The pricing effects of ambiguous private information*

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Abstract

Private information which is ambiguous need not be revealed by market prices in a rational expectations equilibrium. This partial revelation property is due to inertia with respect to information on the part of the reciepient and does not require the presence of noise, unlike in models with unambiguous information. We provide a framework in which to study the conditions for such informational inefficiency to arise and to analyse its implications for market variables. Changes in informational regimes can be endogeneous. We conduct comparative static analysis with respect to wealth share, public information, and individual learning to study shifts in informational regime.

1 Introduction

Asset markets are continually beset by new information. This includes information about counterparty, sovereign, market and other risks. The quality of this information and hence its utility can vary widely both across asset classes and through time. In addition to allocating ownership rights to assets, markets also convey information through the observation of relative prices for the items being transacted.

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This enhanced role of market prices as aggregators and communicators of information underlies the concept of rational expectations equilibrium (REE) as formalized by Allen (1981), Radner (1979), and Grossman and Stiglitz (1980) among others. Existing analyses of REE are primarily in a framework with investors whose decision-making is modeled by the Savage (1954) subjective expected utility (SEU) framework.

As noted by Ellsberg (1961), the SEU framework precludes any role for ambiguity and this has important behavioral implications. Following the seminal work by Ellsberg (1961), several representations for decision-making under ambiguity have been developed. The Gilboa and Schmeidler (1989) representation with multiple priors is one of the most well-known and studied in this class of models.

This paper provides a framework for studying REE with traders who may be ambiguity averse in the sense of Gilboa and Schmeidler (1989). We show that private ambiguous information may not be incorporated and communicated by market prices in REE. This implies that REE are partially revealing and hence the market is informationally inefficient.

The partial revelation property studied here does not rely on the presence of 'noise'. Noise or noise traders are the most commonly used method for constructing partially revealing REE, see for example a recent discussion in Dow and Gorton (2008). In this paper, partial revelation is a direct result of ambiguity aversion as modeled by the Gilboa and Schmeidler (1989) and there are no noise traders.

The model is based on Condie and Ganguli (2010), who showed that the nature of partial revelation considered here has the desirable property of being robust in the context of general financial market economies. A feature which distinguishes partial revelation discussed here from that in some common noise-based models is that information on volume does not change the informational properties of the prices here (see Blume, Easley, and O'Hara (1994) for a discussion).

Incorporating concern for ambiguity in models of financial markets has provided a number of insights that are otherwise puzzling in the standard SEU framework. Epstein and Schneider (2010) is a recent survey of the growing literature on the effects of ambiguity and ambiguity aversion in financial markets. Much of this work is conducted in the context of representative agent or homogeneous information models.

There is little work examining the informational efficiency of prices in the presence of ambiguity. Exceptions include Tallon (1998), Caskey (2008), Ozsoylev and Werner (2009), and Mele and Sangiorgi (2009). However, in these papers, the partial revelation property is driven by noise traders.

The paper proceeds as follows. We first develop the financial market model in section 2. Section 3 describes the nature of partial revelation in this framework and the conditions needed for partial revelation to be possible. Section 4 discusses some comparative statics which illustrate the properties of this form of partial revelation further.

2 A model of ambiguous information

There are two types of investors in the market indexed by $n \in \mathcal{N} = \{A, E\}$ who live for 3 periods and trade assets. Time is indexed by $t = 0, 1, 2$.

There is one asset whose payoff is certain and denoted by V_f, called the risk-free asset or bond.[1] This asset is in zero net supply.

There is another asset whose payoff or terminal value denoted by V is uncertain and it is assumed to have unit net supply. Each investor n is endowed with a fraction $x_0^n > 0$ of the uncertain asset at time 0. Trade occurs in period 1 with the resolution of uncertainty occuring in period 2.

We assume that $\log V$, denoted v henceforth, is normally distributed with mean μ and variance σ^2. In period 0 all investors have identical information about the expected value of the uncertain assets. However, the two types of traders differ in their prior beliefs and in their perception of information as we describe next.

Prior beliefs of A and E investors. Both types of investors believe that v is normally distributed with variance σ^2. However, both types are uncertain about the mean of v. Type E investors' prior beliefs over the mean log payoff, i.e. the mean of v, are given by a normal distribution that has mean μ_0^E and precision ρ_0.

On the other hand, type A investors' have ambiguous beliefs about the distribution of the uncertain asset. These investors' prior beliefs over the mean of v are characterized by a set M_{pr}^A of normal distributions. Each of these prior distributions has precision ρ_0 but its mean is in the set $[\underline{\mu_0}, \overline{\mu_0}]$, i.e. M_{pr}^A is indexed by $[\underline{\mu_0}, \overline{\mu_0}]$. A generic element of $[\underline{\mu_0}, \overline{\mu_0}]$ is denoted by μ_0^A. We refer to this form of ambiguity as

[1]It would perhaps be more appropriate to use the term 'uncertainty-free' to describe this asset in our setting, but we stay with the usual terminology.

ambiguity about fundamentals.

We assume that $\mu_0^E \in \left[\underline{\mu_0}, \overline{\mu_0}\right]$.

Private and public information. In period 0, each investor may receive *a private signal* that conveys information about the mean log payoff μ. For investor n, the signal takes the form $s^n = \mu + \epsilon^n$ where ϵ^n is distributed normally with precision ρ_s and mean μ_s^n. A similar information structure appears in Peress (2009).[2]

Type E investors believe the signal is unbiased, i.e. $\mu_s^E = 0$. However, type A investors may percieve *ambiguity in the signal* in the sense that they know only that $\mu_s^A \in \left[\underline{\mu}_s, \overline{\mu}_s\right]$ where $\underline{\mu}_s < 0 < \overline{\mu}^s$.

This ambiguity in the signal reflects the possibility that the signal provides biased information about the payoff of the asset. Investors may doubt the unbiasedness of a signal because of concerns about the signal source, because the information is intangible in the sense of Daniel and Titman (2006), because the relationship between the signal and the asset is ambiguous, or for other reasons (see for example, Epstein and Schneider (2008) and Illeditsch (2010)).

