Adverse Selection in Competitive Search Equilibrium[*]

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Abstract

We study economies with adverse selection, plus the frictions in competitive search theory. With competitive search, principals post terms of trade (contracts), then agents choose where to apply, and they match bilaterally. Search allows us to analyze the effects of private information on both the intensive and extensive margins (the terms and probability of trade). There always exists a separating equilibrium where each type applies to a different contract. The equilibrium is unique in terms of payoffs. It is not generally efficient. We provide an algorithm for constructing equilibrium. Three applications illustrate the usefulness of the approach, and contrast our results with those in standard contract and search theory.

1   Introduction

We are interested in equilibrium and efficiency in economies with adverse selection, plus the frictions in competitive search theory. In competitive search, principals on one side of the market post terms of trade—here they post contracts—and agents on the other side choose where to direct their search. Then they match bilaterally, although some principals and agents may fail to find a partner. Agents here have private information concerning their type. Although there is not much work explicitly incorporating competitive search and information frictions simultaneously, with a few exceptions discussed below, we think that such an integration is natural. It is is natural because it allows uninformed principals to try, through the posted terms of trade, to attract certain types and screen out others. The framework yields strong results: we prove that equilibrium always exists, which is not always the case with adverse selection; we prove that equilibrium payoffs are unique; we show that equilibrium is not generally efficient; and we provide an algorithm for constructing equilibrium.

The model is useful for studying a variety of substantive economic issues. For example, in labor markets, an idea discussed in many papers using contract theory is that incentive problems related to private information can distort allocations in terms of hours per worker (the intensive margin). Another idea is that incentive problems may affect the probability that a worker gets a job in the first place (the extensive margin), and for this, search theory is useful. It seems interesting to study models where both margins are operative, to see how incentive problems manifest themselves in terms of distortions in either hours per worker, or unemployment, or both. Similarly, in asset markets it is commonly thought that information frictions may be reflected by discounts in prices (again, the intensive margin). It is sometimes also suggested that these frictions may be reflected in liquidity, or the time it takes to trade (the extensive margin). Again, it is interesting to allow both margins to operate in the same model. To this end, we think it is a good idea to pursue models that integrate information and search theory.[1]

In our framework, although agents observe everything that is posted and search wherever they like, matching is bilateral—each principal meets at most one agent and vice versa. We assume that the number of agents and the distribution of types is fixed, while the number of active principals is determined by free entry. As is standard, principals and agents potentially

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[1]There are too many contributions to provide a survey here, but by way of example, a well-known paper where information and incentive issues distort hours per worker is Green and Kahn (1983), and a one where they distort unemployment is Shapiro and Stiglitz (1984). Papers where informational frictions distort the terms of trade in the asset market include Glosten and Milgrom (1985) and Kyle (1985), and one where they distort the time it takes to trade is Williamson and Wright (1994). We discuss other papers in more detail below.

face a trade-off between the terms of trade and market tightness. For example, a worker might like to apply for a high-wage job, but not if too many others also apply. In our setup, principals also need to form expectations about market composition, i.e. which types of agents search for a given contract. Under mild assumptions, including a generalized single-crossing property, there always exists a separating equilibrium where each principal posts a contract that attracts a single type of agent. Because we get separating equilibria, we can assume without loss of generality that principals post contracts, as opposed to an ostensibly more general situation where they post revelation mechanisms. This simplifies the model. We also show how to characterize equilibrium by solving an algorithm, involving a sequence of optimization problems. This simplifies the analysis a lot.

We provide a series of applications and examples to illustrate the usefulness of the framework and to show how some well-known results in contract theory and in search theory change when we combine elements of both in the same model. The first application is a classic sorting problem (Akerlof, 1976). Suppose that workers are heterogeneous with respect both to their expected productivity and to their cost of working longer hours: more productive workers find long hours less costly. Contracts specify a combination of wages and hours of work but cannot be directly conditioned on a worker's type (it is private information). In equilibrium, firms may require that more productive workers work longer hours than under full information—a version of the rat race. We discuss cases where, although hours are distorted, the probability that a worker gets a job is not, and other cases where there is also over-employment of high types. We find that the equilibrium can be Pareto inefficient if there are few low-productivity agents or the difference in the cost of working is small.

Our second application is a version of the well-known Rothschild and Stiglitz (1976) insurance problem. Consider a labor market interpretation, rather than pure insurance, as in the original model (only because in the labor market our assumption of bilateral meetings may seem more natural). Risk-averse workers and risk-neutral firms can combine to produce output, but only some pairs are productive. Workers differ in the probability that they will be productive once they are matched. Firms can observe whether a worker is productive but cannot observe a worker's type. If a match proves unproductive, the worker is let go. In equilibrium, firms separate workers by only partially insuring them against the probability the match will be unproductive. Indeed, workers are worse off if they find a job and are then let go, than they would be had they never found a job in the first place. We interpret this as an explanation for the fact that firms do not fully insure workers against layoff risk: if they did, they would attract low-productivity applicants.

This example also illustrates how our approach resolves the nonexistence problem in stan-

dard adverse selection models like Rothschild and Stiglitz (1976). When there are relatively few low productivity workers, equilibrium may not exist in that model for the following reason: given any separating contract, profit for an individual firm can be increased by a deviation to a pooling contract that subsidizes low-productivity workers. Here, such a pooling contract will not increase profit. The key difference is that in our model firms can match with at most one worker, so a deviation cannot serve the entire population. Suppose a firm posts a contract designed to attract a representative cross section of agents. The more workers that search for this contract, the less likely it is that any one will match. This discourages some from searching. Critically, it is the most productive workers who are the first to go, because their outside option—trying to obtain a separating contract—is more attractive. Hence, only undesirable types are attracted by the deviation, making it unprofitable.

Our third application, to asset markets, illustrates among other things how adverse selection can sometimes make it harder to trade without affecting the terms of trade. Principals want to buy and agents want to sell apples, meant to represent assets that could be high or low quality.[2] As in Akerlof (1970), some apples are bad: they are lemons. To make the case stark, we can even assume there are no fundamental search frictions, so that everyone on the short side of the market matches. But it is important to understand that here the short side of a market is endogenous. In equilibrium, we show that sellers with good apples trade only probabilistically, which is precisely how buyers screen low quality. This is different from related results in the literature (e.g. Nosal and Wallace (2007)), where lotteries are used to screen. Interestingly, screening through search saves resources, compared to lotteries. Still, equilibrium can be Pareto dominated by a pooling allocation if there are few bad apples. We also show that in some cases the market completely shuts down, an extreme lemons problem perhaps relevant for understanding the recent collapse in some credit markets.

In terms of the literature, many papers propose alternative solutions to the Rothschild and Stiglitz (1976) nonexistence problem. As mentioned, a key difference in our paper is that matching is bilateral, and that each principal can serve at most one agent. This can create distortions along the extensive margin and implies that principals must form expectations about which agents are most attracted to a contract. Ours is not the first paper to highlight the key role of capacity constraints in breaking the nonexistence result. Gale (1996) uses a notion of competitive equilibrium with price-taking principals and agents in an environment with one-sided asymmetric information. He allows for the possibility of rationing in equilibrium and recognizes how this affects the composition of agents seeking particular contracts. We discuss in more detail the relationship between his notion of equilibrium and ours when

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[2]Apples here stand in for claims to long-lived assets (trees) with uncertain dividends (fruit), as in standard asset-pricing theory.

we present the formal definition in Section 2; for now we simply note that Gale (1996) does not prove existence or uniqueness, nor does he provide our simple characterization.

More recently, research by Inderst and Wambach prove the existence of equilibrium in an adverse selection model with capacity constraints and a finite number of principals and agents. Inderst and Wambach (2001) develop a model that is close to our second example, although they find that the equilibrium is not unique due to the possibility of coordination failures from the independent outcomes of mixed search strategies. This is not an issue in our large economy. Inderst and Wambach (2002) develop a model that is a special case of our first example, and they prove existence and uniqueness of equilibrium.[3] Due to the nature of the example, there is no rationing along the equilibrium path, although as they stress, rationing off the equilibrium path is key to sustaining equilibrium. In our third example, there is always rationing in equilibrium, a possibility that appears to be new to this literature. Ours is the first paper to develop a general framework for analyzing competitive search with adverse selection, and to present a variety of applications that stress distortions along both the intensive and extensive margins.

Other papers follow Prescott and Townsend (1984) and study adverse selection in competitive economies, without rationing. In Gale (1992) and Dubey and Geanakoplos (2002), agents are price-takers, and trade contracts at prices that clear the market. Most contracts are not traded and so beliefs about the composition of traders are arbitrary, which makes it easy to sustain equilibrium. They then consider different refinements to ensure that equilibrium is unique. Bisin and Gottardi (2006) also assume that agents are price-takers, but only permit principals to post incentive-compatible contracts. Again, this ensures the existence of a unique outcome, the least cost separating equilibrium. The mechanism that selects equilibrium is quite different in our paper, relying on the willingness of agents to be rationed.

An older literature resolves the Rothschild-Stiglitz non-existence problem by modifying the game. Miyazaki (1977) and Wilson (1977) allow principals to withdraw contracts after other principals deviate. This makes deviations less profitable and typically leads to the existence of a pooling equilibrium where some agents cross-subsidize others. Riley (1979) lets principals add new contracts to the menu after other principals deviate, allowing the least cost separating equilibrium to survive. We remain closer to Rothschild and Stiglitz (1976), allowing principals to commit to contracts, and find that there is no cross-subsidization in equilibrium.

Our paper is also related to a large literature on competitive search (Montgomery, 1991;

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[3]In Inderst and Wambach (2002), there are capacity constraints but no search frictions. A related working paper, Inderst and Müller (1999), also has search frictions.

Peters, 1991; Moen, 1997; Shimer, 1996; Acemoglu and Shimer, 1999; Burdett, Shi and Wright, 2001; Mortensen and Wright, 2002). A few earlier papers proposed extensions of the competitive search framework to an environment with private information (Faig and Jerez, 2005; Guerrieri, 2008; Moen and Rosén, 2006). However, in all these papers agents are ex-ante homogeneous and their private information is about the quality of the match.

The rest of the paper is organized as follows. In Section 2, we develop the general environment, define equilibrium, and discuss some critical assumptions. In Section 3, we show how to find equilibrium by solving a constrained optimization problem, prove that a separating equilibrium always exists, and show that equilibrium payoffs are unique. In Section 4, we define a class of incentive-feasible allocations in order to discuss whether equilibrium outcomes are efficient within this class. In Sections 5–7, we present the applications discussed above, in each case characterizing equilibria and discussing efficiency. Section 8 concludes. Proofs are relegated to the Appendix.

2　The Model

2.1　Preferences and Technology

There is a measure 1 of agents, a fraction $\pi_i > 0$ of whom are of type $i \in \mathbb{I} \equiv \{1, 2, \ldots, I\}$. Type is an agent's private information. It could index, for instance, productivity or preferences, if agents are workers; if they are sellers of assets or commodities, type could represent the quality of their holdings. There is a large set of ex ante homogeneous principals, each of whom may or may not participate in the market: they can enter, which provides a principal an opportunity to match with an agent, if and only if they pay cost $k > 0$.[4] To keep the analysis focused, in this project, we consider an environment where principals and agents have a single opportunity to match, and matching is bilateral. If a principal and an agent match, the pair enter into a relationship described by a contract.

A contract is given by a vector $y \in Y$ that may specify actions for the principal, actions for the agent, transfers between them, and other possibilities. As we make explicit in the applications, it can include lotteries. A contract can also index actions, transfers, etc. by dates and states of nature. That is, relationships can involve dynamic state-contingent contracts, although the search part of the model is, as we said above, static in the sense that principals and agents have a single opportunity to match. All we need to say for now about the set of feasible contracts Y is that it is compact, nonempty, and contained in a metric

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[4]In our model the ex ante investment k is given. See Mailath, Postlewaite and Samuelson (2009) for a model where asymmetric information may lead to inefficient pre-match investments.

space with metric $d(y, y')$ for $y, y' \in Y$. A principal who matches with a type i agent gets a payoff $v_i(y) - k$ from contract y. A principal who does not enter the market gets a payoff normalized to 0, while one who enters but fails to match gets $-k$. A type i agent matched with a principal gets a payoff $u_i(y)$, while an unmatched agent gets a payoff also normalized to 0. For all i, $u_i : Y \mapsto \mathbb{R}$ and $v_i : Y \mapsto \mathbb{R}$ are assumed to be continuous.

When we say that a principal enters the market, we mean he posts a mechanism, which is a menu of contracts. In the simplest labor market context, a firm enters the market by posting a vacancy—a willingness to hire at a given wage—where by assumption it cannot post multiple vacancies, and even if many workers apply (direct their search to this vacancy), at most one worker ever contacts the firm.[5] Generally, by posting we mean that the principal announces and commits to a mechanism, and all agents can see what every principal posts. By the revelation principle, without loss of generality, we can assume they post direct revelation mechanisms. A posted mechanism is a vector of contracts, $\mathbf{y} = \{y_1, \ldots, y_I\} \in Y^I$, specifying that if a principal and an agent match, the latter (truthfully) announces his type i and they implement y_i. The mechanism $\mathbf{y}$ is incentive compatible if $u_i(y_i) \geq u_i(y_j)$ for all i, j.[6] Let $C \subset Y^I$ denote the set of incentive compatible mechanisms. Principals only post mechanisms in C.

We now turn to the matching process. As we said, each agent observes what all the principals post, and then directs his search to any one he likes (although he can only apply to one, he can use a mixed strategy to decide which one). Let $\Theta^m(\mathbf{y})$ denote the principal-agent ratio, or market tightness, associated with a mechanism $\mathbf{y}$, defined as the measure of principals posting $\mathbf{y}$ divided by the measure of agents applying to $\mathbf{y}$, $\Theta^m : C \mapsto [0, \infty]$. Let $\gamma_i^m(\mathbf{y})$ denote the share of agents applying to $\mathbf{y}$ that are type i, with $\Gamma^m(\mathbf{y}) \equiv \{\gamma_1^m(\mathbf{y}), \ldots, \gamma_i^m(\mathbf{y}), \ldots, \gamma_I^m(\mathbf{y})\} \in \Delta^I$, the I-dimensional unit simplex. That is, $\Gamma^m(\mathbf{y})$ satisfies $\gamma_i^m(\mathbf{y}) \geq 0$ for all i and $\sum_i \gamma_i^m(\mathbf{y}) = 1$, and $\Gamma^m : C \mapsto \Delta^I$. The functions Θ^m and Γ^m are determined endogenously in equilibrium, and as we discuss below, they are defined for all incentive-compatible mechanisms, not only the ones that are posted in equilibrium.

