Information Heterogeneity in the Macroeconomy*

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Abstract

This paper considers the role that information heterogeneity can play in generating wealth inequality. We solve a model where households face both aggregate and idiosyncratic shocks to returns and wages under two assumptions about information – fully-informed (FI) economies have agents who observe all states while partially-informed (PI) economies have agents that must rely on the Kalman filter to extract estimates of the states based on observed prices. We find that the PI economy has higher aggregate activity (output, consumption, investment) and larger fluctuations in output and investment. There are three key differences between PI and FI economies: PI agents are generally incorrect in their estimate of the state, PI agents face larger risks in returns and wages, and PI agents have heterogeneous forecasts. Quantitatively, the first effect is dominant, while the other two have only limited effect.

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1. Introduction

This paper studies the role of information in a dynamic economy, particularly the role of information asymmetries in generating inequality. We study an economy of incomplete insurance and business cycles – modified from Krusell and Smith (1998) to include idiosyncratic shocks to earnings and returns – under two assumptions about the information available to agents. The first economy is populated by agents we label 'fully-informed' or 'FI', because they observe the aggregate and idiosyncratic components of the prices separately.[1] We show that this type of agent can uncover the exact values of the aggregate states relevant for forecasting prices (the aggregate capital stock and the level of technology) from these four pieces of information. FI agents therefore agree on the distribution of future prices, which we will label as 'beliefs' about prices. Furthermore, the FI agents also agree on the point estimates of these prices.

In the second economy, the 'partially-informed' or 'PI' economy, agents do not observe the idiosyncratic and aggregate components of the prices separately. Using the Kalman filter PI agents construct estimates of the four unobserved states (aggregate capital, technology, and the idiosyncratic wage and return shocks) using only two signals, the total wage and the total return on assets; as a result, their inference is incomplete. Furthermore, since observations are idiosyncratic beliefs become idiosyncratic – an agent with a high current realization of the idiosyncratic shock will perceive the distribution of future prices differently than one with a low current realization, even without correlation between the aggregate and idiosyncratic shocks. As a result, heterogeneous beliefs arise in equilibrium among the PI agents.[2] In particular, PI agents will disagree about the mean of future prices.

Our interest in this model is driven by several considerations. First, we find it advisable to consider the possibility that agents in our models are no better informed about the aggregate capital stock than the model-builder is, for three reasons. One, the aggregate capital stock series published by the BEA is annual and thus not available for real-time decision-making at higher frequencies; two, as shown by An (2006), the published capital stock series diverges from one filtered from a medium-scale New Keynesian model that fits observed fluctuations; and three, the

[1] Measured idiosyncratic shocks to labor earnings are quite large (Storesletten, Telmer, and Yaron 2004, Guvenen 2007) and are a standard part of incomplete insurance models. Idiosyncratic shocks to returns are not standard; we interpret idiosyncratic shocks to returns as reflecting the types of investment errors present in Campbell (2006).

[2] Our paper is related to the vast literature on rational expectations in models with heterogeneous information (see Grossman 1991 and the many references within) but with signals that are driven by idiosyncratic shocks (similar to Lucas 1972 or Graham and Wright 2007).

published capital stock series is subject to large and frequent revisions, as noted by Rupert (2007). Therefore, it seems reasonable to assume that agents don't have good information about aggregate capital, forcing them to filter it using knowledge of the model economy and their observations about prices. Information about the distribution of wealth, which is the "true" state of the world, is even worse in the actual economy, as measurements of this distribution (the Surveys of Consumer Finances and the Panel Study of Income Dynamics) are infrequent and difficult to reconcile with National Income and Product Accounts.

Second, the data on trading volume (DeJong and Espino 2006) is difficult to reconcile with standard models of consumption-smoothing in the presence of uninsurable idiosyncratic earnings risk – income moves too slowly to generate the large transactions that occur at high frequencies in equity markets, for example.[3] One direction for theory to proceed is the introduction of 'speculative trade' driven by heterogeneous beliefs; our environment is a natural one for embedding heterogeneous beliefs and it does not require rule-of-thumb agents or noise traders (PI agents have naturally heterogeneous beliefs, although FI agents do not).[4] Finally, we know from previous work (Aiyagari 1994, Guvenen 2005) that this economy produces savings behavior that is highly sensitive to perceived returns on assets; our model naturally produces differences across individuals that could dramatically alter their savings behavior. In the interest of full disclosure, we note here that we do not pursue this direction in this paper, as simply explaining the mechanisms in our model with one asset is sufficiently involved as to make the paper quite long as it is. We intend to investigate trading volume data in future work, mainly to assess whether models that capture trading volume data are in any way different from those in which trading volume data is ignored.

Models with asymmetric information that preserve heterogeneity of beliefs in the limit are not straightforward to construct. Work by Pearlman and Sargent (2005), Kasa (2000), Walker (2007), and Kasa, Walker, and Whiteman (2007) demonstrate some of the technical obstacles that must be surmounted to prevent agents from using past observations of prices to infer the private information of other agents, although this literature is typically concerned with the conditions under which equal numbers of states and signals does not produce full revelation. In our model the number of signals

[3] Heterogeneity in complete markets is known to generate no asset trading volume; see Judd, Kubler, and Schmedders (2003). Thus, the failure of the incomplete market model to resolve the issue is of primary importance. Interestingly, heterogeneity of beliefs is not sufficient either, as shown by Beker and Espino (2008).

[4] Bomfim (2001a) studies agents who use biased rules of thumb to forecast prices; he finds little effect without strategic complementarities between the output decisions of biased and unbiased agents. Porapakkarm and Young (2009) finds only limited effects in a model similar to the one used here. In contrast, Krusell and Smith (1996) find large effects of including agents who use unsophisticated rules of thumb to make savings decisions. It seems that the forecasting equations are not too important, provided the expectations are not biased.

given to each agent is two (total wage and total return), and there are infinitely-many agents receiving different signals at any point in time. Without the idiosyncratic shocks to returns, the fact that idiosyncratic shocks do not enter into the law of motion for the aggregate states would imply that, with enough data, the entire time series of capital and technology could be recovered with arbitrary precision. Idiosyncratic shocks to returns prevent this unraveling, leading to an economy in which private information persists forever.[5] Different from those papers, which attempt to solve the infinite regress problem identified by Townsend (1983) and studied recently by Nimark (2007), we are interested in whether we can ignore it.

We are able to provide some analytical results about the PI economy relative to the FI economy. First, PI agents have biased beliefs about the current state of the world. In the presence of a common shock, we show that even a continuum of observers do not produce unbiased average estimates; thus, the distribution of beliefs in the PI economy is centered around the wrong mean. It turns out that this bias also moves significantly over time – the standard deviation of the average bias in the idiosyncratic wage shock is about 1/3 of the standard deviation of the wage shock itself. PI agents also perceive expected risks differently, although we cannot prove that their perceived risk is actually higher due to general equilibrium effects. We therefore calibrate and solve the model numerically to obtain quantitative answers.

Our first quantitative result involves a comparison of the aggregate behavior of the FI economy to the PI economy; this question extends the main question of Krusell and Smith (1998) – does the absence of insurance markets for idiosyncratic risk have important implications for aggregate dynamics? – to a world of asymmetric and disparate information. We find that the PI economy displays several strikingly different outcomes. First, aggregate activity is higher in the PI economy (output, consumption, and investment), not only on average but in every period we observe. Second, the PI economy displays larger fluctuations in output and investment than the FI economy, but (slightly) smaller fluctuations in aggregate consumption. Third, the correlation between aggregate capital returns and all aggregates drops significantly; in particular, the correlation between aggregate returns and aggregate wages drops from significantly positive to slightly negative. A low correlation between returns and wages is consistent with the joint behavior of wages and stock returns in US data. Finally, aggregate consumption in the PI economy moves sluggishly relative to

[5] Note that the preservation of private information in the limit is not merely an issue of counting signals and states. Depending on the model, private information could disappear in the limit even if the number of states exceeds the number of signals, as happens in our model when idiosyncratic return shocks are shut down.

the FI economy; with filtering, the response of the perceived state naturally lags behind that of the true state, leading to movements in consumption that lag behind the corresponding full information values.

We then examine individual agent behavior in order to provide intuition for the first two results. We show that three mechanisms are at play in generating the differential saving behavior of PI agents, based on PI agents viewing essentially all movements in prices as driven by the much larger idiosyncratic shocks. Because idiosyncratic return shocks are not persistent, PI agents view increases in returns as purely transitory; thus, PI agents do not have any incentive to raise their savings, although FI agents do. On the other hand, two effects work to raise the saving of PI agents. First, PI agents also perceive that wages are not as persistent, meaning that they expect lower wages in the future; pure consumption smoothing incentives imply that they should increase saving today to spread their transitory good luck over many periods. Second, PI agents underestimate the power of decreasing returns to saving; because they believe the shocks are idiosyncratic, they also believe that their increased demand for saving will be matched by a decline for some other individual. As a result, they misperceive the evolution of average returns and end up saving more.

Since our model features dispersed information, it is worthwhile to examine the dynamics of higher-order expectations. Here, two results from Nimark (2007) are key. First, as the order of expectation n increases the difference between the n and $n+1$-order expectations declines; put another way, the sequence of higher-order expectations is convergent. Averaging the individual higher-order expectations therefore implies that the average belief also converges. Second, we find that the time series variance of the average of higher-order expectations is decreasing in the order n. That is, higher-order expectations converge to constants. The intuition is straightforward. Individual expectations are smoothed versions of actual realizations, and averaging smooths them even more, so higher-order expectations become smoother. We find that the volatility of the first-order expectation is small, and the orthogonal component of these variables (relative to the information already used by the PI agents) is also small, leading to almost no forecasting gain when these variables are used to predict future capital. That is, current prices convey nearly all the information PI agents need to forecast future prices. As a result our model displays an "approximate aggregation" result that is much stronger (and perhaps more surprising) than that found in Krusell and Smith (1998).[6]

[6] We say that it is more surprising because there is no obvious reason to expect decision rules to be linear in the directions needed for this aggregation to obtain, unlike Krusell and Smith (1998) where aggregation obtains due to

Our next experiment compares the wealth concentration in the FI and PI economies. Contrary to our intuition, the PI economy displays less wealth concentration, not more. There are two mechanisms at work in the model that affect wealth concentration. First, since PI agents have beliefs that are symmetrically distributed about the mean belief (not the true value), agents who are 'optimistic' regarding returns will tend to save more; given that savings functions are concave in expected returns, the effect of the upward-biased belief is larger than the effect of the downward-biased symmetric belief, leading to wealth concentration. Second, the higher precautionary savings by the PI agents leads to less wealth concentration, since the poor are not so poor and their additional savings reduces the return to the wealthy. The precautionary savings by PI agents is driven by two effects. First, partial information raises the volatility of individual wages by roughly 10 percent; following standard consumer choice of uncertainty theory, this increase in volatility causes households to save more, and this increase is stronger for those agents who are most risk averse (the poor). Second, partial information lowers the correlation between wages and returns, making capital a good hedge against background risk; again, standard theory predicts an increase in the demand for capital will result.[7] We find that the second effect is dominant in our economy, but make no claims about its generality; presumably, this ranking could change if we introduce transfer programs to the poor that are means-tested.

For our model to be a good tool for policy work, it should replicate some direct measures of consumer expectations. We consider two such measures – the Michigan Survey of Consumer Sentiment and the Consumer Confidence Survey from the Board of Governors. In each case, the measure of consumer confidence displays a strong positive correlation with Industrial Production (a measure available at monthly frequencies) and roughly the same variance. In contrast, constructing measures of confidence in our model – such as the average expected value of period $t+1$ output – has only a small variance and very low correlation with output. The low variance is a consequence of the filtering procedure, which naturally smooths estimated states; the low correlation with output is generated because households view changes as idiosyncratic and therefore disconnected from the aggregate economy. Precisely what these surveys measure is the object of study by a number of recent papers – such as Barsky and Sims (2008) – and they conclude that consumer confidence is an indicator of news about future productivity. Since our model does not feature such shocks, we

known theoretical results (see Bewley 1977).

[7] The second effect has been noted in other models, such as models of habit formation (Díaz, Pijoan-Mas, and Ríos-Rull 2003) or the spirit of capitalism (Luo and Young 2007a).

are perhaps mistaken in comparing the two variables. Embedding news shocks in our model would substantially increase the already large computational burden and further lengthen the paper (as well as obscuring our discussion of effects of information and signal extraction), so we confine it to future research.

In concluding the introduction we link our work here to a related paper by Graham and Wright (2007). That paper studies a island economy in which households are uninformed about the source of fluctuations in returns and wages, as in our environment. However, Graham and Wright (2007) solve their model by linearizing and completing the state space with a hierarchy of expectations (as in Nimark 2007), whereas we attempt to solve the infinite regress problem approximately. There are known pitfalls associated with linearization, particularly in incomplete markets (the wealth of a given agent follows a random walk locally and linearization removes the bounds that ensure the existence of a stationary distribution, welfare rankings may be reversed). In contrast, our model maintains nonlinearity where it is needed to obtain stationarity (at the individual level) and only imposes it on household expectations, where it appears to be present even when not explicitly imposed (see the discussion in Krusell and Smith 2006 or Young 2007). A side effect of linearization is the elimination of precautionary savings motives; while we find the added effect of information heterogeneity to be small in our economy, it need not be in general.

We also want to point out connections between our work and Lorenzoni (2009), who uses signals to generate private information about aggregate productivity. In his model, signals are pure noise – they do not alter the budget sets of households in any real way. In contrast, we assume that the signals agents receive are budget-relevant, because they are movements in prices. It is feasible for us to introduce noisy signals into our model – Porapakkarm (2009) examines the effects of noisy indicators of aggregate productivity, finding them minor – but we choose not to, so that our focus is kept tightly on the effects of signals generated only through price movements. A more complete model is feasible, if computationally burdensome, and our future research is directed along those lines (we discuss extensions in our concluding remarks).

2. Model

The model economy is populated by a continuum of households and a continuum of firms, both with unit measure. The production sector is represented by a stand-in firm that operates a Cobb-Douglas production technology,

$$Y_t = \exp(z_t) K_t^{\alpha} N_t^{1-\alpha},$$

where K_t and N_t are aggregate capital and labor inputs in the economy and $\alpha \in (0,1)$ is capital's share of income. The aggregate shock in the economy is the technology shock z_t, which evolves as

$$z_{t+1} = \rho_z z_t + e_{t+1}; \quad e_t \sim \text{iid } N\left(0, \sigma_e^2\right); \tag{2.1}$$

we assume $|\rho_z| < 1$. With competitive factor markets the factor prices would satisfy

$$\begin{aligned} \log\left(MPK_t\right) &= \log\left(\alpha\right) + \left(1-\alpha\right)\log\left(N_t\right) + z_t + \left(\alpha - 1\right)\log\left(K_t\right) \\ \log\left(MPN_t\right) &= \log\left(1-\alpha\right) - \alpha\log\left(N_t\right) + z_t + \alpha\log\left(K_t\right), \end{aligned} \tag{2.2}$$

where MPK_t and MPN_t are marginal product of capital and labor. $\delta \in [0,1]$ is a fixed depreciation rate.

The other sector of the economy is represented by a continuum of infinitely-lived households with total measure 1. These agents are heterogeneous *ex post* along three dimensions: their uninsurable idiosyncratic shocks ε_t^i and η_t^i to wages and returns, their accumulated cash on hand m_t^i, and their information sets Ω_t^i. ε_t^i evolves according to an exogenous AR(1) process

$$\varepsilon_{t+1}^i = \mu_\varepsilon + \rho_\varepsilon \varepsilon_t^i + \nu_{t+1}^i; \quad \nu_t^i \sim \text{iid } N\left(0, \sigma_\nu^2\right). \tag{2.3}$$

Under the assumption that households are perfect substitutes in terms of labor input, individual wages are given by $w_t^i = \exp\left(\varepsilon_t^i\right) MPN_t$. We assume that $|\rho_\varepsilon| < 1$. η_t^i evolves according to the process

$$\eta_{t+1}^i = \mu_\eta + \rho_\eta \eta_t^i + \zeta_{t+1}^i; \quad \mu_\eta = -\frac{\sigma_\zeta^2}{2\left(1+\rho_\eta\right)}; \quad \zeta_t^i \sim \text{iid } N\left(0, \sigma_\zeta^2\right), \tag{2.4}$$

where $\left|\rho_\eta\right| < 1$. μ_η is defined such that unconditional mean of $\exp\left(\eta_t^i\right)$ is one. The individual return to saving is then given by $R_t^i = \exp\left(\eta_t^i\right) MPK_t$. The covariance matrix of the exogenous shocks $\left(e_t, \nu_t^i, \eta_t^i\right)$ is denoted

$$\mathbf{\Sigma} = \begin{bmatrix} \sigma_e^2 & \rho_{e\nu}\sigma_e\sigma_\nu & \rho_{e\zeta}\sigma_e\sigma_\zeta \\ \rho_{e\nu}\sigma_e\sigma_\nu & \sigma_\nu^2 & \rho_{\nu\zeta}\sigma_\nu\sigma_\zeta \\ \rho_{e\zeta}\sigma_e\sigma_\zeta & \rho_{\nu\zeta}\sigma_\nu\sigma_\zeta & \sigma_\zeta^2 \end{bmatrix}.$$

Since we will assume inelastic labor supply by households, N_t will be a constant (denoted $\overline{N}$).[8]

[8] Elastic labor supply would be straightforward to introduce, provided that the equilibrium function used to forecast

2.1. Information Structure

We now discuss the informational assumptions we make – that is, what agents observe and how they make inference about what they do not observe. For future reference all variables that may not be common across households are indexed by a superscript i.

Definition 2.1. *The observation set* Υ_t^i *is the set of* (i) *variables that* i *directly observes up to period* t *and* (ii) *the model structure and all parameters.*

Definition 2.2. *The information set* Ω_t^i *is the union of* (i) *the observable set* Υ_t^i *and* (ii) *the set of variables that* i *can infer by using* Υ_t^i*.*

Under an assumption of full information, the definitions are redundant since inferable variables can always be assumed as directly observable; hence $\Upsilon_t^i = \Omega_t^i$. In contrast, under partial information an agent directly observes only parts of the economy; however, she can construct an inference about other unobserved parts, implying that $\Upsilon_t^i \subset \Omega_t^i$.

We will confine our study to two economies; the FI economy where everyone is *fully-informed (FI)* and PI economy where everyone is *partially-informed (PI)*.[9] Both FI and PI agents are identical except for their observable sets; in particular, we assume that they face the same processes for the idiosyncratic shocks. The PI agents are forced to use the Kalman filter to extract signals about $\left(K_t, z_t, \varepsilon_t^i, \eta_t^i\right)$ from observations $\left\{R_\tau^i, w_\tau^i\right\}_{\tau \leq t}$, whereas the FI agent directly observes these values. One key point is that PI agents do not observe their individual shock ε_t^i, but only their "paycheck" $w_t^i = \exp\left(\varepsilon_t^i\right) MPN_t$. Similarly, they do not observe their idiosyncratic return shock η_t^i, but only the total return $R_t^i = \exp\left(\eta_t^i\right) MPK_t$.[10]

N is log-linear:

$$\log\left(N_t\right) = b_0 + b_1 z_t + b_2 \log\left(K_t\right).$$

A serious complication this setup poses is the need to solve endogenously for N_t at each point in time by clearing the labor market; since the burden of the model is already large and elastic labor supply only complicates the discussion, we abstract from it.

[9] In Krusell and Smith (1998) every household is fully-informed; their equilibrium can be approximated by allowing households to use only information in current period. We also show here that in an economy where not all households are fully-informed, if there exists a zero mass fully-informed agent her knowledge of current period values is sufficient to accurately forecast the evolution of aggregate capital.

[10] We leave aside the issue of how wages are determined when individual and aggregate wage components are not observable. We could assume that households observe part of their earnings – namely, the variations in their hours worked. This modification would have no qualitative effect and would tend to weaken our quantitative results; since those quantitative results depend most critically on the return shocks, we expect them to be robust to this change.

2.1.1. FI Economy

In the FI economy, all agents are fully-informed about the relevant state variables. Their information set is therefore given by

$$\Upsilon_t^{FI} \equiv \left\{k_\tau^i, \varepsilon_\tau^i, \eta_\tau^i, \Gamma_\tau\left(k, \varepsilon, \theta\right), z_\tau, R_\tau^i, w_\tau^i, \Xi\right\}_{\tau \leq t},$$

where k^i is individual capital, $\Gamma_t\left(\cdot\right)$ is the cross-sectional distribution of households, and Ξ is the model structure and all parameters. The FI agent's recursive problem is

$$V^{FI}\left(k_t^i, \varepsilon_t^i, \eta_t^i, \Gamma_t, z_t\right) = \max_{k_{t+1}^i \in \left[0, m_t^i\right]} \left\{u\left(m_t^i - k_{t+1}^i\right) + \beta E\left[V^{FI}\left(k_{t+1}^i, \varepsilon_{t+1}^i, \eta_{t+1}^i, \Gamma_{t+1}, z_{t+1}\right) | \Omega_t^{FI}\right]\right\} \tag{2.5}$$

subject to the budget constraint and law of motion for Γ,

$$\begin{aligned} m_t^i &= k_t^i\left(1 + R_t^i - \delta\right) + w_t^i \overline{h} \\ \Gamma_{t+1} &= F\left(\Gamma_t, z_t, z_{t+1}\right), \end{aligned}$$

and the shock processes (2.1), (2.3), and (2.4). k_{t+1}^i is individual savings in capital and $E\left[\cdot | \Omega_t^{FI}\right]$ is the expectation operator conditioned on information set Ω_t^{FI}.[11] Note that $K_{t+1} \in \Omega_t^{FI}$ since it is an aggregation of current savings.[12] Following the approximate aggregation result in Krusell and Smith (1998) and Young (2007) the only relevant aggregate variables are K_t and z_t; other moments of Γ_t do not contribute to forecasting future prices.[13] We therefore parameterize the law of motion for K_{t+1} as

$$\log\left(K_{t+1}\right) = a_0 + a_1 z_t + a_2 \log\left(K_t\right); \tag{2.6}$$

this assumption is based on results in Young (2007) that show more flexible functional forms do not alter the aggregate dynamics of the model.[14]

[11] Asset markets are incomplete here, since there exists only one asset (a claim to capital). In a complete market environment, belief heterogeneity is formally equivalent to discount factor heterogeneity and therefore leads to a degenerate wealth distribution (see Tsyrennikov 2006).

