^2 = (e_1^2, e_3^2; e_{11}^2, e_{13}^2, e_{21}^2, e_{23}^2) = (1, 0.1; 1.2, 0, 2, 0).$$

Money supply in this model is endogenous $M = (p_3 - p_2)\omega \sum_h e_3^h$.

Nominal interest rate: $i = 0.1$.

Taxes θ_s^h are: $\theta^1 = (\theta_1^1, \theta_2^1) = (0.5, 0.5)$; $\theta^2 = (\theta_1^2, \theta_2^2) = (0.5, 0.5)$.

In the first period the price of non-durable good p_{s1} is fixed: $p_{11} = 1$; $p_{21} = 1$.

The collateral requirements are: $C_1 = 0.45$; $C_2 = 0.25$.

To illustrate the effect on equilibrium we consider different points in utility space with random choice of the purchases by central bank ω between 0 and 1 (see figures 8 and 9)

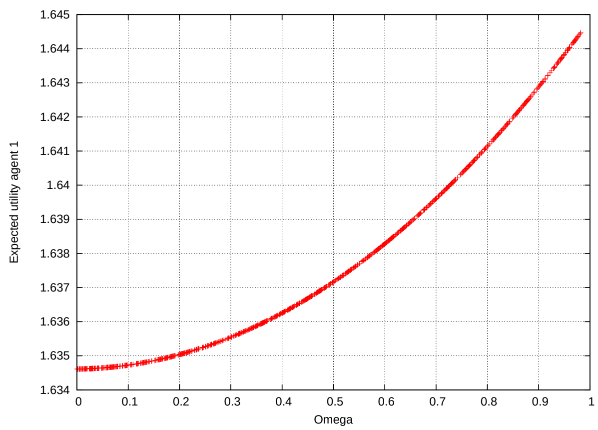


FIGURE 8. Purchases of durable good by central bank's effect on utility of agent 1

Remark 1. When markets are complete and the collateral constraints are not binding for all agents, purchases of the durable good by the central bank have no effect on the equilibrium allocation of resources.

3.2. **Monetary model with flexible prices and three periods.** We consider a similar economy as section 3.1 with rental markets and purchases of durable good by central bank, but now we have three time periods $t = 0, 1, 2$ with uncertainty over the state of nature denoted by the subscript $s \in \mathcal{S} = \{1, \ldots, S\}$ in the second period and $n \in \mathcal{N} = \{1, \ldots, N\}$ in

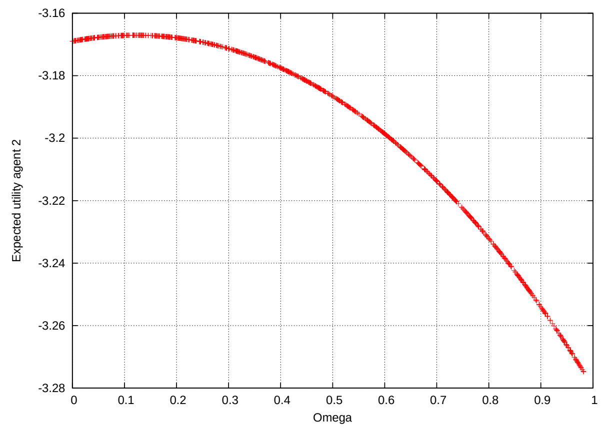


Figure 9. Purchases of durable good by central bank's effect on utility of agent 2

the third period. In this model the central bank purchases durable good in the second period, denoted by $\omega_s \in (0,1]$ that represent a ratio over aggregate durable good, i.e. $\omega_s \sum_h (e^h_{s3} + e^h_3)$ and each agent h receives a transfer of money (private endowments of money) in the amount m^h in period 0, where $\sum_h m^h = M > 0$ is the money supply and m^h_s in period 1, where $\sum_h m^h_s = M_s > 0$.