An agent who applies to $\mathbf{y}$ matches with a principal with probability $\mu(\Theta^m(\mathbf{y}))$, independent of type, where the matching function $\mu : [0, \infty] \mapsto [0, 1]$ is nondecreasing. A principal offering $\mathbf{y}$ matches with a type i agent with probability $\eta(\Theta^m(\mathbf{y}))\gamma_i^m(\mathbf{y})$, where $\eta : [0, \infty] \mapsto [0, 1]$ is nonincreasing. We impose $\mu(\theta) = \theta\eta(\theta)$ for all θ, since the left hand side is the matching probability of an agent and the right hand side is the matching probability

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[5]This assumption renders moot the usual notion of capacity constraints, which is the notion that a firm can only hire a limited number of workers. Here they never contact more than one worker.

[6]Since we are not concerned with moral hazard, we assume that any $y \in Y$ can be implemented by any principal-agent pair.

of a principal times the principal-agent ratio. Together with the monotonicity of μ and η, this implies both functions are continuous. It is convenient to let $\bar{\eta} \equiv \eta(0) > 0$ denote the highest probability with which a principal can meet an agent, obtained when the principal-agent ratio is 0. Similarly let $\bar{\mu} \equiv \mu(\infty) > 0$ denote the highest probability with which an agent can meet a principal. Conversely, $\mu(\theta) = \theta\eta(\theta)$ ensures that $\eta(\infty) = \mu(0) = 0$.

2.2 Assumptions

Throughout our analysis, we make three assumptions on preferences. First, let

$$\bar{Y}_i \equiv \big\{y \in Y \mid \bar{\eta}v_i(y) \geq k \text{ and } u_i(y) \geq 0\big\}$$

be the set of contracts that deliver nonnegative utility to a type i agent while permitting the principal to make nonnegative profits if the principal-agent ratio is 0, and

$$\bar{Y} \equiv \bigcup_i \bar{Y}_i.$$

In equilibrium, contracts that are not in $\bar{Y}$ are not traded, because there is no way to simultaneously cover the fixed cost k and attract agents.

Our first formal assumption is quite mild, and simply says that for any given contract principals weakly prefer higher types.

Assumption A1 *Monotonicity: for all $y \in \bar{Y}$,*

$$v_1(y) \leq v_2(y) \leq \ldots \leq v_I(y).$$

For the next assumptions, let $B_\varepsilon(y) \equiv \{y' \in Y \mid d(y, y') < \varepsilon\}$ be a ball of radius ε around y.

Assumption A2 *Local non-satiation: for all $i \in \mathbb{I}$, $y \in \bar{Y}_i$, and $\varepsilon > 0$, there exists a $y' \in B_\varepsilon(y)$ such that $v_i(y') > v_i(y)$ and $u_j(y') \leq u_j(y)$ for all $j < i$.*

Another mild assumption, A2 is satisfied in any application where contracts allow transfers.[7]

Our final assumption guarantees that it is possible to design contracts that attract some agents without attracting less desirable agents.

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[7]Moreover, we use A2 only in the proof of Proposition 2 below, and nowhere else. We use it to establish that it is possible to make a principal better off while not improving the well-being of agents. If η is strictly decreasing, we could do this by adjusting market tightness, but since for some examples it is interesting to have η only weakly decreasing, we introduce A2.

Assumption A3 *Sorting: for all $i \in \mathbb{I}$, $y \in \bar{Y}_i$, and $\varepsilon > 0$, there exists a $y' \in B_\varepsilon(y)$ such that*

$$u_j(y') > u_j(y) \text{ for all } j \geq i \text{ and } u_j(y') < u_j(y) \text{ for all } j < i.$$

A3 is a generalized version of a standard single-crossing condition. Assume for illustration that: $y \equiv (y_1, y_2) \in \mathbb{R}^2$; $\bar{Y}$ is open; and the functions $u_i(y_1, y_2)$ are differentiable. Then A3 holds if the marginal rate of substitution between y_1 and y_2 is higher for higher types—i.e., if

$$\frac{\partial u_i(y_1, y_2)/\partial y_1}{\partial u_i(y_1, y_2)/\partial y_2}$$

is monotone in i.[8] If this assumption fails, principals may be unable to screen agents and the equilibrium can involve pooling.

2.3 Mechanisms versus Contracts

Under A1-A3, we may without loss of generality assume that each principal posts a single contract y, rather than a mechanism $\mathbf{y}$ offering a (potentially different) contract to each type i that may show up. In this case, the principal-agent ratio and share of type i agents are defined for every contract, $\Theta : Y \mapsto [0, \infty]$ and $\Gamma : Y \mapsto \Delta^I$. This reduces the notation considerably.

When we say that we can assume without loss of generality that each principal posts a contract, rather than a mechanism, we mean the following: Proposition 5 establishes that any equilibrium under the restriction that each principal posts a contract generates an equilibrium for the model where principals post mechanisms. In this equilibrium, each principal posts a degenerate mechanism, offering the same contract to all agents, $\mathbf{y} = \{y, \ldots, y\}$. Conversely, any equilibrium for the general model is payoff-equivalent to one for the model where each principal posts a single contract.[9] This is not too surprising once one sees that equilibria in both environments have the feature that principals separate agents, in the sense that different types trade different contracts.

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[8] Actually, A3 is more general. For example, suppose

$$u_i(y_1, y_2) = \left(\tfrac{1}{2}y_1^{\rho_i} + \tfrac{1}{2}y_2^{\rho_i}\right)^{1/\rho_i} - \frac{y_1^2 + y_2^2}{2} - \frac{1}{8}.$$

for $\rho_i < 1$, and ρ_i is higher for higher types. Then the elasticity of substitution between y_1 and y_2, $1/(1 - \rho_i)$, is increasing in i. Let $Y = [0, 1]^2$, a superset of the points where $u_i(y_1, y_2) > 0$. Any point (y_1, y_2) on the boundary of Y is not in $\bar{Y}$ since $u_i(y_1, y_2) < 0$. This example fails the single-crossing condition when $y_1 = y_2$, but satisfies A3, since it is possible to increase the spread between y_1 and y_2 and attract higher types while repelling lower ones.

[9] In fact, under some conditions, principals might be willing to post menus that attract multiple types and offer them different contracts. But even in such a case, there is always a payoff-equivalent equilibrium in which all principals offer degenerate mechanisms.

Given this, we proceed by assuming for now that each principal posts a single contract. To summarize where we are, the expected utility of a principal who posts contract y is

$$\eta(\Theta(y)) \sum_i \gamma_i(y) v_i(y) - k.$$

The expected utility of a type i agent who applies to contract y is

$$\mu(\Theta(y)) u_i(y).$$

The functions Θ and Γ are defined over the set of feasible contracts Y and are determined in equilibrium.

2.4 Equilibrium

In equilibrium, principals post profit maximizing contracts and earn zero profit; and conditional on the contracts posted and the search behavior of other agents, each agent directs his search to a preferred contract. In practice, many contracts are not posted in equilibrium, but it is still necessary to define beliefs about the principal-agent ratio and the types of agents that would apply for those contracts if they were offered. We propose the following definition of equilibrium, and argue below that the implied beliefs are reasonable.

Definition 1 *A competitive search equilibrium is a vector $\bar{U} = \{\bar{U}_i\}_{i \in \mathbb{I}} \in \mathbb{R}_+^I$, a measure λ on Y with support Y^P, a function $\Theta : Y \mapsto [0, \infty]$, and a function $\Gamma : Y \mapsto \Delta^I$ satisfying:*

(i) *principals' **profit maximization** and **free-entry**: for any $y \in Y$,*

$$\eta(\Theta(y)) \sum_i \gamma_i(y) v_i(y) \leq k,$$

with equality if $y \in Y^P$;

(ii) *agents' **optimal search**: let*

$$\bar{U}_i = \max\left\{0, \max_{y' \in Y^P} \mu(\Theta(y')) u_i(y')\right\}$$

and $\bar{U}_i = 0$ if $Y^P = \varnothing$; then for any $y \in Y$ and i,

$$\bar{U}_i \geq \mu(\Theta(y)) u_i(y),$$

with equality if $\Theta(y) < \infty$ and $\gamma_i(y) > 0$; moreover, if $u_i(y) < 0$, either $\Theta(y) = \infty$ or $\gamma_i(y) = 0$;

(iii) ***market clearing***:

$$\int_{Y^P} \frac{\gamma_i(y)}{\Theta(y)} d\lambda(\{y\}) \leq \pi_i \text{ for any } i,$$

with equality if $\bar{U}_i > 0$.

To understand the equilibrium concept, first consider contracts that are actually posted in equilibrium, $y \in Y^P$. Part (i) of the definition implies that principals earn zero profits from any such contract. Since $\eta(\infty) = 0 < k$, it must be that $\Theta(y) < \infty$. Part (ii) then implies that if type i agents apply for any such contract, that is, $\gamma_i(y) > 0$, they cannot earn a higher level of utility from any other posted contract. And part (iii) guarantees that all type i agents apply to some contract, unless they are indifferent about participating in the market, which gives them the outside option $\bar{U}_i = 0$.

Now consider how our definition imposes restrictions on contracts that are not posted in equilibrium. First, we assume that each principal anticipates that he cannot affect the utility of any type of agent, taking $\bar{U} = \{\bar{U}_i\}_{i \in \mathbb{I}}$ as given. Second, we use a version of subgame perfection. If a principal posts a deviating contract y, he anticipates that the principal-agent ratio $\Theta(y)$ will be such that one type of agent is indifferent about applying for the contract and all other types weakly prefer some other posted contracts. Moreover, his belief about the distribution $\Gamma(y)$ places all its weight on types that are indifferent about applying for the contract. Of course, some contracts may be unattractive to all agents for any principal-agent ratio, in which case we impose $\Theta(y) = \infty$. This is all captured in part (ii) of the definition.[10] Equilibrium condition (i) then imposes that, for principals not to post y, they must not earn positive profit from it, given $\Theta(y)$ and $\Gamma(y)$.

One can think heuristically about $\Theta(y)$ and $\Gamma(y)$ through a hypothetical adjustment process. When a principal considers posting a deviating contract y, he initially imagines an infinite principal-agent ratio. Some contracts will not be able to attract agents even at that ratio, $\bar{U}_i \geq \bar{\mu} u_i(y)$ for all i, in which case $\Theta(y) = \infty$ and the choice of $\Gamma(y)$ is arbitrary and immaterial. Otherwise, some agents would be attracted to the contract, pulling down the principal-agent ratio. This adjustment process stops at the value of $\Theta(y)$ such that $\bar{U}_i = \mu(\Theta(y)) u_i(y)$ for some i and $\bar{U}_j \geq \mu(\Theta(y)) u_j(y)$ for all j. Moreover, only agents with $\bar{U}_i = \mu(\Theta(y)) u_i(y)$ would be attracted to the contract, restricting $\Gamma(y)$.

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[10]The requirement that if $u_i(y) < 0$ then either $\Theta(y) = \infty$ or $\gamma_i(y) = 0$ rules out the possibility that type i agents earn zero utility, but apply for contract y with the expectation they will not be able to get it, $\Theta(y) = 0$. Other agents with $\bar{U}_j > 0$ would then not apply for the contract. Such a belief might make a deviation unprofitable, if $\bar{\eta} v_i(y) < k$, but we find it implausible. In particular, it is inconsistent with the adjustment process described in the next paragraph.

Our definition of equilibrium is related to the "refined equilibrium" concept in Gale (1996). That model initially allows principals to have arbitrary beliefs about the composition of agents attracted to contracts $y \notin Y^P$. In particular, principals may anticipate attracting only undesirable agents, which raises the required principal-agent ratio. At a high principal-agent ratio, the contract may be unattractive to all types of agents, and so the pessimistic beliefs are never confirmed. Gale (1996) notes that this can create multiple equilibria and then argues for a refinement, where the principal believes that the type of agent attracted to y is the type who is willing to endure the lowest principal-agent ratio (p. 220), and suggests that this is equivalent to the "universal divinity" concept (Banks and Sobel, 1987). This refinement is similar to our restriction on Γ for contracts that are not posted in equilibrium.

3   Characterization

We now show how to construct an equilibrium as the solution to a set of optimization problems. For any type i, consider the following problem:

$$\max_{\theta \in [0,\infty], y \in Y} \mu(\theta) u_i(y) \tag{P-i}$$

$$\text{s.t. } \eta(\theta) v_i(y) \geq k,$$

$$\text{and } \mu(\theta) u_j(y) \leq \bar{U}_j \text{ for all } j < i.$$

In terms of economics, (P-i) chooses market tightness θ and a contract y to maximize the expected utility of type i subject to a principal making nonnegative profits when only type i agents apply, and subject to types lower than i not wanting to apply.

Now consider the larger problem (P) of solving (P-i) for all i. More precisely, we say that a set $\mathbb{I}^* \subset \mathbb{I}$ and three vectors $\{\bar{U}_i\}_{i \in \mathbb{I}}$, $\{\theta_i\}_{i \in \mathbb{I}^*}$, and $\{y_i\}_{i \in \mathbb{I}^*}$ solve (P) if:

1. $\mathbb{I}^*$ denotes the set of i such that the constraint set of (P-i) is non empty and the maximized value is strictly positive, given $(\bar{U}_1, \ldots, \bar{U}_{i-1})$;

2. for any $i \in \mathbb{I}^*$, the pair (θ_i, y_i) solves problem (P-i) given $(\bar{U}_1, \ldots, \bar{U}_{i-1})$, and $\bar{U}_i = \mu(\theta_i) u_i(y_i)$

3. for any $i \notin \mathbb{I}^*$, $\bar{U}_i = 0$.

Lemma 1 below claims that there exists a solution to (P). Then, Proposition 1 says that we can find any equilibrium by solving (P); conversely, Proposition 2 says that any solution to (P) generates an equilibrium. Existence and uniqueness of equilibrium (Proposition 3) follows directly. We then find conditions which ensure that $\bar{U}_i > 0$ for all i, and so $\mathbb{I}^* = \mathbb{I}$,

in Proposition 4. Finally, Proposition 5 shows that our results are not sensitive to the restriction that principals post contracts rather than revelation mechanisms.

