The Relative Contributions of Private Information Sharing and Public Information Releases to Information Aggregation

Darrell Duffie
Stanford

Semyon Malamud
ETH Zurich

Gustavo Manso
MIT

JET Symposium on Search Theory - September 2009

Information Aggregation in Markets

Information Transmission in Markets: Hayek (1945), Grossman (1976), Grossman and Stiglitz (1981).

- Centralized Exchanges: Wilson (1977), Townsend (1978), Milgrom (1981), Vives (1993), Pesendorfer and Swinkels (1997), and Reny and Perry (2006).
- Decentralized Markets:
  - Wolinsky (1990), Blouin and Serrano (2002), Golosov, Lorenzoni, and Tsyvinski (2009).
  - Duffie and Manso (2007), Duffie, Giroux, and Manso (2008),

Outline of the Talk

1. Purely Private Learning (DM, DGM)
2. Market Setting
3. Public and Private Learning
4. Rates of Convergence

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Duffie and Manso (2007) and Duffie, Giroux, and Manso (2008)

- Private information sharing
- Closed-form solution for the evolution of cross-sectional beliefs
- Beliefs converge exponentially to a common posterior
- Extreme decentralization result: the rate of convergence does not depend on the number of agents in each meeting

Model Primitives

- Continuum of agents
- Two possible states of nature: $X = \begin{cases} H & \text{with probability } p_H \\ L & \text{with probability } p_L \end{cases}$
- Each agent is initially endowed with signals $S = \{s_1, \ldots, s_n\}$ s.t. $\mathrm{P}(s_i = 1 \mid H) \geq \mathrm{P}(s_i = 1 \mid L)$
- For every pair agents, their initial signals are X-conditionally independent
- Random matching, intensity λ.

Initial Information Endowment

After observing signals signals $S = \{s_1, \ldots, s_n\}$, the logarithm of the likelihood ratio between states H and L is by Bayes' rule:

$$\log \frac{\mathrm{P}(X = H \mid s_1, \ldots, s_n)}{\mathrm{P}(X = L \mid s_1, \ldots, s_n)} = \log \frac{p_H}{p_L} + \sum_{i=1}^{n} \log \frac{\mathrm{P}(s_i \mid H)}{\mathrm{P}(s_i \mid L)}.$$

We say that the "type" θ associated with this set of signals is

$$\theta = \sum_{i=1}^{n} \log \frac{\mathrm{P}(s_i \mid H)}{\mathrm{P}(s_i \mid L)}.$$

What Happens in a Meeting?

- Upon meeting agents participate in an auction/bargaining game.
- If bids are strictly increasing in the type associated with the signals agents have collected, then bids reveal type.

Information is Additive in Type Space

Proposition: Let $S = \{s_1, \ldots, s_n\}$ and $R = \{r_1, \ldots, r_m\}$ be independent sets of signals, with associated types θ and ϕ. If two agents with types θ and ϕ reveal their their types to each other, then both agents achieve the posterior type $\theta + \phi$.

This follows from Bayes' rule, by which

$$\begin{aligned} \log \frac{\mathrm{P}(X = H \mid S, R, \theta + \phi)}{\mathrm{P}(X = L \mid S, R, \theta + \phi)} &= \log \frac{p_H}{p_L} + \theta + \phi, \\ &= \log \frac{\mathrm{P}(X = H \mid \theta + \phi)}{\mathrm{P}(X = L \mid \theta + \phi)} \end{aligned}$$

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&= \log \frac{P(X = H \mid \theta + \phi)}{P(X = L \mid \theta + \phi)}
\end{aligned}$$

By induction, this property holds for all subsequent meetings.

Solution for Cross-Sectional Distribution of Information

The Boltzmann equation for the cross-sectional distribution μ_t of types is

$$\frac{d}{dt}\mu_t = -\lambda\,\mu_t + \lambda\,\mu_t * \mu_t. \qquad (1)$$

with a given initial distribution of types μ_0.