Given $p \in \mathbb{R}^{3(S+SN+1)}_{++}$, and $q \in \mathbb{R}^{J+SJ}_{+}$ the agent h aims to maximize his utility function under the budget constraints:

$$
\begin{aligned}
&\max_{x\geq 0,\psi\geq 0,\varphi\geq 0,\mu\geq 0} u^h(x^h) \quad \text{s.t.}\\
&p_1(x_1^h - e_1^h) + p_2(x_2^h - x_3^h) + p_3(x_3^h - e_3^h) + \sum_j q_j(\psi_j^h - \varphi_j^h) + \mu^h - m^h \leq 0;\\
&p_{s1}(x_{s1}^h - e_{s1}^h) + p_{s2}(x_{s2}^h - x_{s3}^h) + p_{s3}(x_{s3}^h - e_{s3}^h - x_3^h)\\
&-\sum_j(\psi_j^h - \varphi_j^h)\min\{1, p_{s3}C_j\} + (1+i)(\theta_s^h M - \mu^h) + \mu_s^h - m_s^h \leq 0; \quad \forall s \in \mathcal{S}\\
&p_{sn1}(x_{sn1}^h - e_{sn1}^h) + p_{sn2}(x_{sn2}^h - x_{sn3}^h) + p_{sn3}(x_{sn3}^h - e_{sn3}^h - x_{s3}^h)\\
&+\theta_{sn}^h[(1+i_s)(M_s + (p_{s3} - p_{s2})\omega_s \sum_h (e_{s3}^h + e_3^h)) - p_{sn3}\omega_s \sum_h (e_{s3}^h + e_3^h)]\\
&-\sum_j(\psi_j^h - \varphi_j^h)\min\{1, p_{sn3}C_{sj}\} - (1+i_s)\mu_s^h \leq 0; \quad \forall sn \in \mathcal{SN}\\
&x_3^h - \sum_j \varphi_j^h C_j \geq 0;\\
&x_{s3}^h - \sum_j \varphi_{sj}^h C_{sj} \geq 0.
\end{aligned}
\tag{3.2}
$$

A competitive equilibrium is defined by agents' optimality and market clearing.

An equilibrium for the economy E is a vector $[(\overline{x}, \overline{\psi}, \overline{\varphi}, \overline{\mu}); (\overline{p}, \overline{q})]$, such that:

(i) $(\overline{x}^h, \overline{\psi}^h, \overline{\varphi}^h, \overline{\mu}^h)$ solves problem 3.2.

(ii) $\sum_{h=1}^H (\overline{x}_1^h - e_1^h) = 0$

(iii) $\sum_{h=1}^H (\overline{x}_2^h - e_3^h) = 0$

(iv) $\sum_{h=1}^H (\overline{x}_3^h - e_3^h) = 0$

(v) $\sum_{h=1}^H (\overline{x}_{s1}^h - e_{s1}^h) = 0$

(vi) $\sum_{h=1}^H (\overline{x}_{s2}^h - e_{s3}^h - e_3^h) = 0$

(vii) $\sum_{h=1}^H \overline{x}_{s3}^h - (1-\omega_s)\sum_{h=1}^H (e_{s3}^h + e_3^h) = 0$

(viii) $\sum_{h=1}^H (\overline{x}_{sn1}^h - e_{sn1}^h) = 0$

(ix) $\sum_{h=1}^H (\overline{x}_{sn2}^h - e_{sn3}^h - e_{s3}^h - e_3^h) = 0$

(x) $\sum_{h=1}^{H}(\overline{x}_{sn3}^{h} - e_{sn3}^{h} - e_{s3}^{h} - e_{3}^{h}) = 0$

(xi) $\sum_{h=1}^{H}(\overline{\psi}_{j}^{h} - \overline{\varphi}_{j}^{h}) = 0$

(xii) $\sum_{h=1}^{H}(\overline{\psi}_{sj}^{h} - \overline{\varphi}_{sj}^{h}) = 0$

(xiii) $\sum_{h=1}^{H}\overline{\mu}^{h} - M = 0$

(xiv) $\sum_{h=1}^{H}\overline{\mu}_{s}^{h} - M_s - (p_{s3} - p_{s2})\omega_s\sum_{h=1}^{H}(e_{s3}^{h} - e_{3}^{h}) = 0$

We now describe two numerical examples one that illustrate the potential welfare effects of the addition dimension of central-bank policy.

