Sunspots and Multiplicity

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Abstract

This paper proves that, in a general financial model with incomplete markets, the multiplicity of certainty equilibria is not necessary for the existence of sunspot effects. These effects are present, by definition, when real economic variables differ across realizations of extrinsic uncertainty. In a financial model of incomplete markets with identical payouts across realizations of extrinsic uncertainty, the past literature has concluded that multiplicity of certainty equilibria is necessary for sunspot effects. However, all these papers model extrinsic uncertainty using expected utility. In this paper, I consider a generalization of expected utility, a form that Dave Cass labeled UBU (utility-based utility). With this form, my result proves that the class of rational expectations equilibria exhibiting sunspot effects is much broader than the class of multiple certainty equilibria.

JEL Classifications: D61; D81; D91

Key words: sunspots; extrinsic uncertainty; incomplete markets; determinacy; intertemporal consistency; variance aversion; Cass-Shell Immunity Theorem

1 Introduction

The idea of sunspots, far simpler than a new model or a new solution concept, has been extensively studied by economists, because sunspots are a formal representation of seemingly irrational behavior exhibited in financial markets. In these markets, traders respond to irrelevant information that has no bearing on "fundamentals." These responses, if adopted as

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the "market psychology," are self-fulfilling optimal actions of traders. In terms of economic modeling, sunspots are realizations of extrinsic uncertainty, or uncertainty that does not impact the fundamentals (endowments, preferences, payouts) of the economy. As was shown thirty years ago (Cass and Shell (1983), Balasko (1983), and Azariadis (1981)), even when the tenet of rational expectations is maintained, sunspots can affect the real equilibrium variables.

Economists, for better or worse, seek meaning in the intuition behind an economic result. In the case of sunspot effects, the intuitive explanation has limited the potential scope for sunspot phenomena and has truly been a disservice.[1] Consider a typical exchange between a sunspot expert (economist A) and one who is in the dark about sunspots (economist B). Economist A says that beliefs can be self-fulfilling in equilibrium. Economist B states that this is only possible if the rational expectations assumption is removed. Economist A counters by citing sunspot effects as outcomes of rational expectations models in which beliefs can have real effects. Economist B appears interested and questions how sunspots can have such effects. Economist A responds by asking B to envision an economy with multiple certainty equilibria in which sunspots serve to coordinate the beliefs of agents on one vector of certainty equilibrium prices.

This intuitive explanation is certainly misleading. I prove, within a class of models described below, that sunspots can have real effects independent of the number of certainty equilibria. To study sunspot effects, I must consider a dynamic model of uncertainty and this paper, in fact, considers the most classical version of such a model: a two-period general financial model with incomplete markets. The model contains extrinsic uncertainty in the final period, but not intrinsic uncertainty. The financial element of the model is the trading of numeraire assets, whose payouts in the final period are identical across realizations of extrinsic uncertainty. Without intrinsic uncertainty, the only independent numeraire asset is a risk-free bond whose payouts are equal in all realizations of extrinsic uncertainty and normalized to one.

A friction is required for sunspot effects to occur (Cass-Shell Immunity Theorem, 1983). In this paper, the friction is incomplete markets. A mature literature considers a different friction, restricted market participation, which is often caused by imposing a demographic structure on the model. This literature begins with the two-period model of Cass and Shell (1983), proceeds to the more general overlapping generations (OLG) model of Azariadis and Guesnerie (1986), and culminates with the most recent papers (particularly, Dávila, Gottardi, and Kajii, 2007) to study sunspot equilibria in OLG economies. Each of the first two papers proves, for its respective model, that multiplicity of certainty equilibria is not necessary for sunspot effects.

Within the class of two-period financial models with incomplete markets, this paper considers numeraire assets, which have far different implications for sunspot equilibria than other asset types. With nominal assets, Cass (1992) proves that sunspot equilibria are generically indeterminate when there are fewer assets than states of extrinsic uncertainty. This result only requires the reader to count. With incomplete markets and nominal assets, equilibria

[1] The reasons behind the disservice can perhaps be traced to the seminal Cass and Shell (1983) paper. Most readers have tended to focus on the canonical example in the body of the paper, in which multiplicity of certainty equilibria appears to be necessary for sunspot effects, rather than on the example in the appendix.

are generically indeterminate, but the set of equilibria without sunspot effects (namely those with identical real variables in all states) is generically finite. With real assets, Gottardi and Kajii (1999) prove that for a generic subset of economies, sunspot effects occur without a multiplicity of certainty equilibria. The asset yields, in their model, are identical across realizations of extrinsic uncertainty, but the asset payouts, as functions of endogenous commodity prices, are not. Thus, their result states that a "potential multiplicity" of certainty equilibria is necessary for sunspot effects. With numeraire assets, the sunspot equilibria are not generically indeterminate and the asset payouts are parameters, independent of endogenous commodity prices.

Within the narrower class of two-period financial models with incomplete markets and numeraire assets, Hens (2000) proves that multiple certainty equilibria are necessary for sunspot effects, except when the asset payouts differ across realizations of extrinsic uncertainty. Obviously, this exceptional case violates the definition of extrinsic uncertainty itself as fundamental elements of the economy, the asset payouts, are impacted. Thus, to prove my result that multiple certainty equilibria are not necessary for sunspot effects, I set the asset payouts (parameters) equal for all realizations of extrinsic uncertainty.

To explain the apparent contradiction that Hens (2000), with identical payouts across realizations of extrinsic uncertainty, proves that multiplicity of certainty equilibria is necessary for sunspot effects, while this paper proves the opposite, consider the utility form used by Hens (2000). Not just Hens (2000), but in fact the entire literature on sunspots, including all of the above-cited works, assume that preferences are represented by the expected utility form with identical objective probabilities for all households. In this paper, I employ a different utility form that generalizes expected utility. This generalized form is free from the restriction that uncertainty is simply risk characterized by the assignment of objective probabilities to states in the final period, and instead is defined in terms of the broader concept of a filtration of a state space. In Cass (2008), this generalized utility form is introduced together with the assumptions required for extrinsic uncertainty and is labeled UBU (utility-based utility).

I introduce the UBU form for a general finite horizon model, before detailing the specifics for my two-period model. The UBU form is the representation of choice over a menu of consumption in the current period and prospective utilities over all vaguely uncertain states in the following period. UBU is a special case of the recursive tree-structure of utility functions introduced by Johnsen and Donaldson (1985). Both forms are generalizations of expected utility that satisfy intertemporal consistency, specifically (i) all prior contingent consumption plans are optimal once the current state is realized and (ii) given the current realized state, current choices are independent of contingent choices made for states that cannot be realized. Skiadas (1998) extends the idea of Johnsen and Donaldson (1985) so as to include choice over both contingent consumption plans and information filtrations. These two papers and the UBU form are in sharp contrast to the "standard" modeling of uncertain realizations over time as epitomized by Epstein and Zin (1989). The fact that Epstein-Zin preferences are defined over lotteries across realizations, rather than directly on realizations as in the UBU form, verifies that this "standard" modeling is still wed to probabilities.

Regarding a two-period financial model of uncertainty, the utility in the UBU form is a function of the initial period consumption and the prospective utilities for all states of uncertainty in the final period. For each state, the prospective utility is the value of utility only for consumption in that state, not unlike the Bernoulli utility in the expected utility form. For my purposes, in showing that multiplicity of certainty equilibria is not necessary for sunspot effects, it is essential that the initial period consumption and the prospective utilities in the UBU form are not additively separable. With this dependence between initial period consumption and prospective utilities, sunspots leading to different final period consumptions affect the initial period consumption and ultimately the asset choice. With expected utility, the asset choice can be fixed and then sunspots have no effect on initial period consumption.