Aggregate private information in the economy is denoted by the joint signal $\mathbf{s} = (s^n)_{n \in \{A,E\}}$. We denote the set of joint signals by $\mathbf{S}$. We are primarily interested in how ambiguity in the signals affects the informational efficiency of prices and what implications it has for various financial market variables.

In addition to the private signals received by investors, all investors receive information from *a public signal* as well. This signal takes the form $\zeta = \mu + \epsilon_\zeta$, where ϵ_ζ is normally distributed with mean 0 and precision ρ_ζ. For simplicity, we assume that there is no ambiguity in the public signal given that ambiguity may be present in the prior beliefs and in the private signals. Moreover, since we focus on the effects of partially-revealing prices that do not reveal ambiguous private signals and the public signal will always be revealed, this additional generality is not added.

2.1 Decision making

The investors maximize the expected utility of terminal wealth W_2. Their von Neumann-Morgenstern utility denoted by u^n is in the constant relative risk aversion class

[2]Peress (2009) corrects an error in Peress (2004).

(CRRA) with common CRRA coefficient γ, i.e.

$$u^n(W_2) = \frac{W_2^{1-\gamma}}{1-\gamma}. \tag{1}$$

If investors percieve ambiguity after incorporating all information from private and public signals and from prices, their decision-making is modeled using using the Gilboa and Schmeidler (1989) representation. Denoting by M^n the set of distributions representing investor n's beliefs given his information, the utility from a portfolio θ^n is

$$U^n(\theta) = \min_{m\in M^n} \mathbb{E}_m[u^n(W_2)] = \min_{m\in M^n} \mathbb{E}_m\left(\frac{W_2^{1-\gamma}}{1-\gamma}\right) \tag{2}$$

2.2 Market prices and rational expectations equilibria

Trade in the assets occurs in period 1 and equilibrium requires that markets for all assets clear. Market prices play the role of information aggregators and communicators through a price function.

A price function Φ is a mapping between $\mathbf{S}$ and the set of prices, i.e. $\Phi(\mathbf{s}) = (P, R_f)$, where P denotes the price of the uncertain asset while R_f denotes the return on the risk-free asset. Information is revealed through prices when the prevailing market prices under two joint signals that convey different information are different, i.e. the function Φ is invertible. When this occurs for all joint signals, market participants can correctly infer the information in the joint signal by observing the prices in the market and the price function Φ is said to be *fully-revealing*.

The market price may not reveal all privately held information if the function from joint signal information into equilibrium prices is not invertible. In this case, the function is said to be *partially revealing*. When prices are partially revealing, more than one joint signal may be consistent with the observed price. In general, upon observing the market prices (P, R_f), each investor knows that the joint signal $\mathbf{s}$ is in the set $\Phi^{-1}(P, R_f)$.

The holdings of investor n in the uncertain and risk-free assets are x_t^n and $b_t^n, t = 0, 1$, respectively. Hence, initial wealth for investor n at price P is $W_0^n = x_0^n P$, whereas the terminal or period 2 wealth of investor n at time 2 given choices in period 1 is $W_2^n = x_1^n P + b_1^n$. The fraction of wealth put into the risky asset at time 1 is labeled θ_1^n.

By definition

$$x_1^n = \frac{\theta_1^n W_0^n}{P} \tag{3}$$

The market clearing conditions for the assets are

$$\begin{aligned} &\sum_n \frac{\theta_1^n}{P} W_0^n = 1 \\ &\sum_n (1-\theta_1^n) R_f W_0^n = 0. \end{aligned} \tag{4}$$

We now provide a definition of rational expectations equilibrium (REE) for this setting.

Definition 1. A *rational expectations equilibrium* is a set of portfolio weights $\{\theta_1^n(\mathbf{s})\}_{n\in\mathcal{N}}$ and a price function Φ, which specifies prices $P(\mathbf{s})$ and $R_f\mathbf{s}$ for each information signal $\mathbf{s}$ such that the following hold for each joint signal σ.

1. Each investor n has information s^n and $\Phi^{-1}(P(\mathbf{s}), R_f(\mathbf{s}))$ and chooses a portfolio θ_1^n that satisfies
$$\theta_1^n \in \arg\max U^n(\theta | s^n, \Phi^{-1}(P(\mathbf{s}), R_f(\mathbf{s}))) \tag{5}$$

2. The market clearing equations given in (4) are satisfied.

Given this definition an REE is said to be *fully-revealing* when the equilibrium price function is fully-revealing and it is said to be *partially-revealing* otherwise. A fully-revealing REE represents (strong-form) market efficiency. We are primarily interested in the case of market inefficiency, i.e. partially-revealing REE.

2.3 An approximate solution to the model.

To solve this model we use the method developed by Campbell and Viciera (2002) to approximate returns. As noted there, this solution method becomes exact as the discrete time interval shrinks to zero. We discuss here the approximation as applicable to type A investors, since this covers the case of type E investors also. Let M^n denote the set of distributions that represent the beliefs of investor n conditional on any information that she may have received. Let σ^2 denote the conditional variance of the investor's log portfolio payoff.

We begin by approximating the return on initial wealth W_0^n as a function of the returns to the individual assets. Throughout, lowercase letters represent the natural log of model variables. Given n's portfolio $(\theta_1^n, 1-\theta_1^n)$ and using $R = V/P$ to denote the return on the uncertain asset,

$$W_2^n = W_0^n(\theta_1^n R + (1-\theta_1^n)R_f). \tag{6}$$

Expressing the returns in natural logs implies

$$\begin{aligned}\log \frac{\theta_1^n R + (1-\theta_1^n)R_f}{R_f} &= \log\left(\theta_1^n \frac{R}{R_f} + 1 - \theta_1^n\right) \\ &= \log(1 + \theta_1^n \left(e^{r-r_f} - 1\right))\end{aligned} \tag{7}$$

We then form a second order Taylor series approximation around the point $r - r_f = 0$ to get

$$\theta_1^n \left(r - r_f\right) + \frac{1}{2}\theta_1^n (r - r_f)^2. \tag{8}$$

We use the unconditional expectation and obtain

$$\theta_1^n \left(r - r_f\right) + \frac{1}{2}\theta_1^n (1-\theta_1^n)\sigma^2 \tag{9}$$

as our approximation of returns.