3.2.1. *Example: Effectiveness of central-bank purchases of the durable good.* We consider an example with two states in period 1 $S = 2$; two states in period 2 $N = 2$ and two agents, $H = 2$, each with identical logarithm utility function. The probabilities in period 2 are: $\varepsilon_s = 0.5\,\forall s$ and in period 3 are: $\varepsilon_{sn} = 0.25\,\forall s, n$.

We suppose that endowments are:

$e^1 = (4,4)$, $e_1^1 = (5,0)$, $e_2^1 = (2,0)$, $e_{11}^1 = (5,0)$, $e_{12}^1 = (7,0)$, $e_{21}^1 = (1,0)$, $e_{22}^1 = (2,0)$;
$e^2 = (2,1)$, $e_1^2 = (6,0)$, $e_2^2 = (1,0)$, $e_{11}^2 = (2,0)$, $e_{12}^2 = (6,0)$, $e_{21}^2 = (3,0)$, $e_{22}^2 = (5,0)$.

In the first period agent 1 is rich and agent 2 is poor. In the second period both agents have higher endowments of good 1 in $s = 1$ (good state) and lower endowments of good 1 in $s = 2$ (bad state).

Money m^h in the first period are: $m^1 = 2$ and $m^2 = 1$ and in the second period are: $m_s^h = 0 \quad \forall h, s$. Nominal interest rate are: $i = 0.05$, $i_1 = 0.05$ and $i_2 = 0$ (bad state in $t = 1$). Taxes are: $\theta_s^h = 0.5 \quad \forall h, s$ and $\theta_{sn}^1 = (0.6, 0.5; 0.5, 0.4)$; $\quad \theta_{sn}^2 = (0.4, 0.5; 0.5, 0.6)$. The price of perishable good in the last period p_{sn1} is fixed for all s and n: $p_{sn1} = 1$.

In this economy there are two assets in the first period and four assets in the second period.Collateral requirements are: $C_j = (0.1, 0.2)$; $C_{sj} = (0.71, 0.38, 1.25, 0.71)$. In this case agents default in the second period and default in some assets in the third period.

We solved 10.000 samples with random choice of the central bank purchases of durable good (ω_1 and ω_2) between 0 and 1.

The ratio of central bank purchases that maximize utility for agent 1 is $\omega_1 = 0$ and $\omega_2 = 0$ (see Figure 10), and for agent 2 is $\omega_1 = 0.25$ and $\omega_2 = 0.4$ (see Figure 11).

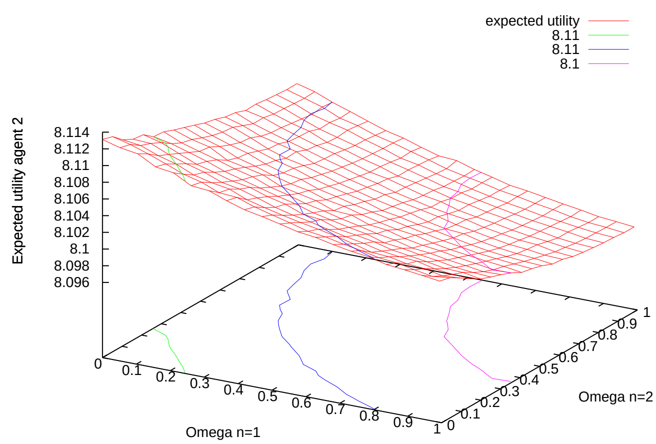


FIGURE 10. central bank purchases of durable good effect on Expected utility of agent 1

3.2.2. *Example: Central-bank purchases of the durable good can lead to a Pareto-improvement.* We consider the same probabilities, endowments, collateral requirements, money and interest rate as in the previous example, but now the taxes are: $\theta_s^h = 0.5 \quad \forall h, s$ and $\theta_{sn}^1 = 0.8; \quad \theta_{sn}^2 = 0.2. \quad \forall s, n.$

We solved 10.000 samples with random choice of the central bank purchases of durable good (ω_1 and ω_2) between 0 and 1.