This paper is organized as follows. Section 2 introduces the UBU functional form and specifies the assumptions required to define extrinsic uncertainty. Section 3 provides a robust example in which sunspots have effects in a general financial model with incomplete markets and numeraire assets. Section 4 proves that the sunspot equilibrium is not the randomization over certainty equilibria and additionally that a unique certainty equilibrium exists for this example. Section 5 concludes with general observations about the example.

2 UBU

I consider a general financial model with two time periods and extrinsic uncertainty in the final period. The extrinsic uncertainty is modeled as a finite number of states $s \in \mathcal{S} = \{1, ..., S\}$ that can be realized in the final period. By convention, the initial period is state $s = 0$. In all states, a finite number of households $h \in \mathcal{H} = \{1, ..., H\}$ trade and consume a finite number of physical commodities $l \in \{1, .., L\}$. The model is a financial model, because assets may be included to allow households to transfer wealth between states.

For this paper, the uncertainty is only extrinsic. Extrinsic uncertainty means that the fundamentals (household endowments and utility functions and asset payouts) in the states $s \in \mathcal{S}$ are identical. A sunspot, by definition, is the realization of extrinsic uncertainty. Sunspot effects occur if real economic variables differ across realizations of extrinsic uncertainty.

Let h be any household. The consumption of a bundle of L physical commodities by h in state $s = 0$ is $x^h(0)$ and in state $s \in \mathcal{S}$ is $x^h(s)$. Consumption over all states is defined by the vector $x^h = \left(x^h(0), \left(x^h(s)\right)_{\forall s \in \mathcal{S}}\right)$. The consumption set is defined by $X^h = \mathbb{R}_+^{L(S+1)}$ and the utility function by $u^h : X^h \to \mathbb{R}$. The UBU form defines u^h as a function of $S + 1$ prospective utility functions $\left(v_0^h, \left(v_s^h\right)_{\forall s \in \mathcal{S}}\right)$:

$$u^h(x^h) = v_0^h \left(x^h(0), \left(v_s^h(x^h(s))\right)_{\forall s \in \mathcal{S}}\right)$$

where $v_s^h : \mathbb{R}_+^L \to \mathbb{R} \ \forall s \in \mathcal{S}$ and $v_0^h : \mathbb{R}_+^L \times \underset{s \in \mathcal{S}}{\times} v_s^h \left(\mathbb{R}_+^L\right) \to \mathbb{R}$. By appropriate restrictions on the function v_0^h, the expected utility form is obtained.

The following assumptions are imposed on the prospective utility functions $\left(v_0^h, \left(v_s^h\right)_{\forall s \in \mathcal{S}}\right)$:

1. v_s^h is C^0, concave, and locally non-satiated $\forall s \in \mathcal{S}$.

2. v_0^h is C^0, concave, the function $\bar{v}_0^h : \underset{s \in \mathcal{S}}{\times} v_s^h \left(\mathbb{R}_+^L\right) \to \mathbb{R}$, defined by holding $x^h(0) \in \mathbb{R}_+^L$

fixed at any level, is strictly increasing, and the function $\hat{v}_0^h : \mathbb{R}_+^L \to \mathbb{R}$, defined by holding $\left(v_s^h(x^h(s))\right)_{\forall s\in\mathcal{S}} \in \underset{s\in\mathcal{S}}{\times} v_s^h\left(\mathbb{R}_+^L\right)$ fixed at any level, is locally non-satiated.

From the continuity assumptions in 1 and 2, u^h is C^0. From the monotonicity assumptions in 1 and 2, u^h is locally non-satiated. Further, the contingent consumption plans are intertemporally consistent: (i) contingent plans $\left(x^h(s)\right)_{\forall s\in\mathcal{S}}$ made in the initial period are optimal when any state $s \in \mathcal{S}$ is realized and (ii) choices $x^h(s)$ are independent of choices $x^h(s')$ for $s' \neq s$.[2] From the concavity assumptions in 1 and 2, u^h is concave. To verify this, take any $x^h, y^h \in X^h$ and consider the convex combination $\theta x^h + (1-\theta)y^h$ for $\theta \in [0,1]$:

$$\begin{aligned}
u^h\left(\theta x^h + (1-\theta)y^h\right) &= v_0^h\left(\theta x^h(0) + (1-\theta)y^h(0), \left(v_s^h(\theta x^h(s) + (1-\theta)y^h(s))\right)_{\forall s\in\mathcal{S}}\right) \\
&\geq v_0^h\left(\theta x^h(0) + (1-\theta)y^h(0), \left(\theta v_s^h(x^h(s)) + (1-\theta)v_s^h(y^h(s))\right)_{\forall s\in\mathcal{S}}\right) \text{ by 1} \\
&\geq \theta v_0^h\left(x^h(0), \left(v_s^h(x^h(s))\right)_{\forall s\in\mathcal{S}}\right) + (1-\theta)v_0^h\left(y^h(0), \left(v_s^h(y^h(s))\right)_{\forall s\in\mathcal{S}}\right) \text{ by 2} \\
&= \theta u^h\left(x^h\right) + (1-\theta)u^h\left(y^h\right).
\end{aligned}$$

Extrinsic uncertainty

By definition, for all possible realizations of extrinsic uncertainty, the fundamentals remain unchanged. Unchanging fundamentals means more than just holding household parameters (endowments and prospective utility functions) constant across states $s \in \mathcal{S}$. Restrictions need to be made so that the overall utility function u^h satisfies *variance aversion*, a property that is defined below and proven to hold for the UBU form in theorem 1.

Sunspots, the realizations of extrinsic uncertainty, have effects when they lead to different consumptions choices. If sunspots have effects, they are said to matter.

Definition 1 *Sunspots have effects iff* $\exists h \in \mathcal{H}$ *s.t.* $x^h(s) \neq x^h(s')$ *for some* $s, s' \in \mathcal{S}$.

Define the permutation $\sigma : \mathcal{S} \times \mathcal{S} \to \mathcal{S}$ by $s \longmapsto \sigma(s,s') = \left\{\begin{matrix} s+s' & \text{if } s+s' \leq S \\ s+s'-S & \text{if } s+s' > S \end{matrix}\right\}$. The key concept of circular symmetry is borrowed from Cass (2008) and defined below.

Definition 2 *A function* v_0^h *satisfies circular symmetry iff* $\forall s' = 1, ..., S-1$:

$$v_0^h\left(x^h(0), \left(v_s^h(x^h(s))\right)_{\forall s\in\mathcal{S}}\right) = v_0^h\left(x^h(0), \left(v_s^h(x^h(\sigma(s,s')))\right)_{\forall s\in\mathcal{S}}\right).$$

The assumptions for extrinsic uncertainty are given by:

3. $v_s^h = v^h \quad \forall s \in \mathcal{S}$.

4. v^h is strictly concave.

5. v_0^h satisfies circular symmetry.

[2] The utility form of Johnsen and Donaldson (1985) is the most general form satisfying intertemporal consistency. UBU is a special case of their form.