If terminal wealth W_2^n is lognormally distributed then the solution to the individual's optimization problem is equivalent to the solution to

$$\max_{\theta} \min_{m \in M^n} \log \mathbb{E}_m \left[\frac{(W_2^n)^{1-\gamma}}{1-\gamma}\right]. \tag{10}$$

The term $\log \mathbb{E}_m \left[(W_2^n)^{1-\gamma}\right]$ by the lognormality of W_2^n can be rewritten as

$$\begin{aligned}\mathbb{E}_m \log(W_2^n)^{1-\gamma} + \frac{1}{2} Var \log(W_2^n)^{1-\gamma} &= \\ (1-\gamma)\mathbb{E}_m w_2^n + \frac{1}{2}(1-\gamma)^2 Var \log W_2^n &= \\ (1-\gamma)\mathbb{E}_m w_0^n + \log(\theta_1^n R + (1-\theta_1^n)R_f) + \frac{1}{2}(1-\gamma)^2\sigma^2&.\end{aligned} \tag{11}$$

Since w_0^n and r_f are both non-stochastic and $1-\gamma$ is a scale factor that won't affect the solution to the problem, solving the optimization problem is equivalent to solving

$$\max_{\theta} \min_{m \in M^n} \mathbb{E}_m \log(\theta R + (1 - \theta R_f)) - r_f + \frac{1-\gamma}{2}\sigma^2. \tag{12}$$

Using the approximation given in (9), we can rewrite the optimization problem as

$$\max_{\theta}\min_{m\in M^n}\mathbb{E}_m\theta\left(r-r_f\right)+\frac{\theta(1-\theta)}{2}\sigma^2+\frac{(1-\gamma)\theta^2}{2}\sigma^2. \tag{13}$$

The first order conditions for investor n are given by

$$0\in\left\{\mathbb{E}_m r-r_f+\frac{1}{2}\sigma^2-\gamma\theta\sigma^2 : m\in M^n\right\}. \tag{14}$$

Therefore, investor n will not hold a position in the risky asset if and only if

$$\min_{m\in M^n}\mathbb{E}_m v-r_f+\frac{1}{2}\sigma^2-p\le 0\le\max_{m\in M^n}\mathbb{E}_m v-r_f+\frac{1}{2}\sigma^2-p. \tag{15}$$

Using $[\underline{\mu}^n,\overline{\mu}^n]$ to denote the interval of means given by the set of distributions M, investor n will not hold the uncertain asset if

$$\underline{\mu}^n-r_f+\frac{1}{2}\sigma^2\le p\le\overline{\mu}^n-r_f+\frac{1}{2}\sigma^2 \tag{16}$$

This implies that the optimal portfolio weight for the uncertain asset is given by

$$\theta_1^n(M^n)=\begin{cases}\frac{1}{\gamma\sigma^2}\left(\underline{\mu}-r_f+\frac{1}{2}\sigma^2-p\right) & \underline{\mu}-r_f+\frac{1}{2}\sigma^2-p>0\\ 0 & \underline{\mu}-r_f\le p-\frac{1}{2}\sigma^2\le\overline{\mu}-r_f\\ \frac{1}{\gamma\sigma^2}\left(\overline{\mu}-r_f+\frac{1}{2}\sigma^2-p\right) & \overline{\mu}-r_f+\frac{1}{2}\sigma^2-p<0\end{cases} \tag{17}$$

2.4 Information and updating

In order to understand the value of private information and the nature of partial revelation in this model, we must first specify how this information is incorporated into the beliefs of investors. In the present framework, information is processed and incorporated using an updating rule developed in Epstein and Schneider (2007). This updating rule includes standard Bayesian updating with unambiguous beliefs as a special case.

In Bayesian updating, conditional beliefs about the probability of an uncertain event A are found through the use of a prior distribution that reflects the beliefs of the decision maker prior to receiving the information and a likelihood function that expresses the relationship between the signal and the parameter that the decision maker uses in making decisions. In particular, assume that the probability of the event A depends on a parameter B over which the decision maker has a prior denoted $f(B)$. Suppose further that given a parameter value B_0, the likelihood of receiving a signal s is given by $l(s|B_0)$. Bayes' rule then indicates that the updated distribution of B conditional on having observed the signal s is

$$f(B|s) = \frac{f(B)l(s|B)}{\int l(s|B)dB}. \tag{18}$$

Epstein and Schneider (2007) look at the set of possible Bayes updates that arise from the set of possible likelihoods and the set of possible priors. For this example, if the set of priors is denoted $\{f_i(B)\}_{i\in I}$ for some index set I and the set of likelihoods is $\{l_j(s|B)\}_{j\in J}$ for some index set J, then the set of updated beliefs is given by

$$\{f(B|s)\} = \left\{\frac{f_i(B)l_j(s|B)}{\int l_j(s|B)dB} : i \in I, j \in J\right\} \tag{19}$$

For the model with ambiguous beliefs presented here, the set of priors over the mean of v is the set of all normal distributions with mean in the set $\left[\underline{\mu}_0, \overline{\mu}_0\right]$. For any $\mu_0 \in \left[\underline{\mu}_0, \overline{\mu}_0\right]$, the set of likelihoods is the set of normal distributions with mean $\mu_0 + \mu_s$ with μ_s in the set $\left[\underline{\mu}_s, \overline{\mu}_s\right]$.

