Competition in Financial Innovation*

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Abstract

This paper examines the incentives offered by frictionless markets with short sales to entrepreneurs for innovating securities. In economies with symmetric investor utilities, we develop the conditions under which competition provides insufficient incentives to innovate securities and, in equilibrium, markets are incomplete in all (pure strategy) equilibria, even when innovation is costless. Thereby we provide an alternative to Allen and Gale's (1991) classic foundation for endogenous market incompleteness.

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The development of capital markets has been associated with an unprecedented number of asset innovations. Many of the financial products used in contemporaneous markets—such as straps, swaps, or mortgage-backed securities—have only been introduced in the past three decades. The choice as to which securities to offer in financial markets is fundamental to central banks, treasury departments and other institutional investors. Understanding incentives to innovate assets is at least as relevant from a macroeconomic perspective: Financial innovation determines whether markets are complete or incomplete and, hence, is critical for efficiency and determining how markets respond to various economic shocks. Indeed, missing markets and the resulting exposure to systemic risk have been central to recent discussions of the 2008 financial crisis.[1]

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[1] To monitor financial innovation, in September 2009, the Security and Exchange Commission (SEC) created the Division of Risk, Strategy and Financial Innovation, the first new division created at the SEC in 37 years.

In the literature, the traditional general equilibrium framework has been extended to study the economic factors that determine which assets are traded in the economy. The literature recognizes several potential determinants of innovation, the most primitive being a "spanning" motive: by introducing new securities, innovators benefit from trader demand for risk-sharing. Other strands of literature attribute innovation to incentives to mitigate frictions: asymmetric information (between trading parties or due to imperfect monitoring of performance) or transaction costs.[2] This paper contributes to the literature on the spanning motive. We are interested in understanding entrepreneurs' incentives to innovate in *frictionless* markets in which short sales are allowed.[3] In an influential paper, Allen and Gale (1991, henceforth AG) introduce a framework in which entrepreneurs, who each own a firm (a real asset), can innovate securities and asset structure is determined endogenously in equilibrium. In addition, AG establish abstract results about equilibrium properties, such as existence and provide a seminal numerical example that has become crucial to current understanding of the issue. However, as emphasized by Duffie and Rahi (1995), spanning theory has few concrete normative or predictive results, and these have been demonstrated in specific numerical examples.

A key question in the research on asset innovation is whether competition among entrepreneurs provides sufficient incentives to innovate so that large markets are complete. An important lesson from the spanning literature, most notably from AG, is that market incompleteness need not be inconsistent with equilibrium innovation, even if a complete financial structure maximizes each firm's market value, as long as security innovation is costly. Specifically, AG construct an economy in which, in equilibrium, entrepreneurs randomize over financial structures and all outcomes, in particular, incomplete markets are observed with positive probability. In such case, market incompleteness can be attributed to entrepreneurs' inability to coordinate on a complete financial structure, given independent randomization. Nonetheless, the mechanism presented by AG holds in a specific example and need not apply to all equilibria, or for other primitives. This paper seeks to understand in which environments endogenous innovation—modeled as rational behavior—leads to complete markets.

In a general model with competitive investors who have *symmetric* preferences over consumption, we provide sharp predictions regarding endogenous market incompleteness when entrepreneurs strategically choose which securities to issue. These predictions depend on the (shape) of investors' *marginal* utility function and hold in all pure strategy equilibria. When marginal utility is concave, a complete financial structure maximizes the firms' market values and provided innovation costs are not prohibitively high, the endogenous financial structure is complete. In economies with convex marginal utility (e.g., CARA or CRRA utility), any incomplete financial structure dominates a complete financial structure in terms of market value and markets are incomplete in equilibrium. Thus, we identify natural primitive conditions on a

[2] Allen and Gale (1994) and Duffie and Rahi (1995) provide literature surveys.

[3] Thus, in our analysis, value to securitization of an issuer's asset does not derive from short sales constraints. Allen and Gale (1991, 1994), Chen (1995) and Pesendorfer (1995) examine short sales restrictions as a profit source for corporate issuers.

class of economies that give rise to incompleteness which, unlike AG, does not occur as a result of entrepreneurs' miscoordination of their innovation activities. Our mechanism operates even if innovation is costless, for any market size (including large markets), under simultaneous or sequential modes of competition, for an arbitrary number of states of the world, with arbitrary endowment distributions and arbitrary, possibly idiosyncratic, returns to real assets. The economic mechanism involves the shape of investor marginal utility, which changes the very nature of competition and, hence, incentives to innovate; competition in innovation can be seen as a problem of provision of a public good or a public "bad."

Predictions can be strengthened for economies with convex marginal utility in which market value is monotone in security span. As we demonstrate, such economies include markets with two states and economies with CARA utilities and riskless real assets. Here, issuing a single security (equity) is a strictly dominant strategy and consequently, in the *unique* equilibrium, the financial structure has minimal span. When uncertainty is rich enough (that is, when there are more than two states), strict dominance need not hold in an arbitrary model with symmetric utilities. We provide an example of an economy with convex marginal utility in which complete markets are observed with positive probability in a mixed strategy equilibrium, even though (any) incomplete market structure is individually strictly optimal and innovation is costly. Therefore, the AG mechanism applies more broadly; an undesirable outcome from the entrepreneurs' perspective occurs due to their inability to coordinate on an optimal (in this case, incomplete) financial structure.

This paper offers two technical contributions. First, we characterize the comparative statics of the market value of a real asset with respect to security span. Permitting unrestricted choice sets of entrepreneurs and unlimited short sales along with the quasilinearity of investor utility gives tractability to our approach; we recast the maximization of firm value over financial structures as an optimization problem over spans. More generally—to the best of our knowledge—this paper is the first to study the class of games in which players' strategy sets are collections of all linear subspaces (spans) of a common linear space. Apart from the financial application in this paper, these types of games arise naturally in competition in bundling of commodities or in design of product lines.[4] The results developed here extend directly and contribute to these contexts. The difficulty that stems from strategies being linear subspaces is that collections of all linear subspaces are not convex sets and payoffs are discontinuous in subspace dimensionality. Thus, standard techniques do not apply.

The paper is organized as follows: Section 1 reviews the classic example from AG; Section 2 presents the model of financial innovation; Section 3 establishes equilibrium properties; Section 4 derives comparative statics of a firm market value; Section 5 characterizes the endogenous financial structure of the economy; Section 6 concludes. All proofs appear in the Appendix.

[4]The problem of an entrepreneur issuing securities to sell a return on a real asset is mathematically equivalent to the problem of a producer choosing a portfolio of bundles to sell an inventory of commodities or design of product lines in which consumers have utility over multidimensional characteristics and producers decide what vectors of characteristics to build into their products.

1 Motivation: an Example by Allen and Gale (1991)

In the literature, normative predictions regarding entrepreneurs' incentives to innovate in frictionless markets are based on (versions of) the following example by AG. Consider a two-period economy with uncertainty in which there are two possible states of the world at date 2. Suppose that this economy is populated by N entrepreneurs and a continuum of investors. Each entrepreneur is endowed with a real asset (a firm), which in the second period gives random return $z = (0.5, 2.5)$ in terms of numèraire, contingent on the resulting state of the world.

In the first period, the entrepreneurs, who only derive utility from consumption in period 1, sell their claims to the second-period return to two types of investors. As a function of their consumption in the first period, c^1, and their (random) consumption in the second period, c^2, half of the investors have preferences

$$5 + c^1 - \mathrm{E}[\exp(-10c^2)],$$

while the other half have preferences

$$5 + c^1 + \mathrm{E}[\ln(c^2)].$$

The mass of each type is normalized to $N/2$.

In order to sell their date-2 returns, all entrepreneurs simultaneously choose from two financial structures: each can *costlessly* issue equity, in which case one market opens and shares of the entrepreneur's firm are traded; or alternatively, the entrepreneur can *innovate* by issuing, *at a cost*, two state-contingent claims, in which case, two markets open. There are no other assets in the economy, so if all entrepreneurs choose to issue equity, financial markets are incomplete. If one or more entrepreneur innovates, however, financial markets are complete.