Define the average consumption bundle across states $s \in \mathcal{S}$ as:

$$\bar{x}^h = \frac{1}{S} \sum\nolimits_{s \in \mathcal{S}} x^h(s).$$

Definition 3 *The utility function u^h satisfies variance aversion iff $u^h(x^h(0), \bar{x}^h, ..., \bar{x}^h) \geq u^h(x^h(0), (x^h(s))_{\forall s \in \mathcal{S}})$, with strict inequality if $x^h(s) \neq x^h(s')$ for some $s, s' \in \mathcal{S}$.*

Theorem 1 *Under assumptions 1 thru 5, u^h satisfies variance aversion.*

Proof. Define $\bar{x}^h = \frac{1}{S} \sum\nolimits_{s \in \mathcal{S}} x^h(s)$. Then by assumption 4:

$$v^h \left(\bar{x}^h\right) \geq \frac{1}{S} \sum\nolimits_{s \in \mathcal{S}} v^h \left(x^h(s)\right), \tag{1}$$

with strict inequality if $x^h(s) \neq x^h(s')$ for some $s, s' \in \mathcal{S}$. Since v_0^h is concave (assumption 2):

$$\begin{array}{l} v_0^h \left(x^h(0), \frac{1}{S} \sum\nolimits_{s \in \mathcal{S}} v^h \left(x^h(s)\right), ..., \frac{1}{S} \sum\nolimits_{s \in \mathcal{S}} v^h \left(x^h(s)\right)\right) \geq \\ \frac{1}{S} \sum\nolimits_{s \in \mathcal{S}} v_0^h \left(x^h(0), v^h \left(x^h(s)\right), v^h \left(x^h(\sigma\left(s, 1\right))\right), ..., v^h \left(x^h(\sigma\left(s, S-1\right))\right)\right). \end{array} \tag{2}$$

For $s = 1$, no further steps are needed. For $s > 1$, define $s' = s - 1$. Then $s = \sigma\left(1, s'\right)$ and for $\tilde{s} \in \{1, .., S-1\}$:

$$\sigma\left(\sigma\left(1, s'\right), \tilde{s}\right) = \sigma\left(1 + \tilde{s}, s'\right) = \left\{ \begin{array}{cl} 1 + \tilde{s} + s' & \text{if } 1 + \tilde{s} + s' \leq S \\ 1 + \tilde{s} + s' - S & \text{if } 1 + \tilde{s} + s' > S \end{array} \right\}$$

by definition of σ and

$$\begin{array}{l} \left(v^h \left(x^h(s)\right), v^h \left(x^h(\sigma\left(s, 1\right))\right), ..., v^h \left(x^h(\sigma\left(s, S-1\right))\right)\right) = \\ \left(v^h \left(x^h(\sigma\left(1, s'\right))\right), v^h \left(x^h(\sigma\left(2, s'\right))\right), ..., v^h \left(x^h(\sigma\left(S, s'\right))\right)\right) = \left(v^h(x^h(\sigma\left(\psi, s'\right)))\right)_{\forall \psi \in \mathcal{S}} \end{array}. \tag{3}$$

Thus, $\forall s \in \mathcal{S}$, using equation (3) and the circular symmetry assumption 5:

$$\begin{array}{l} \left(v^h \left(x^h(s)\right), v^h \left(x^h(\sigma\left(s, 1\right))\right), ..., v^h \left(x^h(\sigma\left(s, S-1\right))\right)\right) = \\ \left(v^h(x^h(\sigma\left(\psi, s'\right)))\right)_{\forall \psi \in \mathcal{S}} = \left(v^h(x^h(\psi))\right)_{\forall \psi \in \mathcal{S}}. \end{array} \tag{4}$$

Using equations (1), (2), and (4) and the assumption 2 that v_0^h is strictly increasing in $\left(v_s^h\right)_{\forall s \in \mathcal{S}}$:

$$v_0^h \left(x^h(0), v^h \left(\bar{x}^h\right), ..., v^h \left(\bar{x}^h\right)\right) \geq v_0^h \left(x^h(0), \left(v^h(x^h(s))\right)_{\forall s \in \mathcal{S}}\right),$$

with strict inequality if $x^h(s) \neq x^h(s')$ for some $s, s' \in \mathcal{S}$. Therefore, $u^h(x^h(0), \bar{x}^h, ..., \bar{x}^h) \geq u^h(x^h(0), (x^h(s))_{\forall s \in \mathcal{S}})$, with strict inequality if $x^h(s) \neq x^h(s')$ for some $s, s' \in \mathcal{S}$. ■

Corollary 1 *If the assumptions of the first basic welfare theorem are met (no frictions present), then the Cass-Shell Immunity Theorem (Cass and Shell, 1983) is applicable and $x^h(s) = x^h(s')$ $\forall s, s' \in \mathcal{S}$ and $\forall h \in \mathcal{H}$. Thus, sunspots cannot have any effects; they do not matter.*

Proof. The proof is immediate given the property of *variance aversion*, but a proof is provided in Cass (2008). ■

3 A Sunspot Example

I define the remaining variables and parameters and introduce the final assumptions required for extrinsic uncertainty.

The household endowments are defined by $e^h = \left(e^h(0), \left(e^h(s)\right)_{\forall s \in \mathcal{S}}\right)$ and are assumed to be strictly positive: $e^h >> 0$.[3] The spot commodity prices are defined by $p = \left(p(0), (p(s))_{\forall s \in \mathcal{S}}\right)$ where $p(0) \in \mathbb{R}^L_+ \backslash \{0\}$ and $p(s) \in \mathbb{R}^L_+ \backslash \{0\}$ $\forall s \in \mathcal{S}$ by assumptions 1 and 2. The prices are normalized so that $p(s) = \left(\left(\frac{p_l(s)}{p_L(s)}\right)_{\forall l < L}, 1\right)$.

With $J < S$ numeraire assets, markets are incomplete. The payout of asset j in state $s \in \mathcal{S}$ in terms of the commodity $l = L$ is given by $r_j(s)$. In terms of the real economic variables, it is innocuous to assume that the payout matrix $R = \begin{bmatrix} r_1(1) & ... & r_J(1) \\ : & : & : \\ r_1(S) & ... & r_J(S) \end{bmatrix}$ has full column rank.

The assumptions for extrinsic uncertainty include:

6. $e^h(s) = e^h(1) \;\; \forall s \in \mathcal{S}$.
7. $r_j(s) = r_j(1) \;\; \forall s \in \mathcal{S}$.

Assumption 7 together with R full column rank implies that only one numeraire asset exists ($J = 1$) and it has a risk-free payout normalized to 1 in all states. The asset choice by household h is denoted by $z^h \in \mathbb{R}$ and the price of the asset is $q \in \mathbb{R}_+$.

I define a *financial equilibrium with extrinsic uncertainty*, referred to as a *sunspot equilibrium.*

Definition 4 *A financial equilibrium with extrinsic uncertainty is a vector* $\left(\left(x^h, z^h\right)_{\forall h \in \mathcal{H}}, p, q\right)$ *such that*

1. given (p, q), $\forall h \in \mathcal{H}$:

$$\begin{array}{rcl} \left(x^h, z^h\right) & \in & \arg\max u^h\left(x^h\right) \\ & & \textit{subj. to} \begin{array}{c} p(0)\left(x^h(0) - e^h(0)\right) + qz^h \leq 0 \\ \left(p(s)\left(x^h(s) - e^h(1)\right) - z^h \leq 0\right)_{\forall s \in \mathcal{S}} \end{array} \end{array}.$$

2. markets clear:

$$\begin{array}{l} \sum_{h \in \mathcal{H}} x^h_l(0) - e^h_l(0) = 0 \;\; \forall l \\ \sum_{h \in \mathcal{H}} x^h_l(s) - e^h_l(1) = 0 \;\; \forall l, \forall s \in \mathcal{S} \\ \sum_{h \in \mathcal{H}} z^h = 0 \end{array}.$$

Given assumptions 1 thru 7 and $e^h >> 0$ $\forall h \in \mathcal{H}$, a *sunspot equilibrium* always exists.

[3] By convention, for $x, y \in \mathbb{R}^n$, (i) $x >> y$ if $x_i > y_i$ $\forall i$, (ii) $x \geq y$ if $x_i \geq y_i$ $\forall i$, and (iii) $x > y$ if $x \geq y$ and $x \neq y$.