Standard results on Bayesian updating with normal distributions imply that for any $\mu_0 \in \left[\underline{\mu_0}, \overline{\mu_0}\right]$ and $\mu_s \in \left[\underline{\mu_s}, \overline{\mu_s}\right]$, the mean of v, conditional on having observed the signal s is normally distributed with mean

$$\mu|s = \frac{\rho_0\mu_0 + \rho_s\left(s + \mu_s\right)}{\rho_0 + \rho_s} \tag{20}$$

and precision

$$\rho|s = \rho_0 + \rho_s. \tag{21}$$

Therefore, the set of updated priors representing the ambiguity of an investor is the set of normal distributions with precision $\rho_0 + \rho_s$ and means

$$\{\mu|s\} = \left\{\frac{\rho_0\mu_0 + \rho_s\left(s + \mu_s\right)}{\rho_0 + \rho_s} : \mu_0 \in \left[\underline{\mu_0}, \overline{\mu_0}\right], \mu_s \in \left[\underline{\mu_s}, \overline{\mu_s}\right]\right\} \tag{22}$$

This can be extended to the case of observing both private and public signals. The updated set of priors is the set of all normal distributions with precision $\rho_0 + \rho_s + \rho_\zeta$ and means

$$\{\mu|s,\zeta\} = \left\{ \frac{\rho_0\mu_0 + \rho_\zeta\zeta + \rho_s\left(s+\mu_s\right)}{\rho_0+\rho_\zeta+\rho_s} : \mu_0 \in \left[\underline{\mu_0}, \overline{\mu_0}\right], \mu_s \in \left[\underline{\mu}_s, \overline{\mu}_s\right] \right\} \tag{23}$$

In what follows, we will denote the set of distributions after observing the private signal s and public signal ζ by $M^n(s)$ for investor n. We suppress the dependence on the public signal for simplicity of exposition since it is observed by all investors.

3 Partial revelation

To discuss partial revelation, we use $M^n(s)$ to denote the set of distributions for investor n after she observes private signal s and use $[\underline{\mu}|s, \overline{\mu}|s]$ to denote the corresponding interval of means (of v). For ease of exposition in this section, we will not consider the reciept of a public signal.

Hence, given the optimal portfolio expression provided in (17), investor n will hold no position in the uncertain asset under both signal realisations s and s' if $[\underline{\mu}|s, \overline{\mu}|s] \cap [\underline{\mu}|s', \overline{\mu}|s']$ is non-empty and

$$\max\{\underline{\mu}|s, \underline{\mu}|s'\} \le p_1 + r_f - \frac{1}{2}\sigma^2 \le \min\{\overline{\mu}|s, \overline{\mu}|s'\}. \tag{24}$$

This observation is key in the existence of partially-revealing equilibria.

3.1 A benchmark: unambiguous information and full revelation

To see this, first consider a market that begins with all investors having homogeneous information and participating in the market or equivalently, assume that all information is unambiguous and so each investor's beliefs can be represented by a single probability distribution.

For now, assume that the investors are long in the uncertain asset. The uncertain-asset portfolio weight for investor n is

$$\theta_1^n = \frac{1}{\gamma\sigma^2}\left(\mu^n + \frac{1}{2}\sigma^2 - p\right) \tag{25}$$

where μ^n denotes the mean of v under investor n's beliefs.

This implies that the market clearing price P^0 of the uncertain asset is characterized by the equation

$$\sum_{n\in\{A,E\}} \frac{x_0^n P^0}{P^0 \gamma \sigma^2}\left(\mu^n + \frac{1}{2}\sigma^2 - P^0\right) = 1. \tag{26}$$

Hence, with $p^0 = \log P^0$ and $\mu^n = \underline{\mu}$ under the current assumption of homogeneous beliefs, we have

$$p^0 = \underline{\mu} + \frac{1-2\gamma}{2}\sigma^2. \tag{27}$$

This price demonstrates features common to many asset pricing models. In particular, recall that μ denotes the true, but unknown, mean of $v(=\log V)$. Then the expected gross return on the risky asset is

$$\begin{aligned} ER &= E\left(\frac{V}{P^0}\right) \\ &= \frac{e^{\mu+\frac{1}{2}}}{\sigma^2} e^{\underline{\mu}+\frac{1-2\gamma}{2}\sigma^2} \\ &= e^{\mu-\underline{\mu}-\gamma\sigma^2} \end{aligned} \tag{28}$$

Taking the natural log of this expected gross return shows that the return incorporates an ambiguity premium in the amount of $\mu - \underline{\mu}$ that compensates investors for the ambiguity about model primitives (see Cao, Wang, and Zhang (2005), Easley and O'Hara (2009), and Ui (2010) for discussions about ambiguity premia).

There is also a term representing the risk premium $\gamma\sigma^2$ which as expected depends positively on the degree of risk aversion γ and positively on the variance σ^2 of the asset payoff.

Now consider the case when some investors, say type A, observe a private signal s. As a benchmark, first assume that the signal is known by all market participants to be unambiguous and hence unbiased.[3] Further assume that there is no ambiguity about the fundamentals.

Since $\underline{\mu_s} = \overline{\mu_s} = 0$ and $\underline{\mu_0} = \overline{\mu_0} = \mu_0$, the updating rule given in equation (22) collapses to

$$E\left[\mu|s\right] = \frac{\rho_0\mu_0 + \rho_s s}{\rho_0 + \rho_s} \tag{29}$$

[3]That is, we assume that even those who have not observed the signal know that it is unbiased.