From the perspective of the insurance opportunities available in markets, the literature on asset innovation aims to determine whether competition among entrepreneurs gives rise to sufficient innovation to complete large markets. AG demonstrate that, in equilibrium, markets can be incomplete with positive probability for an arbitrary market size: Arbitrage ensures that firms with identical returns have the same market value. In this economy, if we denote by V_C the market value of each entrepreneur's firm when markets are complete and by V_I its value when only equity is issued, AG obtain that

$$V_C = 0.58603 > 0.58583 = V_I,$$

so that market value *is greater under complete markets*[5] (See Tables 1 and 2 in AG, pp. 1052-1053.) Thus, completing the market is essentially a *public good*: all entrepreneurs are better off if one pays an innovation cost to introduce contingent claims. AG focus on the symmetric mixed

[5]The feature that market value is greater under complete markets is shared by all examples in AG.

strategy equilibrium, in which each entrepreneur chooses to innovate with positive probability and as a result, all outcomes, including incomplete markets, occur with positive probability.[6]

A general lesson from this example is that, in the presence of innovation costs, large frictionless markets may be incomplete due to miscoordination among entrepreneurs, resulting from independent randomization. Clearly, costly innovation is necessary for the free riding mechanism to operate, since otherwise markets are complete.

As a starting point of our analysis, we observe that, if the utility of the first type of investors above is instead given by

$$5 + c^1 + \mathrm{E}[\ln(c^2 + 2)],$$

then the predictions that we obtain change dramatically. Each firm's market value is maximized in *incomplete* markets, for now

$$V_C = 2.0952 < 2.3228 = V_I.$$

Financial innovation is then no longer a public good from the point of view of the entrepreneurs. Rather, it becomes a *public "bad,"* as all entrepreneurs are worse off if one or more of them innovates. As a result, issuing equity is a strictly dominant strategy and, in the unique equilibrium, markets are *incomplete with probability 1, even if asset innovation is costless.*[7]

Both examples describe markets with plausible investor preferences. Yet, the corresponding predictions regarding endogenous market incompleteness differ markedly. In this paper, we attempt to identify the economic mechanisms that underlie distinct equilibrium predictions. Our primary result is the determination of such a mechanism: we offer sharp predictions about the form of the endogenous financial structure in a general model in which investors value future consumption equally.

2 Investors, Entrepreneurs and Equilibrium

As in AG, we consider a two-period economy with uncertainty. In the second period, there are S states of the world. All the agents in this economy, whom we describe next, agree that the probability of state $s = 1, \ldots, S$ is $\pi^s > 0$ and denote the probability distribution over these future states by $\pi = (\pi^1, \ldots, \pi^S)$.

2.1 Investors

Financial assets are demanded by a continuum of investors who derive utility from consumption of numèraire in both periods of the economy (and across states of the world in the second

[6]In fact, with a larger number of entrepreneurs, the free-riding problem becomes more severe: *ceteris paribus*, for each entrepreneur, the probability that at least one other entrepreneur introduces contingent claims increases. This reduces individual incentives to innovate and the probability that one or more entrepreneur innovates is bounded away from 1.

[7]A similar inequality reversal may also occur if one changes real asset returns or investor endowments instead.

period). These investors have different needs for insurance against date-2 income risk. That is, there are K types of investors that we index by $k = 1, \dots, K$, and these types differ in their endowments of wealth at date 2. In state s, investor k will have wealth $e_k^s \geq 0$; the random variable $e_k := (e_k^1, \dots, e_k^S)$ denotes investor k's future wealth. The mass of type k investors is denoted by $\theta_k > 0$ and the mass of all investors is $\theta = \sum_k \theta_k$.

While their endowments of future wealth may differ, all types of investors have the same preferences over consumption and their utilities are quasilinear and von Neumann-Morgenstern in the second period consumption. That is, for all types, the expected utility derived from present consumption $c^1 \in \mathbb{R}$ and a state-contingent future consumption $c^2 = (c^{2,1}, \dots c^{2,S}) \in \mathbb{R}_+^S$ is given by

$$c^1 + U(c^2),$$

where

$$U(c^2) := \mathrm{E}_\pi[u(c^2)] = \sum_s \pi^s u(c^{2,s})$$

for a $\mathbb{C}^2$ Bernoulli index $u : \mathbb{R}_+ \to \mathbb{R}$ that satisfies the standard assumptions of strict monotonicity and strict concavity as well as the Inada condition that $\lim_{c \to 0} u'(c) = \infty$.

Thus, although investors have common preferences *over consumption*, they exhibit different preferences across types for financial trades, which arises from heterogeneity in their initial endowments.

2.2 Entrepreneurs

Financial assets are issued by a group of entrepreneurs, each of whom has future wealth that may depend on the state of the world and who wants to 'sell' that future wealth in exchange for present consumption. Specifically, suppose that there is a finite number, N, of strategic entrepreneurs who are indexed by $n = 1, \dots, N$. Entrepreneur n owns a real asset (e.g., a firm) that pays $z_n^s > 0$ units of the numèraire at date 2, if state of the world s is realized, but he does not care about future consumption and his utility is given by the date-1 revenue that he can raise from selling the future return on his real asset. That is, the random variable $z_n := (z_n^1, \dots, z_n^S)$ is the return to the real asset that entrepreneur n wants to sell in exchange for numèraire in the first period.

Entrepreneurs do not know the future endowments of the investors and hold probabilistic beliefs over the profile $(e_1, \dots, e_K)$. These beliefs are common to all entrepreneurs and given by the joint distribution function G, which is defined over $\mathbb{R}_+^{S \times K}$. For some of our results, we assume that distribution G is absolutely continuous with respect to the Lebesgue measure, but no other restrictions are placed on G. In particular, the associated marginal distributions can differ across investor types and the joint distribution G can feature an arbitrary interdependence of endowments so long as the correlations are not perfect, which is the case given absolute continuity.

2.3 Equilibrium

To sell claims to the return from their real assets (z_n), entrepreneurs simultaneously issue securities at date 1; at date 2, payments against the issued securities are made and investors consume numèraire.

Each entrepreneur can choose from a wide variety of alternative selling strategies. One possibility is opening an equity market to sell shares. Another alternative is to issue S claims, one for each state, paying z_n^s units of the numèraire in the corresponding state s and 0 otherwise. More generally, entrepreneur n can issue a portfolio that comprises an arbitrary finite number, $I_n = 1, 2, ...,$ of securities: a *financial structure* F_n specifies payments of issued securities, $F_n := \{f_n^1, \ldots, f_n^{I_n}\}$, where the random variable $f_n^i \in \mathbb{R}^S$ is the promised, state-contingent payment of security i in units of numèraire. As in AG, the financial structure F_n exhausts the returns to entrepreneur n's real asset so that he is solvent in the second period at all states and, without loss of generality, the supply of each security issued by entrepreneur n is equal to 1: $F_n\mathbf{1} = z_n$, where we treat F_n as an $S \times I_n$ matrix and where $\mathbf{1} := (1, \ldots, 1)$.[8]

Considering all entrepreneurs, $I := \sum_n I_n$ markets open at date 1, for securities $F = \{F_1, \ldots, F_N\}$, which we treat as an $S \times I$ matrix. Denote by $\mathcal{F}$ the set of all financial structures that satisfy the solvency condition for any n (i.e., $F_n\mathbf{1} = z_n$). All investors are non-atomic, so the prices at which trade of securities takes place are given by the competitive equilibrium prices of the economy under financial structure F, where we assume that investors can sell the issued securities short, but cannot issue other securities.[9] For entrepreneur n, the prices that are relevant are those that correspond to his securities, which we denote by p_n, so the market value of his firm is given by $V_n := p_n \cdot \mathbf{1}$. Since the competitive equilibrium prices depend on the profile of future wealth of the investors and since this profile is not known by the entrepreneurs, at date 1 entrepreneur n's gross payoff is

$$\mathrm{E}_G[V_n(F)] = \mathrm{E}_G[p_n \cdot \mathbf{1}].$$

Since all entrepreneurs choose their financial structure simultaneously, they behave *à la* Nash, so that each takes the structures chosen by the other entrepreneurs as given and chooses F_n so as to maximize $V_n(F)$, net of the issuance costs he incurs.

[8]More generally, apart from choosing a financial structure F_n, entrepreneur n could also decide about the supply of each security, Q_n^i, for each $i = 1, \ldots, I_n$. This, for instance, would allow the entrepreneur to issue z_n^s units of the standard elementary security for state s. All of our results straightforwardly extend to this case.

[9]For a complete presentation, if the assets issued are structure F, a *competitive equilibrium* comprises security prices $p \in \mathbb{R}^I$ and an allocation $(t_1, \ldots, t_K) \in \mathbb{R}^{I \times K}$ of financial securities across investor types, such that each t_k solves

$$\max_t U(e_k + Ft) - p \cdot t,$$

while $\sum_k \theta_k t_k = \mathbf{1}$. Under our assumptions on preferences, the previous condition of optimality on t_k can be replaced by the requirement that $F^\top DU(e_k + Ft_k) = p$. We will observe in Section 3.3 that equilibrium prices exist and are unique so that our reference to *the* equilibrium prices under structure F is justified.