The following example is one in which sunspot effects occur. Let $H = L = S = 2$. The utility function $u^h : \mathbb{R}^6_+ \to \mathbb{R}$ is defined in terms of two utility functions $\left(v^h_0, v^h\right)$:

$$u^h\left(x^h\right) = v^h_0\left(x^h(0), v^h\left(x^h(1)\right), v^h\left(x^h(2)\right)\right)$$

where $v^h : \mathbb{R}^2_+ \to \mathbb{R}$ is defined by:

$$\begin{aligned} v^1\left(x^1(s)\right) &= 1 + \left(\tfrac{1}{8}\right)^{-3} \frac{\left(x^1_1(s)\right)^{-2}}{-2} + \left(\tfrac{7}{8}\right)^{-3} \frac{\left(x^1_2(s)\right)^{-2}}{-2} \quad \forall s \in \mathcal{S}. \\ v^2\left(x^2(s)\right) &= 1 + \left(\tfrac{7}{8}\right)^{-3} \frac{\left(x^2_1(s)\right)^{-2}}{-2} + \left(\tfrac{1}{8}\right)^{-3} \frac{\left(x^2_2(s)\right)^{-2}}{-2} \quad \forall s \in \mathcal{S}. \end{aligned}$$

These prospective utility functions satisfy assumptions 1 and 4. For use later, all equilibria (and any convex combination of equilibria using concavity) satisfy $v^h\left(x^h(s)\right) > 0 \ \forall h, s$. Thus, $v^h : \mathbb{R}^2_+ \to \mathbb{R}_{++} \ \forall h$.

The initial period utility function $v^h_0 : \mathbb{R}^2_+ \times \mathbb{R}^2_{++} \to \mathbb{R}$ is defined by:

$$v^h_0\left(x^h(0), \left(v^h\left(x^h(s)\right)\right)_{\forall s \in \mathcal{S}}\right) = x^h_1(0) + \left(x^h_2(0)\right)^{\alpha^h_1} \left(v^h\left(x^h(1)\right)\right)^{\frac{\alpha^h_2}{2}} \left(v^h\left(x^h(2)\right)\right)^{\frac{\alpha^h_2}{2}} \ \forall h.$$

If $\alpha^h_1 + \alpha^h_2 \leq 1$, then v^h_0 is concave. Set:

$$\begin{aligned} \left(\alpha^1_1, \alpha^1_2\right) &= (0.3, 0.08). \\ \left(\alpha^2_1, \alpha^2_2\right) &= (0.3, 0.5756). \end{aligned}$$

With these values, plus given $v^h : \mathbb{R}^2_+ \to \mathbb{R}_{++} \ \forall h$, then v^h_0 satisfies assumptions 2 and 5.[4]

The endowments for both households are given by:

	$e^h_1(0)$	$e^h_2(0)$	$e^h_1(1)$	$e^h_2(1)$
$h = 1$	16	32	47	2
$h = 2$	32	16	1	46

Recall from assumption 6 that $e^h(s) = e^h(1) \ \forall s \in \mathcal{S}$.

As the prospective utility functions are differentiable, concave, and strictly monotonic, then (a) the first order conditions with respect to the variables x^h and z^h are necessary and sufficient conditions for an optimal solution to the household problem given in definition 4 and (b) the budget constraints in the household problem hold with equality. Solving these system of equations together with the market-clearing conditions yields the following *sunspot equilibrium:*

initial	$\frac{p_1(0)}{p_2(0)} = 33.55$		$q = 0.50$	
period	$x^1(0) = (16.67, 26.26)$	$x^2(0) = (31.33, 21.74)$	$z^1 = -1$	$z^2 = 1$

final	$\frac{p_1(1)}{p_2(1)} = \frac{1}{8}$		$\frac{p_1(2)}{p_2(2)} = 8$	
period	$x^1(1) = (35, 2.5)$	$v^1\left(x^1(1)\right) = 0.672$	$x^1(2) = (45.5, 13)$	$v^1\left(x^1(2)\right) = 0.872$
	$x^2(1) = (13, 45.5)$	$v^2\left(x^2(1)\right) = 0.872$	$x^2(2) = (2.5, 35)$	$v^2\left(x^2(2)\right) = 0.672$

[4]The utility function for v^h_0 is of the general form: $v^h_0\left(x^h(0), \left(v^h\left(x^h(s)\right)\right)_{\forall s}\right) = \sum_l \left(x^h_l(0)\right)^{\gamma^h_l} \left[\prod_s \left(v^h\left(x^h(s)\right)\right)^{\delta^h_l}\right]$ with $\gamma^h = \left(.., \gamma^h_l, ..\right) > 0 \ \ \forall h$, $\delta^h = \left(.., \delta^h_l, ..\right) > 0 \ \ \forall h$, and $\gamma^h_l + S \cdot \delta^h_l < 1 \ \ \forall h, \forall l$.

In this example, the sunspots have real effects since:

$$x^h(1) \neq x^h(2) \quad \forall h.$$

4 Multiple Certainty Equilibria Not Necessary

A *financial equilibrium without extrinsic uncertainty* (a *certainty equilibrium*, for short) is defined by setting $S = 1$. The consumption set is $\tilde{X}^h = \mathbb{R}^{2L}_+$, consumption over all states is defined by a vector $\tilde{x}^h = \left(\tilde{x}^h(0), \tilde{x}^h(1)\right)$, endowments are defined by $e^h = \left(e^h(0), e^h(1)\right) >> 0$, and the utility function $\tilde{u}^h : \tilde{X}^h \to \mathbb{R}$ is defined in terms of two functions $\left(\tilde{v}^h_0, v^h\right)$:

$$\tilde{u}^h(\tilde{x}^h) = \tilde{v}^h_0\left(\tilde{x}^h(0), v^h(\tilde{x}^h(1))\right)$$

where $v^h : \mathbb{R}^L_+ \to \mathbb{R}$ satisfies assumptions 1 and 4 and $\tilde{v}^h_0 : \mathbb{R}^L_+ \times v^h\left(\mathbb{R}^L_+\right) \to \mathbb{R}$ satisfies assumptions 2 and 5. The function v^h remain unchanged, but I use the notation $\tilde{u}^h$ and $\tilde{v}^h_0$ as the dimension of the domain for these functions is strictly less when compared to their counterparts u^h and v^h_0. The economy with the utility form used with extrinsic uncertainty (the previous section) and the economy with the utility form used without extrinsic uncertainty (this section) are only *comparable* if the utility forms satisfy the following definition.

Definition 5 *The economy with and without extrinsic uncertainty are comparable iff, for* $y^h = \left(\tilde{x}^h(1), ..., \tilde{x}^h(1)\right) \in \mathbb{R}^{LS}_+$ *and* $w^h = \left(v^h(\tilde{x}^h(1)), ..., v^h(\tilde{x}^h(1))\right) \in \left(v^h\left(\mathbb{R}^L_+\right)\right)^S$, *the utility forms* $\left(u^h, \tilde{u}^h, v^h_0, \tilde{v}^h_0\right)$ *satisfy* $\tilde{u}^h\left(\tilde{x}^h(0), \tilde{x}^h(1)\right) = u^h\left(\tilde{x}^h(0), y^h\right)$ *and* $\tilde{v}^h_0\left(\tilde{x}^h(0), v^h\left(\tilde{x}^h(1)\right)\right) = v^h_0\left(\tilde{x}^h(0), w^h\right)$ *for any* $\left(\tilde{x}^h(0), \tilde{x}^h(1)\right)$.

The spot commodity prices are defined by $\tilde{p} = (\tilde{p}(0), \tilde{p}(1))$ where $\tilde{p}(0), \tilde{p}(1) \in \mathbb{R}^L_+ \backslash \{0\}$ by assumptions 1 and 2. As above, the prices are normalized so that $\tilde{p}(s) = \left(\left(\frac{\tilde{p}_l(s)}{\tilde{p}_L(s)}\right)_{\forall l<L}, 1\right)$. The household asset choice is denoted by $\tilde{z}^h \in \mathbb{R}$ and the asset price by $\tilde{q} \in \mathbb{R}_+$.