[12] The presence of z_{t+1} in the law of motion for Γ_t reflects the law of large numbers requirement for the idiosyncratic shocks.

[13] This result is due to the near-linearity of the optimal saving function k_{t+1}^i with respect to m_t^i combined with the fact that changes in the aggregate states linearly displace the savings function. Also contributing to the approximate aggregation result is the fact that only agents with low m_t^i have nonlinear savings rules and they are both small in measure and contribute negligible amounts to aggregate saving. Extensive discussions of this point can be found in Krusell and Smith (1998), Krusell and Smith (2006), and Young (2007).

[14] While these conjectures (approximate aggregation and linearity of the aggregate law of motion) are based on a different model, we verify that they hold here. Linearity is critical, since it permits us to apply the Kalman filter in

It is obvious that any FI agent who knows $\left(R_t^i, w_t^i, \varepsilon_t^i, \eta_t^i\right)$ can compute (K_t, z_t) by using (2.2); thus prices fully reveal the relevant state variables in our setting. To make comparisons across agents simple and relatively free of numerical error, we rewrite the FI agent problem using $\left(R_t^i, w_t^i\right)$ as state variables rather than (K_t, z_t). The recursive problem of an FI agent is therefore

$$V^{FI}\left(s_t^{i,FI}\right) = \max_{k_{t+1}^i \in 0, m_t^i)} \left\{ u\left(m_t^i - k_{t+1}^i\right) + \beta \int_{\Psi_{t+1}^i} V^{FI}\left(s_{t+1}^{i,FI}\right) dF\left(\Psi_{t+1}^i\right) \right\} \tag{2.7}$$

subject to

$$\begin{aligned} m_t^i &= k_t^i\left(1 + R_t^i - \delta\right) + w_t^i \overline{h} \\ \log\left(R_{t+1}^i\right) &= A_0 + A_1 \log\left(R_t^i\right) + A_2 \log\left(w_t^i\right) - A_2 \varepsilon_t^i + \left(\rho_\eta - A_1\right) \eta_t^i + e_{t+1} + \zeta_{t+1}^i \\ \log\left(w_{t+1}^i\right) &= A_3 + A_4 \log\left(R_t^i\right) + A_5 \log\left(w_t^i\right) + \left(\rho_\varepsilon - A_5\right) \varepsilon_t^i - A_4 \eta_t^i + e_{t+1} + \nu_{t+1}^i \end{aligned}$$

and idiosyncratic processes (2.3), and (2.4), where $\Psi_t^i = \left\{e_{t+1}, \nu_{t+1}^i, \zeta_{t+1}^i\right\}$ and

$$s_t^{i,FI} = \left\{k_t^i, \varepsilon_t^i, \eta_t^i, \log\left(R_t^i\right), \log\left(w_t^i\right)\right\}.$$

Appendix A shows that the dynamic equations for R_t^i and w_t^i shown above can be derived from equations (2.2), (2.1), and (2.6), where the coefficients $\{A_i\}_0^5$ will be determined endogenously in equilibrium. For ease of comparison with the PI agent, the evolution $\left(\log\left(R_t^i\right), \log\left(w_t^i\right)\right)$ can be written as

$$\begin{bmatrix} \log\left(R_{t+1}^i\right) \\ \log\left(w_{t+1}^i\right) \end{bmatrix} \sim N\left(\begin{bmatrix} E_t^{FI}\left[\log\left(R_{t+1}^i\right)\right] \\ E_t^{FI}\left[\log\left(w_{t+1}^i\right)\right] \end{bmatrix}, \mathbf{V}_2 \right) \tag{2.8}$$

$$E_t^{FI} \begin{bmatrix} \log\left(R_{t+1}^i\right) \\ \log\left(w_{t+1}^i\right) \end{bmatrix} = \begin{bmatrix} A_0 \\ A_3 \end{bmatrix} + \begin{bmatrix} A_1 & A_2 \\ A_4 & A_5 \end{bmatrix} \begin{bmatrix} \log\left(R_t^i\right) \\ \log\left(w_t^i\right) \end{bmatrix} + \begin{bmatrix} -A_2 & \rho_\eta - A_1 \\ \rho_\varepsilon - A_5 & -A_4 \end{bmatrix} \begin{bmatrix} \varepsilon_t^i \\ \eta_t^i \end{bmatrix}$$

where $\mathbf{V}_2$ is a constant covariance matrix defined in Appendix A.

It is helpful here to note a property of $\mathbf{V}_2$: it does not depend on the values of (a_0, a_1, a_2) in the law of motion for K_t. Thus, FI agents who are confronted with a different aggregate law of motion – such as the one that arises from an economy populated entirely by PI agents – will know that returns and wages carry the same risk one period ahead. However, the predicted conditional

the PI economy.

means do depend on these values, so that agents facing different aggregate laws of motion will have different predictions (that is, different beliefs).

2.1.2. PI Economy

In the PI economy each agent is disparately informed. Specifically, PI agent i's observation set is defined by

$$\Upsilon_t^{PI} = \left\{k_\tau^i, R_\tau^i, w_\tau^i, \Xi\right\} \subset \Upsilon_t^{FI}.$$

That is, PI agents do not observe the aggregate state of the world, nor do they observe their idiosyncratic shocks $\left(\varepsilon_t^i, \eta_t^i\right)$ separately from their total wage and total return. In addition, the observations $\left\{R_\tau^i, w_\tau^i\right\}_{\tau \leq t}$ are private information.

This information structure creates the problem of forecasting other's forecasts and the number of state variables explodes to infinity.[15] Intuitively, aggregate K_{t+1} is the summation over individual k_{t+1}^i, which is itself a function of i's private information $\left(R_t^i, w_t^i\right)$. Thus to forecast K_{t+1}, agent i needs to forecast every other agent's $\left(R_t^j, w_t^j\right)$. Thus, k_{t+1}^i will depend not only agent i's private information $\left(R_t^i, w_t^i\right)$ but also on what she expects $\left(R_t^j, w_t^j\right)$ to be for all j. Consequently, K_{t+1} will be a function of individual i's expectation of other's $\left(R_t^j, w_t^j\right)$, denoted the first-order expectation.[16] We can repeat the same induction and introduce an infinite array of higher-order expectations into the PI agent's problem. To avoid an infinite-dimensional problem, we impose Assumptions (2.3) and (2.4) below.

Assumption 2.3. *PI agents ignore all the higher order expectations when they make their savings decisions.*

Assumption 2.4. *PI agents believe that the law of motion of aggregate capital is captured by (2.6).*

Note that Assumption (2.3) does not remove the higher order expectations from Ω_t^{PI}; a PI agent can construct a belief about the higher-order expectations from Υ_t^{PI}. Assumption (2.3), exogenously imposed here, allows us to remove all higher-order expectations from the state vector of the household. Like the FI problem, the approximate aggregation embedded in Assumption (2.4) is a computational technique to avoid keeping track of the whole distribution of households;

[15] Townsend (1983) is the first statement of this problem.
[16] See Nimark (2007).

in addition, here it is a restriction needed to consistently apply approximate aggregation in the PI economy, since the approximate law of motion for K_{t+1} cannot contain variables not contained in Ω_t^i. Proposition (2.5) states the consistency restriction formally.

Proposition 2.5. *Any linear approximate law of motion for K_t is limited to linear combinations of $\mathcal{M}_s\left[x_t^i\right]$, where $x_t^i \in \Omega_t^i$ and $\mathcal{M}_s\left[x^i\right]$ is an aggregate operator over x^i (for example moments, percentiles, Gini coefficient).*

Proof *The policy function k_{t+1}^i is a function of $\left\{x_1^i, x_2^i, x_3^i, ..\right\}$, where $x^i \in \Omega_t^i$. By aggregation, $K_{t+1} = \int k_{t+1}^i \Gamma_t\left(\cdot\right)$. Consequently K_{t+1} is a function of some moments of $\left\{x_1^i, x_2^i, x_3^i, ...\right\}$.*

Applying the consistency restriction in the FI economy is trivial; under full information, any variable x^i can be used in the approximate law of motion since all variables are in Ω_t^{FI}. However, in the PI economy only aggregation over $x^i \in \Omega_t^{PI}$ can be used. After we state the PI agent's problem, we argue that although z_t and K_t are not in Ω_t^{PI} (2.6) can still be used to approximate the one that is consistent with Proposition (2.5).

Given Assumptions (2.3) and (2.4), a PI agent can use the Kalman filter to estimate $\left\{K_t, z_t, \varepsilon_t^i, \eta_t^i\right\}$ and construct forecasted R_{t+1}^i and w_{t+1}^i. The state and measurement equations are

$$\begin{bmatrix} z_{t+1} \\ \varepsilon_{t+1}^i \\ \eta_{t+1}^i \\ \log\left(K_{t+1}\right) \end{bmatrix} = \begin{bmatrix} 0 \\ \mu_\varepsilon \\ \mu_\eta \\ a_0 \end{bmatrix} + \begin{bmatrix} \rho_z & 0 & 0 & 0 \\ 0 & \rho_\varepsilon & 0 & 0 \\ 0 & 0 & \rho_\eta & 0 \\ a_1 & 0 & 0 & a_2 \end{bmatrix} \begin{bmatrix} z_t \\ \varepsilon_t^i \\ \eta_t^i \\ \log\left(K_t\right) \end{bmatrix} + \begin{bmatrix} e_{t+1} \\ \nu_{t+1}^i \\ \zeta_{t+1}^i \\ 0 \end{bmatrix} \tag{2.9}$$

and

$$\begin{bmatrix} \log\left(R_t^i\right) \\ \log\left(w_t^i\right) \end{bmatrix} = \begin{bmatrix} \log\left(\alpha\right) + \left(1-\alpha\right)\log\left(\overline{N}\right) \\ \log\left(1-\alpha\right) - \alpha\log\left(\overline{N}\right) \end{bmatrix} + \begin{bmatrix} 1 & 0 & 1 & \alpha - 1 \\ 1 & 1 & 0 & \alpha \end{bmatrix} \begin{bmatrix} z_t \\ \varepsilon_t^i \\ \eta_t^i \\ \log\left(K_t\right) \end{bmatrix}. \tag{2.10}$$

In Appendix B, we show that PI agents need to filter only $\left(\varepsilon_t^i, \eta_t^i\right)$ since the measurement equations are an exact linear combination of the state variables. Denote the belief about $\left(\varepsilon_t^i, \eta_t^i\right)$ given information up to period t by $\left(\varepsilon^i_{t|t}, \eta^i_{t|t}\right)$ and the prior by

$$\begin{bmatrix} \varepsilon^i_{t|t} \\ \eta^i_{t|t} \end{bmatrix} \sim N\left(\begin{bmatrix} \overline{\varepsilon}^i_{t|t} \\ \overline{\eta}^i_{t|t} \end{bmatrix}, \overline{\mathbf{P}} \right);$$

here we assume that the agents are using the stationary covariance matrix $\overline{\mathbf{P}}$ from the Kalman filter.[17]

The recursive problem of PI agent is

$$V^{PI}\left(s^{i,PI}_t\right) = \max_{k^i_{t+1}\in\left(\left(0,m^i_t\right)\right.} \left\{ u\left(m^i_t - k^i_{t+1}\right) + \beta \int_{\log\left(R^i_{t+1}\right),\log\left(w^i_{t+1}\right)} V^{PI}\left(s^{i,PI}_{t+1}\right) d\Phi_t\left(\log\left(R^i_{t+1}\right), \log\left(w^i_{t+1}\right) | \Omega^{PI}_t\right) \right\}$$

subject to

$$m^i_{t+1} = k^i_t\left(1 + R^i_t - \delta\right) + w^i_t \overline{h},$$

where

$$s^{i,PI}_t = \left\{ k^i_t, \overline{\varepsilon}^i_{t|t}, \overline{\eta}^i_{t|t}, \log\left(R^i_t\right), \log\left(w^i_t\right) \right\}.$$

The Kalman filter (derived explicitly in Appendix B) yields the following forecast rules:

$$\begin{bmatrix} \log\left(R^i_{t+1}\right) \\ \log\left(w^i_{t+1}\right) \end{bmatrix} \sim N\left(\begin{bmatrix} E^{PI}_t\left[\log\left(R^i_{t+1}\right)\right] \\ E^{PI}_t\left[\log\left(w^i_{t+1}\right)\right] \end{bmatrix}, \mathbf{B}_3\overline{\mathbf{P}}\mathbf{B}_3^T + \mathbf{V}_2 \right) \tag{2.11}$$

$$E^{PI}_t \begin{bmatrix} \log\left(R^i_{t+1}\right) \\ \log\left(w^i_{t+1}\right) \end{bmatrix} = \begin{bmatrix} A_0 \\ A_3 \end{bmatrix} + \begin{bmatrix} A_1 & A_2 \\ A_4 & A_5 \end{bmatrix} \begin{bmatrix} \log\left(R^i_t\right) \\ \log\left(w^i_t\right) \end{bmatrix} + \begin{bmatrix} -A_2 & \rho_\eta - A_1 \\ \rho_\varepsilon - A_5 & -A_4 \end{bmatrix} \begin{bmatrix} \overline{\varepsilon}^i_{t|t} \\ \overline{\eta}^i_{t|t} \end{bmatrix}$$

where the dynamics of the estimated idiosyncratic shocks follow

$$\begin{bmatrix} \overline{\varepsilon}^i_{t+1|t+1} \\ \overline{\eta}^i_{t+1|t+1} \end{bmatrix} = \begin{bmatrix} \mu_\varepsilon \\ \mu_\eta \end{bmatrix} + \begin{bmatrix} \rho_\varepsilon \overline{\varepsilon}^i_{t|t} \\ \rho_\eta \overline{\eta}^i_{t|t} \end{bmatrix} + \mathbf{G}_3\left(\begin{bmatrix} \log\left(R^i_{t+1}\right) - E_t\left[\log\left(R^i_{t+1}\right)\right] \\ \log\left(w^i_{t+1}\right) - E_t\left[\log\left(w^i_{t+1}\right)\right] \end{bmatrix} \right); \tag{2.12}$$

$\{A_i\}_0^5$ are the same coefficients as in (2.8). $\mathbf{B}_3$ and $\mathbf{G}_3$ are constant matrices derived from the filtering problem, while $\mathbf{V}_2$ is the same matrix as in FI forecast rule (2.8). The presence of the term $\mathbf{B}_3\overline{\mathbf{P}}\mathbf{B}_3^T$ implies two things about the PI agents' forecasts – they are riskier than the FI agents' forecasts (since $\mathbf{B}_3\overline{\mathbf{P}}\mathbf{B}_3^T$ is positive semidefinite) and depend on the coefficients (a_0, a_1, a_2). Thus,

[17] Having the agents use an evolving variance-covariance matrix introduces an additional state variable into the problem, the "age" of the agent. Porapakkarm (2009) examines this issue and finds it is quantitatively small.

PI agents who find themselves confronted with the law of motion from an economy populated entirely by FI agents will perceive that the riskiness of returns and wages has changed.

Propositions (2.6) and (2.7) show the consistency of Assumption (2.4) with Proposition (2.5).

Proposition 2.6. *Assume Assumption (2.3) and restrict the aggregate operator $\mathcal{M}_s[\cdot]$ only to the first moment. Then a log-linear approximate law of motion for K_t, if it exists, takes the form*

$$\log(K_{t+1}) = \alpha_0 + \alpha_1 z_t + \alpha_2 \log(K_t) + \alpha_3 \mathbf{E}^i\left[\overline{\varepsilon}^i_{t|t}\right] + \alpha_4 \mathbf{E}^i\left[\overline{\eta}^i_{t|t}\right], \tag{2.13}$$

where $\mathbf{E}^i_t\left[x^i\right]$ is cross-sectional expectation over x^i.

Proposition 2.7. *If $\frac{\sigma_\varepsilon}{\sigma_z} \to \infty$ and $\frac{\sigma_\eta}{\sigma_z} \to \infty$, (2.6) and (2.13) coincide.*

The proofs of Propositions (2.6) and (2.7) are contained in Appendix C. Intuitively, when both ε^i_t and η^i_t dominate z_t, PI agents will infer that any unexpected change in R^i_t and w^i_t is almost entirely due to ε^i_t and η^i_t. In this case, $\left(\overline{\varepsilon}^i_{t|t} - \varepsilon^i_t\right)$ and $\left(\overline{\eta}^i_{t|t} - \eta^i_t\right)$ will converge to zero. Consequently $\mathbf{E}^i\left[\overline{\varepsilon}^i_{t|t}\right]$ and $\mathbf{E}^i\left[\overline{\eta}^i_{t|t}\right]$ will converge to their unconditional mean and (2.13) will reduce to (2.6). A trivial case is when $\sigma_z \to 0$ but σ_ε and σ_η are fixed. Since there is no aggregate shock, both (2.6) and (2.13) degenerate to $K_{t+1} = K_t = \overline{K}$ (the model becomes Aiyagari 1994 with return shocks).

There are two additional aspects of the PI problem that we need to make clear. First, we are *assuming* that every PI agent uses the state space given above. If some agents were to believe that other variables were states, we cannot guarantee that the equilibrium would not be different.[18] Second, the coefficients $\left(a, \rho_z, \rho_\varepsilon, \rho_\eta, \mathbf{\Sigma}\right)$ are *identified*; with an infinite amount of data, any PI agent would be able to estimate them with arbitrary accuracy. We implicitly are assuming that this estimation occurred in the distant past, so any transient dynamics associated with learning these parameters has ceased having any effect on the economy.

Finally, we note that the PI economy does not have a "representative agent" analogue. In a world of complete markets, the idiosyncratic shocks $\left(\varepsilon^i, \eta^i\right)$ would be insured away, leaving the PI economy (which has two signals and two underlying states) equivalent to the FI economy. Thus, the comparison between FI and PI is necessarily complicated by the incomplete market assumption, since both types of agents are living in constrained inefficient environments (see Davila *et al.* 2005);

[18] Young (2007) shows that heterogeneity of the state space has no impact in an FI economy.

Graham and Wright (2008) contains a related discussion about the inconsistency between private information and complete asset markets. The implication of the incomplete market assumption is that welfare comparisons are more difficult; we cannot say definitively that FI agents are better off than PI agents, since price effects matter. Porapakkarm (2009) contains some results regarding the welfare effects of more information in an OLG setup that permits a more definitive statement; in his model, newborns are better off in the FI world, but the gain is trivial.

2.2. Market Clearing

In our economy equilibrium requires that supply and demand are equated in the markets for capital, labor, and goods. The first two markets will be in equilibrium if

$$\begin{aligned} \overline{N} &= \overline{h} \int \exp\left(\varepsilon_t^i\right) \Gamma_t\left(\cdot\right) \\ K_t &= \int k_t^i \Gamma_t\left(\cdot\right), \end{aligned}$$

where $\Gamma_t(\cdot) = \Gamma_t\left(k_t^i, \varepsilon_t^i, \eta_t^i\right)$ in the FI economy and $\Gamma_t(\cdot) = \Gamma_t\left(k_t^i, \varepsilon_t^i, \eta_t^i, \overline{\varepsilon}_{t|t}^i, \overline{\eta}_{t|t}^i\right)$ in the PI economy. Agents will also differ along the $\left(\overline{z}_{t|t}^i, \log\left(\overline{K}_{t|t}^i\right)\right)$ dimensions, but we do not need to keep track of these variables explicitly because they are exact linear combinations of the other states and the observed prices. Note that we could define the domain of Γ_t as the same for both PI and FI agents, since for FI agents we have $\varepsilon_t^i = \overline{\varepsilon}_{t|t}^i$ and $\eta_t^i = \overline{\eta}_{t|t}^i$.

To obtain the goods market clearing condition we integrate the budget constraints to obtain

$$\int m_t^i \Gamma_t\left(\cdot\right) = \int k_{t+1}^i \Gamma_t\left(\cdot\right) + \int c_t^i \Gamma_t\left(\cdot\right).$$

We then note that

$$m_t^i = k_t^i \left(1 + \exp\left(\eta_t^i\right) MPK_t - \delta\right) + MPN_t \exp\left(\varepsilon_t^i\right) \overline{h},$$

so that

$$\int \left[k_t^i \left(1 + \exp\left(\eta_t^i\right) MPK_t - \delta\right) + MPN_t \exp\left(\varepsilon_t^i\right) \overline{h}\right] \Gamma_t\left(\cdot\right) = \int k_{t+1}^i \Gamma_t\left(\cdot\right) + \int c_t^i \Gamma_t\left(\cdot\right).$$

Rearranging yields

$$(1-\delta)\int k_t^i\Gamma_t(\cdot)+MPK_t\int k_t^i\exp\left(\eta_t^i\right)\Gamma_t(\cdot)+MPN_t\overline{h}\int\exp\left(\varepsilon_t^i\right)\Gamma_t(\cdot)=\int k_{t+1}^i\Gamma_t(\cdot)+\int c_t^i\Gamma_t(\cdot).$$

Since there is no correlation between k_t^i and the realization of η_t^i, we can separate that integral into the product of two integrals,

$$MPK_t\int k_t^i\exp\left(\eta_t^i\right)\Gamma_t(\cdot)=MPK_t\left(\int k_t^i\Gamma_t(\cdot)\right)\left(\int\exp\left(\eta_t^i\right)\Gamma_t(\cdot)\right).$$

Then we use the definition of aggregates and the fact that $\int\exp\left(\eta_t^i\right)\Gamma_t(\cdot)=1$ to obtain

$$(1-\delta)K_t+MPK_tK_t+MPN_t\overline{N}=K_{t+1}+C_t.$$

From the firm's first-order conditions we get the goods market clearing condition:

$$z_tK_t^\alpha N_t^{1-\alpha}+(1-\delta)K_t=K_{t+1}+C_t.$$

Therefore, if the labor and capital markets both clear, the goods market will automatically clear.