In terms of related approaches in the literature, Moen (1997) first defined competitive search equilibrium and showed that it is equivalent to the solution to a constrained optimization problem, like our problem (P-1). We extend this to an environment with adverse selection, and show that for all types $i > 1$, this introduces the additional constraints $\mu(\theta)u_j(y) \leq \bar{U}_j$ for all $j < i$. Essentially these constraints ensure that lower types are not attracted to the contract designed for type i. We establish in Lemma 1 below that the appropriate constraint for higher types $j > i$ are also satisfied, so only downward incentive constraints bind in equilibrium.

The solution to problem (P) is essentially the least-cost separating equilibrium: it maximizes the utility of each type of agent subject to principals earning non-negative profits and subject to worse types of agents not attempting to get the contract. The explanation for why an equilibrium must be separating is standard, although the application to an environment where the trading probabilities μ and η are endogenous is more novel. If there were pooling in equilibrium, the sorting condition A3 ensures that principals could perturb the contract to attract only the more desirable type of agent, breaking the proposed equilibrium. Similarly, any other separating contract, e.g. one that strictly excludes lower types, is dominated by a less distortionary contract, e.g. one that leaves lower types indifferent about the contract.

The question remains why a least-cost separating equilibrium always exists. For example, in Rothschild and Stiglitz (1976) such a proposed equilibrium may be broken by a non-distorting pooling contract if there are sufficiently few low types. Such a deviation is never optimal here because of the endogenous composition of the searchers attracted to an off-the-equilibrium-path contract. Part (ii) of the definition of equilibrium ensures that if an agent is attracted to a deviating contract, he earns the same expected payoff from the contract as from his most-preferred equilibrium contract. All other types do better sticking with their equilibrium contract. Since pooling deviations cross-subsidize low types at the expense of high types, low types typically have more to gain from the deviation and are therefore the ones who actually search for the deviating contract. We stress that beliefs about the composition of searchers for contracts that are not posted in equilibrium, Γ, are not arbitrary; if they were, there may be many equilibria supported by the belief that only bad types search for deviating contracts, as in Gale (1996) before he introduces his refinement. When beliefs about Γ are rationally determined by which type of agent has the strongest incentive to apply for a deviating contract, the equilibrium is unique.

We now proceed to show all of this formally. As a preliminary step, we prove that (P) has a solution and provide a partial characterization, by showing that the zero profit condition

binds and that higher types are not attracted by (θ, y).

Lemma 1 *There exists* $\mathbb{I}^*$, $\{\bar{U}_i\}_{i \in \mathbb{I}}$, $\{\theta_i\}_{i \in \mathbb{I}^*}$, *and* $\{y_i\}_{i \in \mathbb{I}^*}$ *that solve (P). At any solution,*

$$\eta(\theta_i)v_i(y_i) = k \text{ for all } i \in \mathbb{I}^*,$$

$$\mu(\theta_i)u_j(y_i) \leq \bar{U}_j \text{ for all } j \in \mathbb{I} \text{ and } i \in \mathbb{I}^*.$$

All formal proofs are in the Appendix, but the existence proof comes directly from noticing that (P) has a recursive structure. As a first step, (P-1) depends only on exogenous variables and thus determines $\bar{U}_1$. In general, at step i, (P-i) depends on the previously determined values of $\bar{U}_j$ for $j < i$ and determines $\bar{U}_i$. Thus, we can solve (P) in I iterative steps.

We now show that a solution to (P) can be used to construct an equilibrium in which some principals offer a contract to attract type $i \in \mathbb{I}^*$, while keeping out other types. The relevant contract is suggested by the solution to (P), but we must also show that no other contract gives positive profit.

Proposition 1 *Suppose* $\mathbb{I}^*$, $\{\bar{U}_i\}_{i \in \mathbb{I}}$, $\{\theta_i\}_{i \in \mathbb{I}^*}$, *and* $\{y_i\}_{i \in \mathbb{I}^*}$ *solve (P). Then there exists a competitive search equilibrium* $\{\bar{U}, \lambda, Y^P, \Theta, \Gamma\}$ *with* $\bar{U} = \{\bar{U}_i\}_{i \in \mathbb{I}}$, $Y^P = \{y_i\}_{i \in \mathbb{I}^*}$, $\Theta(y_i) = \theta_i$, *and* $\gamma_i(y_i) = 1$.

The next result establishes that any equilibrium can be characterized using (P). The proof is based on a variational argument, showing that if (θ_i, y_i) does not solve (P), it cannot be part of an equilibrium.

Proposition 2 *Let* $\{\bar{U}, \lambda, Y^P, \Theta, \Gamma\}$ *be a competitive search equilibrium. Let* $\{\bar{U}_i\}_{i \in \mathbb{I}} = \bar{U}$ *and* $\mathbb{I}^* = \{i \in I | \bar{U}_i > 0\}$. *For each* $i \in \mathbb{I}^*$, *there exists a contract* $y \in Y^P$ *with* $\Theta(y) < \infty$ *and* $\gamma_i(y) > 0$. *Moreover, take any* $\{y_i\}_{i \in \mathbb{I}^*}$ *and* $\{\theta_i\}_{i \in \mathbb{I}^*}$ *with* $\gamma_i(y_i) > 0$ *and* $\theta_i = \Theta(y_i) < \infty$. *Then* $\mathbb{I}^*$, $\{\bar{U}_i\}_{i \in \mathbb{I}}$, $\{\theta_i\}_{i \in \mathbb{I}^*}$, *and* $\{y_i\}_{i \in \mathbb{I}^*}$ *solve (P).*

The above results imply that in equilibrium any contract y that attracts type i solves (P-i), in the sense that the solution to the problem has $\theta_i = \Theta(y)$ and $y_i = y$. The existence of equilibrium and uniqueness of equilibrium payoffs now follow immediately.

Proposition 3 *Competitive search equilibrium exists, and the equilibrium* $\bar{U}$ *is unique.*

The next result shows that, when there are strict gains from trade for all types, all agents get strictly positive utility.

Proposition 4 *Assume that for all* i *there exists* $y \in Y$ *with* $\bar{\eta}v_i(y) > k$ *and* $u_i(y) > 0$. *Then in any competitive search equilibrium,* $\bar{U}_i > 0$ *for all* i, *and in particular there exists a contract* $y \in Y$ *with* $\Theta(y) < \infty$ *and* $\gamma_i(y) > 0$ *for all* i.

The proof follows by showing that the maximized value of any (P-i) is positive as long as $\bar{U}_j > 0$ for $j < i$. One might imagine a stronger claim, that if there are strict gains from trade for any type i then $\bar{U}_i > 0$, but an example in Section 7 shows that this may not be the case. In particular, if there are no gains from trade for some type $j < i$, so $\bar{U}_j = 0$, it may be that $\bar{U}_i = 0$ even though there would be gains from trade for type i with full information.

Finally, as stated before, we show that our restriction to contract posting is without loss of generality in the following sense:

Proposition 5 *Any competitive search equilibrium with contract posting is a competitive search equilibrium with revelation mechanisms. Conversely, any competitive search equilibrium with revelation mechanisms is payoff-equivalent to a competitive search equilibrium with contract posting.*

The proof in the appendix includes the definition of competitive search equilibrium with revelation mechanisms.

4 Feasible Allocations

To set the stage for studying efficiency, we define a feasible allocation. We begin by defining an allocation, by which we basically mean a description of the posted contracts together with the implied search behavior and payoffs of agents.

Definition 2 *An allocation is a vector $\bar{U}$ of expected utilities for the agents, a measure λ over the set of feasible contracts Y with support Y^P, a function $\tilde{\Theta} : Y^P \mapsto [0, \infty]$, and a function $\tilde{\Gamma} : Y^P \mapsto \Delta^I$.*

Note that $\tilde{\Theta}$ and $\tilde{\Gamma}$ are different from the Θ and Γ in the definition of equilibrium, because the former are defined only over the set of posted contracts, while the latter are defined for all feasible contracts.

An allocation is feasible whenever: (1) each posted contract offers the maximal expected utility to agents who direct their search for that contract and no more to those who do not; (2) each posted contract generates zero profits; and (3) markets clear. Any feasible allocation could be implemented through legal or other restrictions on the contracts that can be offered; however, some feasible allocations may not correspond to equilibria because principals may want to offer contracts that are not posted. More formally, we have:

Definition 3 *An allocation $\{\bar{U}, \lambda, Y^P, \tilde{\Theta}, \tilde{\Gamma}\}$ is feasible if*

1. for any $y \in Y^P$ and i such that $\tilde{\gamma}_i(y) > 0$ and $\tilde{\Theta}(y) < \infty$,

$$\bar{U}_i = \mu(\tilde{\Theta}(y))u_i(y),$$

where

$$\bar{U}_i \equiv \max\left\{0, \max_{y' \in Y^P} \mu(\tilde{\Theta}(y'))u_i(y')\right\}$$

and $\bar{U}_i = 0$ if $Y^P = \varnothing$.

2. for any $y \in Y^P$, $\eta(\tilde{\Theta}(y))\sum_i \tilde{\gamma}_i(y)v_i(y) = k$;

3. for all $i \in \mathbb{I}$, $\int \frac{\tilde{\gamma}_i(y)}{\tilde{\Theta}(y)}d\lambda(\{y\}) \leq \pi_i$, with equality if $\bar{U}_i > 0$.

5   Application I: The Rat Race

We now proceed with the first of our three main applications, a version of the rat race (Akerlof, 1976). For concreteness, think of agents here as workers who are heterogeneous in terms of both their productivity and their preference over consumption and working hours, and principals as firms that are willing to pay more for high-productivity (good) workers and can observe hours but not productivity. We prove that if the disutility of hours is lower for good workers, a separating equilibrium may require them to work more hours than in the first best. In addition, if longer hours produce more output, then the constrained optimum features over-employment of good workers.[11] Finally, under a regularity condition on the matching function, good workers get more consumption when employed than they would get under full information.

5.1   Setup

A contract here is $y = (c, h)$, where c is the worker's consumption and $h \geq 0$ is the amount of work hours. We assume there are $I = 2$ types. The payoff of a type i worker who applies to (c, h) and is matched is

$$u_i(c, h) = c - \phi_i(h),$$

where ϕ_i is a differentiable, increasing, strictly convex function with $\phi_i(0) = \phi_i'(0) = 0$. We assume that $\phi_1(h) = \kappa\phi_2(h)$ for all h and impose assumption A3, which here amounts to

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[11] A static version of Inderst and Müller (1999) is a special case of this example, with output independent of hours worked. In this case, asymmetric information does not distort employment.

$\kappa > 1$. The payoff of a firm posting (c, h) matched with a type i worker is

$$v_i(c, h) = f_i(h) - c,$$

where f_i is a differentiable, nonnegative, nondecreasing, weakly concave production function.

Assumption A1 requires $f_2(h) \geq f_1(h)$ for all h. We assume $\bar{\eta}(f_1(h) - \phi_1(h)) > k$ for some h, which ensures that there are gains from trade for both types; Proposition 4 implies $\bar{U}_1 > 0$ and $\bar{U}_2 > 0$. We also assume there is a $\bar{h} > 0$ such that $f_2(\bar{h}) = \phi_2(\bar{h})$. Equilibrium contracts never set $h > \bar{h}$. We then restrict the set of contracts to $Y = [-\varepsilon, f_2(\bar{h})] \times [0, \bar{h}]$ for some $\varepsilon > 0$. The maximum effort level ensures Y is compact, but is otherwise irrelevant. The possibility of a small negative transfer ensures assumption A2 holds, while no firm would ever pay more than $f_2(\bar{h})$. Finally, we assume the matching function μ is strictly concave and continuously differentiable. Then we find equilibrium by solving problem (P), where problem (P-i) is

$$\bar{U}_i = \max_{\theta \in [0, \infty], (c, h) \in Y} \mu(\theta)(c - \phi_i(h))$$

$$\text{s.t. } \mu(\theta)(f_i(h) - c) \geq \theta k,$$

$$\text{and } \mu(\theta)(c - \phi_j(h)) \leq \bar{U}_j \text{ for } j \leq i.$$

5.2 Equilibrium

Before characterizing equilibrium, we describe the allocation that would arise under full information. For type i, this is given by the solution to problem (P-i), but ignoring the constraint $\mu(\theta)(c - \phi_j(h)) \leq \bar{U}_j$ for all $j < i$. One can prove that the zero profit condition always binds, $\eta(\theta)(f_i(h) - c) = k$, and use that to eliminate c from the objective function. Then the unconstrained values of hours and recruiting solve

$$(h_i^*, \theta_i^*) = \arg\max_{h, \theta} \mu(\theta)(f_i(h) - \phi_i(h)) - \theta k.$$

The level of consumption can be derived from the zero profit condition, $c_i^* = f_i(h_i^*) - k\theta_i^*/\mu(\theta_i^*)$.

Under asymmetric information, there are no additional constraints in problem (P-1), and so equilibrium consumption, hours, and market tightness are unchanged for bad workers, $c_1 = c_1^*$, $h_1 = h_1^*$, and $\theta_1 = \theta_1^*$. Problem (P-1) also defines $\bar{U}_1$. Now turn to the good worker's problem. Under some conditions, this problem may also be unconstrained. For example, if the difference in the disutility of work is large relative to the difference in productivity,

bad workers may prefer their unconstrained contract to the unconstrained contract for good workers. This is the case if and only if

$$\mu(\theta_2^*)(f_2(h_2^*) - \phi_1(h_2^*)) - \theta_2^* k \leq \mu(\theta_1^*)(f_1(h_1^*) - \phi_1(h_1^*)) - \theta_1^* k. \tag{1}$$

We are interested in cases where this constraint is violated. We prove that the contract for good workers specifies that they work too many hours ($h_2 > h_2^*$) and, as long as $f_2(h)$ is strictly increasing, over-employs good workers ($\theta_2 > \theta_2^*$). Under an additional mild restriction on the matching function μ,[12] employed good workers are also over-paid ($c_2 > c_2^*$). In short, the equilibrium is distorted along both the intensive and extensive margins.

Result 1 *Assume condition* (1) *is not satisfied. There exists a competitive search equilibrium with $h_2 > h_2^*$. If f_2 is constant, $\theta_2 = \theta_2^*$ and $c_2 = c_2^*$. If f_2 is strictly increasing, $\theta_2 > \theta_2^*$. If in addition the elasticity of the matching function μ is non-increasing in θ, $c_2 > c_2^*$.*

As usual, the proof is in the appendix.