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Proposition: The unique solution of (1) is the Wild sum

$$\mu_t = \sum_{n\geq 1} e^{-\lambda t}(1 - e^{-\lambda t})^{n-1}\mu_0^{*n}. \qquad (2)$$

Proof of Wild Summation

Taking the Fourier transform $\varphi(\,\cdot\,,t)$ of μ_t of the Boltzmann equation

$$\frac{d}{dt}\mu_t = -\lambda\,\mu_t + \lambda\,\mu_t * \mu_t.$$

we obtain the following ODE

$$\frac{\partial\varphi(s,t)}{\partial t} = -\lambda\varphi(s,t) + \lambda\varphi^2(s,t).$$

whose solution is

$$\varphi(s,t) = \frac{\varphi(s,0)}{e^{\lambda t}(1-\varphi(s,0)) + \varphi(s,0)}.$$

This solution can be expanded as

$$\varphi(s,t) = \sum_{n\geq 1} e^{-\lambda t}(1-e^{-\lambda t})^{n-1}\varphi^n(s,0),$$

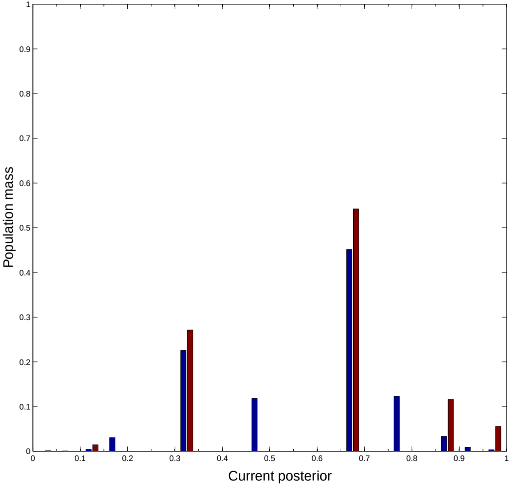
which is the Fourier transform of the Wild sum (2).

Using the Same Techniques, DM and DGM . . .

1. generalize results to m agents at each meeting.

2. show that convergence to perfect information is exponential and that the rate of convergence does not depend on the number m of agents in each meeting.

Groups of 2 (blue) versus Groups of 3 (red)



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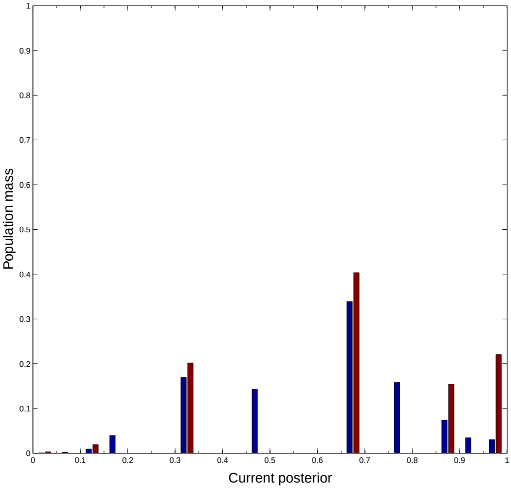

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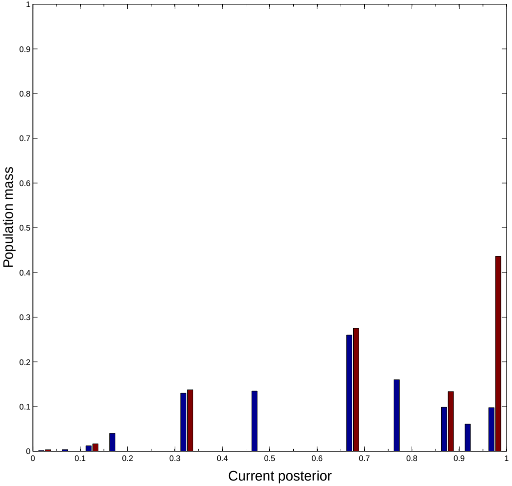

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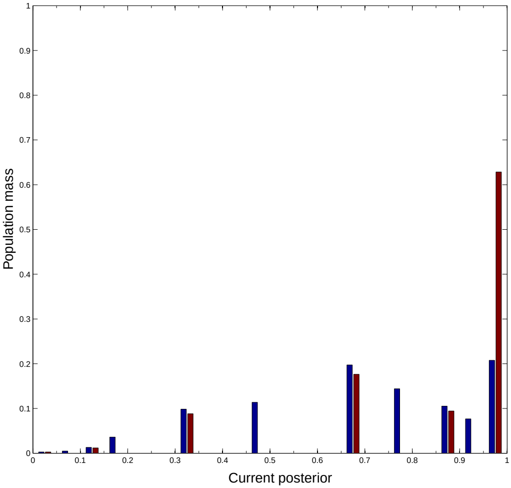