3. Effects of Unobservable State Variables

Before we present numerical results, we discuss the effects of unobservable common state variables in the PI economy. The difference between a PI agent and an FI agent is captured by the difference in their forecast rules (2.8) and (2.11). There are two points to address here. First by having less information, PI agents perceive higher risks than their FI counterparts. Since the matrix $\mathbf{B}_3\overline{\mathbf{P}}\mathbf{B}_3^T$ is positive semidefinite, PI forecasts have higher variance (in the matrix sense). Second, there is endogenous heterogeneity in beliefs and forecasts among PI agents, since $\overline{\varepsilon}_{t|t}^i$ and $\overline{\eta}_{t|t}^i$ depend on the whole history of $\left\{R_\tau^i,w_\tau^i\right\}_{\tau<t}$. Thus PI agents who receive the same R_t^i and w_t^i will typically infer different state variables depending on their current beliefs, which translates into different expected R_{t+1}^i and w_{t+1}^i. On the contrary, FI agents with the same R_t^i and w_t^i will always have the same expected $\left(R_{t+1}^i,w_{t+1}^i\right)$. As the second point will play an important role in determining the aggregate behavior of the PI economy, we discuss this result in some detail. Since we are assuming a convergent covariance matrix in the Kalman filter, the terms belief and expectation will be used interchangeably.

3.1. Degeneration of PI Beliefs

We can view both FI and PI agents as possessing priors over the values of four variables, $\left\{MPK_t, MPN_t, \varepsilon_t^i, \eta_t^i\right\}$. For the PI agent these priors are nondegenerate normals with means and variances derived from the Kalman filter. For the FI agent these priors are degenerate at the current value, so we can view them as normals with a mean equal to the true value and zero variance. One important issue to consider is whether PI beliefs converge to FI beliefs as the sample size gets large; since we permit agents to use an infinite amount of past data when estimating the states, this convergence is undesirable.[19]

Proposition 3.1. *Assume that either (i) $\sigma_\zeta = 0$ or (ii) $\sigma_\nu = 0$. Any PI agent observing the time series of $\left\{R_\tau^i, w_\tau^i\right\}_{\tau=-\infty}^{t}$ can recover all relevant state variables and PI beliefs degenerate into FI beliefs.*

To see how this degeneration obtains, suppose that K_0 is known with certainty and that $\sigma_\zeta = 0$; that is, agents do not receive idiosyncratic shocks to their returns $\left(R_t^i = MPK_t\right)$. Observing MPK_0 is sufficient to determine z_0 exactly:

$$z_0 = \frac{MPK_0}{\alpha K_0^{\alpha-1}\overline{N}^{1-\alpha}}.$$

Now we can use the law of motion for capital to obtain K_1:

$$K_1 = \exp\left(a_0 + a_1 z_0 + a_2 \log\left(K_0\right)\right).$$

Given K_1 and R_1 we can obtain z_1 and repeat indefinitely, constructing a sequence $\{z_t, K_t\}_{t=0}^{\infty}$ that depends on K_0 as a unique parameter (assuming that households know α and the coefficients from the law of motion); given the infinite length of the sample and the stationarity of z, we can estimate K_0 with arbitrary precision. From this sequence it is straightforward to obtain $\left\{MPN_t, \varepsilon_t^i\right\}_{t=0}^{\infty}$ given observations on $\left\{w_t^i\right\}_{t=0}^{\infty}$. Thus, while this economy still has more underlying states than signals, it implies full information in the limit.[20] If instead $\sigma_\nu = 0$ simply switching the roles of w_t^i and R_t^i in the discussion suffices to prove the assertion.[21]

[19] There may be a connection between our result and the notion of asymptotic invertibility from Baxter, Graham, and Wright (2008), but we have not attempted to make it here.

[20] Our result is the first one that we are aware of that implies more states than signals still leads to complete revelation; the literature on incomplete information revelation typically focuses on the case where the number of signals equals the number of states.

[21] If labor supply is elastic, the result will still hold provided total labor input is an invertible function of K_t.

We find it helpful here to point out that the degeneration of beliefs will not occur if there is another asset in the economy. For concreteness, suppose there exists a risk-free bond with price q_t. Following Krusell and Smith (1997), suppose we conjecture that the bond price takes the form

$$q_t = b_0 + b_1 z_t + b_2 \log(K_t).$$

Then observations on the bond price would permit the households to unravel the exact time series for $\{z_0, K_0\}$. In this case, to preserve informational heterogeneity we would need to add another shock (such as a shock to bond supply, a common tool in noise trader models and used in Krusell, Smith, and Young 2009).

3.2. Cross-Sectional Beliefs

In this subsection, we compare cross-sectional beliefs in the PI economy with the FI economy, focusing on two characteristics: the cross-sectional average and cross-sectional variance of beliefs. For a given period t, the first variable measures the average perception of agents over the states of the economy while the second measures the dispersion of their perceptions. We show that unobservable common state variables always induce a bias in cross-sectional average of beliefs and heterogeneity in individuals' forecasts. To illustrate, we digress and discuss a simple filtering problem with similar properties.

3.2.1. A Simple Filtering Problem

Consider the linear state-space system

$$\begin{aligned} y_{t+1} &= \rho y_t + e_{t+1} \\ x_t^i &= y_t + \varepsilon_t^i \\ e_{t+1} &\sim \text{iid } N\left(0, \sigma_e^2\right), \varepsilon_{t+1}^i \sim \text{iid } N\left(0, \sigma_\varepsilon^2\right) \end{aligned} \tag{3.1}$$

where y_t is a common unobservable variable and x_t^i is individual i's measurement of y_t corrupted with classical measurement error; we assume $|\rho| < 1$ to guarantee the filter converges to a steady state. Given prior $y^i_{t|t} \sim N\left(\overline{y}^i_{t|t}, p\right)$ we can write

$$\begin{bmatrix} y^i_{t+1|t} \\ x^i_{t+1|t} \end{bmatrix} \sim N\left(\begin{bmatrix} \rho\overline{y}^i_{t|t} \\ \rho\overline{y}^i_{t|t} \end{bmatrix}, \begin{bmatrix} p+\sigma_e^2 & p+\sigma_e^2 \\ p+\sigma_e^2 & p+\sigma_e^2+\sigma_\varepsilon^2 \end{bmatrix}\right)$$

where p is steady state variance of the filter. After observing x^i_{t+1}, the posterior is given by

$$\begin{aligned} y^i_{t+1|t+1} &\sim N\left\{\rho y^i_{t|t} + \beta\left(x^i_t - \rho\overline{y}^i_{t|t}\right), \left(p+\sigma_e^2\right) - \frac{\left(p+\sigma_e^2\right)^2}{p+\sigma_e^2+\sigma_\varepsilon^2}\right\}, \\ \beta &= \frac{p+\sigma_e^2}{p+\sigma_e^2+\sigma_\varepsilon^2}. \end{aligned}$$

Therefore, the dynamics of i's belief are given by

$$\overline{y}^i_{t+1|t+1} = \rho\overline{y}^i_{t|t} + \beta\left(x^i_{t+1} - \rho\overline{y}^i_{t|t}\right), \tag{3.2}$$

where $\beta \in [0,1]$ is the Kalman gain and where $\overline{y}^i_{t|t}$ is the expectation by agent i about y_t given $\left\{x^i_\tau\right\}_{\tau \leq t}$. We define two operators for convenience: $\mathbf{E}^i$ is the cross-sectional expectation operator and $\mathbf{V}^i$ is the cross-sectional variance operator. We have the following propositions that establish some key properties of the cross-section of beliefs about the unobserved state.[22]

Proposition 3.2. *The cross-sectional distribution of beliefs $\overline{y}^i_{t+1|t+1}$ is*

$$\begin{aligned} \overline{y}^i_{t+1|t+1} &\sim N\left(\mathbf{E}^i\left[\overline{y}^i_{t+1|t+1}\right], \mathbf{V}^i\left[\overline{y}^i\right]\right) \\ \mathbf{E}^i\left[\overline{y}^i_{t+1|t+1}\right] &= (1-\beta)\,\rho\mathbf{E}^i\left[\overline{y}^i_{t|t}\right] + \beta\left(\rho y_t + e_{t+1}\right) \\ \mathbf{V}^i\left[\overline{y}^i\right] &= \frac{\beta^2\sigma_\varepsilon^2}{1-(1-\beta)^2\rho^2}. \end{aligned} \tag{3.3}$$

Proof *Substituting out x^i_{t+1} from (3.2) and rearranging we obtain the individual belief*

$$\overline{y}^i_{t+1|t+1} = (1-\beta)\,\rho\overline{y}^i_{t|t} + \beta y_{t+1} + \beta\varepsilon^i_{t+1}.$$

Applying $\mathbf{E}^i[\cdot]$ and $\mathbf{V}^i[\cdot]$ to both sides, we get the process of the cross-sectional average of beliefs

[22] We are unaware of any results in the filtering literature regarding the properties of the cross-sectional average of filtered variables.

as above and

$$\mathbf{V}^i\left[\overline{y}^i_{t+1|t+1}\right] = (1-\beta)^2\rho^2\mathbf{V}^i\left[\overline{y}^i_{t|t}\right] + \beta^2\sigma^2_{\varepsilon}.$$

Since $(1-\beta)^2\rho^2 < 1$, there exists a time-independent variance of $\overline{y}^i_{t|t}$ given by (3.3). ε^i_t is normally distributed across i and y_{t+1} is fixed for all i. Thus the distribution of $\overline{y}^i_{t|t}$ converges to a normal distribution.

Proposition 3.3. *The average belief about the common state variable y_t is biased almost surely even with a continuum of individuals.*

Proof *Subtracting y_{t+1} from both sides of (3.2) we obtain*

$$\begin{aligned}
\overline{y}^i_{t+1|t+1} - y_{t+1} &= \rho\overline{y}^i_{t|t} + \beta\left(x^i_{t+1} - \rho\overline{y}^i_{t|t}\right) - y_{t+1} \\
&= (1-\beta)\rho\left(\overline{y}^i_{t|t} - y_t\right) + \beta\varepsilon^i_{t+1} - (1-\beta)e_{t+1}.
\end{aligned}$$

Applying $\mathbf{E}^i[\cdot]$ on both sides yields

$$\mathbf{E}^i\left[\overline{y}^i_{t+1|t+1}\right] - y_{t+1} = (1-\beta)\rho\left(\mathbf{E}^i\left[\overline{y}^i_{t|t}\right] - y_t\right) - (1-\beta)e_{t+1}.$$

The RHS is a measure of the cross-sectional bias; since e_{t+1} is not zero almost surely, this estimate is almost surely biased.

If y_t is observable, the dynamics of $\mathbf{E}^i\left[\overline{y}^i_{t|t}\right]$ will coincide with y_t, leading to homogenous beliefs about y_t. In contrast, Propositions (3.2) and (3.3) show that unobservable y_t generates heterogenous beliefs $\overline{y}^i_{t|t}$ that are *biased* on average, even with a continuum of observers. We can also establish one fact about the cross-section of beliefs about future observables.

Proposition 1. *The cross-sectional variance of the forecast $\overline{x}^i_{t+1|t}$ is given by*

$$\mathbf{V}^i\left[\overline{x}^i_{t+1|t}\right] = \frac{\beta^2\sigma^2_{\varepsilon}}{1-(1-\beta)^2} + \sigma^2_e.$$

Proof Using $\mathbf{V}^i\left[\overline{y}^i\right]$ in (3.3) and the measurement equations, we obtain immediately

$$\mathbf{V}^i\left[\overline{x}^i_{t+1|t}\right] = \frac{\beta^2\sigma^2_{\varepsilon}}{1-(1-\beta)^2} + \sigma^2_e.$$

If $\sigma_\varepsilon^2 \to 0$ (no measurement error) then $\beta \to 1$ and $\mathbf{V}^i\left[\overline{x}^i_{t+1|t}\right] \to \sigma_e^2$, the conditional variance of y_{t+1}; it is easy to see that $\mathbf{V}^i\left[\overline{x}^i_{t+1|t}\right]$ is increasing in σ_ε^2, so that unobservable y_t increases the heterogeneity in forecasts $\overline{x}^i_{t+1|t}$.

3.2.2. Beliefs in the PI Economy

In the previous section, we illustrated that an unobservable common state variable induces bias and heterogeneity in cross-sectional beliefs. Here we show that the same effects occur in the PI economy. Denote the vectors of state and observed variables by $\mathbf{Y}^i_t = [z_t \;\; \varepsilon^i_t \;\; \eta^i_t \;\; \log(K_t)]^T$ and $\mathbf{X}^i_t = \left[\log\left(R^i_t\right) \;\; \log\left(w^i_t\right)\right]^T$, respectively. Using matrix notation, we can write the state-space system (2.9) as

$$\begin{aligned} \mathbf{Y}^i_{t+1} &= \mathcal{C} + \mathcal{D}\mathbf{Y}^i_t + \mathbf{\Psi}^i_{t+1}, \\ \mathbf{X}^i_t &= \mathcal{E} + \mathcal{F}\mathbf{Y}^i_t, \\ \mathbf{\Psi}^i_{t+1} &\sim N\left(\mathbf{0}, \mathbf{\Sigma}\right), \end{aligned} \tag{3.4}$$

where $\mathbf{\Psi}^i_{t+1} = \left[e_{t+1} \;\; \upsilon^i_{t+1} \;\; \zeta^i_{t+1} \;\; 0\right]^T$. The following propositions characterize the cross-sectional beliefs in the PI economy.

Proposition 3.4. *Let $\overline{\mathbf{Y}}^i_{t|t}$ be i's expectation of $\mathbf{Y}^i_t$ given observation $\left\{\mathbf{X}^i_\tau\right\}_{\tau \le t}$. Define the covariance matrix of $\mathbf{Y}^i_{t+1|t}$ as $\widehat{\mathbf{P}} = \mathcal{D}\mathbf{P}\mathcal{D}' + \mathbf{\Sigma}$, where $\mathbf{P}$ is the convergent covariance matrix of $\mathbf{Y}^i_{t|t}$ from the Kalman filter. The recursive expectation of $\mathbf{Y}^i_t$ is*

$$\overline{\mathbf{Y}}^i_{t+1|t+1} = \mathcal{C} + (\mathcal{I} - \mathcal{K})\,\mathcal{D}\overline{\mathbf{Y}}^i_{t|t} + \mathcal{K}\mathcal{D}\mathbf{Y}^i_t + \mathcal{K}\mathbf{\Psi}^i_{t+1}, \tag{3.5}$$

where $\mathcal{K} = \widehat{\mathbf{P}}\mathcal{F}'\left(\mathcal{F}\widehat{\mathbf{P}}\mathcal{F}'\right)^{-1}\mathcal{F}$ and $\mathcal{I}$ is the conformable identity matrix. The cross-sectional belief $\overline{\mathbf{Y}}^i_{t|t}$ has a joint-normal distribution,

$$\begin{aligned} \overline{\mathbf{Y}}^i_{t|t} &\sim N\left(\mathbf{E}^i\left[\overline{\mathbf{Y}}^i_{t|t},\right], \mathbf{V}^i\left[\overline{\mathbf{Y}}^i_{t|t}\right]\right), \\ \mathbf{E}^i\left[\overline{\mathbf{Y}}^i_{t|t}\right] &= \mathcal{C} + (\mathcal{I} - \mathcal{K})\,\mathcal{D}\mathbf{E}^i\left[\overline{\mathbf{Y}}^i_{t-1|t-1}\right] + \mathcal{K}\mathcal{D}\mathbf{E}^i\left[\mathbf{Y}^i_{t-1}\right] + \mathcal{K}_{\circ 1}e_t, \\ \mathbf{V}^i\left[\overline{\mathbf{Y}}^i_{t|t}\right] &= (\mathcal{I} - \mathcal{K})\,\mathcal{D}\mathbf{V}^c\left[\overline{\mathbf{Y}}^i_{t-1|t-1}\right]\mathcal{D}^T(\mathcal{I} - \mathcal{K})^T + \mathbf{Q}, \end{aligned} \tag{3.6}$$

where $X_{\circ 1}$ denotes the first column of X, $\mathbf{Q}$ is a constant matrix, and $\mathbf{\Sigma}_Y$ is the cross-sectional variance of $\mathbf{Y}^i_t$.

Proposition 3.5. *The cross-sectional belief* $\mathbf{E}^i\left[\overline{\mathbf{Y}}^i_{t|t}\right]$ *in the PI economy is biased almost surely.*

The proofs of Propositions (3.4) and (3.5) follow the same steps as in the above simple filtering problem and are thus relegated to Appendix D. As shown in the simple problem, the presence of a common state variable produces *biased* average beliefs in the PI economy. Equation (3.6) shows that aggregate shock e_t is the driving force in the stochastic process of average beliefs. Thus, any economy with private information and an aggregate shock will produce biased beliefs on average. The convergence of $\mathbf{V}^i\left[\overline{\mathbf{Y}}^i_{t|t}\right]$ depends on the eigenvalues of the matrix $(\mathcal{I}-\mathcal{K})\,\mathcal{D}$. Due to nonlinearity, we cannot show that $\mathbf{V}^i\left[\overline{\mathbf{Y}}^i_{t|t}\right]$ converges in general. However, we can argue that $\mathbf{V}^i\left[\overline{\mathbf{Y}}^i_{t|t}\right]$ must remain finite. i' beliefs can drift away from the average belief only if the observations $\left(\log\left(R^i_t\right), \log\left(w^i_t\right)\right)$ drift away from their cross-sectional averages. Since both ε^i_t and η^i_t are mean-reverting processes, this drifting cannot continue indefinitely. Thus $\mathbf{V}^i\left[\overline{\mathbf{Y}}^i_{t|t}\right]$ converges to a unique matrix or a limit cycle.[23]

Turning to the observables, we derive the expression for the cross-sectional variance of the forecasts.

Proposition 3.6. *Denote* $\overline{\mathbf{X}}^i_{t+1|t} = \left[E_t\left[\log\left(R^i_{t+1}\right)\right] \quad E_t\left[\log\left(w^i_{t+1}\right)\right]\right]^T$. *In the PI economy the cross-sectional variance of* $\overline{\mathbf{X}}^i_{t+1|t}$ *is*

$$\mathbf{V}^i\left[\overline{\mathbf{X}}^i_{t+1|t}\right] = \mathcal{F}\mathcal{D}\mathbf{V}^i\left[\overline{\mathbf{Y}}^i_{t|t}\right]\mathcal{D}^T\mathcal{F}^T,$$

and in the FI economy the cross-sectional variance of forecasts is

$$\begin{aligned}
\mathbf{V}^i\left(E_t\left[\log\left(R^i_{t+1|t}\right)\right]\right) &= \rho^2_\eta\frac{\sigma^2_\zeta}{1-\rho^2_\eta}, \\
\mathbf{V}^i\left(E_t\left[\log\left(w^i_{t+1|t}\right)\right]\right) &= \rho^2_\varepsilon\frac{\sigma^2_\upsilon}{1-\rho^2_\varepsilon}.
\end{aligned}$$

Proof *The cross-sectional variance in the PI economy is derived by applying* $\mathbf{V}^i\left[\overline{\mathbf{Y}}^i_{t|t}\right]$ *to the measurement equations. In the FI economy, agents observe both* z_t *and* K_t *and thus have homogenous forecasts for* MPK_{t+1} *and* MPN_{t+1}. *Since* $\log\left(R^i_{t+1}\right) = \log\left(MPK_{t+1}\right) + \eta^i_{t+1}$ *and* $\log\left(w^i_{t+1}\right) = \log\left(MPN_{t+1}\right) + \varepsilon^i_{t+1}$, *the cross-sectional variance of FI forecasts is determined by the cross-sectional variance of* ε^i_{t+1} *and* η^i_{t+1} *only.*

[23] In all of our numerical experiments it converged to a unique matrix.

Proposition (3.6) shows that any heterogeneity in forecasts in the FI economy depends only on the exogenous idiosyncratic shocks. On the contrary, in the PI economy there is endogenous heterogeneity in the forecasts induced by unobservable state variables. Unlike the simple filtering problem, however, we cannot analytically show that the cross-section variance of forecasts in the PI economy is higher than that in the FI economy, since it depends endogenously on the behavior of the aggregate capital stock through the matrices $\mathcal{D}$ and $\mathcal{F}$. Thus we defer further discussion to the next section after we present numerical results.

Our final comments in this section involve the law of motion for the aggregate capital stock. Under FI, our result that only K_t matters for the determination of K_{t+1} is parallel to that found in Krusell and Smith (1998), so we do not elaborate further on it. Under PI, however, the result is even stronger – forecasting is sufficiently accurate if one ignores not only higher moments of the distribution of wealth but also higher-order expectations. We show below that if the law of motion for $\log(K_t)$ is independent of first-order expectations, then endowing one agent with $\mathbf{E}^i\left[\overline{\mathbf{Y}}^i_{t|t}\right]$ (their true values, not an individual's belief about them) does not improve her prediction about $\log(K_{t+1})$. This result is a form of approximate aggregation that may open an even wider class of models to study.

4. Quantitative Results

This section of the paper is divided into five subsections. The first subsection discusses the calibration and computation of the FI and PI models. In the second subsection we compare the aggregate dynamics of the two economies; our interest here is determining whether the assumption of homogeneous information is critically important for the standard battery of business cycle statistics. The third subsection discusses the differences in individual behavior that give rise to the different aggregate dynamics – essentially, we ask what makes a PI agent different from an FI agent. In the fourth subsection we discuss the wealth distributions produced by the two economies. Finally, in the fifth subsection we discuss the connections between average expectations of PI agents and empirical measures of consumer confidence and expectations.

4.1. Calibration and Computation

We assume one model period corresponds to one quarter. The felicity function $u(c)$ is chosen to be $\log(c)$, so that relative risk aversion equals one. The chosen parameters of the model are $\beta = 0.99$,

$\alpha = 0.36$, $\delta = 0.0217$, and $\overline{h} = 0.3271$; these values yield aggregate outcomes generally consistent with US data on capital/output and investment/output ratios and capital's share of income. The log of technology shock z_t is estimated from the annual series of GDP and tangible assets from the National Income and Product Accounts and then converted into a quarterly process, yielding $\rho_z = 0.96429$ and $\sigma_e^2 = 0.0071^2$. We use the estimates of Storesletten, Telmer, and Yaron (2004) for the ε process, although for convenience we ignore the countercyclical variance movements that is the main focus of their paper. For the process for η we have little information – we therefore assume that the process has $\rho_\eta = 0$ and $\sigma_\zeta^2 = 0.05$; consistent with our interpretation of η as reflecting poorly-diversified portfolios, it seems natural to assume that they are not persistent errors. We also present a discussion of how our results depend on ρ_η.