One may also be interested in comparing outcomes across individuals in equilibrium. It is straightforward to prove that good workers are employed more often than bad workers, $\theta_2 > \theta_1$. To see this, recall that first best hours h_i^* maximize $f_i(h) - \phi_i(h)$. Revealed preference then implies $f_2(h_2^*) - \phi_2(h_2^*) \geq f_2(h_1^*) - \phi_2(h_1^*)$. Since also $f_2(h) \geq f_1(h)$ and $\phi_2(h) < \phi_1(h)$ for all $h > 0$, $f_2(h_1^*) - \phi_2(h_1^*) > f_1(h_1^*) - \phi_1(h_1^*)$. This proves $f_2(h_2^*) - \phi_2(h_2^*) > f_1(h_1^*) - \phi_1(h_1^*)$. In addition, the first best market tightness θ_i^* maximizes $\mu(\theta)(f_i(h_i^*) - \phi_i(h_i^*)) - \theta k$, proving $\theta_2^* > \theta_1^*$. Since $\theta_2 \geq \theta_2^*$ and $\theta_1 = \theta_1^*$, the result follows. On the other hand, without an additional regularity condition, $f_1'(h) \leq f_2'(h)$ for all h, we cannot generally establish whether good workers work more and are paid more than bad workers.

The notion that separating good workers from bad ones may require over-working the good workers dates back to Akerlof (1976). In a model without frictions, zero profit then implies that good workers are compensated with higher wages. Our novel results are that a separating equilibrium always exists and that, under mild conditions, it involves over employment of good workers, both relative to bad workers and relative to the first best. Competitive search is central to these conclusions.

5.3 Efficiency

Consider an allocation that treats the two types identically. All firms post the contract $(c, h) = (c^p, h^p)$ and the market tightness is $\tilde{\Theta}(c^p, h^p) = \theta^p$, where $h^p = h_2^*$ is the first-best

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[12]We require that the elasticity of μ is non-increasing. The elasticity is nonnegative since μ is nondecreasing. Since μ is bounded above, the elasticity converges to zero in the limit as θ converges to infinity.

level of hours for type 2 workers, $\theta^p = \theta_2^*$ is the first-best principal-agent ratio for type 2, and c^p ensures firms earn zero expected profits:

$$f_2'(h^p) = \phi_2'(h^p),$$
$$\mu'(\theta^p)\big(f_2(h^p) - \phi_2(h^p)\big) = k,$$
$$c^p = \pi_1 f_1(h^p) + \pi_2 f_2(h^p) - \frac{\theta^p k}{\mu(\theta^p)}.$$

The share of type i agents searching for this contract satisfies $\tilde{\gamma}_i(c^p, h^p) = \pi_i$ and there are enough of these contracts to be consistent with market clearing.

It is straightforward to verify that this allocation is feasible. We claim that if there are sufficiently few bad productivity workers, it is also a Pareto improvement over the equilibrium:

Result 2 *Assume condition* (1) *is not satisfied. For fixed values of the other parameters, there exists a $\bar{\pi} > 0$ such that if $\pi_1 < \bar{\pi}$, the equilibrium is Pareto dominated by the pooling allocation where all firms post (c^p, h^p) with associated market tightness θ^p.*

In equilibrium, firms that want to attract type 2 workers need to screen out type 1 workers. The cost of screening is independent of the share of type 1 workers, while the collective benefit of screening depends on the share of type 1 workers. Type 2 workers may collectively prefer to cross-subsidize type 1 workers to avoid costly screening. However, this is inconsistent with equilibrium since any individual worker would prefer a contract that screens out the bad types.

6   Application II: Insurance

Our next application is based on Rothschild and Stiglitz (1976), where risk neutral principals offer insurance to risk averse agents who are heterogeneous in their probability of a loss. This illustrates several features. First, we show here that even if a pooling allocation does not Pareto dominate the equilibrium, a partial-pooling allocation may. Second, to illustrate that traditional search frictions are not necessary for existence, in this example we assume that the short side of the market, which is determined endogenously, matches for sure: $\mu(\theta) = \min\{\theta, 1\}$.

6.1 Setup

We again specify the model in terms of worker-firm matching.[13] Now the productivity of a match is initially unknown by both the worker and firm. Some workers are more likely than others to generate productive matches, but firms cannot observe this: type i produce 1 unit of output with probability p_i and 0 otherwise, and p_i is the agent's private information. A contract specifies a transfer to the worker conditional on realized productivity. Workers are risk averse and firms are risk neutral. In the absence of adverse selection, full insurance equates the marginal utility of agents across states. We show that firms here do not provide full insurance, because incomplete insurance helps keep undesirable workers from applying.

A contract consists of a pair of consumption levels, conditional on employment and unemployment after match productivity has been realized, $y = (c_e, c_u)$. The payoff of a type i worker who applies to (c_e, c_u) and is matched is

$$u_i(c_e, c_u) = p_i \phi(c_e) + (1 - p_i)\phi(c_u),$$

where $p_1 < p_2 < \cdots < p_I < 1$ and the utility function $\phi : [\underline{c}, \infty) \to \mathbb{R}$ is increasing and strictly concave with $\lim_{c \to \underline{c}} \phi(c) = -\infty$ for some $\underline{c} < 0$ and $\phi(0) = 0$. The payoff of a firm posting (c_e, c_u) matched with type i is

$$v_i(c_e, c_u) = p_i(1 - c_e) - (1 - p_i)c_u.$$

To ensure A1 is satisfied, we restrict the set of feasible contracts to $Y = \{(c_e, c_u) | c_u + 1 \geq c_e \geq \underline{c}$ and $c_u \geq \underline{c}\}$. The assumption $\lim_{c \to \underline{c}} \phi(c) = -\infty$ ensures that actions of the form $(c_e, \underline{c})$ yield negative utility for all types and so are not in $\bar{Y}$. Then, since a reduction in c_u raises $v_i(c_e, c_u)$ and lowers $u_j(c_e, c_u)$ for any $j < i$, and is feasible for all $y \in \bar{Y}$, A2 is satisfied. To verify A3, consider an incremental increase in c_e to $c_e + dc_e$ and an incremental reduction in c_u to $c_u - dc_u$ for some $dc_e > 0$ and $dc_u > 0$. For a type i worker, this raises utility by approximately $p_i \phi'(c_e)dc_e - (1 - p_i)\phi'(c_u)dc_u$, which is positive if and only if

$$\frac{dc_e}{dc_u} > \frac{1 - p_i}{p_i} \frac{\phi'(c_u)}{\phi'(c_e)}.$$

Since $(1 - p_i)/p_i$ is decreasing in i, an appropriate choice of dc_e/dc_u yields an increase in utility if and only if $j \geq i$, which verifies A3. Finally, assume $p_1 \leq k < p_I$, which ensures that there are no gains from employing the lowest type, even in the absence of asymmetric information,

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[13]We frame the discussion in terms of labor markets, rather than general insurance, because it seems more reasonable to assume an employer only wants to hire a fraction of the available workforce than to assume an insurance company only wants to serve a fraction of its potential customers.

but there are gains from trade for higher types, say by setting $c_e = c_u = p_I - k > 0$. Let i^* denote the lowest type without gains from trade, so $p_{i^*} \leq k < p_{i^*+1}$.[14]

6.2 Equilibrium

We can again characterize equilibrium using (P), leading to:

Result 3 *There exists a competitive search equilibrium where for all $i \leq i^*$, $\bar{U}_i = 0$; and for all $i > i^*$, $\theta_i = 1$, $\bar{U}_i > 0$, and $c_{e,i} > c_{e,i-1}$ and $c_{u,i} < c_{u,i-1}$ are the unique solution to*

$$p_i(1 - c_{e,i}) - (1 - p_i)c_{u,i} = k$$
$$\text{and } p_{i-1}\phi(c_{e,i}) + (1 - p_{i-1})\phi(c_{u,i}) = p_{i-1}\phi(c_{e,i-1}) + (1 - p_{i-1})\phi(c_{u,i-1}),$$

where $c_{e,i^*} = c_{u,i^*} = 0$.

In this case, adverse selection distorts the intensive margin—firms do not offer full insurance—but does not affect the extensive margin, since all workers and firms match if there are gains from trade. One interesting feature of equilibrium here is that $c_{u,i} < 0$ for all $i > i^*$, and therefore a worker is worse off when he matches and turns out to be unproductive than he would be if did not match in the first place. If one interprets a bad match as a layoff, contracts give laid-off workers lower payoffs than those who never match, because this keeps inferior workers from applying for the job.[15]

6.3 Efficiency

Again we show equilibrium may not be efficient. First note that a worker with p_i close to 1 suffers little from the distortions introduced by the information problem. At the extreme, if $p_I = 1$, $c_{u,I} = \underline{c}$ excludes other workers without distorting the type I contract at all. Generally, adverse selection has the biggest impact on the utility of workers with an intermediate value of p_i. We show that a Pareto improvement may result from partial pooling. Consider $p_1 = 1/4$, $p_2 = 1/2$, and $p_3 = 3/4$ and suppose there are equal numbers of type 1 and 3, so that half of all matches are productive. Set $\phi(c) = \log(1 + c)$ and $k = 3/8$. Then in equilibrium, $\bar{U}_1 = 0$; $c_{e,2} = 0.344$, $c_{u,2} = -0.094$, and $\bar{U}_2 = \phi(0.104)$; and $c_{e,3} = 0.576$, $c_{u,3} = -0.227$, and $\bar{U}_3 = \phi(0.319)$.

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[14]The results extend to the case $k < p_1$ by defining $i^* = 1$. The lowest type obtains full insurance at an actuarially fair price, $c_{e,1} = c_{u,1} = p_1 - k$, and the inductive characterization for $i > i^*$ is unchanged.

[15]To interpret a bad match as a layoff, it might help to imagine that an agent actually must work for some time before productivity is realized, as in Jovanovic (1979).

Pooling all three types, the best incentive-feasible allocation involves $c_e = c_u = 1/8$ and $\bar{U}_i = \phi(1/8)$. Compared to the equilibrium, this raises the utility of type 1 and type 2 workers, but reduces the utility of type 3 workers. Now consider an allocation that pools types 1 and 2. If there are sufficiently few type 1 workers, it is feasible to set $c_e = c_u > 0.104$, delivering greater utility to types 1 and 2. For example, suppose $\pi_1 = \pi_3 = 0.01$ and $\pi_2 = 0.98$. Then the utility of types 1 and 2 rises to $\phi(0.122)$. By raising the utility of type 2, it is easier to exclude them from type 3 contracts, reducing the requisite inefficiency of those contracts. This raises the utility of type 3, in this case, to $\phi(0.325)$.

6.4 Relationship to Rothschild-Stiglitz

Rothschild and Stiglitz (1976, p. 630) "consider an individual who will have income of size W if he is lucky enough to avoid accident. In the event an accident occurs, his income will be only $W - d$. The individual can insure against this accident by paying to an insurance company a premium α_1 in return for which he will be paid $\hat{\alpha}_2$ if an accident occurs. Without insurance his income in the two states, 'accident,' 'no accident,' was $(W, W - d)$; with insurance it is now $(W - \alpha_1, W - d + \alpha_2)$ where $\alpha_2 = \hat{\alpha}_2 - \alpha_1$."

We can normalize the utility of an uninsured individual to zero and express the utility of one who anticipates an accident with probability p_i as

$$u_i(\alpha_1, \alpha_2) = p_i \phi(W - \alpha_1) + (1 - p_i)\phi(W - d + \alpha_2) - \kappa_i,$$

where $\kappa_i \equiv p_i \phi(W) + (1 - p_i)\phi(W - d)$. Setting $W = d = 1$ and defining $c_e = 1 - \alpha_1$ and $c_u = \alpha_2$, this is equivalent to our example. Our results apply to their setup, with one wrinkle: our fixed cost of posting contracts.

Rothschild and Stiglitz (1976) show that in any equilibrium, principals who attract type i agents, $i > 1$, offer incomplete insurance to deter type $i - 1$ agents, which is of course very similar to our finding. Under some conditions, however, their equilibrium does not exist. Starting from a configuration of separating contracts, suppose one principal deviates by offering a pooling contract to attract multiple types. In their setup, this is profitable if the least-cost separating contract is Pareto inefficient. But such a deviation is never profitable in our environment.

The key difference is that in the original model a deviating principal can capture all the agents in the economy, or at least a representative cross-section, while in our model a principal cannot serve all the agents who are potentially attracted to a contract, given that matching is bilateral. Instead, agents are rationed thorough the endogenous movement in market tightness θ. Whether such a deviation is profitable depends on which agents are

21

most willing to accept a decline in θ. What we find is that high types are the first to give up on the pooling contract when applications get crowded by low types. A lower type, with an inferior outside option $\bar{U}_{i-1} < \bar{U}_i$, is more inclined to stick it out. Hence, a principal who tries to offer a pooling contract will end up with a long queue of type 1 agents—the worst possible outcome. For this reason, a deviation to a pooling contract is not profitable, and equilibrium with separating contracts always exists.

7   Application III: Asset Markets

A feature of the previous two examples is that the posted contracts are designed to screen agents. We now present a model where instead the market tightness can be used to screen bad types. In other words, distortions occur only along the extensive margin. Although the results hold more generally, to stress the point, we again abstract from traditional search frictions and assume $\mu(\theta) = \min\{\theta, 1\}$, so that matching is determined by the short side of the market.

7.1   Setup

Consider an asset market with lemons, in the sense of Akerlof (1970). Buyers (principals) always value an asset more than sellers (agents) value it, but some assets are better than others and their values are private information to the seller. Market tightness, or probabilistic trading, seems in principle a good way to screen out low quality asset holders, since sellers with more valuable assets are more willing to accept a low probability of trade at any given price. This model shows how an illiquid asset market may have a useful role as a screening device.

Each type i seller is endowed with one indivisible asset, which we call an apple, of type i, with value $a_i^S > 0$ to the seller and $a_i^B > 0$ to the buyer, both expressed in units of a numeraire good. A contract is a pair (α, t), where α is the probability that the seller gives the buyer the apple and t is the transfer in terms of numeraire to the seller.[16] The payoff of a seller of type i matched with a buyer posting (α, t) is

$$u_i(\alpha, t) = t - \alpha a_i^S,$$

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[16]Given that apples are indivisible, it may be efficient to use lotteries, with α the probability apples change hands, as e.g. in Prescott and Townsend (1984) and Rogerson (1988). Nosal and Wallace (2007) provide a related model of an asset (money) market where probabilistic trade is useful due to private information, but they use random and not directed search, leading to quite different results. It would actually be equivalent here to assume apples are perfectly divisible and preferences are linear, with α reinterpreted as the fraction traded, but we like the indivisibility since it allows us to contrast our results with models that use lotteries.

while the payoff of a buyer posting (α, t) matched with a type i seller is

$$v_i(\alpha, t) = \alpha a_i^B - t.$$

Note that we have normalized the no-trade payoff to 0.