The segment of the market that has received this information will seek to trade on it and as such will have demand given by

$$\theta_1^n = \frac{1}{\gamma\sigma^2}\left(\frac{\rho_0\mu_0 + \rho_s s}{\rho_0 + \rho_s} + \frac{1}{2}\sigma^2 - p\right). \tag{30}$$

Suppose that the segment of the market that has not received the signal, i.e. type E, is not be able to infer the signal from observation of the market price and hence forms asset demand using its prior uninformed beliefs. Under this assumption, with P^1 denoting price and $p^1 = \log P^1$, the market clearing condition for the uncertain asset is

$$\frac{x_0^A P^1}{P^1\gamma\sigma^2}\left(\frac{\rho_0\mu_0 + \rho_s s}{\rho_0 + \rho_s} + \frac{1}{2}\sigma^2 - p^1\right) + \frac{x_0^E P^1}{P^1\gamma\sigma^2}\left(\mu_0 + \frac{1}{2}\sigma^2 - p^1\right) = 1 \tag{31}$$

It can be directly verified that unless $s = \mu_0$, the (log) price p^0 given in (27) will not clear the market for the risky asset at time 1. That is, $p^0 = p^1$ is not an equilibrium. As such, for all signals that induce a change in demand by a portion of the market, the equilibrium price will change. A similar observation appears in Grossman (1981) for investors with unambiguous beliefs.

Therefore, the observation of market prices, as envisaged in the REE framework, typically allows uninformed investors to infer the signal received by informed investors when that signal is unambiguous. With this understanding, the equilibrium price can be calculated as

$$p^1 = \frac{\rho_0\mu_0 + \rho_s s}{\rho_0 + \rho} + \frac{1-2\gamma}{2}\sigma^2. \tag{32}$$

The change in log expected gross returns as a result of the information that has been fully revealed is

$$\log\mathbb{E}R^1 - \log\mathbb{E}R^0 = \frac{\rho_s(s-\mu_0)}{\rho_0 + \rho_s} \tag{33}$$

where $R^1 = V/P^1$ and $R^0 = V/P^0$.

This difference is the change in log expected return that comes from the fact that the expected payoff has changed. As usual, this term depends on the degree to which the signal s represents a surprise relative to the prior expected log payoff μ_0. When s represents very good news about the asset ($s - \mu_0$ is large), the expected return to the asset increases significantly. An analogous result holds for large negative surprises.

3.2 Ambiguous information and partial revelation

Now consider the receipt of an ambiguous private signal by A traders. Using $[\underline{\mu}|s, \overline{\mu}|s]$ to denote the interval of means given by the updating rule provided in (23) and the optimal portfolio expression provided in (17), the demand for the uncertain asset from type A traders is

$$\theta_1^A(s) = \begin{cases} \frac{1}{\gamma\sigma^2}\left(\underline{\mu}|s + \frac{1}{2}\sigma^2 - p\right) & \underline{\mu}|s + \frac{1}{2}\sigma^2 - p > 0 \\ 0 & \underline{\mu}|s \leq p - \frac{1}{2}\sigma^2 \leq \overline{\mu}|s \\ \frac{1}{\gamma\sigma^2}\left(\overline{\mu}|s + \frac{1}{2}\sigma^2 - p\right) & \overline{\mu}|s + \frac{1}{2}\sigma^2 - p < 0 \end{cases} \tag{34}$$

Let the beliefs of type E investors, who have not recieved the signal, about the mean of v be denoted μ_{PR}^E. We will be explicit about the construction of these beliefs shortly.

3.2.1 Non participation and partial revelation

In the form of partial revelation to be discussed here those that have received the ambiguous signal will not participate in the market. Therefore, the market clearing price P_1 and $p_1 = \log P_1$ will satisfy

$$\frac{x_0^E P_1}{P_1\gamma\sigma^2}\left(\mu_{PR}^E + \frac{1}{2}\sigma^2 - p_1\right) = 1 \tag{35}$$

Hence, the equilibrium price is given by

$$p_1 = \mu_{PR}^E + \frac{x_0^E - 2\gamma}{2x_0^E}\sigma^2 \tag{36}$$

This price, in addition to bearing the typical ambiguity and risk aversion components, also is affected by the non-participation in the market. Since the total number of market participants is now smaller, those that are holding the asset are holding more risk and must be compensated for it. As the wealth share of those who are uninformed approaches unity, the second term in p_1 approaches the second term in p^0 as given in (27).

Note, that since the A investors are endowed with a positive amount of the uncertain asset, i.e. $x_0^A = \alpha^A > 0$, the above is not a no-trade equilibrium. Morever, there is no indeterminacy in equilibrium prices. Also, unlike some commonly used models with noise-based partial revelation, providing investors with information on volume does change the partially revealing equilibria to a fully revealing one, see Blume, Easley, and O'Hara (1994) for a discussion.

The necessary condition for the signal recipients, i.e. type A investors, to not participate in the market in equilbirum can be obtained by using the demand given in equation (34) and the price in equation (36),

$$\begin{aligned} \underline{\mu}|s \leq \mu_{PR}^E + \frac{x_0^E - 2\gamma}{2x_0^E}\sigma^2 - \frac{1}{2}\sigma^2 \leq \overline{\mu}|s \\ \underline{\mu}|s \leq \mu_{PR}^E - \frac{\gamma\sigma^2}{x_0^E} \leq \overline{\mu}|s \end{aligned} \tag{37}$$

These inequalities and the expressions for updated beliefs $\underline{\mu}|s$ and $\overline{\mu}|s$ can be used to characterise the beliefs of the uninformed E investors under partial revelation as we describe next.

3.3 Prices and uninformed beliefs under partial revelation

Rearranging the above equation shows that if the group of uninformed investors notices that the informed investors have dropped out of the market, they can infer a range of signals that may have been received. Using the explicit expressions for $\underline{\mu}|s$ and $\overline{\mu}|s$ and rearranging terms, this range for s is

$$\mu_{PR}^E - \underline{\mu_s} + \frac{\rho_0}{\rho_s}(\mu_{PR}^E - \underline{\mu_0}) - \frac{\rho_0 + \rho_s}{\rho_s}\frac{\gamma\sigma^2}{x_0^E} \geq s \geq \mu_0^E - \overline{\mu_s} + \frac{\rho_0}{\rho_s}(\mu_0^E - \overline{\mu_0}) - \frac{\rho_0 + \rho_s}{\rho_s}\frac{\gamma\sigma^2}{x_0^E} \tag{38}$$

Given the information structure in our model, for the beliefs of type E investors to be consistent with the non-participation of the informed group of A investors, the E investors will know that the signal observed by A investors falls within a bounded interval $[a, b]$ of potential signals.