3 Allocation and Market Value

3.1 Market Completeness

The (column) span of F, which is the linear subspace of $\mathbb{R}^S$ defined as

$$\langle F\rangle := \{y \in \mathbb{R}^S \mid Ft_k = y \text{ for some } t_k \in \mathbb{R}^I\},$$

determines the set of all date-2 numèraire transfers that can result from trades of the securities in F. Financial structure F is said to be *complete* if the rank of F equals S; otherwise, structure F is said to be *incomplete*.

3.2 Characterization of Equilibrium Allocation

This section characterizes the date-2 allocation of numèraire among investors that results from securities trading in competitive financial markets. Lemma 1 asserts that, given the financial structure chosen by the entrepreneurs, competitive financial markets allocate numèraire at date 2 in the same way that a utilitarian fictitious planner would, while restricted to allocations that are feasible under the pre-determined financial structure.

It is useful to consider the set $\mathcal{L}$ of all linear subspaces $L \subseteq \mathbb{R}^S$ of date-2 numèraire transfers that contain the real assets of all entrepreneurs, $\{z_n \mid n = 1, \ldots, N\} \subseteq L$. Let $X(L)$ be the correspondence that gives the set of allocations that result from the numèraire transfers in the linear subspace L,

$$X(L) := \left\{ x \in \mathbb{R}_+^{S\times K} \,\middle|\, \sum_k \theta_k(x_k - e_k) = \sum_n z_n \text{ and } (x_k - e_k) \in L \text{ for all } k \right\}. \tag{1}$$

Thus, for any F, $X(\langle F\rangle)$ is a collection of allocations that are feasible through the trades of securities in F. Also, let

$$\bar{U}(x_1, \ldots, x_K) := \sum_k \theta_k U(x_k),$$

which computes the sum of utilities across investor types at allocation $(x_1, \ldots, x_K)$ of future consumption. Given transferable utility, the following characterization of competitive equilibrium allocations of numèraire holds for any financial structure F.

LEMMA **1** (ALLOCATIVE EQUIVALENCE)**.** *Fix a financial structure F. Let $(\tilde{t}_1, \ldots, \tilde{t}_K)$ be an allocation of the securities F such that $\sum_k \theta_k \tilde{t}_k = \mathbf{1}$. $(\tilde{t}_1, \ldots, \tilde{t}_K)$ is a competitive equilibrium allocation under structure F if, and only if, the resulting allocation of numèraire at date 2, $(\tilde{x}_1, \ldots, \tilde{x}_K)$, which is given by $\tilde{x}_k := e_k + F\tilde{t}_k$, solves problem*

$$\max_{x \in X(\langle F\rangle)} \bar{U}(x). \tag{2}$$

The equivalence between the competitive allocation of numèraire and the solution to Problem (2) has useful implications. First, note that for any financial structure F, the numèraire allocation is uniquely determined in the resulting competitive equilibria, even if the securities trades that yield such allocation are not (as is the case, for instance, for linearly dependent securities). Moreover, the equilibrium allocation of numèraire at date 2 depends on the financial structure only through its span; that is, for any two financial structures F and F', such that $\langle F \rangle = \langle F' \rangle$, the numèraire allocations coincide.[10]

3.3 Market Value

Given our last observation, let $x(L)$ be the date-2 numèraire allocation obtained for *any* financial structure F for which $\langle F \rangle = L$. Define the function

$$\kappa(L) := \frac{1}{\theta} \sum_k \theta_k DU(x_k(L)), \tag{3}$$

which measures the average marginal utility across investors, evaluated at the equilibrium date-2 allocation under financial structure F with $\langle F \rangle = L$.[11] Function κ determines state prices for any financial structure, complete or not. Competitive equilibrium prices under financial structure F are characterized by the equation

$$p^\mathsf{T} = \kappa(\langle F \rangle)^\mathsf{T} F.$$

While the latter is straightforward for a complete financial structure, where equilibrium consumption and individual marginal utilities are the same for all investors, when the structure is incomplete, consumption vectors and, hence, marginal utilities may differ across investors. However, even though state prices are not unique, (3) can nonetheless be used to unambiguously price securities.[12] Lemma 2 characterizes each entrepreneur's market value.

[10]From the lemma, the existence of a competitive equilibrium allocation in the markets that open once entrepreneurs choose the financial structure F follows from the compactness of set $X(\langle F \rangle)$ and the continuity of function $\bar{U}(x)$. Its uniqueness holds by the convexity of $X(\langle F \rangle)$ and the strict concavity of $\bar{U}(x)$. In terms of primitives, the uniqueness results from the quasilinearity of the investor utilities, but does not require their symmetry. Without quasilinearity, a given financial structure can produce multiple equilibrium allocations and non-trivial multiplicity of prices. Comparative statics would then require an equilibrium selection criterion. Finally, the dependence of the numèraire allocation on the financial structure through span alone is obtained because in Problem (2), structure F enters the constraint only through its span, $\langle F \rangle$.

[11]Formally, these last two functions are: $x : \mathcal{L} \to \mathbb{R}^{S \times K}_{++}$ and $\kappa : \mathcal{L} \to \mathbb{R}^{S}_{++}$.

[12]In fact, any vector in the set $\{\kappa(L)\} + L^\perp$ constitutes a valid vector of state prices. In particular, each vector whose average defines $\kappa(L)$ does so; the marginal utilities at equilibrium consumption can differ only in the components that are orthogonal to the security span and their differences are irrelevant for security pricing. Our characterization of $\kappa(L)$ as an average is useful for determining a financial structure that maximizes market value of the entrepreneur.

Lemma 2 (Market Value). *The expected market value of an entrepreneur's n real asset z_n under structure F is given by*

$$\mathrm{E}_G[V_n(F)] = \mathrm{E}_G[\kappa(\langle F \rangle)] \cdot z_n.$$

Two implications of this lemma are immediate. First, for any financial structure, the expected market value is unambiguously defined: any two financial structures from $\mathcal{F}$ can be ranked in terms of their profitability for each entrepreneur. In addition, just as with numèraire allocation, market value depends on the financial structure only up to its span. Therefore, financial structures collections that permit the same numèraire transfers define equivalence classes for market value for an entrepreneur.

4 Financial Structure and Market Value

4.1 Existence of a Value-Maximizing Financial Structure

We now show that within the set of all financial structures $\mathcal{F}$, a financial structure exists that maximizes the expected market value of an entrepreneur's real asset, z_n. There are two difficulties with demonstrating existence: first, even if one restricts attention to financial structures with a fixed number of securities I, the domain over which the entrepreneur optimizes, given by the set of all financial structures in $\mathbb{R}^{S \times I}$, is non-compact and additionally, market value is discontinuous in F.[13] AG do not face these difficulties, since they consider entrepreneurs who choose a financial structure from an exogenously pre-specified, finite set. Here the aim is to characterize the financial structure that maximizes market value from the unconstrained set of all financial structures.

To deal with these two problems, we take the following approach. Since any two financial structures with the same span are equivalent in terms of market value (Lemma 2), optimization over financial structures can be recast as the problem of choosing a span—a linear subspace from the set of all linear subspaces of $\mathbb{R}^S$—that maximizes market value rather than optimizing over financial structures F directly. The optimization problem over linear subspaces is more tractable: for any dimension $D \leq S$, the set of all D-dimensional linear subspaces of $\mathbb{R}^S$ is a compact manifold, known as the Grassmannian and market value V_n is continuous on it.[14] This

[13] To see this in an extreme example, consider the following sequence of financial structures with two securities:

$$F^h = \begin{bmatrix} 1/h & 0 \\ 0 & 1/h \end{bmatrix}, \text{ for all } h \in \mathbb{N}.$$

For any finite h, markets are complete under structure F^h and the set of feasible allocations $X(\langle F^h \rangle)$ comprises all allocations. In the limit as $h \to \infty$, security span collapses to a zero-dimensional subspace and $X(\langle \lim_{h\to\infty} F^h \rangle)$ becomes a singleton set that comprises only the autarky point. Consequently, numèraire allocation, and hence the average marginal utility, are discontinuous.

[14] Heuristically speaking, suppose that $S = 2$ and an entrepreneur chooses among all one-dimensional linear

allows us to recover the compactness of the domain and continuity of the objective function, and establish the existence of a financial structure that maximizes market value.

Lemma 3 (Existence). *Fix an entrepreneur's n real asset z_n. A financial structure $F^* \in \mathcal{F}$ exists such that*

$$E_G[V_n(F^*)] \geq E_G[V_n(F)]$$

for all $F \in \mathcal{F}$.