Appendix E presents an extensive discussion of our solution method. We use monomial rules adapted for normal random variables to compute the integrals and a combination cubic-linear spline interpolation scheme to evaluate the value functions.[24] We solve the household problem using Brent's method and Newton-Raphson iteration on the first-order condition, depending on how close we are to the borrowing constraint. The details of the simulation are also contained in Appendix E; briefly, we use a cross-section of $100,000$ households to approximate the continuum and use Monte Carlo methods to generate the shocks.[25]

4.2. Aggregate Dynamics

In this section, by comparing FI and PI economy, we will show that relaxing the full information assumption significantly changes the aggregate dynamics of our economy. The equilibrium law of

[24]Fortran code to solve the model is available upon request. Typically the code takes several days to converge, even when executed in parallel, and is likely to be unstable for poor initial guesses. In addition, the grids often need to be adjusted to ensure convergence. We make no guarantees about the programs working for parameter values that we ourselves have not examined (such has significantly higher risk aversion, for example).

[25]The method used in Young (2007) would be preferable, but it is too computationally burdensome because agents differ along 5 dimensions and the beliefs cannot be restricted to the points in a finite-state approximation to the shock processes. Our results are not sensitive to increasing the number of agents in the cross-section to $800,000$, except that some graphs are smoothed out (particularly aggregate consumption).

motion of K_t and forecasting rules for the two economies are

$$\begin{aligned}
\log(K_{t+1}) &= 0.0907 + 0.0591 z_t + 0.9699 \log(K_t) \qquad (4.1)\\
E_t\left[\log\left(R^i_{t+1}\right)\right] &= -0.1375 + 0.9542 \log\left(R^i_t\right) - 0.0278 \log\left(w^i_t\right) + 0.0278 \varepsilon^i_t - 0.9542 \eta^i_t\\
E_t\left[\log\left(w^i_{t+1}\right)\right] &= 0.0402 + 0.0056 \log\left(R^i_t\right) + 0.9799 \log\left(w^i_t\right) - 0.0369 \varepsilon^i_t - 0.0056 \eta^i_t\\
V_t\begin{bmatrix} \log\left(R^i_{t+1}\right) \\ \log\left(w^i_{t+1}\right) \end{bmatrix} &= \begin{bmatrix} 0.0025 & 0.0071^2 \\ 0.0071^2 & 0.0283 \end{bmatrix}
\end{aligned}$$

for the FI economy and

$$\begin{aligned}
\log(K_{t+1}) &= 0.0713 + 0.0752 z_t + 0.9768 \log(K_t) \qquad (4.2)\\
E_t\left[\log\left(R^i_{t+1}\right)\right] &= -0.1248 + 0.9550 \log\left(R^i_t\right) - 0.0388 \log\left(w^i_t\right) + 0.0388 \overline{\varepsilon}^i_{t|t} - 0.9550 \overline{\eta}^i_{t|t}\\
E_t\left[\log\left(w^i_{t+1}\right)\right] &= 0.0330 + 0.0052 \log\left(R^i_t\right) + 0.9861 \log\left(w^i_t\right) - 0.0431 \overline{\varepsilon}^i_{t|t} - 0.0052 \overline{\eta}^i_{t|t}\\
V_t\begin{bmatrix} \log\left(R^i_{t+1}\right) \\ \log\left(w^i_{t+1}\right) \end{bmatrix} &= \begin{bmatrix} 0.0028 & 0.0068^2 \\ 0.0068^2 & 0.0283 \end{bmatrix}
\end{aligned}$$

for the PI economy, where V_t is the conditional variance operator.

These laws of motion are conditional on the Assumptions (2.3) and (2.4); accuracy tests are needed to verify whether these assumptions are reasonable. If we compute R^2 for the laws of motion for K_t, in both cases it exceeds 0.99998. Given the arguments in den Haan (2007), we prefer to judge the approximation based on the maximum error of the law of motion in our simulation of 5000 periods: these errors are 1.83×10^{-4} and 5.68×10^{-4} in the FI and PI economies, respectively. Finally, we assess how well one PI agent would forecast aggregate capital using the first-order expectations about the idiosyncratic shocks, yielding the regression

$$\log(K_{t+1}) = 0.1179 + 0.0098 z_t + 0.9617 \log(K_t) + 0.0573 \mathbf{E}^i\left[\overline{\varepsilon}^i_{t|t}\right] - 0.0037 \mathbf{E}^i\left[\overline{\eta}^i_{t|t}\right];$$

the maximum error for this regression is 2.71×10^{-4}. Although adding $\mathbf{E}^i\left[\varepsilon^i_{t|t}\right]$ and $\mathbf{E}^i\left[\eta^i_{t|t}\right]$ does reduce the maximum error, the size of error is already very small; furthermore, the coefficients on these additional terms are not particularly large (nor are the terms themselves). Thus the computational benefit from removing $\mathbf{E}^i\left[\overline{\varepsilon}^i_{t|t}\right]$ and $\mathbf{E}^i\left[\overline{\eta}^i_{t|t}\right]$ from the law of motion, and consequently

PI agents' state variables, should outweigh the cost in terms of computational accuracy.[26]

From (4.1) and (4.2), we first note that aggregate capital is more sensitive to the technology shock in the PI economy; we detail in the next subsection the reasons that underlie both the higher capital stock and the heightened sensitivity observed in the PI economy. Second, PI agents face larger one-period risk in returns, although this effect is small in our calibration. Given that the covariance matrix for the FI agent is independent of the law of motion for K_t, it is immediate that the second effect is driven entirely by the filtering problem of the PI agent; that is, the higher risks faced by PI agents do not have a general equilibrium component.

Table 1 presents the standard battery of business cycle statistics – standard deviations and cross-correlations – for a simulation of 5000 periods.[27] The notable differences between the two economies are bolded in the table. Mean aggregate activity measures (output, consumption, capital, investment) are uniformly higher in the PI economy. The volatility of all variables except consumption is also higher in the PI economy, reflecting the higher coefficient on z_t in the law of motion for K_t. With respect to correlations, there are two key differences. First, in the FI economy aggregate consumption has a higher contemporaneous correlation with aggregate output and the aggregate technology shock than in the PI economy. Second, the cross correlations of MPK with other aggregates are uniformly lower in the PI economy; in particular, the correlation between returns and wages goes from positive (but close to zero) in the FI economy to negative in the PI economy. This result is important for understanding why agents accumulate more capital, and we discuss it below in more detail.

The lower panel in Figure (1) plots the aggregate capital stocks from a simulation for both the FI and PI economy; in the PI economy we find that K_t is uniformly higher and this difference increases when technology increases and decreases when z_t falls; the correlation between $\left(K_t^{PI} - K_t^{FI}\right)$ and z_t is 0.49. Due to the higher sensitivity of capital accumulation to the technology shock, swings in K_t are exaggerated in the PI economy; despite the fact that these swings are larger, the higher

[26]Ideally we would compute another PI economy where we assume the law of motion takes the form (2.13) and includes the first order expectations $\left(\mathbf{E}^i\left[\varepsilon^i_{t|t}\right], \mathbf{E}^i\left[\eta^i_{t|t}\right]\right)$ as state variables. If the results are still the same, we can claim that Assumptions (2.3) and (2.4) are approximately sufficient. The key difference between this test and the one in the main text is that behavior is not being held constant here. Unfortunately, the computational burden is very large for the alternative test and we cannot perform it with current technology. The results in Young (2007) suggest it would not matter – behavior is generally invariant with respect to the process generating forecasts of aggregate capital because it moves too little and too slowly.

[27]The initial distribution is the stationary distribution from a model without aggregate shocks; 1000 periods are dropped before statistics are computed, so the total length of the simulation is 6000 periods. The series are not filtered.

average value prevents capital in the PI economy from dropping below that in the FI economy (our plot is entirely representative and is truncated only for the convenience of presentation, as the two series never switched in a very long simulation).

Figure (2) shows the time path of aggregate consumption and investment in the two economies; one thing that stands out is the 'choppiness' of aggregate consumption in the FI economy relative to the PI economy. Figure (3) decomposes aggregate consumption into three bands – cycles with periods between 2 and 6 quarters, between 6 and 32 quarters, and above 32 quarters. The plot shows that the FI economy has significantly more high and medium frequency variation than the PI economy; we also observe a phase shift at very low frequencies, where the PI aggregate consumption lags behind, although the variances very similar over this band. The delay in the response of consumption accounts for the lower contemporaneous correlation between aggregate consumption and output in the PI economy. A similar decomposition for aggregate investment – not shown for brevity – implies that investment is more volatile in the PI economy across all bands and displays only a small phase shift at low frequencies.

4.3. Dynamics of Cross-sectional Beliefs

In Section (**??**), we showed that the unobservable z_t generates endogenous heterogeneity in beliefs and induces bias in the average belief. In order to assess how important these effects are, we report the quantitative results from our calibrated models. A quick summary of the results of these experiments is that belief dispersion is not significantly higher in the PI economy relative to the FI economy but the PI economy does display a pronounced bias.

The dynamics of beliefs are fully characterized by Equation (3.6); in the calibrated equilibrium the coefficients for the matrices are

$$\mathcal{K} = \begin{bmatrix} 0.0241 & 0.0033 & 0.0208 & -0.0121 \\ 1.0251 & 0.9956 & 0.0295 & 0.3396 \\ 0.8883 & -0.0014 & 0.8898 & -0.5700 \\ -0.1368 & 0.0029 & -0.1397 & 0.0905 \end{bmatrix}$$

$$\mathcal{KD} = \begin{bmatrix} 0.0224 & 0.0031 & 0.0000 & -0.0118 \\ 1.0140 & 0.9389 & 0.0000 & 0.3317 \\ 0.8137 & -0.0014 & 0.0000 & -0.5568 \\ -0.1251 & 0.0028 & 0.0000 & 0.0884 \end{bmatrix}$$

$$(\mathcal{I}-\mathcal{K})\,\mathcal{D} = \begin{bmatrix} 0.9419 & -0.0031 & 0.0000 & 0.0118 \\ -1.0140 & 0.0041 & 0.0000 & -0.3317 \\ -0.8137 & 0.0014 & 0.0000 & 0.5568 \\ 0.2003 & -0.0028 & 0.0000 & 0.8885 \end{bmatrix}.$$

All eigenvalues of $(\mathcal{I}-\mathcal{K})\,\mathcal{D}$ are less than one, implying there exists a convergent covariance matrix $\mathbf{V}^i\left[\overline{\mathbf{Y}}^i_{t|t}\right]$ and validating our assumption. Table 2 reports the cross-section standard deviation in beliefs and forecasts; the dispersion of expectations in PI and FI economies is only slightly different. Obviously, for the FI agent there can be no dispersion in beliefs about (z_t, K_t), since these are common variables; thus, the PI agent accommodates dispersion in beliefs about those variables by reducing (slightly) the dispersion in beliefs about the idiosyncratic shocks. With respect to forecasts, the largest difference occurs in the cross-sectional beliefs about $\log\left(\overline{R}^i_{t+1|t}\right)$ rather than $\log\left(\overline{w}^i_{t+1|t}\right)$. To understand why, we note that the difference in the variance-covariance matrix of the forecast errors, $\Xi = \mathbf{B}_3\overline{\mathbf{P}}\mathbf{B}_3^T$, has the following entries:

$$\begin{aligned}
\mathbf{\Xi}(1,1) &= A_2^2 p_{11} + 2A_2\left(A_1-\rho_\eta\right)p_{12} + \left(\rho_\eta - A_1\right)^2 p_{22} \\
\mathbf{\Xi}(1,2) &= -A_2\left(\rho_\varepsilon - A_5\right)p_{11} + \left(\left(\rho_\eta - A_1\right)\left(\rho_\varepsilon - A_5\right) + A_2A_4\right)p_{12} - A_4\left(\rho_\eta - A_1\right)p_{22} \\
\mathbf{\Xi}(2,2) &= \left(\rho_\varepsilon - A_5\right)^2 p_{11} - 2A_4\left(\rho_\varepsilon - A_5\right)p_{12} + A_4^2 p_{22},
\end{aligned}$$

where p_{ij} is the (i,j)th element of $\overline{\mathbf{P}}$. Both $\mathbf{\Xi}(1,2)$ and $\mathbf{\Xi}(2,2)$ are small, meaning that the consequences of private information for dispersion in wage beliefs is limited.[28] But $\mathbf{\Xi}(1,1)$ is

[28]The key for this result is that ρ_ϵ, ρ_z, and a_2 are all relatively close to each other, meaning that filtering does not

relatively large because A_1 is close to 1 and $\rho_\eta = 0$, leading to larger effects on the beliefs about returns.[29] This result is a key one for our investigation; if we want dispersed information to matter, it must involve filtering processes that have very different persistence.

Figure (4) and (5) plot the cross-sectional average of beliefs: $\mathbf{E}^i\left[\overline{z}^i_{t|t}\right]$, $\mathbf{E}^i\left[\log\left(\overline{K}^i_{t|t}\right)\right]$, $\mathbf{E}^i\left[\varepsilon^i_{t|t}\right]$, and $\mathbf{E}^i\left[\eta^i_{t|t}\right]$; the plots show persistent biases in all variables. Both beliefs about aggregates are smoother than their realized counterparts, reflecting the tendency for agents to attribute movements in prices to idiosyncratic components rather than aggregates. In addition, $\mathbf{E}^i\left[\log\left(\overline{K}^i_{t|t}\right)\right]$ reacts with a delay relative to actual $\log(K_t)$. If we look instead at the realized marginal products, we see from Figure (6) that beliefs about $\log(MPN_t)$ are very smooth relative to the true value, while beliefs about $\log(MPK_t)$ display significant fluctuations. The smoothness in $\mathbf{E}^i\left[\log\left(MPN^i_{t|t}\right)\right]$ manifests itself in the relatively large bias found in $\mathbf{E}^i\left[\varepsilon^i_{t|t}\right]$; since ε^i is a relatively large shock, agents attribute almost all movements in wages to idiosyncratic shocks.

To understand how the average beliefs responds to an unobserved aggregate shock, we apply a single technology impulse to the PI economy.[30] Figure (8) displays the impulse response dynamics in the PI economy. Realized z_t jumps up immediately and converges monotonically to the steady state, consistent with the AR(1) process. However, average estimated z_t does not – it jumps up only by a small amount – since agents rationally attribute most of the increase in wages to ε^i. Average estimated K_t falls, the opposite direction from the realized K_t, and remains persistently below average for many periods before overshooting and returning to the mean. In passing we also note a key difference between our model and Graham and Wright (2007) that deserves some future attention – in their model an aggregate productivity shock drives down aggregate consumption, in direct contrast to both standard models and empirical evidence, whereas our model predicts an increase.

To understand the impulse response paths, note that when the shock hits PI agents observe a

distort beliefs too much.

[29]We have

$$A_1 = \rho_z \alpha + (1-\alpha)(a_2 - \alpha a_1);$$

since $\rho_z \approx a_2$ and $a_1 \approx 0$ we get

$$A_1 \approx a_2$$

which is close to 1. This result holds in a wide class of economies, since it relies only on persistence in capital movements and aggregate technology shocks.

[30]We consider an PI (FI) economy currently in the stationary state – that is, $z_t = 0$ for a long time so that the distribution has settled down to some stationary one. Due to precautionary savings this distribution is not the one produced by a model without aggregate shocks, although it typically is close. We then hit this economy with an increase in e_t of 2.5 standard deviations; the size of the shock is designed to make the graphs easier to read and does not materially affect the results (because the model is close to linear).

rise in both R_t^i and w_t^i. We can decompose the observed prices into the aggregate and idiosyncratic components:

$$\log\left(R_t^i\right) = \log\left(MPK_t\right) + \eta_t^i,$$
$$\log\left(w_t^i\right) = \log\left(MPN_t\right) + \varepsilon_t^i.$$

Since σ_ε and σ_η are much larger than σ_z, when a PI agent observes an unexpected rise in R_t^i and w_t^i the correct inference is that the change is mostly due to a rise in both η_t^i and ε_t^i; that is, they underweight the common component driven by z_t.[31] As a result, average estimated MPK_t and MPN_t will move less than the realized one and the shortfall is made up in the increase in average estimated ε_t^i and η_t^i; since PI agents do not observe these averages, they cannot revise their estimates to eliminate the bias. The smaller change in average estimated MPN_t, compared to MPK_t, is due to our calibration that σ_ε is much larger than σ_η.

There is also another channel that generates confusion for PI agents regarding the current state of the world. New observations not only convey the information about the new innovations, they also have a retrospective effect. A PI agent realizes that her prior might be incorrect, so she will always revise her previous prior once she observes more information (namely the realizations of R_t^i and w_t^i).[32] In this model, PI agents respond to technology shocks by revising downward their current estimate of K_t (on average) even though they know that it cannot adjust today (it is hard to see in Figure (8) because the revision is very small). This downward revision produces a decline in $MPN_{t|t}^i$ and therefore an upward adjustment in $\varepsilon_{t|t}^i$.

We conclude this subsection with some comments on robustness. Our results are driven by the fact that idiosyncratic shocks are much larger than aggregate ones and that return shocks are not persistent. We have noted above that relaxing the second one leads to smaller effects, so we have in a sense provided an upper bound on the role of information heterogeneity in this class of models.

4.4. Average Biases and Difference in Aggregate Dynamics

In Section (**??**), we noted that moving from FI economy to PI economy will have three effects. First, agents will perceive higher risk in both R_t^i and w_t^i. Second, endogenous heterogeneity in

[31] Lucas (1972) contains a similar mechanism – in that model, agents observe the product of an aggregate and island-specific shock to prices and cannot differentiate between the two. More recently Cooley and Hansen (1997) have revived this mechanism.

[32] This backward-looking revision is called smoothing in the Kalman filter literature.

expectations about the future arises. And third, there are biases in cross-sectional expectation of all current unobserved states. We also noted that there is not much difference in terms of perceived risks and heterogeneity in expectations between the two economies. Thus, the key difference between the behavior of the FI and PI economies must come through the beliefs about current states; the purpose of this subsection is to show how this difference generates different aggregate savings.

The savings behavior of agents in the PI economy differs from that in the FI economy for two reasons. First, PI agents have different information, and second, they face a different law of motion for the aggregate capital stock. We construct three experiments in order to separate the myriad of effects present in the model. Experiment (1) introduces into the economy a (measure zero) population of PI agents who take as fixed the PI filtering process and the law of motion for aggregate capital, but are not confused about the current state of the world (that is, each period we reset their beliefs to correspond to the true state). Experiment (2) introduces a (measure zero) population of FI agents who take as given the law of motion for aggregate capital from the PI economy (FI agents living in a PI world); these agents differ from PI agents in terms of their beliefs about the state of the world today and the distribution of possible states tomorrow. Experiment (3) computes the equilibrium of an economy populated entirely by the PI agents from Experiment (1) (they are not confused about the current state). Figure (**??**) plots the aggregate capital stock for the two basic economies – PI and FI – plus the aggregates constructed from these three experiments.

Comparing the results of Experiments (1) and (2), we see that their paths lie very close together. The difference between the two populations is simply their expectations about the state of the world tomorrow; as noted above, FI agents view next period's returns as less risky than the PI agents. We note that the difference is not mean zero, as FI agents always save less than their (not confused) PI counterparts. This difference is driven by precautionary savings motives and is model-specific; in the presence of background risk, increased return risk has an ambiguous effect on savings, particularly when the two are correlated.[33] Because the difference in expected risk is small, the difference between behavior is small as well.

We do not plot the time path from Experiment (3), since it lies on top of the path from the FI economy (they have the same law of motion out to at least three decimal places). The difference between Experiment (1) and Experiment (3) is that agents have different mean expectations (since

[33] See Gollier (2001).

capital evolves deterministically); this effect is large. Note that Experiment (1) and Experiment (2) do not lie on top of each other, as FI and Experiment (3) do, despite both of pairs of experiments involving agents who differ only in terms of their expectations about tomorrow (all of them know the state of the world today and also use the same law of motion for aggregate capital). There are minor differences in the variance-covariance matrices produced by Experiments (1) and (3), but these are of the same size as the differences between Experiment (3) and the FI economy, so they do not account for the gap.

Given that confusion about the current state is critical for generating the different behavior of the PI and FI economies, we now discuss why confusion matters. Figure (**??**) plots the expected profile of beliefs given an impulse to technology at time t; that is, these plots are not time paths but rather forecasts based only on information available at time t. Looking first at the forecasts of $\log\left(R_t^i\right)$, the PI agent believes that any deviation will have no persistence at all; thus, there is no incentive to save for tomorrow since returns will not be higher. In contrast, for the FI agent the persistently high returns driven by the aggregate shock do induce an additional incentive to save. Turning to wages, PI agents also perceive less persistence in $\log\left(w_t^i\right)$ compared to the FI forecast; intertemporal substitution motives for smoothing consumption then induce the PI agent to save more today relative to the FI agent.

There is also a subtle general equilibrium effect that tends to increase the saving of the PI agent. PI agents believe that high returns are idiosyncratic, so in periods where they want to buy assets there must be another agent who wants to sell. As a result, PI agents systematically underestimate the power of decreasing returns to scale at the aggregate level; that is, they foresee aggregate capital tomorrow being lower than it will turn out to be. In the calibrated equilibrium the motives that lead to higher saving dominate.

Thus, PI agents tend to save more at a point in time, accounting for the higher capital stock. To see why technology shocks affect the capital stock more strongly in the PI economy, we need to look at the behavior of agents tomorrow. When the PI agent arrives in the period after the technology shock, on average her wealth will be higher than expected. Savings functions in this model are increasing in wealth, meaning that next period the PI agent will have an incentive to save more again, leading to larger fluctuations in investment and capital.