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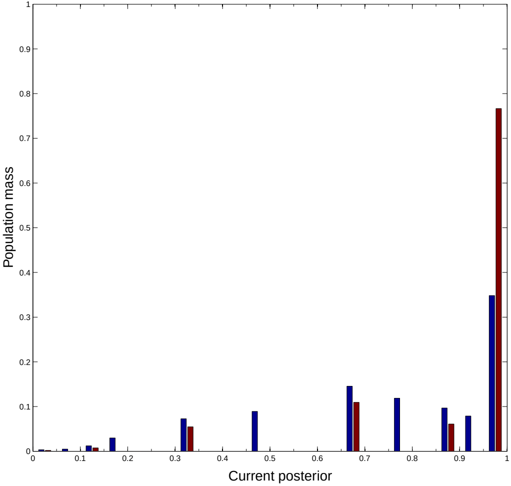

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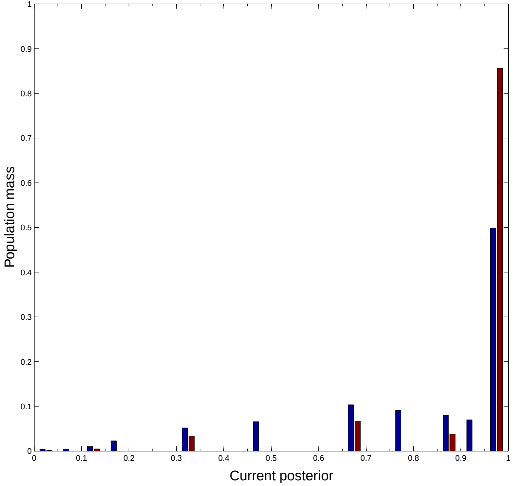

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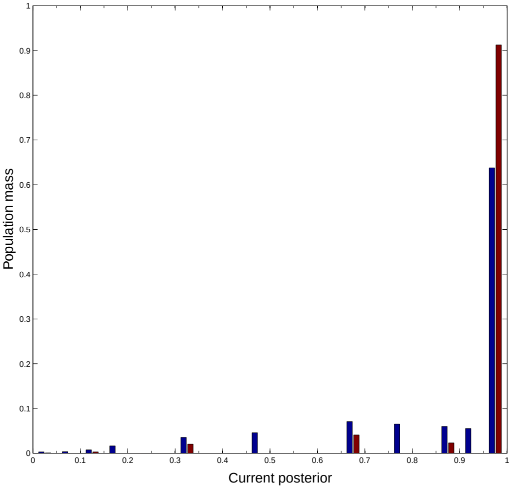

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3. Public and Private Learning
4. Rates of Convergence

Second-Price Auction

- continuum of risk-neutral informed sellers.
- each seller is randomly matched with another informed seller and an uninformed buyer with intensity λ.
- second-price auction to sell to the uninformed buyer an asset that pays 1 if $X = H$ and 0 otherwise.
- unique symmetric equilibrium sellers bid the posterior probability that X is high.

Double Auction

- continuum of risk neutral agents.
- two types of agents with different valuation for the asset. High valuation types are buyers, low valuation types are sellers.
- each seller is randomly matched with a buyer with intensity λ.
- upon meeting, participate in a double auction. If the buyer's bid β is higher than the seller's ask σ, trade occurs.
- conditions under which there exists strictly monotone equilibrium in undominated strategies.

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Model Primitives

Same as DM and DGM, except that:

1. At public information release times $\{T_1, T_2, \ldots\}$, n randomly selected agents have their posterior probabilities revealed to all agents.
2. We allow for random number of agents in each meeting and in each public information release:
   - Meeting group size m: $q_l = \mathbb{P}(m = l)$.
   - Public information release group size n: $p_k = \mathbb{P}(n = k)$.

How Do Public Releases Affect Updates?

- For $t \leq T_1$, evolution is the same as in the case of no public information release.
- At any time $t \in [T_1, T_2)$, any agent's type θ may be viewed as the sum of Z_1 and the privately acquired type $\hat{\theta} = \theta - Z_1$. Thus, at such a time, when agents of respective types $\theta_1, \ldots, \theta_\ell$ meet and exchange their conditional probabilities of the event $\{X = H\}$, the i-th agent knows that the j-th agent's announced type θ_j can be viewed as the sum of the publicly revealed type Z_1 and the privately acquired type $\hat{\theta}_j = \theta_j - Z_1$. Thus, all of the agents leave the meeting with a type equal to Z_1 plus the sum of the privately acquired types $\hat{\theta}_1 + \hat{\theta}_2 + \cdots + \hat{\theta}_\ell$.