We set $I = 2$ and impose a number of restrictions on payoffs. First, both buyers and sellers prefer type 2 apples and both types of agents like apples:

$$a_2^S > a_1^S > 0 \text{ and } a_2^B > a_1^B > 0.$$

Second, there would be gains from trade, including the cost k of posting, if the buyer were sure to trade:

$$a_i^S + k < a_i^B \text{ for } i = 1, 2.$$

The available contracts are $Y = [0, 1] \times [0, a_2^B]$, with $\bar{Y}_i = \{(\alpha, t) \in Y | \alpha a_i^S \leq t \leq \alpha a_i^B - k\}$. Using these restrictions, we verify our three assumptions. As a preliminary step, note that $(\alpha, t) \in \bar{Y}_i$ implies $\alpha \geq k/(a_i^B - a_i^S) > 0$ and $t \geq k a_i^S/(a_i^B - a_i^S) > 0$, so in any equilibrium contract, trades are bounded away from zero.

Since $\alpha > 0$ whenever $(\alpha, t) \in \bar{Y}_i$, the restriction $a_1^B < a_2^B$ implies A1. Also, A2 holds because for any $(\alpha, t) \in \bar{Y}_i$, a movement to $(\alpha, t - \varepsilon)$ with $\varepsilon > 0$ is feasible and raises buyer utility. The important assumption is again A3, which is here guaranteed by $a_1^S < a_2^S$. Fix $(\alpha, t) \in \bar{Y}$ and $\bar{a} \in (a_1^S, a_2^S)$. For arbitrary $\delta > 0$, consider $(\alpha', t') = (\alpha - \delta, t - \bar{a}\delta)$. This is feasible for small δ because $(\alpha, t) \in \bar{Y}$ guarantees that $\alpha > 0$ and $t > 0$. Then

$$u_2(\alpha', t') - u_2(\alpha, t) = \delta(a_2^S - \bar{a}) > 0$$

$$u_1(\alpha', t') - u_1(\alpha, t) = \delta(a_1^S - \bar{a}) < 0.$$

Now for fixed $\varepsilon > 0$, choose $\delta \leq \varepsilon/\sqrt{1 + \bar{a}^2}$. This ensures $(\alpha', t') \in B_\varepsilon(\alpha, t)$, so A3 holds.

7.2   Equilibrium

We again use problem (P) to characterize the equilibrium.

Result 4 *There exists a unique competitive search equilibrium with $\alpha_i = 1$, $t_i = a_i^B - k$, $\theta_1 = 1$, $\bar{U}_1 = a_1^B - a_1^S - k$, $\theta_2 = \dfrac{a_1^B - a_1^S - k}{a_2^B - a_1^S - k} < 1$, and $\bar{U}_2 = \theta_2(a_2^B - a_2^S - k)$.*

With full information, $\theta_2 = 1$ and $\bar{U}_2 = a_2^B - a_2^S - k$. Relative to this benchmark, buyers post too few contracts designed to attract type 2 sellers, and hence too many of them fail to

trade. Since type 2 sellers have better apples than type 1, they are more willing to accept this in return for a better price when they do trade. Agents with inferior assets are less willing to accept a low probability of trade because they do not want to be stuck with their own apple, which is in fact a lemon. Note that the alternative of setting $\theta_2 = 1$ but rationing though the probability of trade in a match, $\alpha_2 < 1$, wastes resources, because it involves posting more contracts at cost k. In other words, extensive distortions—reducing the matching rate—are more cost effective at screening than intensive distortions—lotteries.

7.3 Efficiency

Consider a pooling allocation, with $\alpha_1 = \alpha_2 = 1$ and $t_1 = t_2 = t$. That is, all buyers post $(1, t)$. Moreover, $\tilde{\Theta}(1, t) = 1$, $\tilde{\gamma}_i(1, t) = \pi_i$, and $\lambda(\{(1, t)\}) = 1$. Finally, set $t = \pi_1 a_1^B + \pi_2 a_2^B - k$. This allocation is feasible, given that all sellers apply to the same contract, the choice of t ensures zero profits, and the choice of λ ensures markets clear. The expected payoff for type i sellers is

$$\bar{U}_i = \pi_1 a_1^B + \pi_2 a_2^B - a_i^S - k.$$

With this pooling allocation, type 1 sellers are always better off than they were in equilibrium since $a_1^B < a_2^B$, and type 2 sellers are better off if and only if

$$\pi_1 a_1^B + \pi_2 a_2^B - a_2^S - k > \frac{(a_2^B - a_2^S - k)(a_1^B - a_1^S - k)}{a_2^B - a_1^S - k}.$$

Since $\pi_2 = 1 - \pi_1$, this reduces to

$$\pi_1 < \frac{a_2^B - a_2^S - k}{a_2^B - a_1^S - k} = \frac{\bar{U}_2}{\bar{U}_1}.$$

Both the numerator and denominator are positive, but the numerator is smaller (gains from trade are smaller for type 2 sellers) because $a_2^S > a_1^S$. Thus type 2 sellers prefer the pooling allocation when there is not too much cross subsidization, or π_1 is small, so that the cost of subsidizing type 1 sellers is worth the increased efficiency of trade.

7.4 No Trade

So far, we have assumed there are gains from trade for both types of sellers. Now suppose there are no gains from trade for type 1 apples, $a_1^B \leq a_1^S + k$. Then not only will type 1 seller fail to trade, in equilibrium, the entire market will shut down:

Result 5 *If $a_1^B \leq a_1^S + k$ then, in any equilibrium, $\bar{U}_1 = \bar{U}_2 = 0$.*

Notice that the market shuts down here even if there are still gains from trade in good apples, $a_2^B > a_2^S + k$. Intuitively, it is only possible to keep bad apples out of the market by reducing the probability of trade in good apples. If there is no market in bad apples, however, agents holding them would accept any probability of trade. Hence we cannot screen out bad apples, and this renders the good apple market inoperative. Whether this is related to the recent collapse in asset-backed securities markets seems worth further exploration.

8 Conclusion

We have developed a tractable general framework to analyze adverse selection in competitive search markets. Under our assumptions, there is a unique equilibrium, where principals post separating contracts. We characterized the equilibrium as the solution to a set of constrained optimization problems, and illustrated the use of the model through a series of examples. We expect that one could extend the framework to dynamic situations, with repeated rounds of posting and search, as seems relevant in many applications—including labor and asset markets. It may also be interesting to study the case opposite to the one analyzed here, where the informed instead of the uninformed parties post contracts. In standard competitive search theory the outcome does not depend on who posts. With asymmetric information, contract posting by informed parties may introduce multiplicity of equilibrium through the usual signaling mechanism (see Delacroix and Shi, 2007). All of this is left for future work.

APPENDIX

Proof of Lemma 1. In the first step, we prove that there exists a solution to (P). The second and third steps establish the stated properties of the solution.

Step 1. Consider $i = 1$. If the constraint set in (P-1) is empty, set $\bar{U}_1 = 0$. Otherwise, (P-1) is well-behaved, as the objective function is continuous and the constraint set compact. Hence, (P-1) has a solution and a unique maximum m_1. If $m_1 \leq 0$ set $\bar{U}_1 = 0$; otherwise set $\bar{U}_1 = m_1$ and let (θ_1, y_1) be one of the maximizers.

We now proceed by induction. Fix $i > 1$ and assume that we have found $\bar{U}_j$ for all $j < i$ and (θ_j, y_j) for all $j \in \mathbb{I}^*$, $j < i$. Consider (P-i). If the constraint set is empty, set $\bar{U}_i = 0$. Otherwise, (P-i) again has a solution and a unique maximum m_i. If $m_i \leq 0$, set $\bar{U}_i = 0$; otherwise let $\bar{U}_i = m_i$ and let (θ_i, y_i) be one of the maximizers.

Step 2. Suppose by way of contradiction that there exists $i \in \mathbb{I}^*$ such that (θ_i, y_i) solves (P-i) but $\eta(\theta_i)v_i(y_i) > k$. This together with $\bar{U}_i = \mu(\theta_i)u_i(y_i) > 0$ implies that $y_i \in \bar{Y}_i$ and $\mu(\theta_i) > 0$. Fix $\varepsilon > 0$ such that $\eta(\theta_i)v_i(y) \geq k$ for all $y \in B_\varepsilon(y_i)$. Then A3 ensures there exists a $y' \in B_\varepsilon(y_i)$ such that

$$u_j(y') > u_j(y_i) \text{ for all } j \geq i,$$
$$u_j(y') < u_j(y_i) \text{ for all } j < i.$$

Then the pair (θ_i, y') satisfies all the constraints of problem (P-i):

1. $\eta(\theta_i)v_i(y') \geq k$ from the choice of ε;

2. $\mu(\theta_i)u_j(y') < \mu(\theta_i)u_j(y_i) \leq \bar{U}_j$ for all $j < i$, where the first inequality is by construction of y' and $\mu(\theta_i) > 0$, while the second holds since (θ_i, y_i) solves (P-i).

Now (θ_i, y') achieves a higher value than (θ_i, y_i) for the objective function in (P-i), given $\mu(\theta_i)u_i(y') > \mu(\theta_i)u_i(y_i)$. Hence, (θ_i, y_i) does not solve (P-i), a contradiction.

Step 3. Fix $i \in \mathbb{I}^*$ and suppose by way of contradiction that there exists $j > i$ such that $\mu(\theta_i)u_j(y_i) > \bar{U}_j$. Let h be the smallest such j. Since $i \in \mathbb{I}^*$, $\mu(\theta_i)u_i(y_i) = \bar{U}_i > 0$, which implies $\mu(\theta_i) > 0$ and $u_i(y_i) > 0$. Also, from the previous step (θ_i, y_i) satisfies $\eta(\theta_i)v_i(y_i) = k$, which ensures $\eta(\theta_i) > 0$ and $v_i(y_i) > 0$. In particular, this implies that $y_i \in \bar{Y}_i$.

The pair (θ_i, y_i) satisfies the constraints of (P-h) since

1. $\eta(\theta_i)v_h(y_i) \geq \eta(\theta_i)v_i(y_i) = k$, where the inequality holds by A1 given $h > i$ and $y_i \in \bar{Y}_i \subset \bar{Y}$, and the equality comes from the previous step;

2. $\mu(\theta_i)u_l(y_i) \leq \bar{U}_l$ for all $l < h$, which holds for

   (a) $l < i$ because (θ_i, y_i) satisfy the constraints of (P-i),

   (b) $l = i$ because $\bar{U}_i = \mu(\theta_i)u_i(y_i)$ since $i \in \mathbb{I}^*$,

   (c) $i < l < h$ by the choice of h as the smallest violation of $\mu(\theta_i)u_j(y_i) > \bar{U}_j$.

Since $\mu(\theta_i)u_h(y_i) > \bar{U}_h \geq 0$, (θ_i, y_i) is in the constraint set of (P-h) and delivers a strictly positive value for the objective function; hence $h \in \mathbb{I}^*$. But then the fact that $\bar{U}_h$ is not the maximized value of (P-h) is a contradiction. ∎

Proof of Proposition 1. We proceed by construction.

- The vector of expected utilities is $\bar{U} = \{\bar{U}_i\}_{i \in \mathbb{I}}$.

- The set of posted contracts is $Y^P = \{y_i\}_{i \in \mathbb{I}^*}$.

- λ is such that $\lambda(\{y_i\}) = \pi_i \Theta(y_i)$ for any $i \in \mathbb{I}^*$.

- For all $i \in \mathbb{I}^*$, $\Theta(y_i) = \theta_i$. For any other $y \in Y$, let $J(y) = \{j | u_j(y) > 0\}$ denote the types that attain positive utility from y. If $J(y) \neq \varnothing$ and $\min_{j \in J(y)} \{\bar{U}_j / u_j(y)\} < \bar{\mu}$ then
$$\mu(\Theta(y)) = \min_{j \in J(y)} \frac{\bar{U}_j}{u_j(y)}.$$
  If this equation is consistent with multiple values of $\Theta(y)$, pick the largest one. Otherwise, if $J(y) = \varnothing$ or $\min_{j \in J(y)} \{\bar{U}_j / u_j(y)\} \geq \bar{\mu}$, then $\Theta(y) = \infty$.

- For all $i \in \mathbb{I}^*$, let $\gamma_i(y_i) = 1$ and so $\gamma_j(y_i) = 0$ for $j \neq i$. For any other $y \in Y$, define $\Gamma(y)$ such that $\gamma_h(y) > 0$ only if $h \in \arg\min_{j \in J(y)} \{\bar{U}_j / u_j(y)\}$. If there are multiple minimizers, let $\gamma_h(y) = 1$ for the smallest such h. If $J(y) = \varnothing$, again choose $\Gamma(y)$ arbitrarily, e.g. $\gamma_1(y) = 1$.

We now verify that all of the equilibrium conditions hold.

Condition (i): For any $i \in \mathbb{I}^*$, (θ_i, y_i) solves (P-i), and Lemma 1 implies $\eta(\theta_i) v_i(y_i) = k$. Thus, profit maximization and free entry hold for any contract y_i, $i \in \mathbb{I}^*$. Now consider an arbitrary contract; we claim that principals' profit maximization and free-entry condition is satisfied. Suppose, to the contrary, that there exists $y \in Y$ with $\eta(\Theta(y)) \sum_i \gamma_i(y) v_i(y) > k$. This implies $\eta(\Theta(y)) > 0$, so $\Theta(y) < \infty$, and there exists some type j with $\gamma_j(y) > 0$ and $\eta(\Theta(y)) v_j(y) > k$. Since $\gamma_j(y) > 0$ and $\Theta(y) < \infty$, our construction of $\Theta(y)$ and $\Gamma(y)$ implies that j is the smallest solution to $\min_{h \in J(y)} \{\bar{U}_h / u_h(y)\}$ and hence $u_j(y) > 0$ and $\bar{U}_j = \mu(\Theta(y)) u_j(y)$. So, by construction, for all $h < j$ with $u_h(y) > 0$, $\bar{U}_h > \mu(\Theta(y)) u_h(y)$. Moreover, if $u_h(y) \leq 0$, $\bar{U}_h \geq \mu(\Theta(y)) u_h(y)$ since $\bar{U}_h \geq 0$. This proves that $(\Theta(y), y)$ satisfies the constraints of (P-j). Then, since $\eta(\Theta(y)) v_j(y) > k$, Lemma 1 implies that there exists (θ', y') that satisfies the constraints of (P-j) but delivers a higher value of the objective function, $\mu(\theta') u_j(y') > \bar{U}_j \geq 0$. Since $\bar{U}_j$ is at least equal to the maximized value of (P-j), this is a contradiction.