Our result raises a concern in empirical econometrics. Confusion induced by unobserved state variables causes the PI agents, on average, to accumulate more capital despite facing the same underlying process for idiosyncratic risk. If an econometrician observes only data from the PI economy and incorrectly imposes the full information assumption, one would infer that PI agents are more averse to (background) risk, which is not true (their underlying preferences are the same; although their indirect preferences over intertemporal wealth gambles, measured as the curvature in the value function, are in fact different the size of the effect is very small). Thus, we caution empirical researchers to consider carefully the information sets of their households when estimating household-level parameters.[34]

4.5. Disparate Information and Wealth Concentration

We noted above that the model tended to concentrate belief heterogeneity in returns; heterogeneity of this sort has the potential to generate significant differences in asset holdings over time. Aiyagari (1994), Chamberlain and Wilson (2000), and Guvenen (2008) show how sensitive savings in the FI model is to small changes in expected returns – as the expected return on savings converges to the time rate of preference, savings becomes infinitely elastic (see Figure 12). Furthermore, as seen in the figure the effect is asymmetric: incrementing expected returns by some small amount increases savings more than decrementing the return by the same amount decreases it.[35] Thus, agents who receive good return shocks will save a lot, leading to a significant concentration of wealth among such agents. On the other hand, the presence of return shocks reduces the elasticity of the savings function. When asset returns are random, large positions in k lead to increased uncertainty about future wealth, discouraging continued accumulation; this effect is strengthened when return shocks are highly persistent.[36]

There are additional effects that are present in the model as well. First, the variance of wages rises by roughly 10 percent between the FI and PI economies; this increase in background risk will make agents appear more risk averse, leading to higher precautionary savings (particularly for poor agents). Furthermore, the correlation between returns and wages falls substantially, from basically zero to significantly negative. As a result, capital is going from an independent risk to a hedge against labor income risk, generating an increase in the demand for savings. The combination will tend to raise the savings of the poor substantially, generating a decline in average expected returns

[34] A related effect can be found in Young (2007) – assuming complete markets implies that the estimated coefficient of relative risk aversion is much smaller than the true one because agents seem willing to accept more variance in consumption.

[35] This asymmetry is a consequence of the concavity of the consumption function; Carroll (2004) provides the theoretical foundation for this concavity and we verify it numerically for our model.

[36] Because the return shocks are purely transitory, the effect on the savings function is very small and cannot be seen in Figure (12). We have numerically computed stationary versions of our model with more persistent return shocks and they display a marked decline in the convexity of the savings function.

and therefore cutting the savings of the rich.

To examine how the model sorts out these various effects quantitatively, we compare the wealth distributions in the FI and PI economies. To give the reader a reference point, in US data wealth is highly concentrated among only the top percentile of wealth distribution; Budría Rodríguez *et al.* (2002) report a Gini coefficient of wealth equal to 0.8 with 25 percent of the wealth (and 90 percent of the financial wealth) held by the top 1 percent of the wealth distribution. As is well known, the model we consider here under FI underpredicts the wealth concentration relative to the data because it fails to produce sufficient numbers of very rich agents (see Castañeda, Díaz-Giménez, and Ríos-Rull 2003 or Krusell and Smith 1998); what we are interested in is how the PI economy compares to this benchmark. Figure (13) shows the Lorenz curve and Gini coefficient for the FI and PI economies: there is actually a *lower* Gini coefficient in the PI economy. In addition, the Lorenz curve for the PI economy lies everywhere above the one for the FI economy, meaning that the PI economy both fewer poor and fewer rich households. The precautionary effects are much stronger than the asymmetric effect.

One modification that would enhance the strength of the asymmetric saving effect is to increase ρ_η. By making the return shocks more persistent, we make the household more willing to dramatically increase saving when expected returns are high. However, this increase also has the effect of shrinking the difference between FI and PI agents, because the agent becomes less interested in decomposing the sum of aggregate and idiosyncratic shocks when they have similar persistence. Thus, we do not foresee any simple modifications making heterogeneous information of the sort used here into a theory of wealth concentration. A more promising direction would be to introduce agents with superior information (nested information sets); it seems likely that agents with better information about returns would tend to accumulate more wealth (see Beker and Espino 2008 and the extensive survival literature). Of course, then we open up the issue of why those households possess better information; in keeping with the spirit of our model, it may be that some households have idiosyncratic shocks that are more correlated with, and thus more informative about, aggregate variables.

4.6. Consumer Confidence

We conclude the results section of the paper by drawing some connection between the expectations produced by our model and direct measures of consumer expectations. Figure (14) shows that there is a strong correlation between the Michigan Survey of Consumer Sentiment and an index of Industrial Production at business cycle frequencies (the correlation is 0.58).[37] The data show also that the sentiment measure is as volatile as IP.[38] Figure (**??**) shows that similar results are obtained using the Board of Governors Consumer Confidence Survey. Note: there is substantial cross-sectional variation in the Consumer Sentiment measure. We do not attempt here to confront our model with this cross-sectional information, which includes individual economic variables that may be predictive regarding expectations; rather, we merely note that there is heterogeneity in expectations evident in the data, and we would like to know whether we have captured the dynamics of the mean.

In Figure (15) we plot a measure of consumer sentiment from our model – the average expectation of output $\mathbf{E}^i\left[Y^i_{t|t}\right]$. This variable is considerably less volatile than output in the model and nearly orthogonal to actual output. Other variables from the model, such as the average expectation of aggregate consumption or investment, show similar patterns. As noted above, agents tend to attribute all movements in prices to idiosyncratic factors; as a result, they decouple beliefs about aggregates from their true values. Thus, it does not appear that our notion of beliefs captures the available empirical measures; for example, Barsky and Sims (2008) demonstrate that Consumer Confidence appears to be a manifestation of information about future productivity rather than an assessment about the current unobserved true state. Unfortunately, our model states that $\mathbf{E}^i\left[z^i_{t+1|t}\right]$ will behave in much the same way as $\mathbf{E}^i\left[Y^i_{t|t}\right]$, so the model will fall short along these lines. Introducing news shocks – shocks realized today that do not alter states until tomorrow – or noise shocks – shocks that have no resource implication – may be fruitful at mimicking the behavior of such expectational measures (see Blanchard, L'Huillier, and Lorenzoni 2009 for an empirical attempt to disentangle news and noise).

5. Conclusion

Our concluding comments will be directed at future research. Our model can be viewed as an attempt to integrate heterogeneity of beliefs (or expectations) into a macroeconomic model without relying on noise traders or rule-of-thumb agents. In the PI economy all heterogeneity in beliefs is endogenous and driven by optimal filtering of individual data. Our model is not as general

[37] Both series are taken from the FRED database and HP-filtered using a smoothing parameter of 144, 400.

[38] Danthine, Donaldson, and Johnsen (1998) is one attempt to connect consumer sentiment to business cycles through a 'peso problem' model of unfalsified expectations. The recent explosion of papers on 'news shocks' also represents attempts to find roles for consumer expectations in business cycle models; Sims (2009) provides an empirical case against such shocks being important.

as possible, of course, so it is important to investigate how robust our results are. Porapakkarm (2009) uses Epstein and Zin (1989) preferences to show that a key parameter is the intertemporal elasticity of substitution; when agents have low IES the differences between FI and PI are larger, and we intend to explore this result more thoroughly. Another extension would build on Saito (2005), who explores the problem of filtering between two aggregate shocks; if we interpret one shock as a Markov chain that shifts the process for idiosyncratic risk while the other is continuous and does not affect idiosyncratic risk, this model poses a different problem for partially-informed agents. Difficulty separating the sources of idiosyncratic risk may also play a role (as in Guvenen 2007), as might permitting more quantities to be observed in equilibrium, perhaps with noise (as in Bomfim 2001b, Aruoba 2006, and Porapakkarm 2009).

One substantive issue that we believe can fruitfully be examined in the context of our model is the trading volume puzzle (existing models do not match either the mean or volatility of asset turnover); with heterogeneous beliefs, agents are likely to trade more than the standard model would predict.[39] To explore the trading volume puzzle we would need to add additional assets into our economy, particularly a risk-free asset; that model would then shed some light on how risk-free debt would be priced in a world of incomplete information.[40] In fact, the model might even be useful for making predictions about which assets would be traded. One particularly interesting extension would introduce additional markets and then permit agents to receive signals only about prices in markets they are actively using (via search, for example); thus, households who are not attempting to find a job would not get any new information about their wage, leading to more heterogeneity in beliefs. That model would require that the filter be operated outside the steady state, however, and introduces nontrivial computational burden.[41]

Since our model predicts that less information leads to larger business cycles, it may be useful for understanding the so-called Great Moderation, the decline in aggregate volatility that has been observed since the early 1980s (see Fernández-Villaverde and Rubio-Ramírez 2007).[42] It is reasonable to think that information about the aggregate economy has been getting better,

[39] DeJong and Espino (2006) examines the trading volume predicted by a heterogeneous-agent economy with complete markets.

[40] Ongoing work (Krusell, Smith, and Young 2009) is examining the role of asymmetric information in an economy where risk-free debt is the only asset traded. Preliminary results based on a three-period model suggest that the price of risk-free debt can be significantly different than the full information benchmark; however, any approximate aggregation results that apply to that economy are more subtle than the one we uncover here, and preliminary results for an infinite-horizon version suggest higher-order expectations matter for bond pricing.

[41] Porapakkarm (2009) studies a life-cycle model in which retirees do not get signals from the labor market and information is not transmitted between generations.

[42] Of course, recent events may lessen the profession's interest in accounting for the Great Moderation.

both because the BEA is better at collecting and producing data and because information about aggregates diffuses more rapidly due to IT improvements. While a complete study of this issue is beyond the scope of this paper, our results do suggest that these informational improvements should reduce aggregate volatility, particularly in investment. The rising labor income risk measured by Krueger and Perri (2006) might work in the opposite direction, however, since it would imply that the compounding effects of confusion would get stronger as agents attach even less weight to the aggregate shock. A quantitative comparison of the two effects would be needed to determine which is stronger.

There is also a massive literature on the effect of sunspots – extrinsic uncertainty – to aggregate dynamics (Farmer 1999 is a good survey). Sunspots could in principle play an important role in our model because they could introduce heterogeneity into the state space representation of different agents. Whether agents could come to believe different state space representations, and whether those differences would matter, is unknown at this time but certainly of interest (the results in Carroll and Young 2009 suggest that multiple sunspots may be difficult to obtain). A related literature investigates the effect of rational inattention, either through limited information processing (Luo and Young 2009b) or costs of adjustment (Mankiw and Reis 2006). Since these environments would naturally deliver more heterogeneity, we hope that the current work will permit researchers to explore them further.

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A. Appendix A

In this appendix we show how to derive the price process for R_t^i and w_t^i used by an FI agent. Given competitive factor markets we can solve for z_t and $\log K_t$ by equating factor prices to marginal products:

$$\begin{aligned}
\log\left(R_t^i\right) - \log\alpha - (1-\alpha)\log\left(\overline{N}\right) &= z_t + (\alpha-1)\log\left(K_t\right) + \eta_t^i \\
\log\left(w_t^i\right) - \log(1-\alpha) + \alpha\log\left(\overline{N}\right) &= z_t + \alpha\log\left(K_t\right) + \varepsilon_t^i,
\end{aligned}$$

which implies

$$\begin{aligned}
z_t &= \alpha\left(\log\left(R_t^i\right) - \log\alpha - (1-\alpha)\log\left(\overline{N}\right)\right) + \\
&\quad (1-\alpha)\left(\log\left(w_t^i\right) - \log(1-\alpha) + \alpha\log\left(\overline{N}\right)\right) - (1-\alpha)\,\varepsilon_t^i - \alpha\eta_t^i \\
\log\left(K_t\right) &= \left(\log\left(w_t^i\right) - \log(1-\alpha) + \alpha\log\left(\overline{N}\right)\right) - \\
&\quad \left(\log\left(R_t^i\right) - \log\alpha - (1-\alpha)\log\left(\overline{N}\right)\right) - \varepsilon_t^i + \eta_t^i.
\end{aligned}$$

Now we substitute the laws of motion for aggregate capital, the technology shock, and the idiosyncratic shock processes into the pricing equations in period $t+1$, obtaining

$$\begin{aligned}
\log\left(R_{t+1}^i\right) - \log\alpha - (1-\alpha)\log\left(\overline{N}\right) &= \rho_z z_t + e_{t+1} + (\alpha-1)\left(a_0 + a_1 z_t + a_2\log\left(K_t\right)\right) + \mu_\eta + \rho_\eta\eta_t^i + \zeta_{t+1}^i \\
\log\left(w_{t+1}^i\right) - \log(1-\alpha) + \alpha\log\left(\overline{N}\right) &= \rho_z z_t + e_{t+1} + \alpha\left(a_0 + a_1 z + a_2\log\left(K_t\right)\right) + \mu_\varepsilon + \rho_\varepsilon\varepsilon_t^i + \nu_{t+1}^i.
\end{aligned}$$

Substituting z_t and $\log K_t$ out, we get the forecast equations

$$\begin{bmatrix} \log\left(R_{t+1}^i\right) \\ \log\left(w_{t+1}^i\right) \end{bmatrix} \sim N\left(\begin{bmatrix} E_t\left[\log\left(R_{t+1}^i\right)\right] \\ E_t\left[\log\left(w_{t+1}^i\right)\right] \end{bmatrix}, \mathbf{V}_2\right) \tag{A.1}$$

$$\begin{aligned}
E_t\left[\log\left(R_{t+1}^i\right)\right] &= A_0 + A_1\log\left(R_t^i\right) + A_2\log\left(w_t^i\right) - A_2\varepsilon_t^i + \left(\rho_\eta - A_1\right)\eta_t^i \\
E_t\left[\log\left(w_{t+1}^i\right)\right] &= A_3 + A_4\log\left(R_t^i\right) + A_5\log\left(w_t^i\right) + \left(\rho_\varepsilon - A_5\right)\varepsilon_t^i - A_4\eta_t^i
\end{aligned}$$

$$\mathbf{V}_2 = \begin{bmatrix} \sigma_e^2 + \sigma_\zeta^2 + 2\rho_{e\zeta}\sigma_e\sigma_\zeta & \sigma_e^2 + \rho_{\nu\zeta}\sigma_\nu\sigma_\zeta + \rho_{e\nu}\sigma_e\sigma_\nu + \rho_{e\zeta}\sigma_e\sigma_\zeta \\ \sigma_e^2 + \rho_{\nu\zeta}\sigma_\nu\sigma_\zeta + \rho_{e\nu}\sigma_e\sigma_\nu + \rho_{e\zeta}\sigma_e\sigma_\zeta & \sigma_e^2 + \sigma_\nu^2 + 2\rho_{e\nu}\sigma_e\sigma_\nu \end{bmatrix} \tag{A.2}$$

where

$$
\begin{aligned}
A_0 &= (\alpha-1)\,a_0+(1-A_1)\log(\alpha)-A_2\log(1-\alpha)+((1-A_1)(1-\alpha)+A_2\alpha)\log\left(\overline{N}\right)+\mu_\eta\\
A_1 &= \rho_z\alpha+(1-\alpha)(a_2-a_1\alpha)\\
A_2 &= (1-\alpha)(\rho_z-(1-\alpha)\,a_1-a_2)\\
A_3 &= \alpha a_0-A_4\log(\alpha)+(1-A_5)\log(1-\alpha)+(\alpha(A_5-1)-(1-\alpha)\,A_4)\log\left(\overline{N}\right)+\mu_\varepsilon\\
A_4 &= (\rho_z+\alpha a_1-a_2)\,\alpha\\
A_5 &= (1-\alpha)\,\rho_z+\alpha\,(a_1(1-\alpha)+a_2)\,.
\end{aligned}
$$

The error in the forecasts of future prices is the innovation in the technology shock and the idiosyncratic shocks.

A detailed derivation is now given. For returns, we have

$$
\begin{aligned}
\log\left(R^i_{t+1}\right)-\log(\alpha)-(1-\alpha)\log\left(\overline{N}\right) &= \rho_z z_t+e_{t+1}+(\alpha-1)(a_0+a_1z_t+a_2\log(K_t))+\mu_\eta+\rho_\eta\eta^i_t+\zeta^i_{t+1}\\
&= (\alpha-1)\,a_0+((\alpha-1)\,a_1+\rho_z)\,z_t+(\alpha-1)\,a_2\log(K_t)+\\
&\quad \mu_\eta+\rho_\eta\eta^i_t+e_{t+1}+\zeta^i_{t+1}\\
&= (\alpha-1)\,a_0+\\
&\quad ((\alpha-1)\,a_1+\rho_z)\,\alpha\left(\log\left(R^i_t\right)-\log(\alpha)-(1-\alpha)\log\left(\overline{N}\right)\right)+\\
&\quad ((\alpha-1)\,a_1+\rho_z)(1-\alpha)\left(\log\left(w^i_t\right)-\log(1-\alpha)-\alpha\log\left(\overline{N}\right)\right)-\\
&\quad ((\alpha-1)\,a_1+\rho_z)(1-\alpha)\,\varepsilon^i_t-((\alpha-1)\,a_1+\rho_z)\,\alpha\eta^i_t+\\
&\quad (\alpha-1)\,a_2\varepsilon^i_t+(\alpha-1)\,a_2\eta^i_t+\mu_\eta+\rho_\eta\eta^i_t+e_{t+1}+\zeta^i_{t+1}\\
&= (\alpha-1)\,a_0+\\
&\quad (\rho_z\alpha+(1-\alpha)(a_2-a_1\alpha))\left(\log\left(R^i_t\right)-\log(\alpha)-(1-\alpha)\log\left(\overline{N}\right)\right)+\\
&\quad (1-\alpha)(\rho_z-(1-\alpha)\,a_1-a_2)\left(\log\left(w^i_t\right)-\log(1-\alpha)+\alpha\log\left(\overline{N}\right)\right)-\\
&\quad (1-\alpha)(\rho_z-(1-\alpha)\,a_1-a_2)\,\varepsilon^i_t+\\
&\quad \left(\rho_\eta-\alpha\rho_z-(1-\alpha)(a_2-\alpha a_1)\right)\eta^i_t+\mu_\eta+e_{t+1}+\zeta^i_{t+1}.
\end{aligned}
$$

For wages, we have

$$
\begin{aligned}
\log\left(w_{t+1}^i\right) - \log(1-\alpha) + \alpha \log\left(\overline{N}\right) &= \rho_z z_t + e_{t+1} + \alpha\left(a_0 + a_1 z + a_2 \log\left(K_t\right)\right) + \mu_\varepsilon + \rho_\varepsilon \varepsilon_t^i + \nu_{t+1}^i \\
&= \alpha a_0 + \left(\rho_z + \alpha a_1\right) z_t + \alpha a_2 \log\left(K_t\right) + \mu_\varepsilon + \rho_\varepsilon \varepsilon_t^i + e_{t+1} + \nu_{t+1}^i \\
&= \alpha a_0 + \left(\rho_z + \alpha a_1\right) \alpha \left(\log\left(R_t^i\right) - \log(\alpha) - (1-\alpha)\log\left(\overline{N}\right)\right) + \\
&\quad \left(\rho_z + \alpha a_1\right)(1-\alpha)\left(\log\left(w_t^i\right) - \log(1-\alpha) + \alpha\log\left(\overline{N}\right)\right) - \\
&\quad \left(\rho_z + \alpha a_1\right)(1-\alpha)\varepsilon_t^i - \left(\rho_z + \alpha a_1\right)\alpha\eta_t^i + \\
&\quad \alpha a_2\left(\log\left(w_t^i\right) - \log(1-\alpha) + \alpha\log\left(\overline{N}\right)\right) - \\
&\quad \alpha a_2\left(\log\left(R_t^i\right) - \log(\alpha) - (1-\alpha)\log\left(\overline{N}\right)\right) - \\
&\quad \alpha a_2 \varepsilon_t^i + \alpha a_2 \eta_t^i + \mu_\varepsilon + \rho_\varepsilon \varepsilon_t^i + e_{t+1} + \nu_{t+1}^i \\
&= \alpha a_0 + \left(\rho_z + \alpha a_1 - a_2\right)\alpha\left(\log\left(R_t^i\right) - \log(\alpha) - (1-\alpha)\log\left(\overline{N}\right)\right) + \\
&\quad \left(\rho_z(1-\alpha) + \alpha\left(a_1(1-\alpha) + a_2\right)\right)\left(\log\left(w_t^i\right) - \log(1-\alpha) + \alpha\log\left(\overline{N}\right)\right) + \\
&\quad \left(\rho_\varepsilon - \rho_z(1-\alpha) - \alpha\left(a_1(1-\alpha) + a_2\right)\right)\varepsilon_t^i - \\
&\quad \left(\rho_z + \alpha a_1 - a_2\right)\eta_t^i + \mu_\varepsilon + e_{t+1} + \nu_{t+1}^i.
\end{aligned}
$$

Defining $\{A_i\}_1^5$ as in the text, we have

$$
\begin{aligned}
\log\left(R_{t+1}^i\right) - \log(\alpha) - (1-\alpha)\log\left(\overline{N}\right) &= (\alpha - 1)a_0 + \mu_\eta + \\
&\quad A_1\left(\log\left(R_t^i\right) - \log(\alpha) - (1-\alpha)\log\left(\overline{N}\right)\right) + \\
&\quad A_2\left(\log\left(w_t^i\right) - \log(1-\alpha) + \alpha\log\left(\overline{N}\right)\right) - \\
&\quad A_2\varepsilon_t^i + \left(\rho_\eta - A_1\right)\eta_t^i + e_{t+1} + \zeta_{t+1}^i \\
\log\left(w_{t+1}^i\right) - \log(1-\alpha) + \alpha\log\left(\overline{N}\right) &= \alpha a_0 + \mu_\varepsilon + \\
&\quad A_4\left(\log\left(R_t^i\right) - \log(\alpha) - (1-\alpha)\log\left(\overline{N}\right)\right) + \\
&\quad A_5\left(\log\left(w_t^i\right) - \log(1-\alpha) + \alpha\log\left(\overline{N}\right)\right) + \\
&\quad \left(\rho_\varepsilon - A_5\right)\varepsilon_t^i - A_4\eta_t^i + e_{t+1} + \nu_{t+1}^i.
\end{aligned}
$$

The equations (A.1) are produced by collecting the constants.