Evolution of the Cross-Sectional Distribution of Types

Theorem: Given the variable X of common concern, the probability distribution of each agent's type at time t is

$$\nu_t = \alpha_t * \beta_t,$$

where $\alpha_t = h(\mu_0, t)$ is the type distribution in a model with no public releases of information, satisfying the differential equation

$$\frac{d\alpha_t}{dt} = -\lambda \alpha_t + \lambda \left(\sum_{l=1}^{\infty} q_l \alpha_t^{*l} \right), \qquad \alpha_0 = \mu_0,$$

and where β_t is the probability distribution over types that solves the differential equation

$$\frac{d\beta_t}{dt} = -\eta\beta_t + \eta\beta_t * \sum_{k=0}^{\infty} p_k \alpha_t^{*k},$$

with initial condition given by the Dirac measure δ_0 at zero.

Step 1: Characterizing α_t

Theorem: The unique solution to the dynamic equation for the distribution of types in a model with no public information is

$$\alpha_t \;=\; e^{-\lambda t} \sum_{n=1}^{\infty} a_n(t)\, \mu_0^{*n}.$$

The coefficients $a_n(t)$ are nonnegative, monotone increasing, and bounded, and can be defined recursively by $a_1(t) \;=\; 1$ and

$$a_j(t) \;=\; \lambda \sum_{k=2}^{j} \int_0^t e^{-\lambda\,(k-1)\,s}\, q_k \sum_{j_1+\cdots+j_k=j} \prod_{h=1}^{k} a_{j_h}(s)\, ds, \quad j \geq 2.$$

Step 2: Characterizing β_t

Proposition: The unique solution to the dynamic equation for the distribution of publicly revealed types is

$$\beta_t \;=\; e^{-\eta t} \sum_{n=0}^{\infty} b_n(t)\, \mu_0^{*n},$$

where $b_0(t) = 1$ and

$$b_n(t) \;=\; \sum_{k=1}^{n} \frac{\eta^k}{k!} \sum_{i_1+\cdots+i_k=n} d_{i_1}(t) \cdots d_{i_k}(t),$$

with

$$d_j(t) \;=\; \sum_{k=1}^{j} p_k \int_0^t \left(e^{-\lambda k s} \sum_{i_1+\cdots+i_k=j} a_{i_1}(s) \cdots a_{i_k}(s) \right) ds.$$

Proof of the Proposition

Taking Fourier transforms:

$$\frac{d\hat{\beta}_t}{dt} = -\eta\hat{\beta}_t + \eta\,\hat{\beta}_t \sum_{k=1}^{\infty} p_k\,\hat{\alpha}_s^k\,ds.$$

This is a linear ordinary differential equation whose unique solution, with $\hat{\beta}_0 = 1$, is

$$\hat{\beta}_t = \exp\left(\eta\left(\int_0^t \sum_{k=1}^{\infty} p_k\,\hat{\alpha}_s^k\,ds - t\right)\right).$$

Using the Taylor series for e^x,

$$\hat{\beta}_t = e^{-\eta\,t} \sum_{n=0}^{\infty} \frac{\eta^n}{n!} \left(\int_0^t \sum_{k=1}^{\infty} p_k\,\hat{\alpha}_s^k\,ds\right)^n.$$

Proof of the Proposition

From the previous theorem,

$$\begin{aligned}(\hat{\alpha}_t)^k &= e^{-\lambda k t} \left(\sum_{l=1}^{\infty} a_l(t)\, \hat{\mu}_0^l \right)^k \\ &= e^{-\lambda k t} \sum_{l=k}^{\infty} \sum_{i_1+\cdots+i_k=l} a_{i_1}(t) \cdots a_{i_k}(t)\, \hat{\mu}_0^l.\end{aligned}$$

Therefore,

$$\hat{\beta}_t = e^{-\eta t} \sum_{n=0}^{\infty} b_n(t)\, \hat{\mu}_0^n.$$

Taking the inverse Fourier transform of this identity, we arrive at the result.