Condition (ii): By construction, Θ and Γ ensure that $\bar{U}_i \geq \mu(\Theta(y)) u_i(y)$ for all contracts $y \in Y$, with equality if $\Theta(y) < \infty$ and $\gamma_i(y) > 0$. Moreover, for any $i \in \mathbb{I}^*$, $\bar{U}_i = \mu(\theta_i) u_i(y_i) > 0$ where $\theta_i = \Theta(y_i)$ and y_i is the equilibrium contract offered to i. Finally, for any $y \in Y$ such that $u_i(y) < 0$ for some i, it must be that $i \notin J(y)$. Then, either $J(y) = \varnothing$, in which case $\Theta(y) = \infty$, or $J(y) \neq \varnothing$, in which case $\gamma_h(y) = 1$ for some $h \in J(y)$

27

and so $\gamma_i(y) = 0$.

Condition (iii): Market clearing obviously holds given the way we construct λ. Since all the equilibrium conditions are satisfied the proof is complete. ∎

Proof of Proposition 2. From equilibrium condition (i), any $y \in Y^P$ has $\eta(\Theta(y)) > 0$, hence $\Theta(y) < \infty$. From condition (iii), $\bar{U}_i > 0$ implies $\gamma_i(y) > 0$ for some $y \in Y^P$. This proves that for each $i \in \mathbb{I}^*$, there exists a contract $y \in Y^P$ with $\Theta(y) < \infty$ and $\gamma_i(y) > 0$.

The remainder of the proof proceeds in five steps. The first four steps show that for any $i \in \mathbb{I}^*$ and $y_i \in Y^P$ with $\theta_i = \Theta(y_i) < \infty$ and $\gamma_i(y_i) > 0$, (θ_i, y_i) solves (P-i). First, we prove that the constraint $\eta(\theta_i)v_i(y_i) \geq k$ is satisfied. Second, we prove that the constraint $\mu(\theta_i)u_j(y_i) \leq \bar{U}_j$ is satisfied for all j. Third, we prove that the pair (θ_i, y_i) delivers $\bar{U}_i$ to type i. Fourth, we prove that (θ_i, y_i) solves (P-i). The fifth and final step shows that for any $i \notin \mathbb{I}^*$, either the constraint set of (P-i) is empty or the maximized value is nonpositive.

Step 1. Take $i \in \mathbb{I}^*$ and $y_i \in Y^P$ with $\theta_i = \Theta(y_i) < \infty$ and $\gamma_i(y_i) > 0$. We claim the constraint $\eta(\theta_i)v_i(y_i) \geq k$ is satisfied in (P-i). Note first that $i \in \mathbb{I}^*$ implies $\bar{U}_i > 0$. By equilibrium condition (ii), $\bar{U}_i = \mu(\theta_i)u_i(y_i)$, and so $\mu(\theta_i) > 0$. To derive a contradiction, assume $\eta(\theta_i)v_i(y_i) < k$. Equilibrium condition (i) implies $\eta(\theta_i)\sum_j \gamma_j(y_i)v_j(y_i) = k$, so there is an h with $\gamma_h(y_i) > 0$ and $\eta(\theta_i)v_h(y_i) > k$. Since $\eta(\theta_i) \leq \bar{\eta}$, $\bar{\eta}v_h(y_i) > k$. Moreover, because $\theta_i = \Theta(y_i) < \infty$ and $\gamma_h(y_i) > 0$, optimal search implies $u_h(y_i) \geq 0$. This proves $y_i \in \bar{Y}_h$.

Next, fix $\varepsilon > 0$ such that $\eta(\theta_i)v_h(y) > k$ for all $y \in B_\varepsilon(y_i)$. Then A3 together with $y_i \in \bar{Y}_h$ guarantees that there exists $y' \in B_\varepsilon(y_i)$ such that

$$u_j(y') > u_j(y_i) \text{ for all } j \geq h,$$
$$u_j(y') < u_j(y_i) \text{ for all } j < h.$$

Notice that $y' \in \bar{Y}_h$ as well, since $u_h(y') > u_h(y_i) \geq 0$ and $\bar{\eta}v_h(y') \geq \eta(\theta_i)v_h(y') > k$.

Now consider $\theta' \equiv \Theta(y')$. Note that

$$\mu(\theta')u_h(y') \leq \bar{U}_h = \mu(\theta_i)u_h(y_i) < \mu(\theta_i)u_h(y'),$$

where the weak inequality follows from optimal search, the equality holds because $\theta_i < \infty$ and $\gamma_h(y_i) > 0$, and the strict inequality holds by the construction of y', since $\mu(\theta_i) > 0$. This implies $\mu(\theta') < \mu(\theta_i) < \infty$, and so $\theta' < \theta_i$. Next observe that for all $j < h$, either

28

$u_j(y') < 0$, in which case $\gamma_j(y') = 0$ by equilibrium condition (ii), or

$$\mu(\theta')u_j(y') < \mu(\theta_i)u_j(y_i) \le \bar{U}_j,$$

where the first inequality uses $\mu(\theta') < \mu(\theta_i)$ and the construction of y', while the second follows from optimal search. Hence, $\gamma_j(y') = 0$ for all $j < h$.

Finally, profits from posting y' are

$$\eta(\theta')\sum_{j=1}^{I}\gamma_j(y')v_j(y') \ge \eta(\theta')v_h(y') \ge \eta(\theta_i)v_h(y') > k.$$

The first inequality follows because $\gamma_j(y') = 0$ if $j < h$ and $v_h(y')$ is nondecreasing in h by A1 together with $y' \in \bar{Y}_h \subset \bar{Y}$, the second follows because $\theta' < \theta_i$, and the last inequality uses the construction of ε. A deviation to posting y' is therefore strictly profitable. This is a contradiction and completes Step 1.

Step 2. Again take $i \in \mathbb{I}^*$ and $y_i \in Y^P$ with $\theta_i = \Theta(y_i) < \infty$ and $\gamma_i(y_i) > 0$. Equilibrium condition (ii) implies $\mu(\theta_i)u_j(y_i) \le \bar{U}_j$ for all j so that the second constraint in (P-i) is satisfied for all j.

Step 3. Again take $i \in \mathbb{I}^*$ and $y_i \in Y^P$ with $\theta_i = \Theta(y_i) < \infty$ and $\gamma_i(y_i) > 0$. Equilibrium condition (ii) implies $\bar{U}_i = \mu(\theta_i)u_i(y_i)$, since $\theta_i < \infty$ and $\gamma_i(y_i) > 0$. Hence (θ_i, y_i) delivers $\bar{U}_i$ to type i.

Step 4. Again take $i \in \mathbb{I}^*$ and $y_i \in Y^P$ with $\theta_i = \Theta(y_i) < \infty$ and $\gamma_i(y_i) > 0$. To find a contradiction, suppose there exists (θ, y) that satisfies the constraints of (P-i) but delivers higher utility. That is, $\eta(\theta)v_i(y) \ge k$, $\mu(\theta)u_j(y) \le \bar{U}_j$ for all $j < i$, and $\mu(\theta)u_i(y) > \bar{U}_i$. We now use A2. Note that $\mu(\theta)u_i(y) > \bar{U}_i > 0$ implies $\mu(\theta) > 0$ and $u_i(y) > 0$, while $\eta(\theta)v_i(y) \ge k$ implies $v_i(y) > 0$ and so $\bar{\eta}v_i(y) \ge k$. In particular, $y \in \bar{Y}_i$. We can therefore fix $\varepsilon > 0$ such that for all $y \in B_\varepsilon(y)$, $\mu(\theta)u_i(y) > \bar{U}_i$, and then choose $y' \in B_\varepsilon(y)$ such that $v_i(y') > v_i(y)$ and $u_j(y') \le u_j(y)$ for all $j < i$. This ensures $\eta(\theta)v_i(y') > k$, $\mu(\theta)u_j(y') \le \bar{U}_j$ for all $j < i$, and $\mu(\theta)u_i(y') > \bar{U}_i$. Note that we still have $y' \in \bar{Y}_i$.

We now use A3. Fix $\varepsilon' > 0$ such that for all $y \in B_{\varepsilon'}(y')$, $\eta(\theta)v_i(y) > k$ and $\mu(\theta)u_i(y) > \bar{U}_i$. Choose $y'' \in B_{\varepsilon'}(y')$ such that

$$u_j(y'') > u_j(y') \text{ for all } j \ge i,$$
$$u_j(y'') < u_j(y') \text{ for all } j < i.$$

This ensures $\eta(\theta)v_i(y'') > k$, $\mu(\theta)u_j(y'') < \bar{U}_j$ for all $j < i$, and $\mu(\theta)u_i(y'') > \bar{U}_i$. Note that we still have $y'' \in \bar{Y}_i$.

Now consider the contract y''. From equilibrium condition (ii), $\mu(\theta)u_i(y'') > \bar{U}_i$ implies that $\mu(\theta) > \mu(\Theta(y''))$, which guarantees $\eta(\Theta(y''))v_i(y'') > k$. This also implies $\Theta(y'') < \infty$.

We next claim $\gamma_j(y'') = 0$ for all $j < i$. Suppose $\gamma_j(y'') > 0$ for some $j < i$. Since $\Theta(y'') < \infty$, equilibrium condition (ii) implies $u_j(y'') \geq 0$. We have already shown that $\mu(\theta) > \mu(\Theta(y''))$, and $u_j(y') > u_j(y'')$ by construction. Thus $\mu(\theta)u_j(y') > \mu(\Theta(y''))u_j(y'')$. Equilibrium condition (ii) requires $\bar{U}_j \geq \mu(\theta)u_j(y')$ and thus $\bar{U}_j > \mu(\Theta(y''))u_j(y'')$. This together with $\Theta(y'') < \infty$ implies $\gamma_j(y'') = 0$, a contradiction.

Now the profit from offering contract y'' is

$$\eta(\Theta(y'')) \sum_{j=1}^{I} \gamma_j(y'')v_j(y'') \geq \eta(\Theta(y''))v_i(y'') > k,$$

where the first inequality uses $\gamma_j(y'') = 0$ for $j < i$ and A1, and the second holds by construction. This contradicts the first condition in the definition of equilibrium, and proves (θ_i, y_i) solves (P-i).

Step 5. Suppose there is an $i \notin \mathbb{I}^*$ for which the constraint set of (P-i) is nonempty and the maximized value is positive. That is, suppose there exists (θ, y) such that $\eta(\theta)v_i(y) \geq k$, $\mu(\theta)u_j(y) \leq \bar{U}_j$ for all $j < i$, and $\mu(\theta)u_i(y) > \bar{U}_i = 0$. Replicating step 4, we can first find y' such that $\mu(\theta)u_i(y') > 0$, $\eta(\theta)v_i(y') > k$, and $\mu(\theta)u_j(y') \leq \bar{U}_j$ for all $j < i$. Then we find y'' such that $\mu(\theta)u_i(y'') > 0$, $\eta(\theta)v_i(y'') > k$, and $\mu(\theta)u_j(y'') < \bar{U}_j$ for all $j < i$. Hence y'' only attracts type i or higher and hence must be profitable and deliver positive utility, a contradiction. ∎

Proof of Proposition 3. By Lemma 1 there is a solution to (P). Proposition 1 shows that if $\mathbb{I}^*$, $\{\bar{U}_i\}_{i \in \mathbb{I}}$, $\{\theta_i\}_{i \in \mathbb{I}^*}$, and $\{y_i\}_{i \in \mathbb{I}^*}$ solve (P), there is an equilibrium $\{\bar{U}, \lambda, Y^P, \Theta, \Gamma\}$ with the same $\bar{U}$, $Y^P = \{y_i\}_{i \in \mathbb{I}^*}$, $\Theta(y_i) = \theta_i$, and $\gamma_i(y_i) = 1$. This proves existence. Proposition 2 shows that in any equilibrium $\{\bar{U}, \lambda, Y^P, \Theta, \Gamma\}$, $\bar{U}_i$ is the maximum value of (P-i) for all $i \in \mathbb{I}^*$, and $\bar{U}_i = 0$ otherwise. Lemma 1 shows there is a unique maximum value $\bar{U}_i$ for (P-i) for all $i \in \mathbb{I}^*$. This proves that the equilibrium $\bar{U}$ is unique. ∎

Proof of Proposition 4. Consider $i = 1$. Fix y satisfying $\bar{\eta}v_1(y) > k$ and $u_1(y) > 0$. Then fix $\theta > 0$ satisfying $\eta(\theta)v_1(y) = k$. These points satisfy the constraints of (P-1) and deliver utility $\mu(\theta)u_1(y) > 0$. This proves $\bar{U}_1 > 0$. Now suppose $\bar{U}_j > 0$ for all $j < i$. We claim $\bar{U}_i > 0$. Again fix y satisfying $\bar{\eta}v_i(y) > k$ and $u_i(y) > 0$. Then fix $\theta > 0$ satisfying

30

$\eta(\theta)v_1(y) \geq k$ and $\mu(\theta)u_j(y) \leq \bar{U}_j$ for all $j < i$; this is feasible since $\bar{U}_j > 0$ and $\mu(0) = 0$. These points satisfy the constraints of (P-i) and deliver utility $\mu(\theta)u_i(y) > 0$, which proves $\bar{U}_i > 0$. By induction, the proof is complete. ∎

Proof of Proposition 5. First, we define a competitive search equilibrium in the general environment where principals can post revelation mechanisms. Recall that $C \subset Y^I$ is the set of revelation mechanisms. The proof uses y_i to denote the contract that a mechanism $\mathbf{y}$ offers to an agent who reports that her type is i, and similarly y_i' for a mechanism $\mathbf{y}'$. In addition, $\bar{\mathbf{y}}$ is a degenerate mechanism that offers everyone a contract y.