3.3.1 The price function

Equation (38) in effect implies that the price function Φ_{PR} under partial revelation is non linear in the signals. While the price function in the commonly used Grossman and Stiglitz (1980) framework is linear, this heavily depends on the assumption on normal distributions and CARA utility with non wealth constraints. Other models of noise-based partial revelation with different distributional or utility assumptions such as Mailath and Sandroni (2003) and Barlevy and Veronesi (2003) or wealth constraints such as Yuan (2005) provide non-linear price functions under partial revelation as well.

3.3.2 Uninformed beliefs

The boundaries of the interval $[a,b]$ are determined endogenously according to equation (38). Were the uninformed investors able to observe the signal s, their updated beliefs would be

$$\mathbb{E}[\mu|s] = \frac{\rho_0\mu_0 + \rho_s s}{\rho_0 + \rho_s} \tag{39}$$

Since the uninformed are not able to observe the signal and are expected utility maximizers, they find the expected value of updated beliefs conditional on the knowledge that the signal observed is in $[a,b]$. If $f(s;a \le s \le b)$ is the marginal probability density function over signals conditional on the signal being between a and b then this expected value is

$$\begin{aligned}\mathbb{E}\left[\mu|a \le s \le b\right] &= \int_a^b \frac{\rho_0\mu_0 + \rho_s s}{\rho_0 + \rho_s} f(s;a \le s \le b)ds \\ &= \frac{1}{\rho_0+\rho_s}\left(\rho_0\mu_0 + \rho_s \int_a^b s f(s;a \le s \le b)ds\right) \\ &= \frac{1}{\rho_0+\rho_s}\left(\rho_0\mu_0 + \rho_s E[s|a \le s \le b]\right)\end{aligned} \tag{40}$$

By definition $s = \mu + \epsilon$ where μ and ϵ are independent, normally distributed random variables. Under the assumption that the uninformed are expected utility maximizers who do think that the signal is unbiased, s is normally distributed with mean μ_0 and precision $\rho_0\rho_s/(\rho_0+\rho_s)$. The expected value of s conditional on s being in the interval $[a,b]$ is therefore

$$\mathbb{E}[s|s \in [a,b]] = \mu_0 + \Delta(a,b) \tag{41}$$

where

$$\Delta(a,b) = \frac{\phi\left(\sqrt{\frac{\rho_0\rho_s}{\rho_0+\rho_s}}(a-\mu_0)\right) - \phi\left(\sqrt{\frac{\rho_0\rho_s}{\rho_0+\rho_s}}(b-\mu_0)\right)}{\Phi\left(\sqrt{\frac{\rho_0\rho_s}{\rho_0+\rho_s}}(b-\mu_0)\right) - \Phi\left(\sqrt{\frac{\rho_0\rho_s}{\rho_0+\rho_s}}(a-\mu_0)\right)}\sqrt{\frac{\rho_0\rho_s}{\rho_0+\rho_s}}. \tag{42}$$

The above is derived from the properties of the truncated normal distribution.

Simplification on expression (40) gives

$$\mu_{PR}^{E} = \mathbb{E}\left[\mu | a \leq s \leq b\right] = \mu_0 + \frac{\rho_s}{\rho_0 + \rho_s}\Delta(a,b). \tag{43}$$

The term

$$\frac{\rho_s}{\rho_0 + \rho_s}\Delta(a,b) \tag{44}$$

represents the change in beliefs of an expected utility maximizing investor when that investor knows only that the ambiguity averse investors received a signal that has caused them to not participate in the market. The cutoffs a and b are derived endogenously in the model.

Plugging these beliefs into the non-participation condition (38) gives

$$\frac{\rho_0+\rho_s}{\rho_s}\left(\mu_0 - \frac{\gamma}{\rho x_0^E}\right) - \frac{\rho_0}{\rho_s}\underline{\mu_0} - \underline{\mu_s} + F(a,b) \geq s \geq \frac{\rho_0+\rho_s}{\rho_s}\left(\mu_0 - \frac{\gamma}{\rho x_0^E}\right) - \frac{\rho_0}{\rho_s}\overline{\mu_0} - \overline{\mu_s} + F(a,b) \tag{45}$$

Since by definition the left hand side of this inequality is b and the far right hand side is a, the signal bounds that are consistent with the behavior of the informed agents are found by solving the system of equations

$$\begin{aligned} \mu_0 - \frac{\rho_0}{\rho_s}\overline{\mu_0} - \overline{\mu_s} + \Delta(a,b) - \frac{\rho_0+\rho_s}{\rho_s}\frac{\gamma\sigma^2}{x_0^E} &= a \\ \mu_0 + \frac{\rho_0}{\rho_s}\underline{\mu_0} + \underline{\mu_s} + \Delta(a,b) - \frac{\rho_0+\rho_s}{\rho_s}\frac{\gamma\sigma^2}{x_0^E} &= b \end{aligned} \tag{46}$$

A few things become apparent from the analysis of this system of equations. The first is that the length of the interval of received signals that the uninformed believe the informed to have received is related to the amount of ambiguity in the signal. To see this, subtract the second equation from the first and rearrange to obtain

$$b - a = \frac{\rho_0}{\rho_s}(\overline{\mu_0} - \underline{\mu_0}) + \overline{\mu_s} - \underline{\mu_s}. \tag{47}$$

From this, this system can be reduced to a system of one equation in one unknown, for example, letting $\chi = b - a$ (which is exogenous) this system becomes

$$F(a) = \mu_0 - \frac{\rho_0}{\rho_s}\overline{\mu_0} - \overline{\mu_s} + \Delta(a, a+\chi) - \frac{\rho_0 + \rho_s}{\rho_s}\frac{\gamma\sigma^2}{x_0^E} - a = 0. \tag{48}$$

This implicit equation in a can be solved numerically. However, it is possible to draw some interesting conclusions without solving the above equation, and we proceed to these next.