In this case, the utility of agent 1 is barely affected (see Figure 12), but it is possible to improve the utility of agent 2 (see Figure 13) with purchases by central bank around 40% in the second state (the bad state where the interest rate is zero).

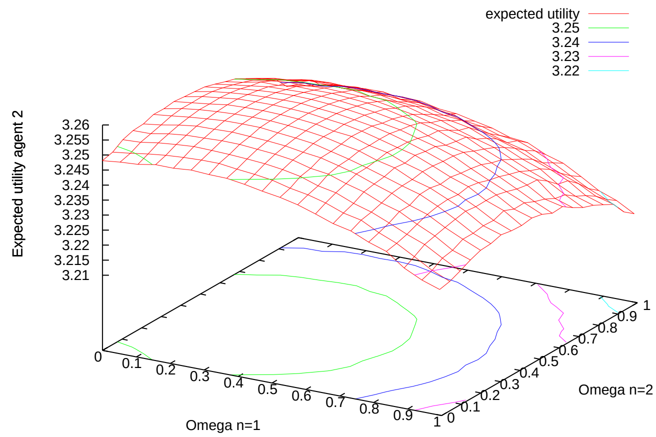


FIGURE 11. Central bank purchases of durable good effect on Expected utility of agent 2

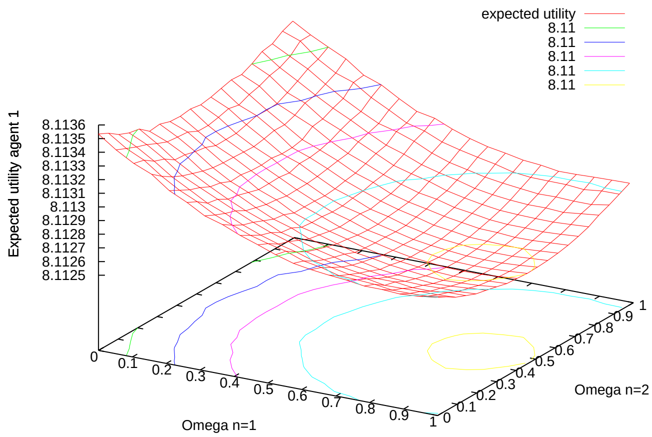


FIGURE 12. Central bank purchases of durable good effect on Expected utility of agent 1

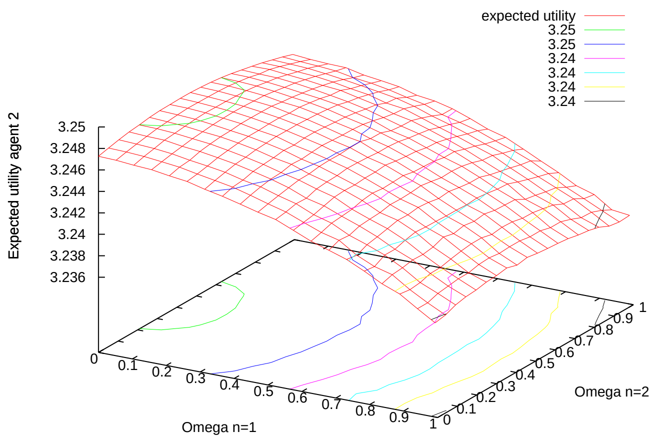


FIGURE 13. Central bank purchases of durable good effect on Expected utility of agent 2

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