Definition 6 *A financial equilibrium without extrinsic uncertainty is a vector* $\left(\left(\tilde{x}^h, \tilde{z}^h\right)_{\forall h \in \mathcal{H}}, \tilde{p}, \tilde{q}\right)$ *such that*

1. *given* $(\tilde{p}, \tilde{q})$, $\forall h \in \mathcal{H}$:

$$\begin{array}{rcl} \left(\tilde{x}^h, \tilde{z}^h\right) & \in & \arg\max \tilde{u}^h\left(\tilde{x}^h\right) \\ & & \textit{subj. to} \begin{array}{c} \tilde{p}(0)\left(\tilde{x}^h(0) - e^h(0)\right) + \tilde{q}\tilde{z}^h \leq 0 \\ \tilde{p}(1)\left(\tilde{x}^h(1) - e^h(1)\right) - \tilde{z}^h \leq 0 \end{array}. \end{array}$$

2. *markets clear:*

$$\begin{array}{l} \sum_{h \in \mathcal{H}} \tilde{x}^h_l(0) - e^h_l(0) = 0 \quad \forall l \\ \sum_{h \in \mathcal{H}} \tilde{x}^h_l(1) - e^h_l(1) = 0 \quad \forall l \\ \sum_{h \in \mathcal{H}} \tilde{z}^h = 0 \end{array}.$$

Returning to the specific example, $H = L = 2$, in order for the two economies to be *comparable*, the function $\tilde{v}_0^h$ must be given by:

$$\tilde{v}_0^h\left(\tilde{x}^h(0), v^h\left(\tilde{x}^h(1)\right)\right) = \tilde{x}_1^h(0) + \left(\tilde{x}_2^h(0)\right)^{\alpha_1^h}\left(v^h\left(\tilde{x}^h(1)\right)\right)^{\alpha_2^h} \quad \forall h,$$

where the parameter values $(\alpha_1^1, \alpha_2^1) = (0.3, 0.08)$ and $(\alpha_1^2, \alpha_2^2) = (0.3, 0.5756)$ remain unchanged.

I show in theorems 2 and 3 that the *sunspot equilibrium* found in section 3 was not obtained as the randomization over multiple *certainty equilibria*. Definition 7 and theorem 3 are valid for any number of states of extrinsic uncertainty, but for expositional simplicity, consider $S = 2$.

Definition 7 *Given two certainty equilibria* $(\tilde{x}_A(0), \tilde{p}_A(0), \tilde{z}_A, \tilde{q}_A, \tilde{x}_A(1), \tilde{p}_A(1))$ *and* $(\tilde{x}_B(0), \tilde{p}_B(0), \tilde{z}_B, \tilde{q}_B, \tilde{x}_B(1), \tilde{p}_B(1))$, *a sunspot equilibrium* $(x(0), p(0), z, q, x(1), p(1), x(2), p(2))$ *is a randomization of the two certainty equilibria iff:*

$$\begin{aligned}
(x(0), p(0), z, q) &= \theta\left(\tilde{x}_A(0), \tilde{p}_A(0), \tilde{z}_A, \tilde{q}_A\right) \\
&\quad +(1-\theta)\left(\tilde{x}_B(0), \tilde{p}_B(0), \tilde{z}_B, \tilde{q}_B\right) \\
(x(1), p(1)) &= (\tilde{x}_A(1), \tilde{p}_A(1)) \\
(x(2), p(2)) &= (\tilde{x}_B(1), \tilde{p}_B(1))
\end{aligned}$$

for some $\theta \in [0, 1]$ *where the designation of states* $s = 1, 2$ *is arbitrary.*

For the *sunspot equilibrium* $(x(0), p(0), z, q, x(1), p(1), x(2), p(2))$ from section 3, there are two possible *randomizations*: (i) $(\tilde{x}_A(0), \tilde{p}_A(0), \tilde{z}_A, \tilde{q}_A) = (\tilde{x}_B(0), \tilde{p}_B(0), \tilde{z}_B, \tilde{q}_B)$ and (ii) $(\tilde{x}_A(0), \tilde{p}_A(0), \tilde{z}_A, \tilde{q}_A) \neq (\tilde{x}_B(0), \tilde{p}_B(0), \tilde{z}_B, \tilde{q}_B)$. Suppose case (i) holds, then

$$(x(0), p(0), z, q) = (\tilde{x}_A(0), \tilde{p}_A(0), \tilde{z}_A, \tilde{q}_A) = (\tilde{x}_B(0), \tilde{p}_B(0), \tilde{z}_B, \tilde{q}_B)$$

and sunspots have effects as $\tilde{x}_A(1) \neq \tilde{x}_B(1)$. The sunspots $s = 1, 2$ serve to coordinate price beliefs on one of the *certainty equilibria* that arises in the final period.

Required for case (i) to hold is that a *certainty equilibrium* (drop the subscript A and B) exists satisfying $\tilde{x}_2^h(0) = x_2^h(0)\ \forall h$, $\tilde{z}^h = z^h\ \forall h$, and $\tilde{q} = q$. Theorem 2 proves that this is not possible.

Theorem 2 *For the economy given in section 3, there does not exists a certainty equilibrium in which* $\tilde{x}_2^h(0) = x_2^h(0)\ \forall h$, $\tilde{z}^h = z^h\ \forall h$, *and* $\tilde{q} = q$.

Proof. Suppose, for contradiction, that $\tilde{x}_2^h(0) = x_2^h(0)\ \forall h$, $\tilde{z}^h = z^h\ \forall h$, and $\tilde{q} = q$. As the utility functions are differentiable, concave, and strictly monotonic, first order conditions are employed to make the argument concise. Let $\tilde{\lambda}^h(0)$ be the Lagrange multiplier associated with the initial period and $\tilde{\lambda}^h(1)$ the Lagrange multiplier associated with the final period. The first order conditions for household h with respect to $\tilde{x}_2^h(0)$ are given by:

$$\alpha_1^h\left(\tilde{x}_2^h(0)\right)^{\alpha_1^h-1}\left(v^h\left(\tilde{x}^h(1)\right)\right)^{\alpha_2^h} - \tilde{\lambda}^h(0) = 0. \tag{5}$$

For the economy with extrinsic uncertainty, after defining the Lagrange multipliers as $\lambda^h(0)$ and $\left(\lambda^h(s)\right)_{\forall s\in\mathcal{S}}$, respectively, the first order conditions for household h with respect to $x_2^h(0)$ are given by:

$$\alpha_1^h \left(x_2^h(0)\right)^{\alpha_1^h-1} \left(v^h\left(x^h(1)\right)\right)^{\frac{\alpha_2^h}{2}} \left(v^h\left(x^h(2)\right)\right)^{\frac{\alpha_2^h}{2}} - \lambda^h(0) = 0. \tag{6}$$

Given $(\tilde{z}^1, \tilde{z}^2) = (z^1, z^2) = (-1, 1)$ by supposition, the prospective utility functions $\left(v^h\right)_{\forall h}$ and endowments $\left(e^h(1)\right)_{\forall h}$ yield exactly three equilibrium prices and allocations in the final period:[5]

(a) $\frac{\tilde{p}_1(1)}{\tilde{p}_2(1)} = \frac{1}{8}$, $\tilde{x}^1(1) = (35, 2.5)$, $v^1\left(\tilde{x}^1(1)\right) = 0.672$, $\tilde{x}^2(1) = (13, 45.5)$, $v^2\left(\tilde{x}^2(1)\right) = 0.872$.

(b) $\frac{\tilde{p}_1(1)}{\tilde{p}_2(1)} = 1$, $\tilde{x}^1(1) = (42, 6)$, $v^1\left(\tilde{x}^1(1)\right) = 0.834$, $\tilde{x}^2(1) = (6, 42)$, $v^2\left(\tilde{x}^2(1)\right) = 0.834$.