4 Comparative statics: wealth share, public signal, learning

The partial revelation conditions can be used for comparative statics exercises with respect to wealth shares, public signals, individual learning to analyse switches between information regimes.

Recall that the conditions for partial revelation are given by

$$\underline{\mu}|s \leq \mu_{PR}^E - \frac{\gamma\sigma^2}{x_0^E} \leq \overline{\mu}|s \tag{49}$$

which can be rearranged to get

$$\frac{\sigma^2\gamma}{\mu_{PR}^E - \underline{\mu}|s} \leq x_0^E \leq \frac{\sigma^2\gamma}{\overline{\mu}|s - \mu_{PR}^E} \tag{50}$$

As can be seen from the above inequalities, the wealth share of type E investors, more generally the uninformed investors, the differences in beliefs between the A and E investors, and the ambiguity percieved by the A investors play a role in determining whether or not the economy is in a partially revealing equilibrium.

4.1 Wealth share

A change in the wealth share of E investors as captured by a change in x_0^E can lead to a switch in informational regimes. Consider the first inequality. If E investors are relatively wealthy compared to A investors, as captured by this inequality, then the economy may have a partial revelation information regime.

The inuition for this is that if A investors are poor enough, then their effect on asset prices is relatively small. This means that in equilibrium it is possible for them to sell off their initial asset endowment and hold only the risk-free asset. E investors hold onto all the uncertainty in the economy and as noted previously are compensated in the form of a risk premium and an ambiguity premium.

4.2 Learning by A investors

So far, we have only discussed the scenario where A investors observe a signal once. However, the analysis extends in a straightforward manner to the case where the A investors observe more than one signal and learn.

The updated beliefs in the case of multiple private signals with the same precision, say K signals, is given by set of normal distributions with precision $\rho_0 + L\rho_s$ and means

$$\{\mu|s_1,\ldots,s_K\} = \left\{ \frac{\rho_0\mu_0 + \rho_s \sum_{k=1}^{K}\left(s_k + \mu_{sk}\right)}{\rho_0 + +K\rho_s} : \mu_0 \in \left[\underline{\mu_0}, \overline{\mu_0}\right], \mu_{sk} \in \left[\underline{\mu_{sk}}, \overline{\mu_{sk}}\right] \text{ for all } k = 1,\ldots,K \right\}. \tag{51}$$

We denote the interval of means of v by $[\underline{\mu}|(s_1,\ldots,s_K), \overline{\mu}|(s_1,\ldots,s_K)]$ in this case. The partial revelation condition then becomes

$$\frac{\sigma^2}{\mu_{PR}^E - \underline{\mu}|(s_1,\ldots,s_K)} \leq x_0^E \leq \frac{\sigma^2}{\overline{\mu}|(s_1,\ldots,s_K) - \mu_{PR}^E} \tag{52}$$

In this case, as information accumulates the interval $[\underline{\mu}|(s_1,\ldots,s_K), \overline{\mu}|(s_1,\ldots,s_K)]$ shrinks. Thus, if the economy is in a partial revelation regime and the A investors are learning, there may be no change in the regime until enough information accumulates and the interval is reduced to a stage where one of the inequalities is violated. This will imply a switch to a full revelation regime.

This shift in information regime accompanied by seeming shift in individual choices is similar to a phenomenon in the presence of adjustment costs noted by Caplin and Leahy (1994). In that setting, adjustment costs meant that individuals did not change choices despite changes in information, while here until there is enough reduction in the ambiguity percieved by investors, there is inertia in response to changes in information.

4.3 Public signals

As noted earlier, the discussion so far has excluded a public signal. Such a signal affects the beliefs of both A and E investors. In particular, it implies that the differences in beliefs as captured by $(\mu_{PR}^E - \underline{\mu}|s)$ and $(\overline{\mu}|s - \mu_{PR}^E)$ are reduced.

When a public signal is received by market participants, the beliefs of both A investors and E investors are changed. The ambiguity averse investors' beliefs about the mean of v after the receipt of a public signal ζ with precision ρ_ζ are given by

$$\{\mu|s,\zeta\} = \left\{ \frac{\rho_0\mu_0 + \rho_\zeta\zeta + \rho_s\left(s + \mu_s\right)}{\rho_0 + \rho_\zeta + \rho_s} : \mu_0 \in \left[\underline{\mu_0}, \overline{\mu_0}\right], \mu_s \in \left[\underline{\mu_{sk}}, \overline{\mu_{sk}}\right] \text{ for all } k \right\} \tag{53}$$

On the other hand the beliefs of expected utility investors E under partial revelation become

$$\{\mu|s,\zeta\} = \frac{\rho_0\mu_0 + \rho_\zeta\zeta + \rho_s\Delta(a,b)}{\rho_0 + \rho_\zeta + \rho_s}. \tag{54}$$

In this framework, a public signal has two important effects on market prices. The direct effect is that the public signal conveys information about the mean log payoff of the risky asset. The second effect is that since the price of the risky asset changes when a positive signal is revealed, the relative wealth of E investors and A investors changes with the receipt of a public signal. This change in relative wealths can then trigger a movement from partial to full-revelation as discussed in section 4.1.

This possibility for transition from partial- to full-revelation means that otherwise anomalous price behavior can occur. In particular, generally the receipt of a public signal that is bad news will lead to a decline in the price of the asset. However, if that bad news lowers the relative wealth share of E investors to the point that A investors now find it optimal to participate in the market, then A's signal will then be revealed to the market and that signal can then influence prices to increase or decrease from where they would be were the economy to remain in partial revelation.

5 Dynamic equilibrium properties (ongoing)

In this section we discuss some of the properties of equilibrium prices when the economy is repeated for multiple trading periods. In each of the economies studied in this section, investors have preferences over terminal payoffs (that is, at the end of the multiple trading periods) that take the form described in the previous sections. The dynamics of the model come from a public signal that is revealed in each period. This public signal provides information on the value of the risky asset that in turn changes the relative wealth shares of investors.