4.2 Completeness of a Value-Maximizing Financial Structure

Next, we characterize the financial structure that maximizes the market value of a real asset. Proposition 1 asserts that the financial structure that dominates in terms of market value depends on the shape of the *marginal* utility, $u'(\cdot)$. Specifically, any incomplete financial structure is superior or inferior to any complete financial structure, depending on whether the marginal utility is convex or concave, respectively, on the relevant part of the domain.

More formally, let $\mathcal{X} \subseteq \mathbb{R}_+$ be a convex set that contains all the possible values of equilibrium consumption allocations, considering all investor types, states of the world in period 2, financial structures and endowment profiles.[15] In Proposition 1, we say that, with regard to entrepreneur n's market value, some structure F strictly dominates an alternative structure F' if $V_n(F) > V_n(F')$, F is not dominated by F' if this inequality is weak and F and F' are equivalent if $V_n(F) = V_n(F')$.

Proposition 1 (Value-Maximizing Financial Structure). *Fix any incomplete financial structure F and any complete structure F'. Then,*

(i) if $u'''(c) > 0$ on $\mathcal{X}$, then, $G-$a.s., structure F strictly dominates structure F' (and F is not dominated by F', surely);

(ii) if $u'''(c) < 0$ on $\mathcal{X}$, then, $G-$a.s., structure F' strictly dominates structure F (and F' is not dominated by F, surely);

(iii) if $u'''(c) = 0$ on $\mathcal{X}$, structures F and F' are equivalent.

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subspaces. Each subspace is represented by a line passing through the origin and is uniquely identified by a point on a semicircle with radius 1 (See Figure 1). A bijection that enlarges the distance of any point on the semicircle by a factor of two (around the circle) translates a semicircle into a full circle. Given such parameterization of linear subspaces, the entrepreneur effectively chooses a point on a circle, a compact set. In addition, the dimension of any linear subspace in the domain of optimization—each represented by a point on the circle—is, by construction, the same and equal to 1; $X(L)$ is a continuous correspondence defined on the circle. By the Maximum Theorem and Lemma 1, the equilibrium numèraire allocation $x(L)$ is continuous and so are state prices, given by the average marginal utility.

[15] That is, let $\mathcal{X}_k^s(e)$ be the projection, over the consumption set for investors of type k in state s, of the image of function $x(L)$ when the endowment profile is e. Let $\mathcal{X}_k^s$ be the union of all the sets $\mathcal{X}_k^s(e)$ over profiles e in the support of G and let $\mathcal{X} = \cup_{k,s}\mathcal{X}_k^s$.

It is important to note that the implication of this proposition holds for almost all realizations of investor endowment profiles, e in the support of G and not merely in expectation. Also, in an economy with only two states of the world at date 2—since there are effectively two choices of financial structures, complete and equity (incomplete)—Proposition 1 fully characterizes the financial structure that maximizes market value, which we highlight as the following corollary.

COROLLARY **1** (TWO-STATE MODEL). *Suppose that $S = 2$. If $u'''(c) > 0$ ($u'''(c) < 0$) on $\mathcal{X}$, then, $G-$a.s., a financial structure maximizes an entrepreneur's market value if, and only if, it consists of equity only (respectively, is complete).*

Recall that, in the example presented by AG, in which $S = 2$, a complete financial structure maximizes market value. This prediction coincides with our symmetric model only if the marginal utility is concave, while for the utility functions often used in macroeconomics and finance, such as CARA or CRRA, an incomplete financial structure brings higher market value.

We now provide a simple example that highlights a key economic intuition behind Proposition 1. Given that the proposition holds almost surely, for transparency of the arguments, the example considers deterministic investor endowments.

EXAMPLE **1.** *Suppose that $S = 2$, there is one entrepreneur with the riskless asset $z_1 = (1, 1)$ and there are two types of investors of equal mass normalized to 1, whose Bernoulli utility function is $u(c^2) = 2\ln(c^2)$. At date 2, the endowments of the investors are $e_1 = (1, 0)$ and $e_2 = (0, 1)$ and the states are equally likely.*

With two states, there are two choices of financial structure: a complete financial structure and equity. With a complete financial structure (e.g., equity and debt), all risk is shared at the equilibrium allocation, $x_1 = x_2 = (1, 1)$; the marginal utility of each investor in each state, given by $1/c^{2,s}$, is the same for both investors and equal to 1; the market value of the entrepreneur's asset is 2.

If only equity is offered instead, each investor obtains half of the claims to z_1, which gives equilibrium allocation of $x_1 = (3/2, 1/2)$ and $x_2 = (1/2, 3/2)$. The average marginal utility in each state is $1\frac{1}{3}$ and the market value equals $2\frac{2}{3}$.

Hence, in this economy, an incomplete financial structure dominates a complete one in terms of market value. It is clear that, when marginal utility is linear, both complete and incomplete financial structures yield the same market value, while when marginal utility is strictly concave, the complete financial structure maximizes market value.

To understand the economic mechanism behind the example, note first that, with a complete financial structure, each investor purchases consumption only in the state for which his initial endowment is 0 and the marginal utilities of investors coincide in each state. When only equity is available, for an investor to obtain consumption in the desired state, he must also purchase the security that pays (the same quantity of) numèraire in the other state. Thus, by introducing a wedge in consumption, an incomplete financial structure creates a wedge in marginal utility between the two investors in each state. With convex marginal utility, the wedge increases the willingness to pay of the investor type with lower equilibrium consumption by more than it reduces the willingness to pay of the investor who consumes more. Therefore, in each state, an incomplete financial structure induces a higher equilibrium average marginal utility as compared to complete markets. Since average willingness to pay for consumption in each state remains high after trade, the equilibrium value of equity remains high as well.

Notice that in Example 1, the market value of the real asset is higher only if the equilibrium allocation of numèraire is inefficient, in the sense that it fails to display full risk-sharing. In general, even with inefficient endowments (which occur $G-$a.s., given that G is absolutely continuous with respect to the Lebesgue measure), the final allocation for an incomplete financial structure may still be efficient. However, for any fixed incomplete financial structure, endowment realizations that give efficient equilibrium allocations are non-generic; the equilibrium allocation is inefficient $G-$a.s.

Since the inequalities in the first two claims of Proposition 1 are strict with $G-$probability 1, it follows that the result is robust to sufficiently small asymmetries in investor utility functions.[16] However, Proposition 1 does not generalize to arbitrary asymmetries in utility functions across investors. Indeed, in the AG example, investor marginal utilities are strictly convex, yet a complete financial structure maximizes the market value of an asset. Considered together, Proposition 1 and the AG example suggest that in markets with asymmetric investor utilities, for convex or concave marginal utility, no general normative predictions based solely on investor preferences can be obtained; the optimality of complete or incomplete financial structures then depends on the model's details, such as endowment or asset return distributions.

The assumptions of the three claims in Proposition 1 are to hold over some convex subset of the respective domains that is large enough to include all the relevant equilibrium allocations of numèraire. We introduce this qualification because otherwise the class of preferences under consideration, for which the third derivative would have to be strictly negative on the whole domain, is vacuous.[17] If distribution G has a bounded support, we can always find a bounded set of outcomes $\mathcal{X}$ to qualify the assumptions on the shape of marginal utilities. Finally, note that Proposition 1 extends to the case of state-dependent investor Bernoulli utilities.

4.3 Monotonicity of Market Value

More generally, with $S \geq 3$, Proposition 1 asserts that a complete financial structure is almost surely dominated by any incomplete financial structure when the marginal utility is convex. We are then led to ask: Does market value monotonically decrease with the hedging possibilities

[16]This is clearly the case for a given financial structure and, by the compactness argument used in the proof of Lemma 3, extends to all incomplete financial structures.

[17]While a global assumption would not be problematic for claim (i), given the Inada assumption about utility, a strictly concave marginal utility function does not exist wherein marginal utilities are always strictly positive and concave or linear.

offered to investors by the financial structures? That is, for any pair of structures F and F', such that $\langle F\rangle \subseteq \langle F'\rangle$, does $V_n(F) \geq V_n(F')$ hold, so that opening a single equity market always maximizes market value for entrepreneur n? Example 2 below demonstrates that, in general, this need not be the case, even for symmetric, quasilinear utilities.

EXAMPLE **2.** *Suppose that $S = 3$, there is one entrepreneur with the riskless asset $z_1 = (1, 1, 1)$ and there are two types of investors of equal mass normalized to 1 whose Bernoulli index is the following $\mathbf{C}^2$ function:*

$$u(c^2) = 3 \times \begin{cases} 2c^2 - \frac{1}{2}(c^2)^2 - \frac{3}{2}, & \text{if } c^2 \leq 1; \\ \ln c^2, & \text{otherwise} \end{cases}. \tag{4}$$

The investor endowments are $e_1 = (1/2, 0, 1)$ and $e_2 = (0, 1/2, 1)$ and the states are equally likely.