Definition 4 *A competitive search equilibrium with revelation mechanisms is a vector $\bar{U} = \{\bar{U}_i\}_{i \in \mathbb{I}} \in \mathbb{R}_+^I$, a measure λ^m on C with support C^P, a function $\Theta^m : C \mapsto [0, \infty]$, and a function $\Gamma^m : C \mapsto \Delta^I$ satisfying:*

(i) *principals' **profit maximization** and **free-entry**: for any $\mathbf{y} \in C$,*

$$\eta(\Theta^m(\mathbf{y})) \sum_i \gamma_i^m(\mathbf{y}) v_i(y_i) \leq k,$$

with equality if $\mathbf{y} \in C^P$;

(ii) *agents' **optimal search**: let*

$$\bar{U}_i = \max \left\{ 0, \max_{\mathbf{y}' \in C^P} \mu(\Theta^m(\mathbf{y}')) u_i(y_i') \right\}$$

and $\bar{U}_i = 0$ if $C^P = \varnothing$; then for any $\mathbf{y} \in C$ and i, $\bar{U}_i \geq \mu(\Theta^m(\mathbf{y})) u_i(y_i)$, with equality if $\Theta^m(\mathbf{y}) < \infty$ and $\gamma_i^m(\mathbf{y}) > 0$; moreover, if $u_i(y_i) < 0$, either $\Theta^m(\mathbf{y}) = \infty$ or $\gamma_i^m(\mathbf{y}) = 0$;

(iii) ***market clearing**: $\int_{C^P} \frac{\gamma_i^m(\mathbf{y})}{\Theta^m(\mathbf{y})} d\lambda^m(\{\mathbf{y}\}) \leq \pi_i$ for any i, with equality if $\bar{U}_i > 0$.*

This is a natural generalization of our definition of competitive search with contract posting. It implicitly assumes that agents always truthfully reveal their type, as a direct revelation mechanism naturally induces them to do.

The proof proceeds in two steps. First, we show that any competitive search equilibrium with contract posting is a competitive search equilibrium with revelation mechanisms. Second, we show that any competitive search equilibrium with revelation mechanisms is payoff-equivalent to a competitive search equilibrium with contract posting.

Step 1. Take a competitive search equilibrium with contract posting $\{\bar{U}, \lambda, Y^P, \Theta, \Gamma\}$. We construct a competitive search equilibrium with revelation mechanisms $\{\bar{U}, \lambda^m, C^P, \Theta^m, \Gamma^m\}$,

where $\lambda^m(\{\bar{\mathbf{y}}\}) = \lambda(\{y\})$, so $y \in Y^P$ if and only if $\bar{\mathbf{y}} \in C^P$; and for all $y \in Y$, $\Theta^m(\bar{\mathbf{y}}) = \Theta(y)$ and $\Gamma^m(\bar{\mathbf{y}}) = \Gamma(y)$. We also need to construct Θ^m and Γ^m for general mechanisms $\mathbf{y}' \in C$. As before, let $J^m(\mathbf{y}') = \{j | u_j(y'_j) > 0\}$ denote the types that attain positive utility from $\mathbf{y}'$. If $J^m(\mathbf{y}') \neq \varnothing$ and $\min_{j \in J^m(\mathbf{y}')}\{\bar{U}_j/u_j(y'_j)\} < \bar{\mu}$ then

$$\mu(\Theta^m(\mathbf{y}')) = \min_{j \in J^m(\mathbf{y}')} \frac{\bar{U}_j}{u_j(y'_j)}.$$

If this equation is consistent with multiple values of $\Theta^m(\mathbf{y}')$, pick the largest one. Also define $\Gamma^m(\mathbf{y}')$ such that $\gamma_h^m(\mathbf{y}') > 0$ only if $h \in \arg\min_{j \in J^m(\mathbf{y}')}\{\bar{U}_j/u_j(y'_j)\}$. If there are multiple minimizers, $\gamma_h^m(\mathbf{y}') = 1$ for the smallest such h. Otherwise, if $J^m(\mathbf{y}') = \varnothing$ or $\min_{j \in J^m(\mathbf{y}')}\{\bar{U}_j/u_j(y'_j)\} \geq \bar{\mu}$, then $\Theta^m(\mathbf{y}') = \infty$ and $\Gamma^m(\mathbf{y}')$ is arbitrary, e.g. $\gamma_1(\mathbf{y}') = 1$.

Conditions (ii) and (iii) are satisfied by construction. Condition (i) is also satisfied for degenerate mechanisms. Now suppose that there exists a general mechanism $\mathbf{y}' \in C$ with $\eta(\Theta^m(\mathbf{y}')) \sum_i \gamma_i^m(\mathbf{y}') v_i(y'_i) > k$. This implies $\Theta^m(\mathbf{y}') < \infty$. In addition, $\Gamma^m(\mathbf{y}')$ puts all its weight on the smallest element of $\arg\min_{j \in J^m(\mathbf{y}')}\{\bar{U}_j/u_j(y'_j)\}$, some type h. That is, $\eta(\Theta^m(\mathbf{y}')) v_h(y'_h) > k$. Now consider the degenerate mechanism $\bar{\mathbf{y}}'_h$. Since incentive compatibility implies $u_i(y'_i) \geq u_i(y'_h)$ for all i, $\Theta^m(\bar{\mathbf{y}}'_h) = \Theta^m(\mathbf{y}')$ and h is also the smallest element of $\arg\min_{j \in J^m(\bar{\mathbf{y}}'_h)}\{\bar{U}_j/u_j(y'_h)\}$. Thus

$$\eta(\Theta^m(\bar{\mathbf{y}}'_h)) \sum_i \gamma_i^m(\bar{\mathbf{y}}'_h) v_i(y'_h) \geq \eta(\Theta^m(\bar{\mathbf{y}}'_h)) v_h(y'_h) = \eta(\Theta^m(\mathbf{y}')) v_h(y'_h) > k.$$

The first inequality uses A1, since h is the worst type attracted to the mechanism. The equality uses $\Theta^m(\bar{\mathbf{y}}'_h) = \Theta^m(\mathbf{y}')$. And the final inequality holds by assumption. In other words, the degenerate mechanism $\bar{\mathbf{y}}'_h$ yields positive profits. But then so would y'_h in the original competitive search equilibrium with contracting posting, a contradiction.

Step 2. Take a competitive search equilibrium with revelation mechanisms $\{\bar{U}, \lambda^m, C^P, \Theta^m, \Gamma^m\}$. We construct a competitive search equilibrium with contract posting, $\{\bar{U}, \lambda, Y^P, \Theta, \Gamma\}$, as follows. First, for each type i with $\bar{U}_i > 0$, select a mechanism $\mathbf{y}$ with $\Theta^m(\mathbf{y}) < \infty$ and $\gamma_i^m(\mathbf{y}) > 0$. Let y_i be the contract offered to i by this mechanism. Then set $\Theta(y_i) = \Theta^m(\mathbf{y})$, $\gamma_i(y_i) = 1$, and $\lambda(\{y_i\}) = \pi_i \Theta(y_i)$. If for two (or more) types i and j, this procedure selects mechanisms $\mathbf{y}$ for i and $\mathbf{y}'$ for j with $y_i = y'_j$, it is straightforward to prove that $\Theta^m(\mathbf{y}) = \Theta^m(\mathbf{y}')$; otherwise both types would prefer the mechanism with the higher principal-agent ratio. Then let $\gamma_i(y_i) = \pi_i/(\pi_i + \pi_j)$ and $\lambda(\{y_i\}) = (\pi_i + \pi_j)\Theta(y_i)$. Finally, for any other degenerate mechanism $\bar{\mathbf{y}}$, $\Theta(y) = \Theta^m(\bar{\mathbf{y}})$ and $\Gamma(y) = \Gamma^m(\bar{\mathbf{y}})$.

Conditions (ii) and (iii) in the definition of equilibrium with contract posting hold by construction. Moreover, no contract $y \in Y$ yields positive profits, since the associated degenerate mechanism $\bar{y}$ could have been offered in the equilibrium with revelation mechanisms. It only remains to verify that principals break even on equilibrium contracts, $\eta(\Theta(y)) \sum_i \gamma_i(y) v_i(y) = k$ for $y \in Y^P$. To prove this, consider any mechanism that is offered in the equilibrium with revelation mechanisms, $\mathbf{y} \in C^P$. Part (i) of the definition of equilibrium imposes $\eta(\Theta^m(\mathbf{y})) \sum_i \gamma_i^m(\mathbf{y}) v_i(y_i) = k$. Suppose for some i with $\gamma_i^m(\mathbf{y}) > 0$, $\eta(\Theta^m(\mathbf{y})) v_i(y_i) > k$. A3 ensures that there is a degenerate mechanism offering a perturbation of contract y_i, say $\bar{\mathbf{y}}'$, which attracts only type i and higher agents and so, by A1, yields positive profits: $\eta(\Theta^m(\bar{\mathbf{y}}')) v_i(y') > k$ for some y' near y_i. This is a contradiction with the definition of competitive search equilibrium with revelation mechanisms, proving that for all $\mathbf{y} \in C^P$ and i with $\gamma_i^m(\mathbf{y}) > 0$, $\eta(\Theta^m(\mathbf{y})) v_i(y_i) = k$. Then by the construction of Θ, Γ, and Y^P, we have that for all $y_i \in Y^P$ with $\gamma_i(y_i) > 0$, $\eta(\Theta(y_i)) v_i(y_i) = k$. This completes the proof. ∎

Proof of Result 1. By Lemma 1 the constraint in problem (P-1) is binding, so we can eliminate c and reduce the problem to

$$\bar{U}_1 = \max_{\theta, h} \mu(\theta) \left(f_1(h) - \phi_1(h) \right) - \theta k.$$

At the solution, h_1 is such that $f_1'(h_1) = \phi_1'(h_1)$ and θ_1 solves $\mu'(\theta_1)(f_1(h_1) - \phi_1(h_1)) = k$. Substituting this into the objective function delivers $\bar{U}_1$, and the constraint delivers c_1.

Next, solve (P-2) using the $\bar{U}_1$ derived in the previous step:

$$\bar{U}_2 = \max_{\theta \in [0, \infty], (t,x) \in Y} \mu(\theta) \left(c - \phi_2(h) \right)$$

$$\text{s.t. } \mu(\theta)(f_2(h) - c) \geq \theta k,$$

$$\text{and } \mu(\theta) \left(c - \phi_1(h) \right) \leq \bar{U}_1.$$

Denoting by λ and ν the Lagrangian multipliers attached respectively to the first and the second constraint, the first order conditions with respect to h and c can be combined to obtain:

$$\lambda = \frac{\phi_1'(h_2) - \phi_2'(h_2)}{\phi_1'(h_2) - f_2'(h_2)} \quad \text{and} \quad \nu = \frac{\phi_2'(h_2) - f_2'(h_2)}{\phi_1'(h_2) - f_2'(h_2)}.$$

Given that we restricted attention to the interesting case where both constraints are binding, both λ and ν are positive. A3 ($\phi_1'(h_2) > \phi_2'(h_2)$) implies the numerator of λ is positive. Then the denominator is positive as well, $\phi_1'(h_2) > f_2'(h_2)$. This is also the denominator of ν, and so its numerator is positive, proving $\phi_2'(h_2) > f_2'(h_2)$. Since $\phi_2'(h_2^*) = f_2'(h_2^*)$, convexity of ϕ_2

and concavity of f_2 imply $h_2 > h_2^*$.

Next eliminate the multipliers from the first order condition with respect θ. Use $\phi_1 = \kappa \phi_2$ to get

$$\mu'(\theta_2)\left(f_2(h_2) - \frac{f_2'(h_2)\phi_2(h_2)}{\phi_2'(h_2)}\right) = k. \tag{2}$$

Note that with $f_2(h) = \bar{f}_2$ for all h, this reduces to $\mu'(\theta_2)\bar{f}_2 = k$. In the economy with symmetric information, we had instead $\mu'(\theta_2^*)(f_2(h_2^*) - \phi_2(h_2^*)) = k$ and $f_2'(h_2^*) = \phi_2'(h_2^*)$ (so if $f_2(h) = \bar{f}_2$, $h_2^* = 0$ and $\mu'(\theta_2^*)\bar{f}_2 = k$). Thus in any case equation (2) holds at $(h_2, \theta_2) = (h_2^*, \theta_2^*)$. Now if $f_2(h) = \bar{f}_2$, equation (2) pins down θ_2, independent of h_2, so $\theta_2 = \theta_2^*$. If f_2 is strictly increasing, equation (2) describes an increasing locus of points in (h_2, θ_2) space, so $h_2 > h_2^*$ implies $\theta_2 > \theta_2^*$.

Finally, consumption satisfies the zero profit condition. Eliminate k using equation (2) to get

$$c_2 = f_2(h_2) - \frac{\theta_2 \mu'(\theta_2)}{\mu(\theta_2)}\left(f_2(h_2) - \frac{f_2'(h_2)\phi_2(h_2)}{\phi_2'(h_2)}\right)$$

Again, if $h_2 = h_2^*$ and $\theta_2 = \theta_2^*$, $c_2 = c_2^*$. If $h_2 > h_2^*$ and $\theta_2 > \theta_2^*$, f_2 strictly increasing and concave, ϕ_2 increasing and convex, and $\frac{\theta_2 \mu'(\theta_2)}{\mu(\theta_2)}$ nonincreasing imply $c_2 > c_2^*$. ∎

Proof of Result 2. We first prove that if the pooling contract raises the utility of type 2 workers relative to the equilibrium level, it raises the utility of type 1 as well. Since the constraint to exclude type 1 workers from the type 2 contract binds in equilibrium, the equilibrium utility of a type 1 worker is

$$\bar{U}_1 = \bar{U}_2 - \mu(\theta_2)(\phi_1(h_2) - \phi_2(h_2)).$$

In the pooling contract, both types work the same hours and have the same market tightness. Therefore,

$$\bar{U}_1^p = \bar{U}_2^p - \mu(\theta^p)(\phi_1(h^p) - \phi_2(h^p)),$$

where $\bar{U}_i^p$ is the utility of a type i worker in the pooling contract. It follows that if $\bar{U}_2^p \geq \bar{U}_2$,

$$\bar{U}_1^p - \bar{U}_1 \geq \mu(\theta_2)(\phi_1(h_2) - \phi_2(h_2)) - \mu(\theta^p)(\phi_1(h^p) - \phi_2(h^p)).$$

Recalling that h^p and θ^p are the first-best levels, result 1 proved $h^p \leq h_2$ and $\theta^p \leq \theta_2$. Therefore $\phi_1(h^p) - \phi_2(h^p) < \phi_1(h_2) - \phi_2(h_2)$ and $\mu(\theta^p) < \mu(\theta_2)$. This proves $\bar{U}_1^p > \bar{U}_1$.