B. Appendix B

In this appendix we present the filtering problem for the PI agent. Using matrix notation, write the state-space system (2.9) as

$$\begin{aligned}
\mathbf{Y}_{t+1} &= \mathcal{C} + \mathcal{D}\mathbf{Y}_t + \mathbf{\Psi}_{t+1} \\
\mathbf{X}_t &= \mathcal{E} + \mathcal{F}\mathbf{Y}_t \\
\mathbf{\Psi}_{t+1} &\sim N(0, \mathbf{\Sigma}).
\end{aligned}$$

Since the observed variables are linear combinations of state variables, we can reduce the number of states. Define

$$\mathcal{H} = \begin{bmatrix} & \mathcal{F} & & \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

and premultiply to obtain

$$\mathcal{H}\mathbf{Y}_{t+1} = \mathcal{H}\mathcal{C} + \mathcal{H}\mathcal{D}\mathcal{H}^{-1}\mathcal{H}\mathbf{Y}_t + \mathcal{H}\mathbf{\Psi}_{t+1}.$$

The first two rows of $\mathcal{H}\mathbf{Y}_t$ are the rows of $\mathbf{X}_t - \mathcal{E}$ and therefore observable in the current period. The unobserved state variables are now only ε_t^i and η_t^i, the idiosyncratic shocks. Rewrite the transformed system as

$$\begin{aligned}
\mathbf{\Lambda}_{t+1}^i &= \mathbf{G}_1 + \mathbf{G}_2\mathbf{\Lambda}_t^i + \mathbf{\Theta}_{t+1}^i \qquad \text{(B.1)} \\
\mathbf{Z}_{t+1}^i &= \mathbf{B}_1 + \mathbf{B}_2\mathbf{Z}_t^i + \mathbf{B}_3\mathbf{\Lambda}_t^i + \mathbf{\Phi}_{t+1}^i
\end{aligned}$$

where

$$
\begin{aligned}
\mathbf{\Lambda}_t^i &= \begin{bmatrix} \varepsilon_t^i \\ \eta_t^i \end{bmatrix} \\
\mathbf{\Theta}_{t+1}^i &= \begin{bmatrix} \nu_{t+1}^i \\ \zeta_{t+1}^i \end{bmatrix} \\
\mathbf{G}_1 &= \begin{bmatrix} \mu_\varepsilon \\ \mu_\eta \end{bmatrix} \\
\mathbf{G}_2 &= \begin{bmatrix} \rho_\varepsilon & 0 \\ 0 & \rho_\eta \end{bmatrix} \\
\mathbf{Z}_t^i &= \begin{bmatrix} \log\left(R_t^i\right) - \log(\alpha) - (1-\alpha)\log\left(\overline{N}\right) \\ \log\left(w_t^i\right) - \log(1-\alpha) + \alpha\log\left(\overline{N}\right) \end{bmatrix} \\
\mathbf{\Phi}_{t+1}^i &= \begin{bmatrix} e_{t+1} + \zeta_{t+1}^i \\ e_{t+1} + \nu_{t+1}^i \end{bmatrix} \\
\mathbf{B}_1 &= \begin{bmatrix} (\alpha-1)\,a_0 + \mu_\eta \\ \alpha a_0 + \mu_\varepsilon \end{bmatrix} \\
\mathbf{B}_2 &= \begin{bmatrix} A_1 & A_2 \\ A_4 & A_5 \end{bmatrix} \\
\mathbf{B}_3 &= \begin{bmatrix} -A_2 & \rho_\eta - A_1 \\ \rho_\varepsilon - A_5 & -A_4 \end{bmatrix}
\end{aligned}
$$

with $\{A_i\}_0^5$ the same terms as in Appendix A. The covariance between $\mathbf{\Theta}_{t+1}^i$ and $\mathbf{\Phi}_{t+1}^i$ is given by the matrix

$$
\mathbf{\Upsilon} = \begin{bmatrix} \mathbf{V}_1 & \mathbf{V}_3^T \\ \mathbf{V}_3 & \mathbf{V}_2 \end{bmatrix},
$$

where

$$\mathbf{V}_1 = \begin{bmatrix} \sigma_\nu^2 & \rho_{\nu\zeta}\sigma_\nu\sigma_\zeta \\ \rho_{\nu\zeta}\sigma_\nu\sigma_\zeta & \sigma_\zeta^2 \end{bmatrix}$$

$$\mathbf{V}_2 = \begin{bmatrix} \sigma_e^2 + \sigma_\zeta^2 + 2\rho_{e\zeta}\sigma_e\sigma_\zeta & \sigma_e^2 + \rho_{\nu\zeta}\sigma_\nu\sigma_\zeta + \rho_{e\nu}\sigma_e\sigma_\nu + \rho_{e\zeta}\sigma_e\sigma_\zeta \\ \sigma_e^2 + \rho_{\nu\zeta}\sigma_\nu\sigma_\zeta + \rho_{e\nu}\sigma_e\sigma_\nu + \rho_{e\zeta}\sigma_e\sigma_\zeta & \sigma_e^2 + \sigma_\nu^2 + 2\rho_{e\nu}\sigma_e\sigma_\nu \end{bmatrix}$$

$$\mathbf{V}_3 = \begin{bmatrix} \rho_{e\nu}\sigma_e\sigma_\nu + \rho_{\nu\zeta}\sigma_\nu\sigma_\zeta & \sigma_\nu^2 + \rho_{e\nu}\sigma_e\sigma_\nu \\ \sigma_\zeta^2 + \rho_{e\zeta}\sigma_e\sigma_\zeta & \rho_{e\zeta}\sigma_e\sigma_\zeta + \rho_{\nu\zeta}\sigma_\nu\sigma_\zeta \end{bmatrix}.$$

Assume that the prior belief over $\mathbf{\Lambda}_t^i$ in period t, $\mathbf{\Lambda}_{t|t}^i$, is independent of $\left\{\mathbf{\Theta}_s^i, \mathbf{\Phi}_s^i\right\}_{s>t}$ and normal:

$$\mathbf{\Lambda}_{t|t}^i \sim N\left(\overline{\mathbf{\Lambda}}_{t|t}^i, \mathbf{P}_{t|t}\right).$$

Conditioning on period t information, we have

$$\begin{bmatrix} \mathbf{\Lambda}_{t+1|t}^i \\ \mathbf{Z}_{t+1|t}^i \end{bmatrix} \sim N\left(\begin{bmatrix} \mathbf{G}_1 + \mathbf{G}_2\overline{\mathbf{\Lambda}}_{t|t}^i \\ \overline{\mathbf{Z}}_{t+1|t}^i \end{bmatrix}, \begin{bmatrix} \mathbf{G}_2\mathbf{P}_{t|t}\mathbf{G}_2^T + \mathbf{V}_1 & \mathbf{G}_2\mathbf{P}_{t|t}\mathbf{B}_3^T + \mathbf{V}_3 \\ \mathbf{B}_3\mathbf{P}_{t|t}\mathbf{G}_2^T + \mathbf{V}_3^T & \mathbf{B}_3\mathbf{P}_{t|t}\mathbf{B}_3^T + \mathbf{V}_2 \end{bmatrix}\right)$$

$$\mathbf{Z}_{t+1|t}^i = \mathbf{B}_1 + \mathbf{B}_2\mathbf{Z}_t^i + \mathbf{B}_3\overline{\mathbf{\Lambda}}_{t|t}^i.$$

After observing $\mathbf{Z}_{t+1}^i$ in period $t+1$, the updated value for $\mathbf{\Lambda}_{t+1|t+1}^i$ will obey

$$\mathbf{\Lambda}_{t+1|t+1}^i \sim N\left(\overline{\mathbf{\Lambda}}_{t+1|t+1}^i, \mathbf{P}_{t+1|t+1}\right)$$

$$\overline{\mathbf{\Lambda}}_{t+1|t+1}^i = \mathbf{G}_1 + \mathbf{G}_2\overline{\mathbf{\Lambda}}_{t|t}^i + \left(\mathbf{G}_2\mathbf{P}_{t|t}\mathbf{B}_3^T + \mathbf{V}_3\right)\left(\mathbf{B}_3\mathbf{P}_{t|t}\mathbf{B}_3^T + \mathbf{V}_2\right)^{-1}\left(\mathbf{Z}_{t+1}^i - \mathbf{B}_1 - \mathbf{B}_2\mathbf{Z}_t^i - \mathbf{B}_3\overline{\mathbf{\Lambda}}_{t|t}^i\right)$$

$$\mathbf{P}_{t+1|t+1} = \left(\mathbf{G}_2\mathbf{P}_{t|t}\mathbf{G}_2^T + \mathbf{V}_1\right) - \left(\mathbf{G}_2\mathbf{P}_{t|t}\mathbf{B}_3^T + \mathbf{V}_3\right)\left(\mathbf{B}_3\mathbf{P}_{t|t}\mathbf{B}_3^T + \mathbf{V}_2\right)^{-1}\left(\mathbf{B}_3\mathbf{P}_{t|t}\mathbf{G}_2^T + \mathbf{V}_3^T\right).$$

From (B.1), conditioned on period t we can write out

$$\begin{aligned} \log\left(R_{t+1}^i\right) - \log(\alpha) - (1-\alpha)\log\left(\overline{N}\right) &= (\alpha-1)\,a_0 + \mu_\eta + A_1\left(\log\left(R_t^i\right) - \log(\alpha) - (1-\alpha)\log\left(\overline{N}\right)\right) + \\ &\quad A_2\left(\log\left(w_t^i\right) - \log(1-\alpha) + \alpha\log\left(\overline{N}\right)\right) - \\ &\quad A_2\epsilon_{t|t}^i + \left(\rho_\eta - A_1\right)\eta_{t|t}^i + e_{t+1} + \zeta_{t+1}^i \\ \log\left(w_{t+1}^i\right) - \log(1-\alpha) + \alpha\log\left(\overline{N}\right) &= \alpha a_0 + \mu_\epsilon + A_4\left(\log\left(R_t^i\right) - \log(\alpha) - (1-\alpha)\log\left(\overline{N}\right)\right) + \\ &\quad A_5\left(\log\left(w_t^i\right) - \log(1-\alpha) + \alpha\log\left(\overline{N}\right)\right) + \\ &\quad \left(\rho_\epsilon - A_5\right)\epsilon_{t|t}^i - A_4\eta_{t|t}^i + e_{t+1} + \nu_{t+1}^i. \end{aligned}$$

The forecast rules are obtained after collecting the constant terms.

Therefore, we obtain the following system of equations that describes the evolution of the prices and beliefs in the PI economy:

$$\begin{aligned}
\begin{bmatrix} \log\left(R^i_{t+1}\right) \\ \log\left(w^i_{t+1}\right) \end{bmatrix} &\sim N\left(\begin{bmatrix} E_t\left[\log\left(R^i_{t+1}\right)\right] \\ E_t\left[\log\left(w^i_{t+1}\right)\right] \end{bmatrix}, \mathbf{B}_3\mathbf{P}_{t|t}\mathbf{B}_3^T + \mathbf{V}_2\right) \\
E_t\left[\log\left(R^i_{t+1}\right)\right] &= A_0 + A_1\log\left(R^i_t\right) + A_2\log\left(w^i_t\right) - A_2\overline{\varepsilon}^i_{t|t} + \left(\rho_\eta - A_1\right)\overline{\eta}^i_{t|t} \\
E_t\left[\log\left(w^i_{t+1}\right)\right] &= A_3 + A_4\log\left(R^i_t\right) + A_5\log\left(w^i_t\right) + \left(\rho_\varepsilon - A_5\right)\overline{\varepsilon}^i_{t|t} - A_4\overline{\eta}^i_{t|t}
\end{aligned} \tag{B.2}$$

$$\begin{bmatrix} \overline{\varepsilon}^i_{t+1|t+1} \\ \overline{\eta}^i_{t+1|t+1} \end{bmatrix} = \begin{bmatrix} \rho_\varepsilon\overline{\varepsilon}^i_{t|t} \\ \rho_\eta\overline{\eta}^i_{t|t} \end{bmatrix} + \mathbf{G}_3\left(\begin{bmatrix} \log\left(R^i_{t+1}\right) \\ \log\left(w^i_{t+1}\right) \end{bmatrix} - \begin{bmatrix} E_t\left[\log\left(R^i_{t+1}\right)\right] \\ E_t\left[\log\left(w^i_{t+1}\right)\right] \end{bmatrix}\right). \tag{B.3}$$

where $\mathbf{G}_3 = \left(\mathbf{G}_2\mathbf{P}_{t|t}\mathbf{B}_3^T + \mathbf{V}_3\right)\left(\mathbf{B}_3\mathbf{P}_{t|t}\mathbf{B}_3^T + \mathbf{V}_2\right)^{-1}$. Letting the process for $\mathbf{P}_{t|t}$ converge to a constant yields the laws of motion in the main body of the paper. Since the Kalman filter recursion depends on endogenous variables its convergence is not ensured (see Baxter, Graham, and Wright 2007). We did not encounter any problems with convergence, however, and do not further pursue this issue here.

C. Appendix C

This section prove Proposition 2.6 and 2.7. Given PI problem (**??**), k^i_{t+1} is a function of $\left\{k^i_t, \overline{\varepsilon}^i_{t|t}, \overline{\eta}^i_{t|t}, \log\left(R^i_t\right), \log\left(w\right.\right.$
In addition, the observation $\left(R^i_t, w^i_t\right)$ imposes two linear restrictions on i's beliefs:

$$\begin{aligned}
\log\left(R^i_t\right) &= \log(\alpha) + (1-\alpha)\log\left(\overline{N}\right) + \overline{z}^i_{t|t} + (\alpha-1)\log\left(\overline{K}^i_{t|t}\right) + \overline{\eta}^i_{t|t}, \\
\log\left(w^i_t\right) &= \log(1-\alpha) - \alpha\log\left(\overline{N}\right) + \overline{z}^i_{t|t} + \alpha\log\left(\overline{K}^i_{t|t}\right) + \overline{\varepsilon}^i_{t|t}.
\end{aligned} \tag{C.1}$$

Thus k^i_{t+1} can be written as a function of $\left\{k^i_t, \overline{z}^i_{t|t}, \log\left(\overline{K}^i_{t|t}\right), \overline{\varepsilon}^i_{t|t}, \overline{\eta}^i_{t|t}\right\}$. Using Proposition (2.5) and restricting $\mathcal{M}_s[\cdot]$ to the first moment, we can write the law of motion for aggregate capital as

$$\log\left(K_{t+1}\right) = \gamma_0 + \gamma_1\log\left(K_t\right) + \gamma_2\mathbf{E}^i\left[\overline{z}^i_{t|t}\right] + \gamma_3\mathbf{E}^i\left[\log\left(\overline{K}^i_{t|t}\right)\right] + \gamma_4\mathbf{E}^i\left[\overline{\varepsilon}^i_{t|t}\right] + \gamma_5\mathbf{E}^i\left[\overline{\eta}^i_{t|t}\right]. \tag{C.2}$$

Next applying $\mathbf{E}^i[\cdot]$ on both sides of (C.1), we obtain the equations

$$\begin{aligned} z_t + (\alpha-1)\log(K_t) &= \mathbf{E}^i\left[\overline{z}^i_{t|t}\right] + (\alpha-1)\,\mathbf{E}^i\left[\log\left(\overline{K}^i_{t|t}\right)\right] + \mathbf{E}^i\left[\overline{\eta}^i_{t|t}\right], \\ z_t + \alpha\log(K_t) &= \mathbf{E}^i\left[\overline{z}^i_{t|t}\right] + \alpha\mathbf{E}^i\left[\log\left(\overline{K}^i_{t|t}\right)\right] + \mathbf{E}^i\left[\overline{\varepsilon}^i_{t|t}\right]. \end{aligned}$$

These two linear restrictions imply that Equation (C.2) is equivalent to Equation (2.13) as in Proposition (2.6).

D. Appendix D

Here we prove Propositions (3.4) and (3.5). For the system (2.9) we can derive the steady state Kalman Filter as

$$\begin{aligned} \mathbf{Y}^i_{t+1|t+1} &\sim N\left(\overline{\mathbf{Y}}^i_{t+1|t+1}, \mathbf{P}\right), \\ \overline{\mathbf{Y}}^i_{t+1|t+1} &= \mathcal{C} + \mathcal{D}\overline{\mathbf{Y}}^i_{t|t} + \widehat{\mathbf{P}}\mathcal{F}'\left(\mathcal{F}\widehat{\mathbf{P}}\mathcal{F}'\right)^{-1}\left(\mathbf{X}^i_{t+1} - \mathcal{E} - \mathcal{F}\left(\mathcal{C} + \mathcal{D}\overline{\mathbf{Y}}^i_{t|t}\right)\right) \\ \widehat{\mathbf{P}} &= \mathcal{D}\mathbf{P}\mathcal{D}' + \mathbf{\Sigma}, \end{aligned} \tag{D.1}$$

where $\mathbf{P}$ is the steady state covariance matrix produced by $\mathbf{P}_{t+1|t+1}$.[43] Substituting out $\mathbf{X}^i_{t+1}$ in (D.1), we have

$$\overline{\mathbf{Y}}^i_{t+1|t+1} = \mathcal{C} + (\mathcal{I} - \mathcal{K})\,\mathcal{D}\overline{\mathbf{Y}}^i_{t|t} + \mathcal{K}\mathcal{D}\mathbf{Y}^i_t + \mathcal{K}\mathbf{\Psi}^i_{t+1} \tag{D.2}$$

where $\mathcal{K} = \widehat{\mathbf{P}}\mathcal{F}'\left(\mathcal{F}\widehat{\mathbf{P}}\mathcal{F}'\right)^{-1}\mathcal{F}$ and $\mathcal{I}$ is the conformable identity matrix. Thus the cross-sectional expectation is multivariate normal with mean

$$\mathbf{E}^c\left[\overline{\mathbf{Y}}^i_{t+1|t+1}\right] = \mathcal{C} + (\mathcal{I} - \mathcal{K})\,\mathcal{D}\mathbf{E}^c\left[\overline{\mathbf{Y}}^i_{t|t}\right] + \mathcal{K}\mathcal{D}\mathbf{E}^c\left[\mathbf{Y}^i_t\right] + \mathcal{K}_{\circ 1}e_{t+1}, \tag{D.3}$$

[43] Given prior in period as $\mathbf{Y}_{t|t} \sim N\left(\overline{\mathbf{Y}}_{t|t}, \mathbf{P}_{t|t}\right)$, we can write the joint normal distribution below

$$\begin{bmatrix} \mathbf{Y}^i_{t+1|t} \\ \mathbf{X}^i_{t+1|t} \end{bmatrix} \sim N\left(\begin{bmatrix} \overline{\mathbf{Y}}^i_{t+1|t} \\ \mathcal{E} + \mathcal{F}\mathbf{Y}^i_{t+1|t} \end{bmatrix}, \begin{bmatrix} \mathbf{P}_{t+1|t} & \mathbf{P}_{t+1|t}\mathcal{F}' \\ \mathcal{F}\mathbf{P}_{t+1|t} & \mathcal{F}\mathbf{P}_{t+1|t}\mathcal{F}' \end{bmatrix}\right),$$

$$\overline{\mathbf{Y}}^i_{t+1|t} = \mathcal{C} + \mathcal{D}\overline{\mathbf{Y}}^i_{t|t},$$

$$\mathbf{P}_{t+1|t} = \mathcal{D}\mathbf{P}_{t|t}\mathcal{D}' + \mathbf{\Sigma}.$$

After observing $\mathbf{X}^i_{t+1}$, the update equation is shown in $(D.1)$.

where $\mathcal{K}_{\circ 1}$ is the first column of $\mathcal{K}$. To get the cross-sectional variance, applying $\mathbf{V}^i[\cdot]$ on both sides of (D.2) yields

$$\begin{aligned}
\mathbf{V}^i\left[\overline{\mathbf{Y}}^i_{t|t}\right] &= (\mathcal{I}-\mathcal{K})\,\mathcal{D}\mathbf{V}^i\left[\overline{\mathbf{Y}}^i_{t-1|t-1}\right]\mathcal{D}^T(\mathcal{I}-\mathcal{K})^T + \mathcal{K}\mathcal{D}\mathbf{V}^i\left[\mathbf{Y}^i_{t-1}\right]\mathcal{D}^T\mathcal{K}^T + \\
&\quad \mathcal{K}\mathbf{V}^i\left[\mathbf{\Psi}^i_t\right]\mathcal{K}^T + (\mathcal{I}-\mathcal{K})\,\mathcal{D}\mathbf{Cov}\left(\overline{\mathbf{Y}}^i_{t-1|t-1}, \overline{\mathbf{Y}}^i_{t-1}\right) + \\
&\quad \mathcal{K}\mathcal{D}\mathbf{Cov}\left(\overline{\mathbf{Y}}^i_{t-1|t-1}, \overline{\mathbf{Y}}^i_{t-1}\right)\mathcal{D}^T\mathcal{K}^T \\
&= (\mathcal{I}-\mathcal{K})\,\mathcal{D}\mathbf{V}^i\left[\overline{\mathbf{Y}}^i_{t-1|t-1}\right]\mathcal{D}^T(\mathcal{I}-\mathcal{K})^T + \mathcal{K}\mathbf{\Sigma}_Y\mathcal{K}^T + \\
&\quad (\mathcal{I}-\mathcal{K})\,\mathcal{D}\left(\sum\nolimits_{j=0}^{\infty}((\mathcal{I}-\mathcal{K})\,\mathcal{D})^j\,\mathcal{K}\mathbf{\Sigma}_Y\left(\mathcal{D}^T\right)^{j+1}\right)\mathcal{K}^T + \\
&\quad \mathcal{K}\left(\sum\nolimits_{j=0}^{\infty}\mathcal{D}^{j+1}\mathbf{\Sigma}_Y\mathcal{K}^T\left(\mathcal{D}^T(\mathcal{I}-\mathcal{K})^T\right)^j\right)\mathcal{D}^T\left(\mathcal{I}-\mathcal{K}^T\right)
\end{aligned}$$

where

$$\begin{aligned}
\mathbf{\Sigma}_Y &= \mathbf{V}^i\left[\mathbf{Y}^i_{t-1}\right] = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & \sigma^2_\varepsilon & 0 & 0 \\ 0 & 0 & \sigma^2_\eta & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \\
\sigma^2_\varepsilon &= \rho^2_\varepsilon\frac{\sigma^2_\upsilon}{1-\rho^2_\varepsilon} \text{ and } \sigma^2_\eta = \rho^2_\eta\frac{\sigma^2_\zeta}{1-\rho^2_\eta}.
\end{aligned}$$

σ^2_ε and σ^2_η are the unconditional variances of ε^i and η^i, respectively. To get the expression for the covariance matrix., resubstitute over $\overline{\mathbf{Y}}^i_{t-j|t-j}$ in (D.2) to obtain

$$\overline{\mathbf{Y}}^i_{t-1|t-1} = \sum_{j=0}^{\infty}((\mathcal{I}-\mathcal{K})\,\mathcal{D})^j\,(\mathcal{I}-\mathcal{K})\,\mathcal{C} + \sum_{j=0}^{\infty}((\mathcal{I}-\mathcal{K})\,\mathcal{D})^j\,\mathcal{K}\mathbf{Y}^i_{t-j-1}.$$

Note that the covariance matrix of $\mathbf{Y}^i_t$ can be written as

$$\mathbf{Cov}\left[\mathbf{Y}^i_{t-j}, \mathbf{Y}^i_t\right] = \mathbf{\Sigma}_Y\left(\mathcal{D}^T\right)^j.$$

Thus

$$\begin{aligned}\mathbf{Cov}\left[\overline{\mathbf{Y}}_{t-1|t-1}^{i},\mathbf{Y}_{t}^{i}\right] &= \sum_{j=0}^{\infty}\left(\left(\mathcal{I}-\mathcal{K}\right)\mathcal{D}\right)^{j}\mathcal{K}\mathbf{Cov}\left[\mathbf{Y}_{t-j-1}^{i},\mathbf{Y}_{t-1}^{i}\right]\mathcal{D}^{T}\\ &= \sum_{j=0}^{\infty}\left(\left(\mathcal{I}-\mathcal{K}\right)\mathcal{D}\right)^{j}\mathcal{K}\mathbf{Cov}\left[\mathbf{Y}_{t-j}^{i},\mathbf{Y}_{t}^{i}\right]\mathcal{D}^{T}\\ &= \sum_{j=0}^{\infty}\left(\left(\mathcal{I}-\mathcal{K}\right)\mathcal{D}\right)^{j}\mathcal{K}\mathbf{\Sigma}_{Y}\left(\mathcal{D}^{T}\right)^{j+1}.\end{aligned}$$

This result concludes the proof of Propositions (3.4).