When only equity is offered, $F = \{(1, 1, 1)\}$, by symmetry, the equilibrium allocation is given by $x_1 = (1, 1/2, 3/2)$ and $x_2 = (1/2, 1, 3/2)$, state prices are $\kappa(\langle F\rangle) = (5/4, 5/4, 2/3)$ and the market value of the entrepreneur's asset is $3\frac{1}{6}$.

Now, consider the following (not necessarily optimal) financial structure with a state-1 contingent claim and a security that pays 1 in states 2 and 3:

$$F' = \{f^{1\prime}, f^{2\prime}\} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 1 \end{pmatrix}.$$

Observe that $\langle F\rangle \subset \langle F'\rangle$. Since security $f^{2\prime}$ pays in the second state, it is relatively more attractive to type-1 investors and, in equilibrium, the allocation of securities is $t_1 \simeq (1/4, 2/3)$ and $t_2 \simeq (3/4, 1/3)$. The implied allocation of numèraire is $x_1 \simeq (3/4, 2/3, 5/3)$ and $x_2 \simeq (3/4, 5/6, 4/3)$, the state prices are $\kappa(\langle F'\rangle) \simeq (5/4, 5/4, 27/40)$ and the market value is $\simeq 3\frac{7}{40} > 3\frac{1}{6}$.

Thus, financial structure F' strictly dominates F in terms of the entrepreneur's market value. Utility function (4) can be perturbed so that marginal utility is strictly convex on the whole domain while F' still yields a strictly higher market value than F.

In Example 2, financial structure F introduces a wedge in numèraire consumption in the first two states, whereas the allocation is efficient with respect to the third state. In the first two states, given that consumption takes place in the domain of quadratic utility, distortion brings no increase in market value relative to complete markets; the average marginal utility remains intact. In contrast, while the two-security financial structure F' improves the efficiency of the first two states' allocations, it introduces a wedge in the third state's allocation. Since consumption in this state is in the domain of a logarithmic function with strictly convex marginal utility, the wedge in the third state increases the state price for that state as well as the firm's market value.

Example 2 demonstrates that, in general, the market value of an asset need not be monotone in a security span. This lack of monotonicity extends to strictly concave marginal utility environments.[18] In the important instance of CARA utility and a riskless real asset, market value is indeed monotone in the span of the financial structure, and the optimal financial structure involves selling a riskless security (e.g., a bond) only.

EXAMPLE **3.** *Consider the case of a single entrepreneur with the riskless asset* $z_1 = (\lambda, \ldots, \lambda)$ *for some* $\lambda > 0$ *and suppose that all investors have CARA Bernoulli utility* $u(c^2) = -e^{-\alpha c^2}$*, while the endowment distribution G is arbitrary. In this case, it is immediate that* $u'(c^2) = -\alpha u(c^2)$*, which implies that*

$$\begin{aligned}
\bar{U}(x(\langle F \rangle)) &= \sum_k (\theta_k \sum_s \pi^s u(x_k^s(\langle F \rangle)) \\
&= -\frac{1}{\alpha} \sum_s \sum_k \theta_k \pi^s u'\left(x_k^s\left(\langle F \rangle\right)\right) \\
&= -\frac{1}{\alpha} \kappa(\langle F \rangle) \cdot \mathbf{1} \\
&= -\frac{1}{\alpha \lambda} V_1(F).
\end{aligned}$$

By Lemma 1, function $\bar{U}(x(\langle F \rangle))$ *is increasing in* $\langle F \rangle$*, so it follows that the market value of* z_1 *is monotonically decreasing in the security span. In particular, opening a market for the riskless asset maximizes its market value.*

4.4 A Geometric Interpretation

We present a geometric interpretation of our results to elucidate the comparative statics of span and equilibrium (market value) as well as the impact of asymmetry in investor preferences on predictions about incentives to innovate. In doing so, we exploit the equivalence of equilibrium numèraire allocation between the entrepreneur's problem and Problem (2) given by Lemma 1. The set of all feasible allocations of numèraire in Example 1 are represented by the Edgeworth box in Figure 2. With a complete financial structure (F), feasible set $X(\langle F \rangle)$ comprises all allocations in the box. If only equity is issued (F'), set $X(\langle F' \rangle)$ is represented by the line segment that connects the endowment points. A fictitious planner welfare function $\bar{U}(x)$ attains its unconstrained maximum at the efficient allocation (where investors consume the same quantities) and decreases for allocations further from the center (Figure 2.A). Thus, if financial markets are complete, allocation is the unconstrained maximum of $\bar{U}$, whereas with only equity, the equilibrium allocation coincides with its constrained maximum on $X(\langle F' \rangle)$.

Figure 2.B depicts an entrepreneur's preference map; each level curve comprises all allocations that give rise to a given firm value $V_1 = [\sum_k \theta_k DU(x_k)] \cdot z_1$. As a result of the symmetry

[18] As the analysis from Section 5.3 implies, the non-monotonicity of market value function does not stem from non-monotonicity of the welfare function $\bar{U}$ in asset span.

of the investor marginal utility, the critical point of the market value function, V_1, is at the efficient allocation too, but whether the efficient allocation yields a minimum or a maximum market value depends on whether the marginal utility and hence, the market value function, is convex (as in Example 1, with a logarithmic utility) or concave.[19] In the case of a quadratic Bernoulli utility function, all allocations in the box are equivalent in terms of market value and entrepreneurs are indifferent to the planner's allocation choice.

In general, the planner preference and market value maps need not overlap, which, in settings with $S > 2$, may result in the non-monotonicity of market value in the security span. In Example 2, by offering two securities F' rather than equity F, the entrepreneur enlarges the feasible set in the direction for which the welfare function $\bar{U}$ increases, and the planner's new optimum also gives rise to higher market value. For CARA utility with a riskless asset, the two maps coincide (See Example 3). Since the constant of proportionality is negative, a smaller security span, and hence a smaller choice set in Problem (2), does not reduce market value. On the other hand, with the exception of CARA utility, it is apparent that one can specify endowments and a real asset return such that increasing the span increases market value.

With asymmetric investor utilities, predictions as to the optimality of an incomplete financial structure depend on the details of the environment. Considering convex marginal utility, then, the efficient allocation and that which minimizes market value do not necessarily coincide. Indeed, this is the case in the example given by AG, as depicted in Figure 3: with equity only, the equilibrium allocation is the point on the line segment that maximizes $\bar{U}$; while with a complete financial structure, it is the unconstrained maximum that yields a higher market value. Thus, with convex investor marginal utility, separation of the efficient and value-minimizing allocations derived from the asymmetry of investor utilities is necessary (but not sufficient) for market completion to be profitable for the entrepreneurs. Similarly, one can construct an example with asymmetric strictly concave marginal utilities in which the market value is maximized by an incomplete financial structure.

5 Competition in Security Innovation

A central question in the literature on financial innovation concerns whether competition provides sufficient incentives to complete the market. Having understood the comparative statics of market value in financial structures, we turn to examining how strategic interactions among entrepreneurs who choose which securities to issue affect endogenous financial structures. By a

[19]In a two-investor economy, the convexity of market value function in allocation in the Edgeworth box is defined as the convexity of the function

$$\frac{1}{2}(DU(x_1) + DU(e_1 + e_2 + z_1 - x_1)) \cdot z_1$$

in x_1. More generally, convexity is defined with respect to consumption of the first $K-1$ investors and consumption of investor K is the residual of the total resources $\sum_k \theta_k e_k + z_1$. If marginal utilities are convex, then market value is convex in this sense as well.

Table 1: Normalized net market values under concave marginal utilities

	E	C
E	$0, 0$	$1, 1-\gamma$
C	$1-\gamma, 1$	$1-\gamma, 1-\gamma$

standard argument (e.g., Kreps (1979)), entrepreneurs can affect prices, even in large markets, so long as they can impact the span of F. We consider a market such as that presented by AG, in which entrepreneurs simultaneously choose portfolios of issued securities. Likewise, we introduce a per-security innovation cost $\gamma \geq 0$ that discourages entrepreneurs from excessive asset innovation.[20]

In Section 5.1, we argue that the nature of competition among entrepreneurs, which determines whether markets are compete or incomplete, is very different depending on whether investor marginal utility is concave or convex. That is, market completion is effectively either a public good or bad.

5.1 An Example

To illustrate how competition affects incentives to innovate, we first examine economies with two states, two entrepreneurs and the riskless real asset $z_n = (1, 1)$. By Lemma 1, it suffices to consider that each entrepreneur chooses between equity (E) or two state-contingent claims (C). Markets are incomplete when both entrepreneurs choose equity and are complete for all other strategy pairs.