Many of the properties of the partially-revealing equilibria discussed in the paper can be seen in figure one. The figure shows one sample path of prices public signals and wealth ratios (with the wealth of the ambiguity averse investor in the numerator) for a fixed, ambiguous signal received at time 0. The shaded areas of the price plot represent periods in which the signal is not revealed. As can be seen, the signal is not revealed in prices for the first 38 (periods 0 through 37) trading periods. It becomes revealed in period 38 because in period 37 a public signal that is bad news is received by all participants. This signal increases A's relative wealth share. This increase in the wealth share of A, compounded with the decrease in the ambiguity of A's information that comes from repeated viewing of an unambiguous public signal narrows the range of signals that will remain hidden in equilibrium to the point that when the public signal in period 38 occurs, it is revealed in market prices. This revelation causes a dramatic increase in market prices even though the public signal that is received in that period is not particularly good news.

Figure 2 gives a representative path when investors of type A receive a new ambiguous signal each period, along with the receipt of the public signal by all market participants. As can be seen, the market moves into and out of periods of partial revelation. This movement happens endogenously as the wealth shares of market investors change in response to both the public signal and the movements between informational regimes.

(To be finished.)

6 Concluding remarks

To be added.

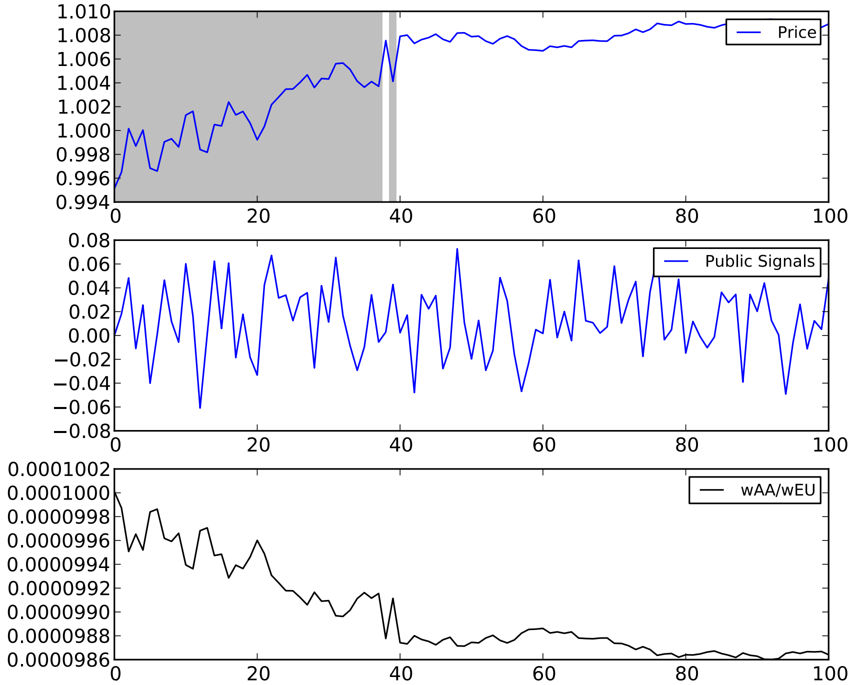

Figure 1: Prices, public signals and relative wealth shares for a single signal received at time 0. Gray areas represent prices for which the privately held signal is not revealed.

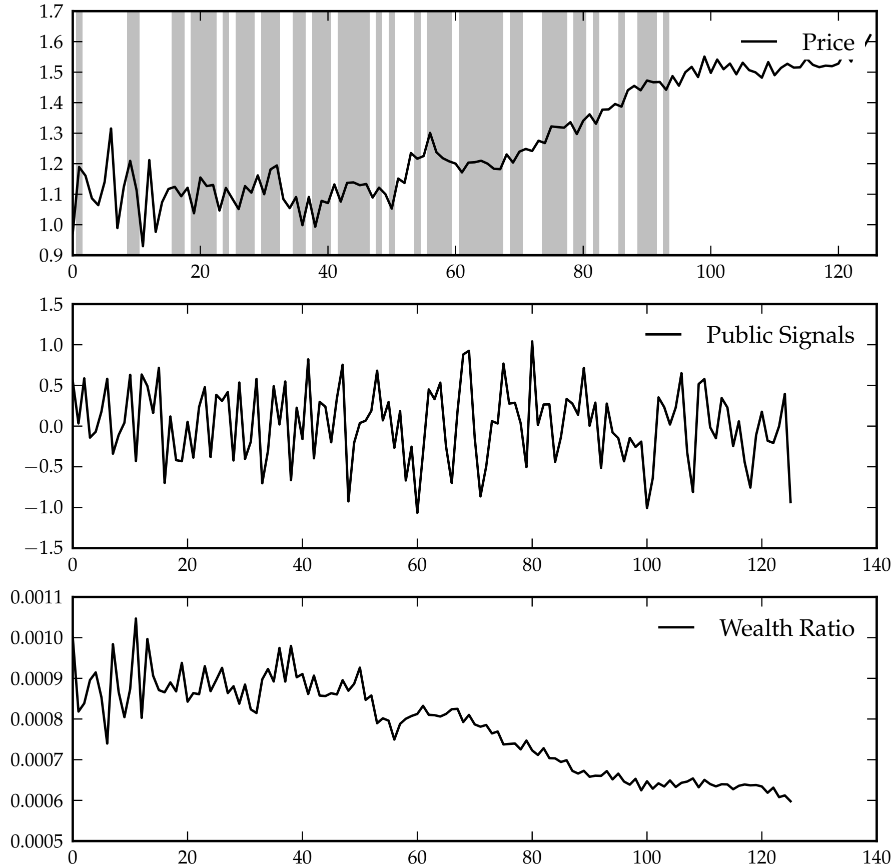

Figure 2: Prices, public signals and relative wealth shares when a new ambiguous signal is received each period. Gray areas represent prices for which the privately held signal is not revealed.

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