(c) $\frac{\tilde{p}_1(1)}{\tilde{p}_2(1)} = 8$, $\tilde{x}^1(1) = (45.5, 13)$, $v^1\left(\tilde{x}^1(1)\right) = 0.872$, $\tilde{x}^2(1) = (2.5, 35)$, $v^2\left(\tilde{x}^2(1)\right) = 0.672$.

For candidate equilibrium (a), consider household $h = 2$, which satisfies:

$$\left(v^2\left(\tilde{x}^2(1)\right)\right)^{\alpha_2^2} > \left(v^2\left(x^2(1)\right)\right)^{\frac{\alpha_2^2}{2}} \left(v^2\left(x^2(2)\right)\right)^{\frac{\alpha_2^2}{2}}. \tag{7}$$

For candidate equilibria (b) and (c), consider household $h = 1$, which satisfies:

$$\left(v^1\left(\tilde{x}^1(1)\right)\right)^{\alpha_2^1} > \left(v^1\left(x^1(1)\right)\right)^{\frac{\alpha_2^1}{2}} \left(v^1\left(x^1(2)\right)\right)^{\frac{\alpha_2^1}{2}}. \tag{8}$$

Using either inequality (7) or (8) (for the above specified household h) and comparing the first order conditions (5) and (6) with $\tilde{x}_2^h(0) = x_2^h(0)\ \forall h$ by supposition, then $\tilde{\lambda}^h(0) > \lambda^h(0)$.

The first order conditions with respect to $\tilde{z}^h$ and z^h are given by:

$$\begin{array}{l} -\tilde{q}\tilde{\lambda}^h(0) + \tilde{\lambda}^h(1) = 0 \\ -q\lambda^h(0) + \lambda^h(1) + \lambda^h(2) = 0 \end{array}.$$

With $\tilde{q} = q$, then $\tilde{\lambda}^h(1) > \lambda^h(1) + \lambda^h(2)$. However, as (7) or (8) hold and v^h is strictly concave, the first order conditions with respect to $\left(\tilde{x}^h(1), x^h(1), x^h(2)\right)$ imply $\tilde{\lambda}^h(1) < \lambda^h(1) + \lambda^h(2)$.[6] This contradiction finishes the proof. ■

The following theorem holds in general, not just for the economy in section 3. Consider the complementary condition to that assumed in theorem 2: $(\tilde{x}_{A,L}(0), \tilde{z}_A, \tilde{q}_A) \neq (\tilde{x}_{B,L}(0), \tilde{z}_B, \tilde{q}_B)$. Theorem 3 proves that *randomization* with this specification is not possible for the general UBU form.

[5] The inspiration for the functional forms $\left(v^h\right)_{\forall h}$ and endowments $\left(e^h(1)\right)_{\forall h}$ leading to this multiplicity was Kubler and Schmedders (2010).

[6] The first order conditions with respect to $\tilde{x}^h(1)$ imply that $\tilde{\lambda}^h(1)$ is proportional to $D_{\tilde{x}_2^h(1)} v^h\left(\tilde{x}^h(2)\right)$. Specifically, they are given by: $\alpha_2^h \left(\tilde{x}_2^h(0)\right)^{\alpha_1^h} \left(v^h\left(\tilde{x}^h(1)\right)\right)^{\alpha_2^h-1} Dv^h\left(\tilde{x}^h(1)\right) - \tilde{\lambda}^h(1)\left(\frac{\tilde{p}_1(1)}{\tilde{p}_2(1)}, 1\right) = 0_{1\times 2}$. A similar relation holds for the first order conditions with respect to $x^h(1)$ and $x^h(2)$.

Theorem 3 *For $S = 2$ and the general UBU utility form*

$$v_0^h\left(x^h(0), v^h\left(x^h(1)\right), v^h\left(x^h(2)\right)\right) = \sum_l \left(x_l^h(0)\right)^{\gamma_l^h}\left(v^h\left(x^h(1)\right)\right)^{\delta_l^h}\left(v^h\left(x^h(2)\right)\right)^{\delta_l^h}$$

with $\gamma_l^h \geq 0\ \forall(h,l)$, $\gamma_L^h > 0\ \forall h$, $\delta_l^h > 0\ \forall(h,l)$, $\delta_L^h > 0\ \forall h$, and $\gamma_l^h + S\cdot\delta_l^h < 1\ \forall(h,l)$, given certainty equilibria $(\tilde{x}_A(0), \tilde{p}_A(0), \tilde{z}_A, \tilde{q}_A, \tilde{x}_A(1), \tilde{p}_A(1))$ and $(\tilde{x}_B(0), \tilde{p}_B(0), \tilde{z}_B, \tilde{q}_B, \tilde{x}_B(1), \tilde{p}_B(1))$ such that $(\tilde{x}_{A,L}(0), \tilde{z}_A, \tilde{q}_A) \neq (\tilde{x}_{B,L}(0), \tilde{z}_B, \tilde{q}_B)$, then $\forall\theta \in [0,1]$, $\nexists$ a sunspot equilibrium $(x(0), p(0), z, q, x(1), p(1), x(2), p(2))$ that is the randomization of the two certainty equilibria.

Theorem 3 states that multiplicity of *certainty equilibria* is not sufficient for sunspot effects, only multiplicity of *certainty equilibria* satisfying $(\tilde{x}_{A,L}(0), \tilde{z}_A, \tilde{q}_A) = (\tilde{x}_{B,L}(0), \tilde{z}_B, \tilde{q}_B)$.

Proof. Suppose, for contradiction, that $\exists\theta \in [0,1]$ such that the equalities defining *randomization* hold:

$$\begin{aligned}(x(0), p(0), z, q) &= \theta\left(\tilde{x}_A(0), \tilde{p}_A(0), \tilde{z}_A, \tilde{q}_A\right) \\ &\quad +(1-\theta)\left(\tilde{x}_B(0), \tilde{p}_B(0), \tilde{z}_B, \tilde{q}_B\right) \\ (x(1), p(1)) &= \left(\tilde{x}_A(1), \tilde{p}_A(1)\right) \\ (x(2), p(2)) &= \left(\tilde{x}_B(1), \tilde{p}_B(1)\right)\end{aligned} \tag{9}$$

The payout of the risk-free bond in the *sunspot equilibrium* must be the same as that in each of the *certainty equilibria*. Thus, $z^h = \tilde{z}_A^h = \tilde{z}_B^h\ \forall h$.

For the *certainty equilibria*, consider the equations combining the first order conditions with respect to the asset with the first order conditions with respect to the commodity $l = L$:[7]

$$\begin{aligned}\tilde{q}_A\cdot\gamma_L^h\left(\tilde{x}_{A,L}^h(0)\right)^{\gamma_L^h-1}\left(v^h\left(\tilde{x}_A^h(1)\right)\right)^{2\delta_L^h} &= 2\delta_L^h\left(\tilde{x}_{A,L}^h(0)\right)^{\gamma_L^h}\left(v^h\left(\tilde{x}_A^h(1)\right)\right)^{2\delta_L^h-1}\cdot D_{x_L(1)}v^h\left(\tilde{x}_A^h(1)\right). \\ \tilde{q}_B\cdot\gamma_L^h\left(\tilde{x}_{B,L}^h(0)\right)^{\gamma_L^h-1}\left(v^h\left(\tilde{x}_B^h(1)\right)\right)^{2\delta_L^h} &= 2\delta_L^h\left(\tilde{x}_{B,L}^h(0)\right)^{\gamma_L^h}\left(v^h\left(\tilde{x}_B^h(1)\right)\right)^{2\delta_L^h-1}\cdot D_{x_L(1)}v^h\left(\tilde{x}_B^h(1)\right).\end{aligned}$$