Under concave marginal utility, competition in asset innovation takes the form of a provision of public good, as in the heterogeneous-utility example of AG. A complete financial structure maximizes market value of both entrepreneurs and they both benefit if one of them innovates. Assuming for simplicity that the market values of the entrepreneurs' real assets are 0 when markets are incomplete and 1 when they are complete (by Proposition 1, $V_I < V_C$), it is useful to summarize the entrepreneurs' reduced form net payoffs in Table 1. Let $\gamma < 1$.

One of the insights from the AG example that holds also in our example with concave marginal utility is that the equilibrium financial structure can be (endogenously) incomplete with positive probability, *even if complete markets maximize the entrepreneurs' market values*: considering Table 1, in the mixed strategy Nash equilibrium, all four outcomes, including incomplete markets (E, E), occur with positive probability. The probability of market incompleteness depends positively on the innovation cost, γ, and tends to 0 as costs become negligible. Market incompleteness can be attributed to the inability of entrepreneurs to coordinate on one

[20]That is, an entrepreneur can offer equity costlessly, but if he chooses a financial structure that contains $I_n > 1$ securities, he pays $(I_n - 1)\gamma$ in innovation costs. In the absence of this cost, trivial equilibria arise in which each entrepreneur chooses a complete financial structure.

Table 2: Normalized net market values under convex marginal utilities

	E	C
E	$1, 1$	$0, 0 - \gamma$
C	$-\gamma, 0$	$-\gamma, -\gamma$

of favorable outcomes ((C, E) or (E, C)) when independently randomizing over two financial structures.[21] The entrepreneurs have *ex post* regret when incomplete markets are realized, each preferring to complete the market, knowing that the other did not.

Importantly, apart from the mixed strategy Nash equilibrium, there are two more equilibria in pure strategies in which one of the two entrepreneurs innovates and markets are complete. Clearly, in a pure strategy equilibrium, the miscoordination that may give rise to market incompleteness does not arise, as each entrepreneur best responds to the financial structure chosen by his opponent.

Next, consider an economy with convex marginal utility. The net values of the entrepreneurs' real assets are as presented in Table 2, where we now assume that the market values are 1 when markets are incomplete and 0 when they are complete (by Proposition 1, $V_I > V_C$). With convex marginal utility, innovation is a public "bad;" issuing equity is a strictly dominant strategy and in the unique Nash equilibrium markets are incomplete.

These examples demonstrate that predictions regarding the incompleteness of endogenous market structure depend on primitive investor preferences, which qualitatively changes incentives to innovate.

5.2 Endogenous Market Completeness

Theorem 1 offers predictions regarding market (in)completeness in a general model. To the extent the miscoordination in financial innovation exhibited by a mixed strategy equilibrium is not a problem, our model provides strong predictions based solely on investor preferences: When investors' marginal utility function is strictly concave, if the innovation costs are not prohibitively high, then markets are complete in all pure strategy Nash equilibria. With convex investors' marginal utility function, the financial structure is incomplete in all pure strategy Nash equilibria, unless the only feasible structures are complete.

[21]The inability to coordinate does not stem from randomization over financial structures *per se*, but from the independence of entrepreneurs' strategies (i.e., the independence of mixed strategy distributions). With public (i.e., perfectly correlated) signals, there are correlated equilibria in which one of the events (C, E) or (E, C) is realized and markets are complete with probability 1.

Theorem **1** (Endogenous Market Completeness). *The following characterize the equilibrium financial structure.*

1. *If $u'''(c) < 0$ on $\mathcal{X}$, then $\bar{\gamma} > 0$ exists such that, for any $0 < \gamma \leq \bar{\gamma}$, in any pure strategy Nash equilibrium, the resulting financial structure F is complete.*

2. *Suppose that $u'''(c) \geq 0$ on $\mathcal{X}$ and that*

$$\langle \{z_n \mid n = 1, \ldots, N\} \rangle \neq \mathbb{R}^S;$$

then, for any $\gamma > 0$, in any pure strategy Nash equilibrium, the resulting financial structure F is incomplete.

The predictions regarding endogenous market (in)completeness are quite robust. They hold (with probability 1) in markets for an arbitrary number of entrepreneurs (that is, regardless of the intensity of competition), any number of states, arbitrary payoff structures of their real assets (with common or idiosyncratic risk) and any (absolutely continuous) joint distribution of investor endowments. The results also extend to dynamic games in which entrepreneurs sequentially choose their financial structures and the solution concept is the Subgame Perfect Nash equilibrium.

Building on the analysis of the monotonicity of an entrepreneur's n market value in the (joint) span of F (Section 4.3), and hence in the span of F_n given the financial structures of entrepreneurs $n' \neq n$, the next result provides sufficient conditions under which, in the unique (dominant strategy) equilibrium, financial structure has minimal span in the sense that

$$F = \{z_n \mid n = 1, \ldots, N\}.$$

Proposition **2** (Equilibrium in Dominant Strategies). *There is a unique Nash equilibrium, in which the resulting financial structure has minimal span (and markets are incomplete), if one of the following two conditions holds:*

1. *There are only two states of the world in period 2 and investor marginal utility function is strictly convex; or*

2. *Investor Bernoulli utility is CARA and all entrepreneurs are endowed with riskless real assets.*

Outside of CARA settings, in a model with convex marginal utility and $S > 2$, issuing equity need not be a dominant strategy and there may be multiple Nash equilibria. By Theorem 1, markets are then incomplete in all pure strategy equilibria.

Further, when marginal utility is convex, in a mixed strategy equilibrium, markets may be *complete* with positive probability even though market value is maximized by an incomplete financial structure and even if innovation is costly. Equilibrium financial structure then involves a set of securities that are individually suboptimal for each entrepreneur; that is, each has *ex post* regret, given the financial structures chosen by the others. Just as in markets with concave marginal utility (See Section 5.1) or in the economy with heterogeneous utilities studied by AG, *ex post* regret occurs when entrepreneurs cannot coordinate their activities—unlike pure strategy simultaneous competition or sequential innovation. The next example demonstrates that the miscoordination mechanism identified by AG carries over to the latter environments.

Example **4.** *Suppose that $S = 3$, there are two entrepreneurs with the riskless asset $z_n = (1, 1, 1)$, $n = 1, 2$, and two types of investors whose utility function and endowments are as in Example 2. Let the mass of each investor type be 1 and let the states be equally likely. In Example 2 and Proposition 1, we demonstrate the existence of a financial structure whose span has dimension 2, which strictly dominates equity and (any) complete financial structure. Therefore, the span of a financial structure F^* that maximizes the market value of the entrepreneurs' assets, which exists by Lemma 3, has dimension 2. Let V^* denote this maximized market value. By continuity of market value on the set of two-dimensional spans, one can find a two-dimensional linear subspace $L^{**} \neq \langle F^* \rangle$, with a corresponding two-asset financial structure F^{**}, that yields market value V^{**} that is arbitrarily close to V^* ($V^{**} \simeq V^*$). By construction, financial structure $F = \{F^*, F^{**}\}$ is complete.*

There is a mixed strategy Nash equilibrium in which entrepreneurs randomize over equity, F^*, and F^{**}. The equilibrium probabilities of choosing F^* and F^{**} are

$$\sigma^* \simeq \sigma^{**} \simeq \frac{1}{3}\left(1 - \frac{\gamma}{V^* - V_C}\right), \tag{5}$$

where V_C is the market value in a complete market.[22] *Since $V^* > V_C$, for a sufficiently small innovation cost γ, probabilities σ^* and σ^{**} are strictly positive. In equilibrium, markets are complete with probability $2\sigma^*\sigma^{**} > 0$.*

For the intuition, the market value in the example is not monotonically decreasing in the security span; each entrepreneur is willing to pay innovation costs in order to partially complete the market—either of the two incomplete financial structures, F^* or F^{**}, gives strictly higher market value than equity. In the described equilibrium, entrepreneurs fail to coordinate on one of F^* and F^{**}, which may result in an undesirable outcome of complete equilibrium financial structure F and $V_C < V^*$.