To prove Proposition (3.5), subtract $\mathbf{Y}_{t+1}^{i}$ and apply $\mathbf{E}^{i}\left[\cdot\right]$ to both sides, yielding

$$\mathbf{E}^{i}\left[\overline{\mathbf{Y}}_{t+1|t+1}^{i}\right]-\mathbf{Y}_{t+1}^{i}=\left(\mathcal{I}-\mathcal{K}\right)\mathcal{D}\left(\mathbf{E}^{i}\left[\overline{\mathbf{Y}}_{t|t}^{i}\right]-\mathbf{Y}_{t}^{i}\right)-\left(\mathcal{I}-\mathcal{K}\right)_{\circ 1}e_{t+1},$$

where $\left(\mathcal{I}-\mathcal{K}\right)_{\circ 1}$ is the first column of $\left(\mathcal{I}-\mathcal{K}\right)$. The presence of the common term e_{t+1} ensures that there is cross-sectional bias in the PI economy.

E. Appendix E

This appendix explains the algorithm to compute the equilibrium of the model. Our algorithm for solving the FI agent's problem is modified from Krusell and Smith (1998) and Young (2007). The objective of the algorithm is to obtain the coefficients in the law of motion for aggregate capital (2.6). We divide the algorithm into three main parts. In summary, the first part is to solve for the value functions $\left(V^{FI},V^{PI}\right)$ over a finite grid of $\left(k^{i},\varepsilon^{i},\eta^{i},R^{i},w^{i}\right)$, given a law of motion (2.6). The second part is to solve for the policy functions k' over a much finer grid of $\left(k^{i},\varepsilon^{i},\eta^{i},R^{i},w^{i}\right)$ using V^{FI} and V^{PI} from the first part. The third part is to simulate the time series of $\{K_t,MPK_t,MPN_t,z_t\}_{t=1}^{T}$ using the policy function from the second part and update the law of motion (2.6). This procedure is iterated from the first part using the updated law of motion until the coefficients (a_0,a_1,a_2) converge. The following subsections explain the algorithm in detail. For the PI agents simply substitute beliefs for actual idiosyncratic shock values and change the laws of motion as appropriate.

E.1. Part 1: Solving for $V^{FI}\left(k^i,\varepsilon^i,\eta^i,\log\left(R^i\right),\log\left(w^i\right)\right)$ and $V^{PI}\left(k^i,\varepsilon^i,\eta^i,\log\left(R^i\right),\log\left(w^i\right)\right)$

1. Discretize the space of $\left\{k^i,\varepsilon^i,\eta^i,\log\left(R^i\right),\log\left(w^i\right)\right\}$ and denote this grid $\{\mathbf{k1},\boldsymbol{\varepsilon}\mathbf{1},\boldsymbol{\eta}\mathbf{1},\mathbf{R1},\mathbf{w1}\}$. The number of grid points are $\{135,11,5,5,11\}$. Since the value functions have more curvature where k^i is close to the borrowing limit, we concentrate our grid points at low values of k^i. In addition, there is curvature over the dimensions of ε^i and $\log\left(w^i\right)$, so we use more points in those directions as well. The value functions in the dimensions of $\log\left(R^i\right)$ and η^i are almost linear, so we use only a small number of grid points.

2. Guess $\{a_j\}_{j=0}^2$ in (2.6). Then compute $\{A_j\}_{j=0}^5$ as shown in Appendix A (and the parameters for belief dynamics of ε^i and η^i in the PI economy as shown in Appendix B).

3. Guess the initial value functions V_0^{FI} and V_0^{PI} on the discretized grids of $\{\mathbf{k1},\boldsymbol{\varepsilon}\mathbf{1},\boldsymbol{\eta}\mathbf{1},\mathbf{R1},\mathbf{w1}\}$.

4. Given the above initial guess, solve the FI and PI agents' problems to get the policy functions for k' and use them to get V_1^{FI} and V_1^{PI}. Iterate until V^{FI} and V^{PI} converge.

E.2. Part 2: Solving for the policy functions $k'=g_k^i\left(k^i,\varepsilon^i,\eta^i,\log\left(R^i\right),\log\left(w^i\right)\right)$

1. Define finer grids $(\mathbf{k2},\boldsymbol{\varepsilon}\mathbf{2},\boldsymbol{\eta}\mathbf{2},\mathbf{R2},\mathbf{w2})$. We use $\{280,21,21,15,25\}$ points, respectively, again with points concentrated near the borrowing constraint in the $\mathbf{k2}$ direction. We put a lot more points over $\boldsymbol{\varepsilon}\mathbf{2}$ and $\mathbf{w2}$ since the optimal k' exhibits significant nonlinearity over these dimensions, while k' is almost linear over $\{\mathbf{k2},\boldsymbol{\eta}\mathbf{2},\mathbf{R2}\}$.

2. Use the resulting value function from the first part in the RHS of the agent's problem and resolve for the policy functions k' for all points in the grids $\{\mathbf{k2},\boldsymbol{\varepsilon}\mathbf{2},\boldsymbol{\eta}\mathbf{2},\mathbf{R2},\mathbf{w2}\}$.

E.3. Part 3: Update law of motion

1. Simulate a long time series of $\{z_t\}_{t=1}^T$. We set $T=6,000$ periods.

2. Assign an initial distribution of 100,000 households whose state variables are $\left\{k_1^i,\varepsilon_1^i,\eta_1^i\right\}$ for the FI economy and $\left\{k_1^i,\varepsilon_1^i,\eta_1^i,\varepsilon_{1|1}^i,\eta_{1|1}^i\right\}$ for the PI economy.[44] For the PI economy we set $\varepsilon_1^i=\varepsilon_{1|1}^i$ and $\eta_1^i=\eta_{1|1}^i$.

[44] The initial distribution $\left\{k_1^i,\varepsilon_1^i,\eta_1^i\right\}$ is the steady state distribution of 100,000 households living in a corresponding FI economy without an aggregate shock.

3. Given the distribution in period 1 and z_1, we can compute $\left\{\log\left(R_1^i\right), \log\left(w_1^i\right)\right\}$ for each households.

4. Simulate next period distribution of realized ε_2^i and η_2^i using (2.3) and (2.4). Then use the policy function k' from the second part to get the next period distribution $\left\{k_2^i, \varepsilon_2^i, \eta_2^i\right\}$. Since the state variables do not generally lie on the grid, we use linear interpolation to evaluate k'. For the PI economy, use z_2 to compute $\log(MPK_2)$, $\log(MPN_2)$, and $\left\{\log\left(R_2^i\right), \log\left(w_2^i\right)\right\}$. Then use (B.3) to update beliefs $\left\{\varepsilon_{2|2}^i, \eta_{2|2}^i\right\}$.

5. Repeat from step 3 for $\{z_t\}_{t=3}^T$.

6. Drop the first $1,000$ simulation periods and use OLS on the equilibrium time series of $\{K_t, z_t\}_{t=1001}^T$ to get a new value for $\{a_j\}_{j=0}^2$.

7. Update these coefficients using the updating rule: $x_{update} = \lambda x_{new} + (1-\lambda)\, x_{old}$ and repeat from step 3 in part 1 until all the coefficients $\{a_j\}_{j=0}^2$ converge.

In the remainder of this section we discuss how we solve the recursive problem in step 4 of Part 1 and step 2 of Part 2. To compute the maximization on the RHS of the Bellman equation we solve the first order condition

$$\begin{aligned} 0 &\geq -u'\left(m_t - k_{t+1}\right) + \beta E\left[\frac{\partial}{\partial k} V\left(k_{t+1}^i, \varepsilon_{t+1}^i, \eta_{t+1}^i, \log\left(R_{t+1}^i\right), \log\left(w_{t+1}^i\right)\right)\right], \\ m_t &= k_t\left(1 + R_{t+1}^i - \delta\right) + w_{t+1}^i \overline{h}, \end{aligned}$$

where $E[\cdot]$ is the conditional expectation according to the corresponding forecast rule. We use bisection and Newton-Raphson procedures to solve for the optimal k', depending on whether we are close to the borrowing limit.[45] To calculate the above integral we use a monomial rule for 5 degree polynomial functions.[46] The integral is three-dimensional for FI agents and two-dimensional for PI agents. Our results are not sensitive to enlarging the size of the population to $800,000$ households.

The last issue is how to approximate V and $\frac{\partial V}{\partial k}$. Since $\left(k^i, \varepsilon^i, \eta^i, \log\left(R^i\right), \log\left(w^i\right)\right)$ will generally not lie on the grids, we use the following steps in the interpolation:

[45] Using the method of endogenous gridpoints (Carroll 2006) helps reduce the computational time by avoiding root finding of the first order condition; we use this method to get us close to the solution.

[46] See Judd (1998).

1. For every grid point of $\mathbf{k1}$, we use linear interpolation over the $(\boldsymbol{\varepsilon}\mathbf{1}, \boldsymbol{\eta}\mathbf{1}, \mathbf{R1}, \mathbf{w1})$ dimensions to obtain $V\left(\mathbf{k1}, \varepsilon^i, \eta^i, \log\left(R^i\right), \log\left(w^i\right)\right)$.

2. We then construct a cubic spline over the $\mathbf{k1}$ dimension to obtain $V\left(k^i, \varepsilon^i, \eta^i, \log\left(R^i\right), \log\left(w^i\right)\right)$. We can then evaluate $\frac{\partial V}{\partial k}$ from the cubic spline; $\frac{\partial V}{\partial k}$ is continuous and smooth in k^i.

Table 1a

Business Cycle Statistics, FI

	z	$\log(Y)$	$\log(K)$	$\log(C)$	$\log(I)$	MPK	MPN
Mean	0.0004	**0.4530**	**3.0174**	**0.1217**	**−0.8135**	**0.0277**	**2.7112**
Std	0.0257	**0.0344**	**0.0333**	0.0277	**0.0628**	0.0006	0.0933
Corr	1.0000	0.9624	0.6237	**0.7884**	0.9863	**0.5971**	0.9624
		1.0000	0.8126	**0.9258**	0.9049	**0.3566**	0.9997
			1.0000	0.9723	0.4879	**−0.2546**	0.8117
				1.0000	0.6771	**−0.0224**	**0.9251**
					1.0000	**0.7191**	0.9049
						1.0000	0.3577
							1.0000
Autocorr	0.9599	0.9778	0.9982	0.9956	0.9498	0.9353	0.9777
	0.9221	0.9562	0.9953	0.9902	0.9030	0.8749	0.9560
	0.8846	0.9343	0.9914	0.9839	0.8570	0.8158	0.9339

Table 1b

Business Cycle Statistics, PI

	z	$\log(Y)$	$\log(K)$	$\log(C)$	$\log(I)$	MPK	MPN
Mean	0.0004	**0.4766**	**3.0830**	**0.1280**	**−0.7485**	**0.0266**	**2.7765**
Std	0.0257	**0.0390**	**0.0501**	0.0275	**0.0875**	0.0007	0.1080
Corr	1.0000	0.9259	0.5773	**0.5476**	0.9927	**0.2624**	0.9262
		1.0000	0.8431	**0.8188**	0.8998	**−0.1216**	0.9996
			1.0000	0.9900	0.5314	**−0.6363**	0.8418
				1.0000	0.4870	**−0.6529**	**0.8173**
					1.0000	**0.3097**	0.8998
						1.0000	**−0.1198**
							1.0000
Autocorr	0.9599	0.9825	0.9983	0.9966	0.9687	0.9605	0.9825
	0.9221	0.9656	0.9957	0.9931	0.9373	0.9227	0.9654
	0.8846	0.9484	0.9923	0.9892	0.9047	0.8847	0.9481

Table 2

Dispersion in Beliefs

	$\overline{z}^i_{t\|t}$	$\log\left(\overline{K}^i_{t\|t}\right)$	$\overline{\varepsilon}^i_{t\|t}$	$\overline{\eta}^i_{t\|t}$	$\log\left(\overline{R}^i_{t+1\|t}\right)$	$\log\left(\overline{w}^i_{t+1\|t}\right)$
PI	0.003	**0.0143**	0.5016	0.0454	**0.0104**	0.4762
FI	0.0	**0.0**	0.5048	0.05	**0.0**	0.4760

Figure 1: Aggregate shock z_t and capital K_t

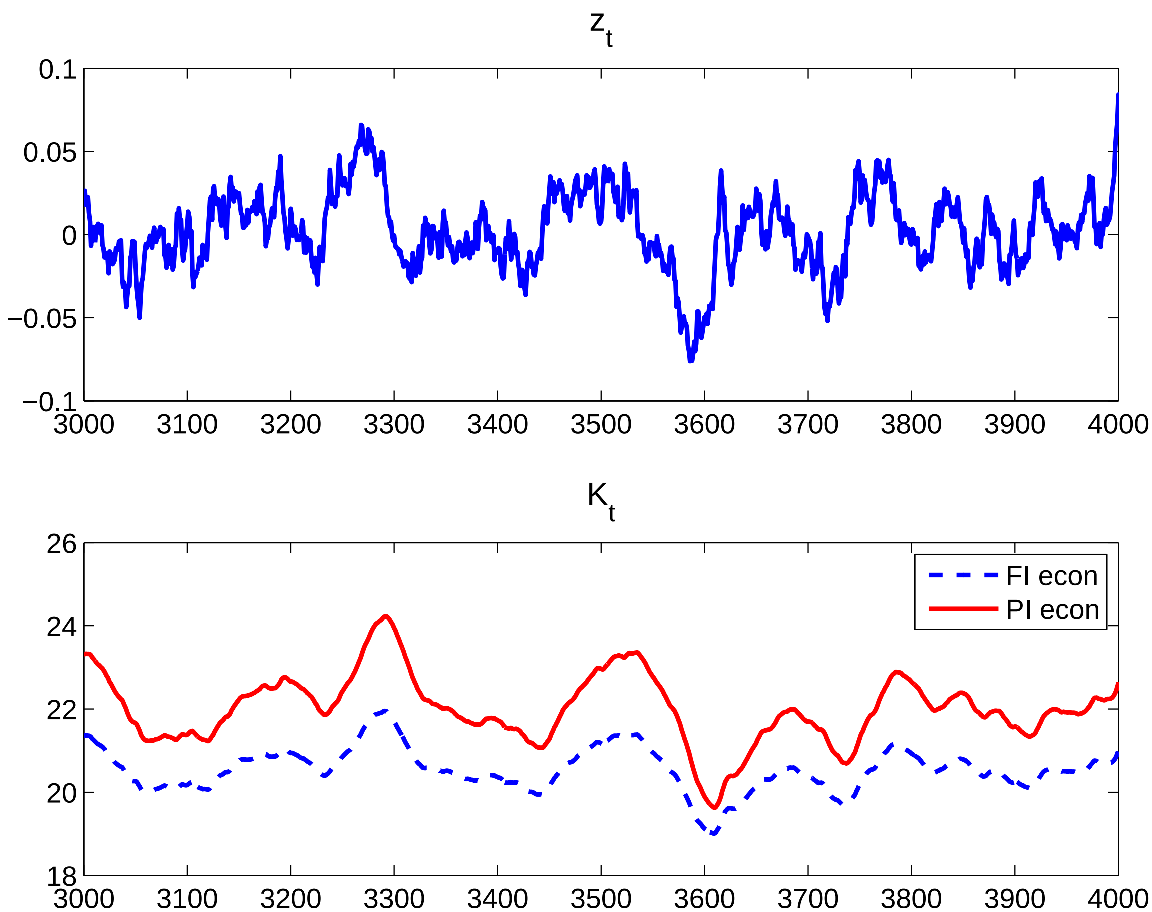

Figure 2: Aggregate consumption and investment

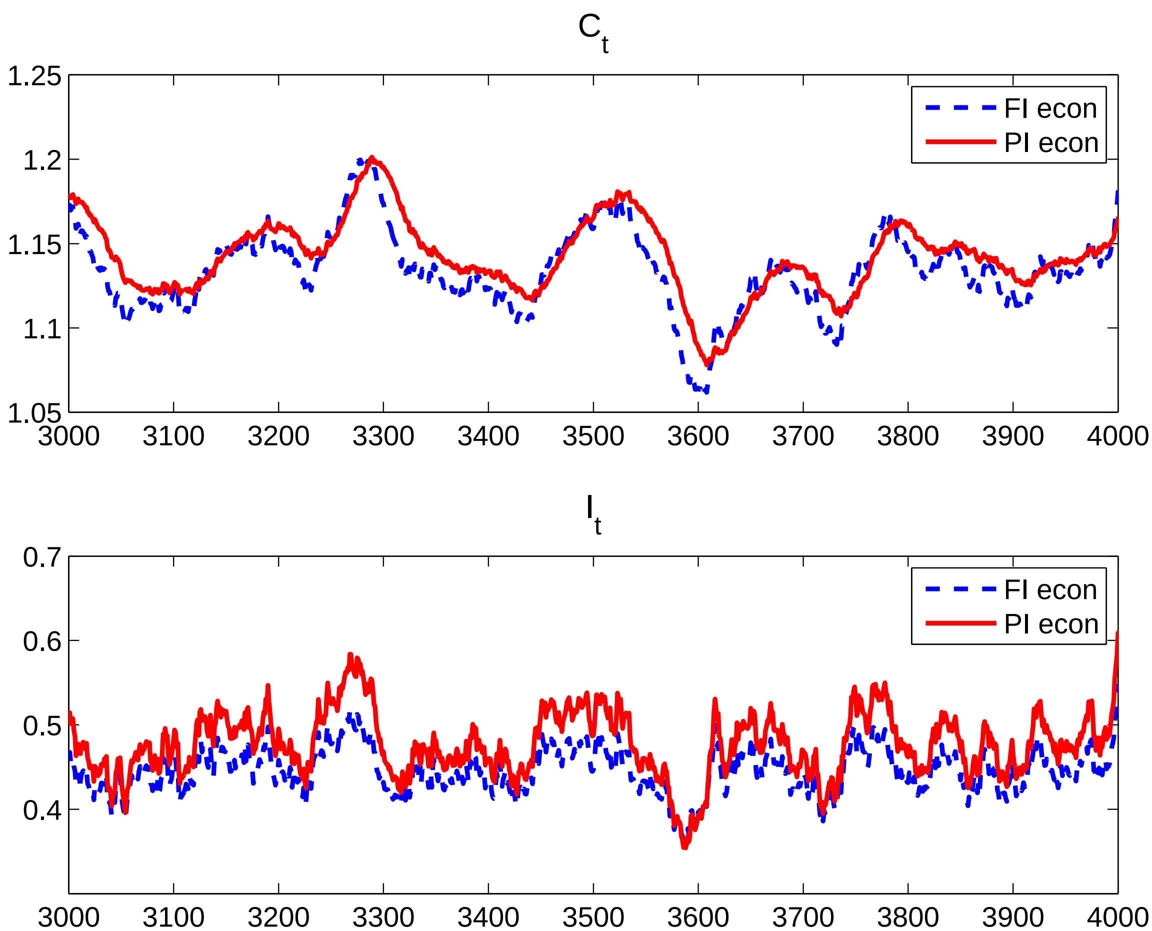

Figure 3: Decomposition of aggregate consumption

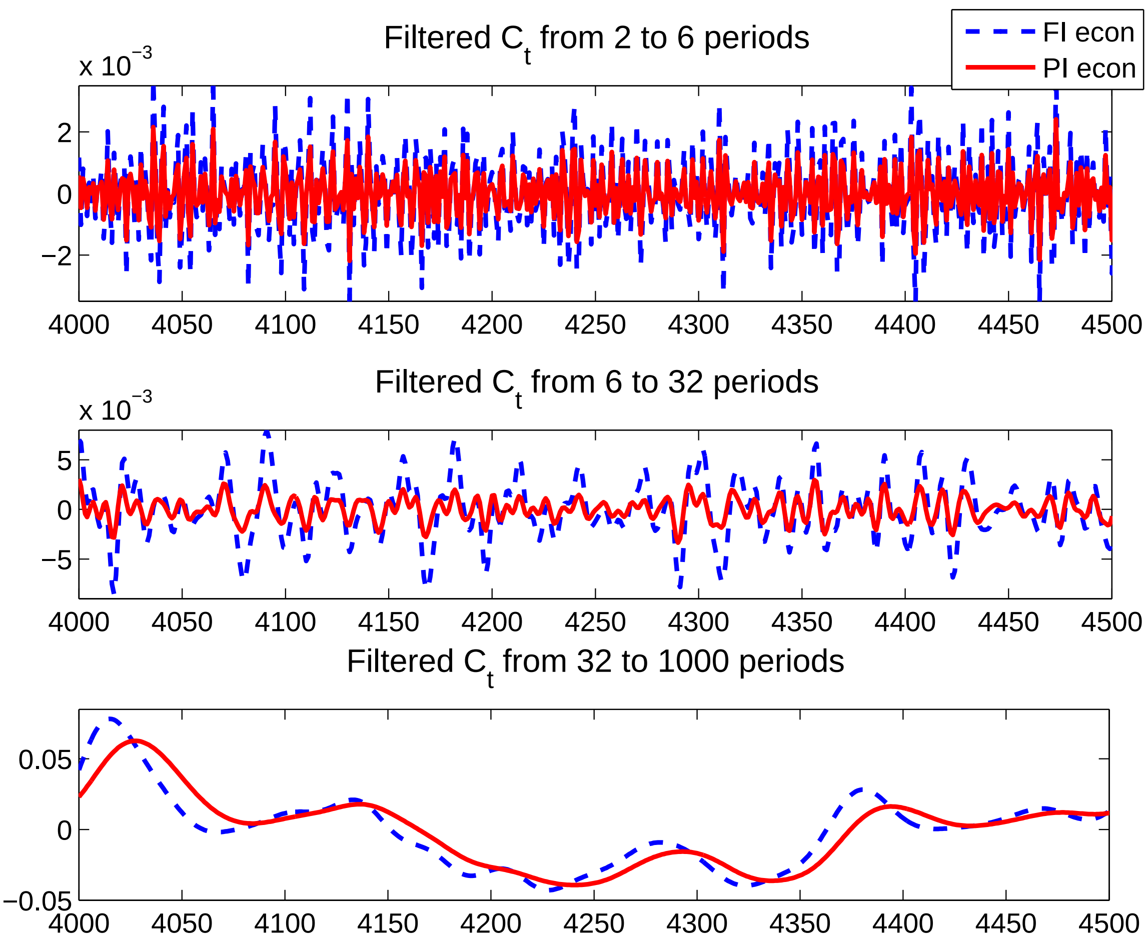

Figure 4: Cross-sectional expectation (1)