The first order conditions can be reduced to:

$$\begin{aligned}\gamma_L^h\frac{\tilde{q}_A}{\tilde{x}_{A,L}^h(0)} &= 2\delta_L^h\frac{D_{x_L(1)}v^h\left(\tilde{x}_A^h(1)\right)}{v^h\left(\tilde{x}_A^h(1)\right)}. \\ \gamma_L^h\frac{\tilde{q}_B}{\tilde{x}_{B,L}^h(0)} &= 2\delta_L^h\frac{D_{x_L(1)}v^h\left(\tilde{x}_B^h(1)\right)}{v^h\left(\tilde{x}_B^h(1)\right)}.\end{aligned} \tag{10}$$

Consider the analogous first order conditions for the *sunspot equilibrium*:

$$\begin{aligned}&q\cdot\gamma_L^h\left(x_L^h(0)\right)^{\gamma_L^h-1}\left(v^h\left(x^h(1)\right)\right)^{\delta_L^h}\left(v^h\left(x^h(2)\right)\right)^{\delta_L^h} = \\ &\delta_L^h\left(x_L^h(0)\right)^{\gamma_L^h}\left(v^h\left(x^h(1)\right)\right)^{\delta_L^h-1}\cdot\left(v^h\left(x^h(2)\right)\right)^{\delta_L^h}\cdot D_{x_L(1)}v^h\left(x^h(1)\right) + \\ &\delta_L^h\left(x_L^h(0)\right)^{\gamma_L^h}\left(v^h\left(x^h(1)\right)\right)^{\delta_L^h}\cdot\left(v^h\left(x^h(2)\right)\right)^{\delta_L^h-1}\cdot D_{x_L(1)}v^h\left(x^h(2)\right).\end{aligned}$$

[7] To obtain these equilibrium conditions, I take advantage of the fact that the utility functions are differentiable, concave, and strictly monotonic. The first order conditions are then necessary and sufficient conditions for an optimal solution to the household problem given in definition 6.

The first order conditions can be reduced to:

$$\gamma_L^h \frac{q}{x_L^h(0)} = \delta_L^h \left(\frac{D_{x_L(1)} v^h \left(x^h(1)\right)}{v^h \left(x^h(1)\right)} + \frac{D_{x_L(1)} v^h \left(x^h(2)\right)}{v^h \left(x^h(2)\right)} \right). \tag{11}$$

Comparing equations (10) and (11), with equalities $v^h \left(x^h(1)\right) = v^h \left(\tilde{x}_A^h(1)\right)$ and $v^h \left(x^h(2)\right) = v^h \left(\tilde{x}_B^h(1)\right)$ using (9), then:

$$\frac{\tilde{q}_A}{\tilde{x}_{A,L}^h(0)} + \frac{\tilde{q}_B}{\tilde{x}_{B,L}^h(0)} = 2 \cdot \frac{q}{x_L^h(0)}. \tag{12}$$

Beginning with (12) and utilizing the definitions $q = \theta\tilde{q}_A + (1-\theta)\tilde{q}_B$ and $x_L^h(0) = \theta\tilde{x}_{A,L}^h(0) + (1-\theta)\tilde{x}_{B,L}^h(0)$, algebra yields:

$$\frac{\tilde{q}_A}{\tilde{x}_{A,L}^h(0)} = \frac{\tilde{q}_B}{\tilde{x}_{B,L}^h(0)}. \tag{13}$$

If $\tilde{q}_A \neq \tilde{q}_B$, without loss of generality $\tilde{q}_A > \tilde{q}_B$, then $\tilde{x}_{A,L}^h(0) > \tilde{x}_{B,L}^h(0)$ $\forall h$. This contradicts market clearing. If $\tilde{q}_A = \tilde{q}_B$, then $\tilde{x}_{A,L}(0) = \left(\tilde{x}_{A,L}^h(0)\right)_{\forall h} = \left(\tilde{x}_{B,L}^h(0)\right)_{\forall h} = \tilde{x}_{B,L}(0)$. This contradicts the supposition in the statement of the theorem that $(\tilde{x}_{A,L}(0), \tilde{z}_A, \tilde{q}_A) \neq (\tilde{x}_{B,L}(0), \tilde{z}_B, \tilde{q}_B)$. ■

Unique certainty equilibrium

The prior result proves that the *sunspot equilibrium* is not the randomization over multiple *certainty equilibria*. In fact, for the specific economy given in section 3, there is only one *certainty equilibrium*:

initial	$\frac{\tilde{p}_1(0)}{\tilde{p}_2(0)} = 32.50$		$\tilde{q} = 0.30$	
period	$\tilde{x}^1(0) = (16.47, 25.09)$	$\tilde{x}^2(0) = (31.53, 22.91)$	$\tilde{z}^1 = -0.859$	$\tilde{z}^2 = 0.859$
final	$\frac{\tilde{p}_1(1)}{\tilde{p}_2(1)} = 0.271$			
period	$\tilde{x}^1(1) = (38.19, 3.53)$	$v^1\left(\tilde{x}^1(1)\right) = 0.765$		
	$\tilde{x}^2(1) = (9.81, 44.47)$	$v^2\left(\tilde{x}^2(1)\right) = 0.863$		

It is straightforward to verify that the above variables satisfy the conditions characterizing *certainty equilibria*.[8] In fact, the equilibrium conditions can be reduced to a system of two equations and this system of two equations only has one solution. The system of two equilibrium equations is constructed in two steps.

I. Final period first order conditions, budget constraints, and market clearing

Taking $\frac{\tilde{p}_1(1)}{\tilde{p}_2(1)}$ and $\tilde{z}^1$ as given, the first order conditions with respect to $\tilde{x}^h(1)$ $\forall h$ yield:

$$\begin{aligned} \left(\frac{1}{8}\right)^{-3} \left(\tilde{x}_1^1(1)\right)^{-3} &= \left(\frac{7}{8}\right)^{-3} \left(\tilde{x}_2^1(1)\right)^{-3} \left(\frac{\tilde{p}_1(1)}{\tilde{p}_2(1)}\right) \\ \left(\frac{7}{8}\right)^{-3} \left(\tilde{x}_1^2(1)\right)^{-3} &= \left(\frac{1}{8}\right)^{-3} \left(\tilde{x}_2^2(1)\right)^{-3} \left(\frac{\tilde{p}_1(1)}{\tilde{p}_2(1)}\right). \end{aligned}$$

[8] See previous footnote.

Using the budget constraints $\left(\frac{\tilde{p}_1(1)}{\tilde{p}_2(1)}\right)\tilde{x}_1^h(1) + \tilde{x}_2^1(1) = \left(\frac{\tilde{p}_1(1)}{\tilde{p}_2(1)}\right)e_1^h(1) + e_2^h(1) + \tilde{z}^h$, the demand functions for commodity $l = 2$ are given by:

$$\begin{aligned}
\tilde{x}_2^1(1) &= \frac{47\left(\frac{\tilde{p}_1(1)}{\tilde{p}_2(1)}\right) + 2 + \tilde{z}^1}{7\left(\frac{\tilde{p}_1(1)}{\tilde{p}_2(1)}\right)^{2/3} + 1} \\
\tilde{x}_2^2(1) &= \frac{\left(\frac{\tilde{p}_1(1)}{\tilde{p}_2(1)}\right) + 46 - \tilde{z}^1}{\frac{1}{7}\left(\frac{\tilde{p}_1(1)}{\tilde{p}_2(1)}\right)^{2/3} + 1}.
\end{aligned}$$

Defining $\rho = \left(\frac{\tilde{p}_1(1)}{\tilde{p}_2(1)}\right)^{1/3}$, the market clearing condition $\tilde{x}_2^1(1) + \tilde{x}_2^2(1) = 48$ reduces to a polynomial of degree 3 :

$$\frac{96}{7}\rho^3 - 48\rho^2 + 48\rho - \frac{144}{7} - \frac{48}{7}\tilde{z}^1 = 0. \tag{14}$$

Equation (14) is the first of two equations characterizing *certainty equilibria.*

II. First order conditions with respect to asset choice

Given the solution to (14), the equilibrium prospective utilities $v^1(\tilde{x}^1(1))$ and $v^2(\tilde{x}^2(1))$ can be calculated.