[22]Suppose entrepreneur n' follows the mixed strategy $(1 - \sigma^* - \sigma^{**}, \sigma^*, \sigma^{**})$ over structures $\{(1, 1, 1)\}$, F^* and F^{**}. Then, the expected profits of entrepreneur n, under the financial structures $\{(1, 1, 1)\}$, F^* and F^{**} are, respectively, $(1 - \sigma^* - \sigma^{**})V_C + \sigma^*V^* + \sigma^{**}V^{**}$, $(1 - \sigma^{**})V^* + \sigma^{**}V_C - \gamma$, and $(1 - \sigma^*)V^{**} + \sigma^*V_C - \gamma$, respectively, where we used that under $\{(1, 1, 1)\}$, market value coincides with V_C (in Example 2, under equity, there is no distortion in the third state consumption and market value is V_C). Equating the three net expected payoffs and taking the limit as $V^{**} \to V^*$ gives $\sigma^* = \sigma^{**}$ as in (5). In the example, with a sufficiently small innovation cost γ, when entrepreneur $n' \neq n$ issues equity $\{(1, 1, 1)\}$, it is optimal for entrepreneur n to choose $F_n = F^*$, in which case the market value equals V^*; it is marginally less profitable to chose F^{**} and obtain V^{**}. However, if entrepreneur n' chooses $F_{n'} = F^*$ or $F_{n'} = F^{**}$, then, given costly innovation, issuing equity alone maximizes his profit.

It is worth noting that, except for predictions concerning miscoordination—with concave marginal utility or in the example presented by AG—the innovation costs are not essential for predictions in the following sense.[23] When innovation costs vanish ($\gamma \to 0$), the probability of market incompleteness tends to 0 in mixed strategy equilibria for markets with concave marginal utility and in the example presented by AG. By contrast, in the limit of any pure strategy equilibria of our model, markets with concave (convex) marginal utility remain complete (incomplete).

5.3 Competition in Asset Innovation and Welfare

The ability to alter the security span and, hence, the allocation of date-2 consumption among investors allows entrepreneurs to affect prices even in markets with large numbers of entrepreneurs. A natural question arises regarding how the power of entrepreneurs to create markets impacts welfare.

Our model has the following implications for the welfare appraisal of asset innovation. Clearly, to achieve efficiency of market outcomes, a policy must induce a full-span portfolio of securities. As suggested by Lemma 1, this recommendation can be strengthened: the introduction of an additional security is never detrimental to welfare, even if asset innovation does not fully complete the financial structure. As monetary transfers add up to 0 across investors and entrepreneurs, then for any pair of structures F and F' such that $\langle F \rangle \subseteq \langle F' \rangle$, by Lemma 1, the change in deadweight loss is equal to

$$\max_{x \in X(\langle F' \rangle)} \bar{U}(x) - \max_{x \in X(\langle F \rangle)} \bar{U}(x).$$

Since $X(\langle F \rangle) \subseteq X(\langle F' \rangle)$, it follows that the deadweight loss is (weakly) decreasing in the span of a financial structure.

By our results, the equilibrium financial structure F necessarily distorts allocation in markets where investor marginal utility is convex: maximization of the market value of an asset by all entrepreneurs requires market incompleteness, which ($G-$a.s.) introduces a wedge in investors' marginal utilities in equilibrium. Indeed, the very mechanism through which market incompleteness provides an effective means to increase entrepreneurs' market values involves the introduction of inefficiency in the allocation of numèraire among investors. Nevertheless, as the analysis from Section 3 implies, entrepreneurs' benefits are not always associated with a loss for investors (See Example 2).[24]

[23] Innovation costs eliminate the (trivial) multiplicity of Nash equilibria that would be present in the model with costless innovation in which entrepreneurs simultaneously choose financial structures: if one entrepreneur chooses a complete financial structure, it is a weak best response for all other entrepreneurs to issue complete financial structures as well, regardless of market primitives (by changing F_n, the entrepreneurs have no impact on financial structure F).

[24] With linear marginal utilities, market value is invariant to financial structure, but among all such financial structures, only those with a full span yield an efficient allocation.

As a more general insight from our analysis, unlike competition in quantities such as the Cournot or Stackelberg models, the allocative efficiency of markets in which entrepreneurs compete in asset innovation (in spans) does not depend on the number of innovators or timing of strategies. Rather, (the shape of) investors' marginal utility is the key determinant of the completeness, and hence efficiency, of frictionless markets.

6 Concluding Remarks

This paper highlights a central role of the shape of investor marginal utility for the incompleteness of frictionless markets. An assessment of whether convex or concave marginal utility is more plausible requires a theory of the third derivative of the utility functions of investors. Empirical evidence developed for theories that recognize the importance of the third derivative, for instance precautionary savings, provides some support in favor of convex marginal utility ("prudence.")[25] Insofar as such preferences (including the standard logarithmic, CARA and CRRA utility functions) describe markets well, a firm's market value is maximized by an incomplete financial structure. In convex marginal utility environments, frictionless markets with unlimited short sales do not offer sufficient incentives for entrepreneurs to complete the financial structure. The incentives to innovate must then originate elsewhere; trading commissions from intermediation, asymmetric information, short sales restrictions or innovation-subsidizing policies, for example. Thus, our results on equilibrium market incompleteness demonstrate and characterize the scope for efficiency improvement through innovation incentives other than the spanning motive and the effectiveness of potential government regulation or intervention. In particular, our research highlights the essential role of intermediary profits from commission fees for completing the market and efficiency (e.g., Chen (1995); Pesendorfer (1995); Bisin (1998)). As a normative implication, strong welfare and regulation recommendations emerge: A policy to encourage innovation to complete markets by reducing innovation costs might be effective if marginal utility is concave, but it is ineffective in markets with convex marginal utility.

[25]Loosely speaking, the mechanism behind the theory of precautionary savings shares the implication of convex marginal utility that lowering consumption increases an agent's marginal utility more than increasing consumption reduces it. However, the precautionary savings effect is present in a single-agent problem, whereas ours operates, crucially, as an *equilibrium* mechanism through heterogeneity across agents. Furthermore, while the precautionary savings phenomenon concerns differences in marginal utilities (and transferring consumption) across states, the conditions for optimality of (in)complete financial structures involve differences in marginal utilities and consumption across agents *within* states.

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Appendix

Proof (Lemma 1, Allocative Equivalence): To prove necessity, suppose that $\tilde{p}$ and $(\tilde{t}_1, \ldots, \tilde{t}_K)$ are the prices and the allocation of securities in a competitive equilibrium under structure F and let x be any allocation of date-2 consumption in $X(\langle F \rangle)$. Let t_k be such that $x_k - e_k = Ft_k$ and $\sum_k F\theta_k t_k = \sum_n z_n$, which exists because $x \in X(\langle F \rangle)$. Since $(\tilde{p}, \tilde{t})$ is a competitive equilibrium, we have that $\tilde{p} = F^\mathsf{T} DU(\tilde{x}_1)$ and, hence,

$$\tilde{p} \cdot \sum_k \theta_k t_k = DU(\tilde{x}_1)^\mathsf{T} \sum_k \theta_k F t_k = DU(\tilde{x}_1)^\mathsf{T} \sum_n z_n = DU(\tilde{x}_1)^\mathsf{T} \sum_k \theta_k F \tilde{t}_k = \tilde{p} \cdot \sum_k \theta_k \tilde{t}_k.$$

In addition, by the optimality of each investor's choice given prices,

$$U(e_k + Ft_k) - \tilde{p} \cdot t_k \leq U(e_k + F\tilde{t}_k) - \tilde{p} \cdot \tilde{t}_k. \tag{6}$$

Aggregating across types, we have that

$$\bar{U}(x) \equiv \sum_k \theta_k U(e_k + Ft_k) \leq \sum_k \theta_k U(e_k + F\tilde{t}_k) \equiv \bar{U}(\tilde{x}).$$

Since $\tilde{x}_k - e_k = F\tilde{t}_k$, it follows that the resulting transfers of numèraire lie in $\langle F \rangle$ and that

$$\sum_k \tilde{x}_k = \sum_k e_k + F\mathbf{1} = \sum_k e_k + \sum_n z_n,$$

which in turn implies that $\tilde{x} := (\tilde{x}_1, \ldots, \tilde{x}_K) \in X(\langle F \rangle)$. This observation and (6) imply that $\tilde{x}$ indeed solves Problem (2).