Next, because condition (1) is not satisfied, the equilibrium expected utility of type 2 workers, $\bar{U}_2$, is strictly less than the first best level, say $\bar{U}_2^*$. In addition, by construction h^p and θ^p are equal to the first best level for type 2, (h_2^*, θ_2^*), and c^p is continuous in π_1 and

34

converges to the first best level c_2^* as π_1 converges to 0. It follows that $\bar{U}_2^p$ is continuous in π_1 and converges to $\bar{U}_2^* > \bar{U}_2$ as π_1 converges to 0. This proves there is a $\bar{\pi}$ such that for all $\pi_1 < \bar{\pi}$, $\bar{U}_2^p > \bar{U}_2$. ∎

Proof of Result 3. For $i \leq i^*$, consider (P-i) without the constraint of keeping out lower types. This relaxed problem should yield a higher payoff

$$\bar{U}_i \leq \max_{\theta \in [0,\infty], (c_e, c_u) \in Y} \min\{\theta, 1\}\big(p_i \phi(c_e) + (1 - p_i)\phi(c_u)\big)$$

$$\text{s.t. } \min\{1, \theta^{-1}\}(p_i(1 - c_e) - (1 - p_i)c_u) \geq k.$$

At the solution, $c_{u,i} = c_{e,i} = c_i$, so this reduces to

$$\bar{U}_i \leq \max_{\theta \in [0,\infty], c \geq \underline{c}} \min\{\theta, 1\}\phi(c)$$

$$\text{s.t. } \min\{1, \theta^{-1}\}(p_i - c) \geq k.$$

Either the constraint set is empty (if $p_i < k + \underline{c}$) or there are no points in the constraint set that give positive utility (given that $p_i - k \leq 0$). In any case, this gives $\bar{U}_i = 0$.

Turn next to a typical problem (P-i), $i > i^*$:

$$\bar{U}_i = \max_{\theta \in [0,\infty], (c_e, c_u) \in Y} \min\{\theta, 1\}\big(p_i \phi(c_e) + (1 - p_i)\phi(c_u)\big)$$

$$\text{s.t. } \min\{1, \theta^{-1}\}(p_i(1 - c_e) - (1 - p_i)c_u) \geq k$$

$$\text{and } \min\{\theta, 1\}\big(p_j \phi(c_e) + (1 - p_j)\phi(c_u)\big) \leq \bar{U}_j \text{ for all } j < i.$$

Note first that the solution sets $\theta_i \leq 1$: if $\theta_i > 1$, reducing θ_i to 1 relaxes the first constraint without otherwise affecting the problem. Hence, we can rewrite the problem as

$$\bar{U}_i = \max_{\theta \leq 1, (c_e, c_u) \in Y} \theta\big(p_i \phi(c_e) + (1 - p_i)\phi(c_u)\big)$$

$$\text{s.t. } p_i(1 - c_e) - (1 - p_i)c_u \geq k$$

$$\text{and } \theta\big(p_j \phi(c_e) + (1 - p_j)\phi(c_u)\big) \leq \bar{U}_j \text{ for all } j < i.$$

Lemma 1 ensures the first constraint is binding, which proves $p_i(1 - c_{e,i}) - (1 - p_i)c_{u,i} = k$. It remains to prove that $\theta_i = 1$, $c_{e,i} > c_{e,i-1}$, $c_{u,i} < c_{u,i-1}$, and the constraint for $j = i - 1$ binds.

We start by establishing these claims for $i = i^* + 1$. In this case, $c_{e,i} = c_{u,i} = 0$ satisfies the constraints but leaves the first one slack. Lemma 1 implies that it is possible to do better,

35

which proves $\bar{U}_i > 0$. On the other hand, consider any $(c_{e,i}, c_{u,i})$ that delivers positive utility and satisfies the last constraint, so

$$p_i\phi(c_{e,i}) + (1 - p_i)\phi(c_{u,i}) > 0 \text{ and } p_j\phi(c_{e,i}) + (1 - p_j)\phi(c_{u,i}) \leq 0$$

for all $j < i$. Subtracting inequalities gives

$$(p_i - p_j)(\phi(c_{e,i}) - \phi(c_{u,i})) > 0,$$

which proves $c_{e,i} > c_{u,i}$. Now if $c_{e,i} > c_{u,i} \geq 0$, $p_j\phi(c_{e,i}) + (1 - p_j)\phi(c_{u,i}) > 0$, so this is infeasible. If $0 \geq c_{e,i} > c_{u,i}$, $p_i\phi(c_{e,i}) + (1 - p_i)\phi(c_{u,i}) < 0$, so this is suboptimal. This proves $c_{e,i} > 0 > c_{u,i}$ when $i = i^* + 1$. Finally, since $p_j\phi(c_{e,i}) + (1 - p_j)\phi(c_{u,i}) \leq 0$ for all $j < i$, setting $\theta_i = 1$ raises the value of the objective function without affecting the constraints and so is optimal.

We now proceed by induction. Fix $i > i^*+1$ and assume that for all $j \in \{i^*+1, \ldots, i-1\}$, we have $c_{e,j} > c_{e,j-1}$, $c_{u,j} < c_{u,j-1}$, $\theta_j = 1$, and

$$p_{j-1}\phi(c_{e,j}) + (1 - p_{j-1})\phi(c_{u,j}) = p_{j-1}\phi(c_{e,j-1}) + (1 - p_{j-1})\phi(c_{u,j-1}) = \bar{U}_{j-1}.$$

We establish the result for i. Setting $c_{e,i} = c_{e,i-1}$, $c_{u,i} = c_{u,i-1}$, and $\theta_i = 1$ satisfies the constraints in (P-i). Since it leaves the first constraint slack, Lemma 1 implies it is possible to do better. Thus

$$\theta_i\big(p_i\phi(c_{e,i}) + (1 - p_i)\phi(c_{u,i})\big) > p_i\phi(c_{e,i-1}) + (1 - p_i)\phi(c_{u,i-1}).$$

On the other hand, the second constraint implies,

$$\theta_i\big(p_{i-1}\phi(c_{e,i}) + (1 - p_{i-1})\phi(c_{u,i})\big) \leq \bar{U}_{i-1} = p_{i-1}\phi(c_{e,i-1}) + (1 - p_{i-1})\phi(c_{u,i-1}).$$

Subtracting inequalities gives $(p_i - p_{i-1})\big(\theta_i\phi(c_{e,i}) - \phi(c_{e,i-1}) - \theta_i\phi(c_{u,i}) + \phi(c_{u,i-1})\big) > 0$. Using $p_i > p_{i-1}$, this implies $\theta_i\phi(c_{e,i}) - \phi(c_{e,i-1}) > \theta_i\phi(c_{u,i}) - \phi(c_{u,i-1})$. As before, we can rule out the possibility that $\theta_i\phi(c_{u,i}) \geq \phi(c_{u,i-1})$, because this is infeasible. We can rule out the possibility that $\theta_i\phi(c_{e,i}) \leq \phi(c_{e,i-1})$, because this is suboptimal. Hence $\theta_i\phi(c_{e,i}) - \phi(c_{e,i-1}) > 0 > \theta_i\phi(c_{u,i}) - \phi(c_{u,i-1})$. Now since $\phi(c_{e,i-1}) > 0$ and $\theta_i \in [0, 1]$, the first inequality implies $c_{e,i} > c_{e,i-1}$. Since $\phi(c_{u,i-1}) < 0$, the second inequality implies $c_{u,i} < c_{u,i-1}$.

Next suppose $\theta_i < 1$ and consider the following variation: raise θ_i to 1 and increase c_e and reduce c_u while keeping both $\theta\big(p_i\phi(c_e) + (1 - p_i)\phi(c_u)\big)$ and $p_ic_e + (1 - p_i)c_u$ unchanged;

i.e. set $c_e > c_{e,i}$ and $c_u < c_{u,i}$ so that

$$\theta_i\big(p_i\phi(c_{e,i}) + (1 - p_i)\phi(c_{u,i})\big) = p_i\phi(c_e) + (1 - p_i)\phi(c_u)$$
$$\text{and } p_i c_{e,i} + (1 - p_i)c_{u,i} = p_i c_e + (1 - p_i)c_u.$$

For all $j < i$, $(p_j - p_i)\big(\theta_i\phi(c_{e,i}) - \phi(c_e) + \phi(c_u) - \theta_i\phi(c_{u,i})\big) > 0$ since $p_j < p_i$, $0 < c_{e,i} < c_e$, $c_u < c_{u,i} < 0$, and $\theta_i < 1$. Add this to $\theta_i\big(p_i\phi(c_{e,i}) + (1-p_i)\phi(c_{u,i})\big) = p_i\phi(c_e) + (1-p_i)\phi(c_u)$ to obtain $\theta_i(p_j\phi(c_{e,i}) + (1 - p_j)\phi(c_{u,i})) > p_j\phi(c_u) + (1 - p_j)\phi(c_e)$. This implies the perturbation relaxes the remaining constraints and so is feasible. This proves $\theta_i = 1$.

Finally, we prove that the constraints for $j < i - 1$ are slack. If not then

$$p_j\phi(c_{e,i}) + (1 - p_j)\phi(c_{u,i}) = \bar{U}_j \geq p_j\phi(c_{e,i-1}) + (1 - p_j)\phi(c_{u,i-1}),$$

where the inequality uses the constraints in problem (P-$(i-1)$). On the other hand,

$$p_{i-1}\phi(c_{e,i}) + (1 - p_{i-1})\phi(c_{u,i}) \leq \bar{U}_{i-1} = p_{i-1}\phi(c_{e,i-1}) + (1 - p_{i-1})\phi(c_{u,i-1}),$$

where the inequality is a constraint in (P-i) and the equality is the definition of $\bar{U}_{i-1}$. Subtracting these equations, $(p_{i-1} - p_j)\big(\phi(c_{e,i-1}) - \phi(c_{e,i}) + \phi(c_{u,i}) - \phi(c_{u,i-1})\big) \geq 0$. Since $p_{i-1} > p_j$, $c_{e,i} > c_{e,i-1}$, and $c_{u,i} < c_{u,i-1}$, we have a contradiction. The constraints for all $j < i - 1$ are slack, while the constraint for $i - 1$ binds, otherwise the solution to (P-i) would have $c_{e,i} = c_{u,i} = p_i - k$. ∎

Proof of Result 4. Write (P-1) as

$$\bar{U}_1 = \max_{\theta\in[0,\infty],(\alpha,t)\in Y} \min\{\theta, 1\}\big(t - \alpha a_1^S\big)$$
$$\text{s.t. } \min\{1, \theta^{-1}\}\big(\alpha a_1^B - t\big) \geq k.$$

By Lemma 1 we can rewrite the problem as

$$\bar{U}_1 = \max_{\theta\in[0,\infty],\alpha\in[0,1]} \min\{\theta, 1\}\alpha\big(a_1^B - a_1^S\big) - \theta k.$$

Since $a_1^B > a_1^S + k$, it is optimal to set $\alpha = \theta = 1$. It follows that $\bar{U}_1 = a_1^B - a_1^S - k$.

Now consider (P-2)

$$\bar{U}_2 = \max_{\theta \in [0,\infty], (\alpha,t) \in Y} \min\{\theta, 1\}(t - \alpha a_2^S)$$

$$\text{s.t. } \min\{1, \theta^{-1}\}(\alpha a_2^B - t) \geq k$$

$$\min\{\theta, 1\}(t - \alpha a_1^S) \leq a_1^B - a_1^S - k.$$

Again, we can eliminate t using the first constraint to write

$$\bar{U}_2 = \max_{\theta \in [0,\infty], \alpha \in [0,1]} \min\{\theta, 1\}\alpha(a_2^B - a_2^S) - \theta k$$

$$\text{s.t. } \min\{\theta, 1\}\alpha(a_2^B - a_1^S) - \theta k = a_1^B - a_1^S - k.$$

Then use the last constraint to eliminate α and write

$$\bar{U}_2 = \max_{\theta \in [0,\infty]} \frac{a_1^B - a_1^S - (1-\theta)k}{a_2^B - a_1^S}(a_2^B - a_2^S) - \theta k$$

$$\text{s.t. } \frac{a_1^B - a_1^S - (1-\theta)k}{\min\{\theta, 1\}(a_2^B - a_1^S)} \in [0, 1],$$

where the constraint here ensures that α is a probability. Since $a_1^S < a_2^S < a_2^B$, the objective function is decreasing in θ, and we set it to the smallest value consistent with the constraints:

$$\theta_2 = \frac{a_1^B - a_1^S - k}{a_2^B - a_1^S - k} < 1.$$

This implies $\alpha_2 = 1$, so the constraint binds. Then $\bar{U}_2$ is easy to compute. ∎

Proof of Result 5. Write problem (P-1) as

$$\bar{U}_1 = \max_{\theta \in [0,\infty], (\alpha,t) \in Y} \min\{\theta, 1\}(t - \alpha a_1^S)$$

$$\text{s.t. } \min\{1, \theta^{-1}\}(\alpha a_1^B - t) \geq k.$$

Again we can eliminate t and rewrite this as

$$\bar{U}_1 = \max_{\theta \in [0,\infty], \alpha \in [0,1]} \min\{\theta, 1\}\alpha(a_1^B - a_1^S) - \theta k.$$

Since $a_1^B \leq a_1^S + k$, $\bar{U}_1 = 0$, which is attained by $\theta = 0$.

Now consider (P-2):

$$\bar{U}_2 = \max_{\theta \in [0,\infty],(\alpha,t) \in Y} \min\{\theta, 1\}\left(t - \alpha a_2^S\right)$$

$$\text{s.t. } \min\{1, \theta^{-1}\}\left(\alpha a_2^B - t\right) \geq k$$

$$\min\{\theta, 1\}\left(t - \alpha a_1^S\right) \leq 0.$$

Eliminating t using the first constraint gives

$$\bar{U}_2 = \max_{\theta \in [0,\infty],\alpha \in [0,1]} \min\{\theta, 1\}\alpha\left(a_2^B - a_2^S\right) - \theta k$$

$$\text{s.t. } \min\{\theta, 1\}\alpha\left(a_2^B - a_1^S\right) - \theta k = 0.$$

Eliminating α using the last constraint gives

$$\bar{U}_2 = \max_{\theta \in [0,\infty]} \frac{a_1^S - a_2^S}{a_2^B - a_1^S}\theta k$$

$$\text{s.t. } \frac{\theta k}{\min\{\theta, 1\}(a_2^B - a_1^S)} \in [0, 1].$$

Since $a_2^B > a_2^S > a_1^S$, the fraction in the objective function is negative. Hence $\theta = 0$ and $\bar{U}_2 = 0$. ∎

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