Wild Sum Representation

Theorem: The probability distribution of any agent's type at time t, given X, is

$$\nu_t = e^{-(\lambda+\eta)t} \sum_{n=1}^{\infty} c_n(t)\, \mu_0^{*n},$$

with coefficients $c_j(t)$ defined by $c_1 = 1$ and

$$c_n(t) = \sum_{k=1}^{n-1} a_k(t)\, b_{n-k}(t).$$

These coefficients are nonnegative and monotone increasing in t. The limit

$$\lim_{t \to +\infty} c_j(t) = \phi_j$$

exists for each j.

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Preliminary Definition

Let F_t denote the cumulative distribution function (CDF) of an arbitrary agents' posterior $p_H(t)$ conditional on the event $\{X = H\}$:

$$F_t(p) = \mathbb{P}(p_H(t) \leq p \,|\, X = H), \quad p \in [0, 1]. \tag{3}$$

Convergence Definition

Definition: We say that the convergence of beliefs to perfect information is exponential at the rate $r > 0$ if there are constants $\kappa_0 > 0$ and κ_1 such that, for any p in $[0, 1]$,

$$e^{-rt}\kappa_0 \leq |F_t(p) - F_\infty(p)| \leq e^{-rt}\kappa_1.$$

Convergence Definition

Definition: We say that the convergence of posterior beliefs to perfect information is exponential almost at the rate $r > 0$ if for any $\varepsilon > 0$ and p in $[0, 1]$, there are constants $\kappa_0 > 0$ and κ_1 such that

$$e^{-(r+\varepsilon)t}\kappa_0 \leq |F_t(p) - F_\infty(p)| \leq e^{-rt}\kappa_1.$$

Public and Private Information

Theorem: If the mean arrival rate λ of an agent's private information meetings is strictly positive, then the convergence of posterior beliefs to perfect information is exponential at the rate $\lambda + \eta$.

Purely Public Information

In this case, $\alpha_t = \mu_0$ for all t, and therefore

$$\nu_t = \mu_0 * \left[\sum_{k=1}^{\infty} \frac{(\eta t)^k}{k!} e^{-\eta t} \left(\sum_{n=1}^{\infty} p_n \, \mu_0^{*n} \right)^{*k} \right].$$

Purely Public Information

Theorem: If $\lambda = 0$ (that is, no private information sharing), then the convergence of posterior beliefs to perfect information is exponential almost at the rate

$$\rho = \eta \left(1 \, - \, \Phi\left(e^{-R}\right)\right). \tag{4}$$

where

$$\Phi(z) \; = \; \sum_{n=1}^{\infty} p_n \, z^n,$$

$$R \; = \; \sup_{\beta \in \mathbb{R}} \left(- \log M(\beta)\right),$$

and M is the moment generating function of μ_0.

Proof of the Theorem

Based on Cramèr's Large Deviations theorem:

$$\mu_0^{*k}(-\infty, a) \;=\; \mathrm{Prob}[Y_1 + \cdots + Y_k > a] \;=\; e^{-k\,(R+o(1))}$$

as $k \to \infty$, and Chernoff (1953) bound:

$$\mathrm{Prob}[Y_1 + \cdots + Y_k > a] \;\leq\; e^{-k\,S(a/k)},$$

where

$$S(x) \;=\; \sup_{\beta \in \mathbb{R}} \left(\beta\, x \;-\; \log E[e^{\beta Y}]\right).$$

Exponential rates of convergence, ρ, for various cases of n, the number of agents whose posteriors are revealed at each arrival of public information.

n	ρ
1	0.025
2	0.049
3	0.073
4	0.097
5	0.120
6	0.142
7	0.164
8	0.185
9	0.205
10	0.226
100	0.923

Table: In this example, we assume no private information sharing ($\lambda = 0$) and take the mean arrival rate η of public releases to be 1. Each agent i is initially endowed with one signal, say s_i, with $P(s_i = 1 \mid H) = 2/3$ and $P(s_i = 1 \mid L) = 1/3.40$

Current Work: Information Percolation in Segmented Markets

- different classes of agents: preferences, initial information, and connectivity.
  - utility for a unit of the asset v_H if $X = H$ and v_i if $X = L$.
  - initial information endowment ψ_{i0}.
  - meetings with intensity λ_i.
  - upon meeting probability κ_{ij} of a class-j counterparty.
- upon meeting, participate in a double auction and learn from counterparty's bid.
- do more connected agents attain higher expected profits?
  - yes, if they can disguise their trades.
  - not necessarily, if characteristics that determine information are commonly observed.
- is it always better for more connected investors to disguise their trades?
  - not necessarily.