For sufficiency, note first that set $X(\langle F\rangle)$, defined in (1), can be alternatively written as

$$X(\langle F\rangle) = \{(e_1 + Ft_1, \ldots, e_K + Ft_K) \mid \textstyle\sum_k \theta^k t^k = \mathbf{1}\},$$

while Problem (2) can be equivalently written as

$$\max_{(t_1,\ldots,t_K)} \sum_k \theta_k U(e_k + Ft_k) : \sum_k \theta_k t_k = \mathbf{1},$$

and, by assumption, $\tilde{t}_k$ is its solution. By the assumption that the Bernoulli index u satisfies the Inada condition and using the fact that return $\sum_n z_n$ is strictly positive in all states and lies in $\langle F\rangle$, we have that $\tilde{x}_k$ is strictly positive in all components and for all investor types. Then, by the Kühn-Tucker Theorem, multipliers $\tilde{p}$ must exist such that, for all k,

$$F^{\mathsf{T}} DU(e_k + F\tilde{t}_k) = \tilde{p}.$$

Since the Bernoulli function u is strictly concave, the latter suffices to imply that $\tilde{t}_k$ solves problem

$$\max_{t_k} U(e_k + Ft_k) - \tilde{p} \cdot t_k$$

and since $\sum_k \theta_k t_k = \mathbf{1}$, securities allocation $(\tilde{t}_1, \ldots, \tilde{t}_K)$ and prices $\tilde{p}$ constitute a competitive equilibrium under structure F. *Q.E.D.*

Proof (Lemma 2, Market Value): Recall that $x(L)$ is the competitive allocation of numèraire for any F such that $\langle F\rangle = L$. Given the profile of endowments of date-2 wealth, since the equilibrium prices of securities satisfy $p = F^{\mathsf{T}} DU(x_k(\langle F\rangle))$ for each investor type k, if we take the average across all investors, we obtain

$$p^T = \frac{1}{\theta} \sum_k \theta_k DU(x_k(\langle F\rangle))^{\mathsf{T}} F \equiv \kappa(\langle F\rangle)^{\mathsf{T}} F.$$

In addition,

$$V_n(F) = p_n \cdot \mathbf{1} = \kappa(\langle F\rangle)^{\mathsf{T}} F_n \mathbf{1} = \kappa(\langle F\rangle) \cdot z_n.$$

The expected market value of entrepreneur n is $E_G[V_n(F)] = E[\kappa(\langle F\rangle)] \cdot z_n$. *Q.E.D.*

Proof (Lemma 3, Existence): Take any linear space L such that $\{z_1, \ldots, z_n\} \subseteq L$. If $\{z_1, \ldots, z_n\}$ contains M linearly independent assets, then the orthogonal complement of $\langle\{z_1, \ldots, z_n\}\rangle$ is a linear subspace of dimension $S - M$ and a basis for L can be constructed by taking the M linearly independent assets and $\dim(L) - M$ linearly independent vectors in $\langle\{z_1, \ldots, z_n\}\rangle^{\perp}$. It follows that, for any $M \leq D \leq S$, the space of D-dimensional spaces of trades that contain $\{z_1, \ldots, z_n\}$ is topologically equivalent to the set of $(D - M)$-dimensional linear subspaces of $\mathbb{R}^{S-M}$. This Grassmannian is a compact manifold (of dimension $(D - M) \times (S - D)$).

Consider the set of all structures F for which the dimension of span $\langle F\rangle$ is D. Over this set, correspondence $X(\langle F\rangle)$ is upper- and lower-semicontinuous. Since function $\bar{U}$ is continuous on $X(\langle F\rangle)$ for any F, it follows by the Theorem of the Maximum that the allocation function, $x(L)$, is continuous on the Grassmannian. Further, it also follows that the expected market value, $\mathrm{E}_G[V_n(L)] = \mathrm{E}_G[\kappa(L)] \cdot z_n$, is continuous as well and, therefore, that a linear space L^* exists that maximizes it over the set of all linear subspaces of dimension D.

Denote by V_n^D the maximized expected market value over the set of structures that span D-dimensional spaces of numèraire transfers. Since S is finite, the seller's program is reduced to finding the maximum of $\{V_n^M, \ldots, V_n^S\}$. *Q.E.D.*

Proof (Proposition 1, Value-Maximizing Financial Structure): With the complete financial structure F', by Lemma 1, the allocation of numèraire is such that each investor consumes the same at date 2, namely that

$$x_k(\langle F' \rangle) = \frac{1}{\theta}\left(\sum_n z_n + \sum_k \theta_k e_k\right),$$

and hence, the resulting market value for entrepreneur n equals

$$V_n(F') = \kappa(\langle F' \rangle)^\mathsf{T} z_n = DU\left(\frac{1}{\theta}\sum_n z_n + \frac{1}{\theta}\sum_k \theta_k e_k\right) \cdot z_n.$$

Next, consider a feasible financial structure F for which $\langle F \rangle \neq \mathbb{R}^S$. Consider the linear subspace of endowment profiles defined by

$$E = \{(e_1, \ldots, e_K) \in \mathbb{R}^{S \times K} \mid \sum_n z_n + \sum_k \theta_k e_k \in \langle F \rangle\}.$$

Since, by feasibility, $\sum_n z_n \in \langle F \rangle$, it follows that E has dimension lower than $S \times K$ and, hence, has zero Lebesgue measure.[26] Since G is absolutely continuous with respect to the Lebesgue measure for $\mathbb{R}^{S\times K}$, it follows that, $G-$a.s., $x_k(\langle F \rangle) \neq x_{k'}(\langle F \rangle)$ for at least two types of investor, k and k'.

For claim (i), note that since function u' is strictly convex over the relevant domain and since, for each state s

$$\sum_k \theta_k x_k^s(\langle F \rangle) = \sum_n z_n^s + \sum_k \theta_k e_k^s,$$

one has that

$$\kappa^s(\langle F \rangle) \equiv \frac{1}{\theta}\sum_k \theta_k u'\left(x_k^s(\langle F \rangle)\right) > u'\left(\frac{1}{\theta}\sum_n z_n^s + \frac{1}{\theta}\sum_k \theta_k e_k^s\right) \equiv \kappa^s(\langle F' \rangle),$$

and hence, $\kappa(\langle F \rangle) \gg \kappa(\langle F' \rangle)$. It follows that

$$V_n(F) = \kappa(\langle F \rangle)^\mathsf{T} z_n > \kappa(\langle F' \rangle)^\mathsf{T} z_n = V_n(F'),$$

[26] To see that this is the case, suppose, by way of contradiction, that $E = \mathbb{R}^{S\times K}$. Let ι_s be the s-th canonical vector in $\mathbb{R}^S$, and construct the following profile of endowments: $e_1 = \frac{1}{\theta_1}\iota_s$ and $e_k = (0, \ldots, 0)$ for every $k \geq 2$. Since this profile lies in E, we have that

$$\sum_n z_n + \iota_s = \sum_n z_n + \sum_k \theta_k e_k \in \langle F \rangle,$$

and then, using $\sum_n z_n \in \langle F \rangle$, we have that $\iota_s \in \langle F \rangle$. But since this is true for all $s = 1, \ldots, S$, we have that $\langle F \rangle = \mathbb{R}^S$.

$G-$a.s. (In the $G-$null set where all investors equate second period consumption, the two levels of market value are equal.)

The arguments for claims (ii) and (iii) are analogous and, hence, omitted. *Q.E.D.*

Proof (Theorem 1, Endogenous Market Completeness): For the first claim, define the cost threshold

$$\bar{\gamma} := \frac{1}{2} \min \left\{ \frac{V_n^S - V_n^D}{S} \mid D = M, \ldots, S-1 \text{ and } n = 1, \ldots N \right\},$$

where M and V_n^D for all $n = 1, \ldots N$ and all $D = M, \ldots, S$ are defined as in the proof of Lemma 3; by Proposition 1, $\bar{\gamma} > 0$. Now, given any $\gamma \leq \bar{\gamma}$, if entrepreneurs other than n chose an incomplete financial structure $F_{-n} := (F_1, ..., F_{n-1}, F_{n+1}, ..., F_N)$, then it is a best response for entrepreneur n to choose F_n such that the resulting structure $\{F_n, F_{-n}\}$ is complete, again by Proposition 1.

For the second claim, let $F = \{F_n \mid n = 1, \ldots, N\}$ be the profile of financial structures chosen at a pure strategy Nash equilibrium. Since, by assumption, the payoff structure of real assets constitutes an incomplete financial structure, at least one entrepreneur exists for whom $F_n \neq \{z_n\}$. For such entrepreneur, $F_n' = \{z_n\}$ gives strictly higher payoff; it is less costly and his asset's market value is at least as high as with F_n. *Q.E.D.*

Proof (Proposition 2, Equilibrium in Dominant Strategies): Consider entrepreneur n and suppose that the strategies chosen by his competitors are F_{-n}. If V_n is decreasing in the span of the financial structure, then, regardless of the rank of F_{-n}, financial structure $F_n = \{z_n\}$ maximizes the market value of z_n and is also the cheapest structure to issue, so it is a unique best response to the arbitrary choices of financial structures by other entrepreneurs.

Since market value is monotone in the security span in economies with two states or CARA utility function and riskless assets, issuing equity is a strictly dominant strategy. The game has the unique (pure strategy) strategy Nash equilibrium and in such equilibrium $F = \{z_n \mid n = 1, \ldots, N\}$. *Q.E.D.*