With the quasilinear utility form for consumption in the initial period (linear in $\tilde{x}_1^h(0)$), the demand functions for commodity $l = 2$ are independent of prices $\frac{\tilde{p}_1(0)}{\tilde{p}_2(0)}$:

$$\begin{aligned}
\tilde{x}_2^1(0) &= 48 \cdot \frac{\left(v^1\left(\tilde{x}^1(1)\right)\right)^{\frac{\alpha_2^1}{1-\alpha_1^1}}}{\left[\left(v^1\left(\tilde{x}^1(1)\right)\right)^{\frac{\alpha_2^1}{1-\alpha_1^1}} + \left(v^2\left(\tilde{x}^2(1)\right)\right)^{\frac{\alpha_2^2}{1-\alpha_1^2}}\right]} \\
\tilde{x}_2^2(0) &= 48 - \tilde{x}_2^1(0).
\end{aligned}$$

Also, the quasilinear utility with equal weight on $\tilde{x}_1^1(0)$ and $\tilde{x}_1^2(0)$ guarantees $\tilde{\lambda}^1(0) = \tilde{\lambda}^2(0)$. Thus, first order conditions with respect to $\tilde{z}^h$ dictate $\tilde{\lambda}^1(1) = \tilde{\lambda}^2(1)$. Using the first order conditions with respect to $\tilde{x}_2^h(1)$, the equality $\tilde{\lambda}^1(1) = \tilde{\lambda}^2(1)$ is equivalently written as:

$$\alpha_2^1\left(\tilde{x}_1^1(0)\right)^{\alpha_1^1}\left(v^1\left(\tilde{x}^1(1)\right)\right)^{\alpha_2^1-1}\left(\frac{7}{8}\right)^{-3}\left(\tilde{x}_2^1(1)\right)^{-3} = \alpha_2^2\left(\tilde{x}_1^2(0)\right)^{\alpha_1^2}\left(v^2\left(\tilde{x}^2(1)\right)\right)^{\alpha_2^2-1}\left(\frac{1}{8}\right)^{-3}\left(\tilde{x}_2^2(1)\right)^{-3}. \tag{15}$$

Equations (14) and (15) are in terms of only two variables: ρ and $\tilde{z}^1$. Once these variables are found, simple linear equations are used to compute all remaining equilibrium variables: $\left(\frac{\tilde{p}_1(0)}{\tilde{p}_2(0)}, \tilde{q}, \tilde{x}^1(0), \tilde{x}^2(0), \tilde{x}^1(1), \tilde{x}^2(1)\right)$. Thus, for multiple *certainty equilibria* to exist, there must exist multiple solutions to the system of two equations (14) and (15).

In particular, as (15) is a linear equation, for multiple *certainty equilibria* to exist, given the equilibrium variable $\tilde{z}^1$, there must exist multiple values for ρ that satisfy (14). Consider the cubic polynomial (14) as a function $f(\rho, \tilde{z}^1) = 0$. The function has two local extrema, a local maximum $f(\rho_1)$ and a local minimum $f(\rho_2)$ with $\rho_1 < \rho_2$. Both ρ_1 and ρ_2 are independent of $\tilde{z}^1$, so define $\tilde{z}_i^1$ such that $f(\rho_i, \tilde{z}_i^1) = 0$ for both i. The function f has three solutions iff $f(\rho_1, \tilde{z}_1^1) \geq 0$ and $f(\rho_2, \tilde{z}_2^1) \leq 0$. Thus, the highest possible ρ such that multiplicity occurs in (14) is $\rho = \overline{\rho}$ such that $f(\overline{\rho}, \tilde{z}_1^1) = 0$ and $f(\rho_1, \tilde{z}_1^1) = 0$. Likewise, the lowest possible ρ such that multiplicity occurs in (14) is $\rho = \underline{\rho}$ such that $f(\underline{\rho}, \tilde{z}_2^1) = 0$ and $f(\rho_2, \tilde{z}_2^1) = 0$. The calculated values provide me with a search window $\rho \in [\underline{\rho}, \overline{\rho}] = [0.285, 2.049]$.

In the range $\rho \in [\underline{\rho}, \overline{\rho}] = [0.285, 2.049]$, use (14) to compute $\tilde{z}^1$ and define the function $g(\rho) = LHS(15) - RHS(15)$. An equilibrium value for ρ is one such that $g(\rho) = 0$. Figure 1 graphs this function over the range $\rho \in [\underline{\rho}, \overline{\rho}] = [0.285, 2.049]$.

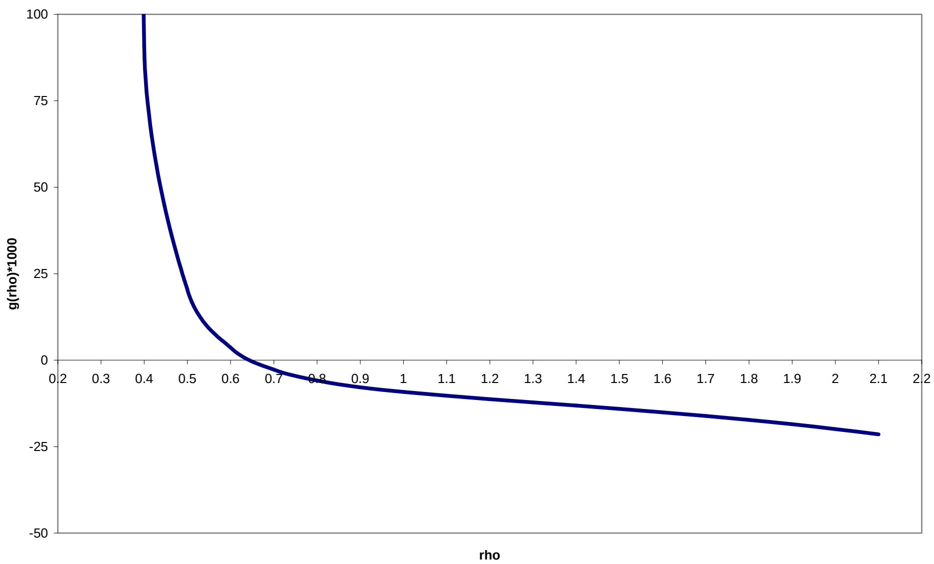

Figure 1: Unique solution to (14) and (15)

The function has a unique solution at the value $\rho = 0.647$. This verifies that the only solution to the system (14) and (15) yields the *certainty equilibrium* previously given.

5 Observations

The example given in the previous two sections, which proves that multiplicity of *certainty equilibria* is not a necessary condition for sunspot effects, is robust to changes in the economy (utility functions and endowments). That is, for an open set of economies around the given one, the same qualitative result holds. Given this result, I can make the following general observation about sunspot effects in financial models with incomplete markets and numeraire assets. Provided that both of the following properties are satisfied for an economy, then sunspots can have real effects independent of the number of underlying *certainty equilibria*:

A. There exists a state of intrinsic uncertainty (in this case the final period) such that, when viewed as a subgame with a fixed initial wealth distribution, the prospective utility functions $\left(v^h\right)_{\forall h}$ and endowments $\left(e^h(1)\right)_{\forall h}$ are such that multiple subgame Walrasian equilibria exist.

B. The UBU utility form is such that the function v_0^h is <u>not</u> additively separable in the vector of initial consumption $x^h(0)$ and the vector of future prospective utility $\left(v^h\left(x^h(s)\right)\right)_{\forall s\in\mathcal{S}}$.

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