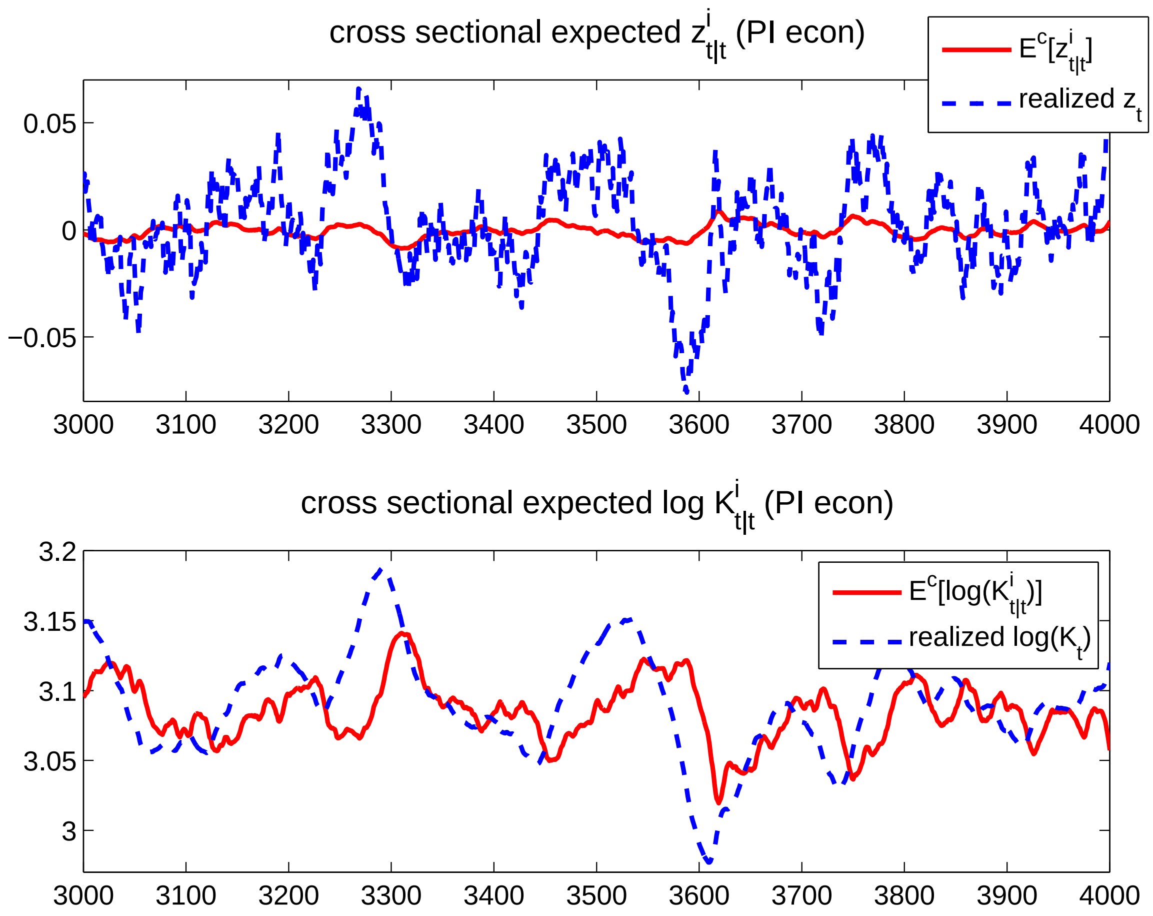

Figure 5: Cross-sectional expectation (2)

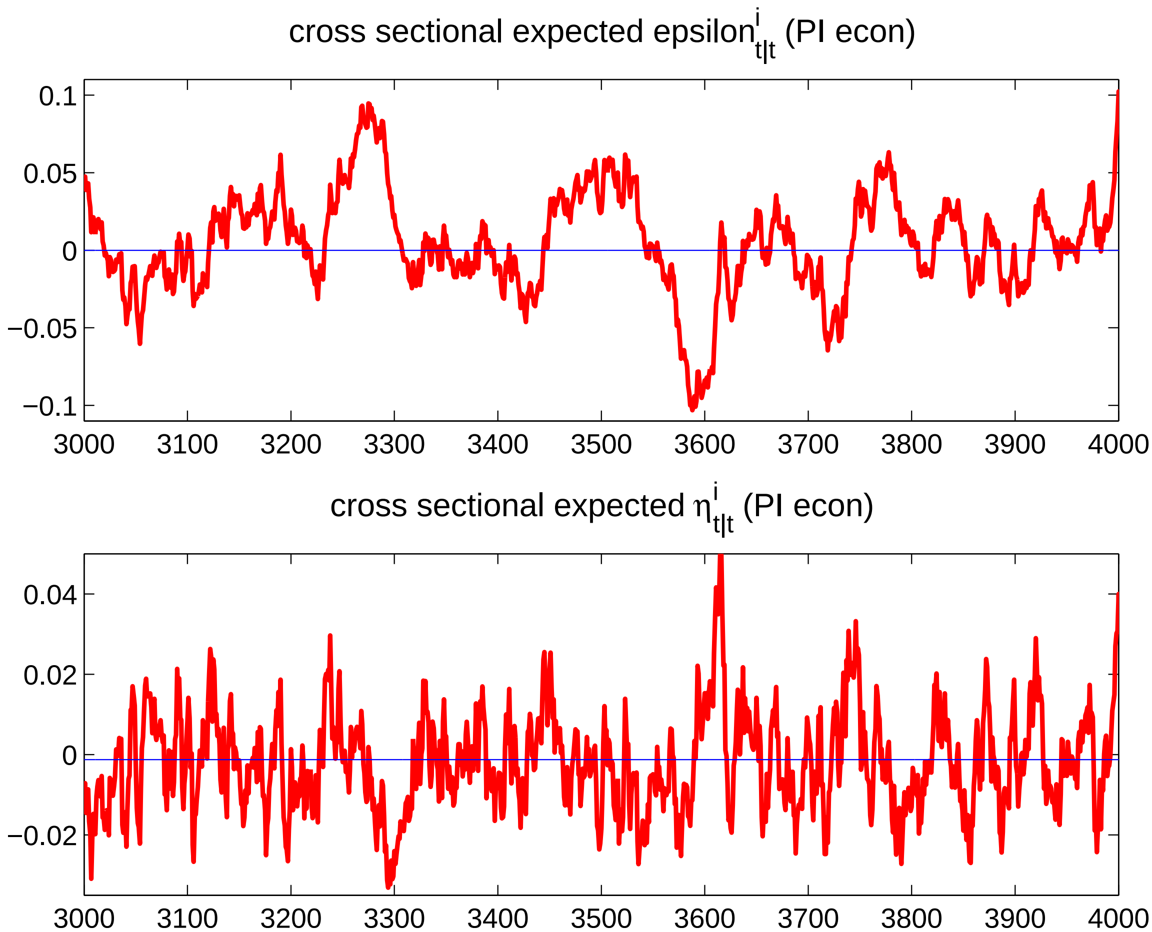

Figure 6: Cross-sectional expectation (3)

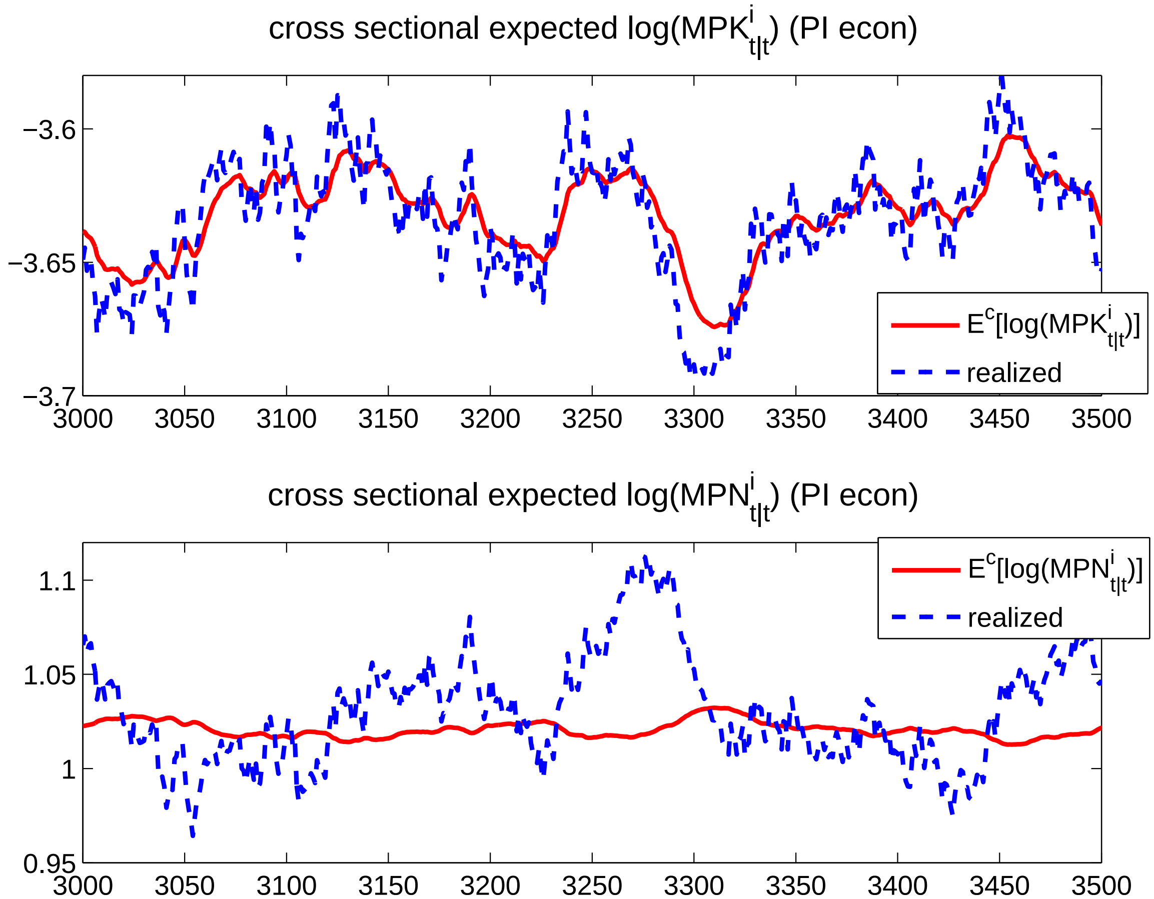

Figure 7: Cross-sectional standard deviation

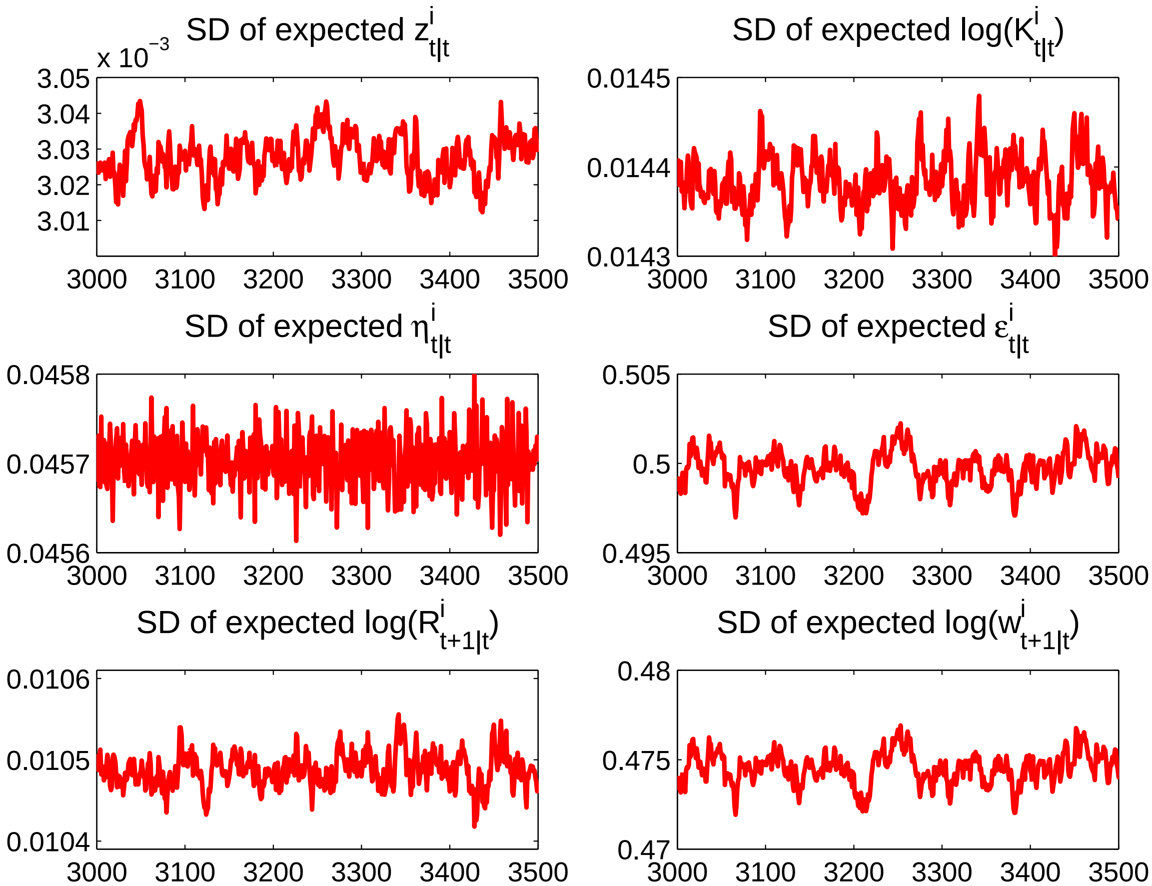

Figure 8: Impulse response in PI economy (1)

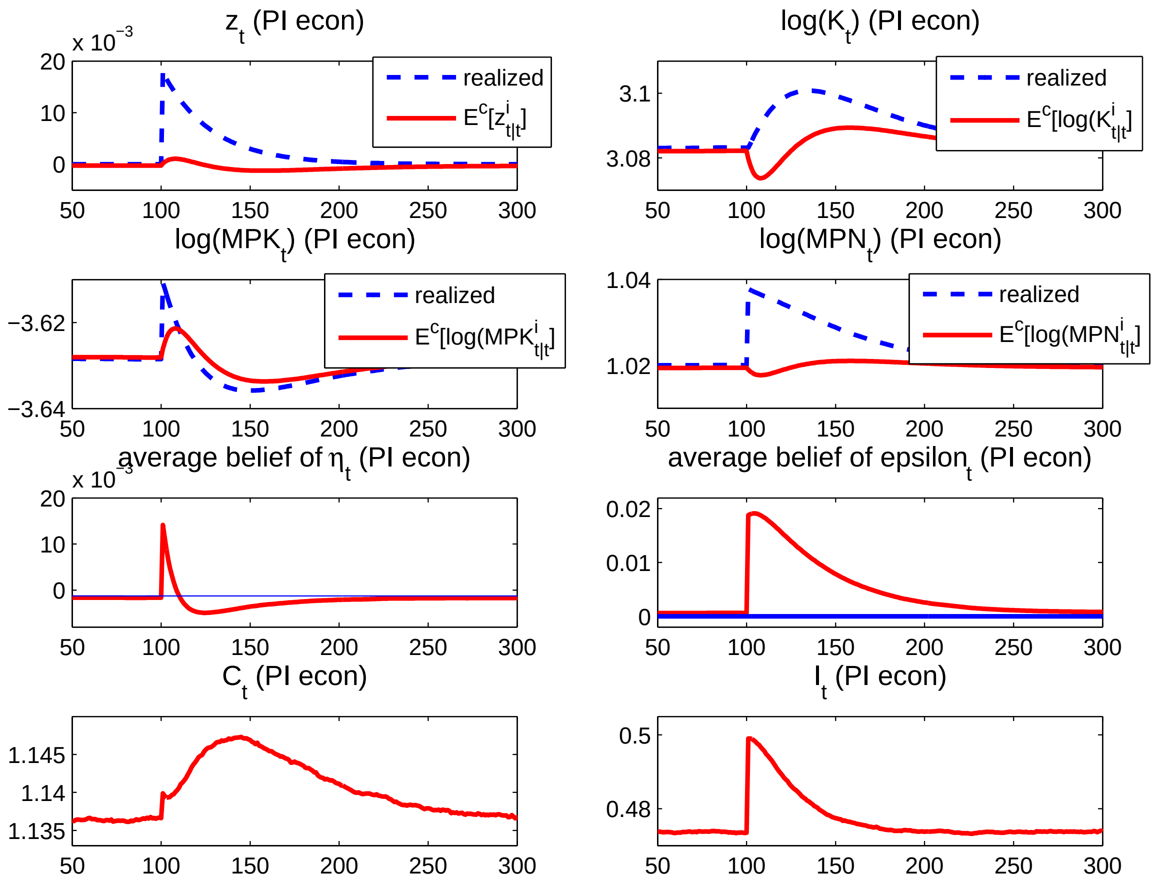

Figure 9: Impulse response in FI economy

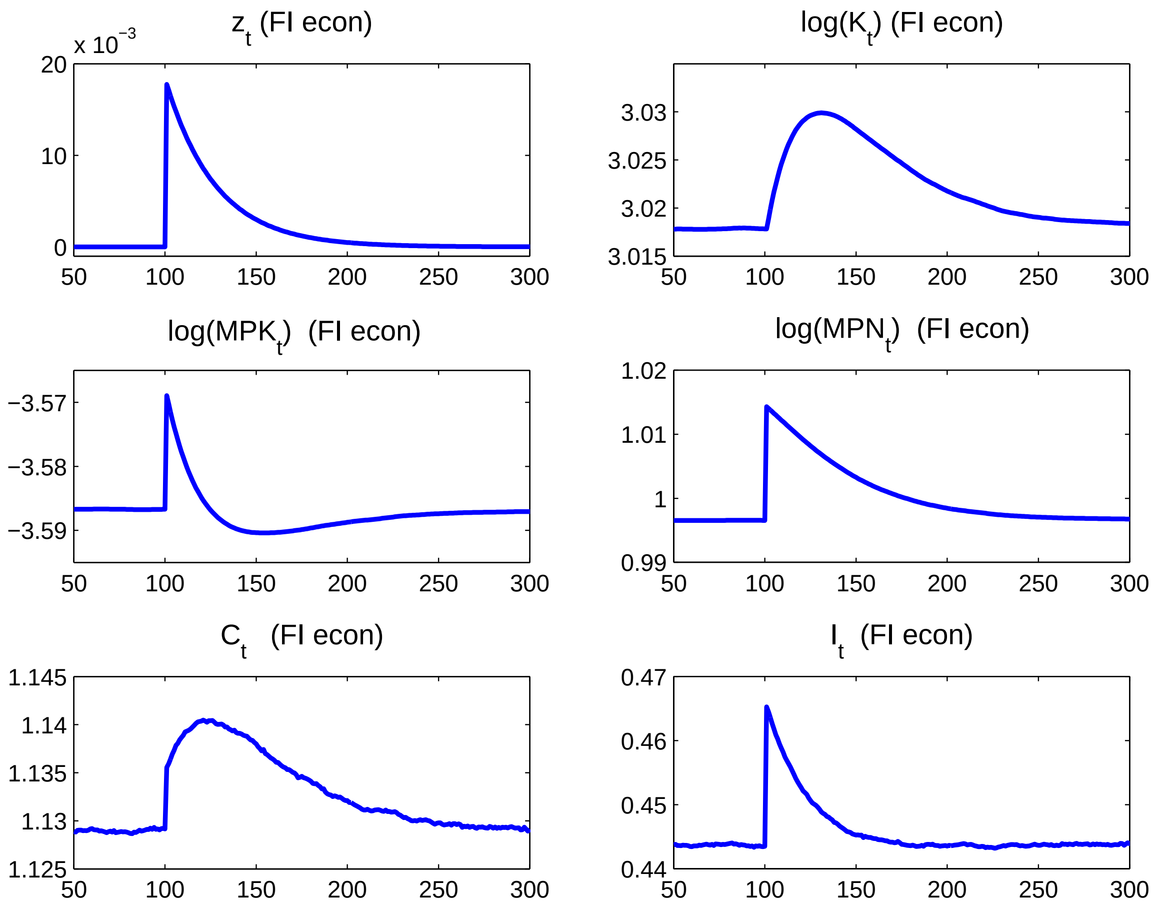

Figure 10: Impulse response in PI economy (2)

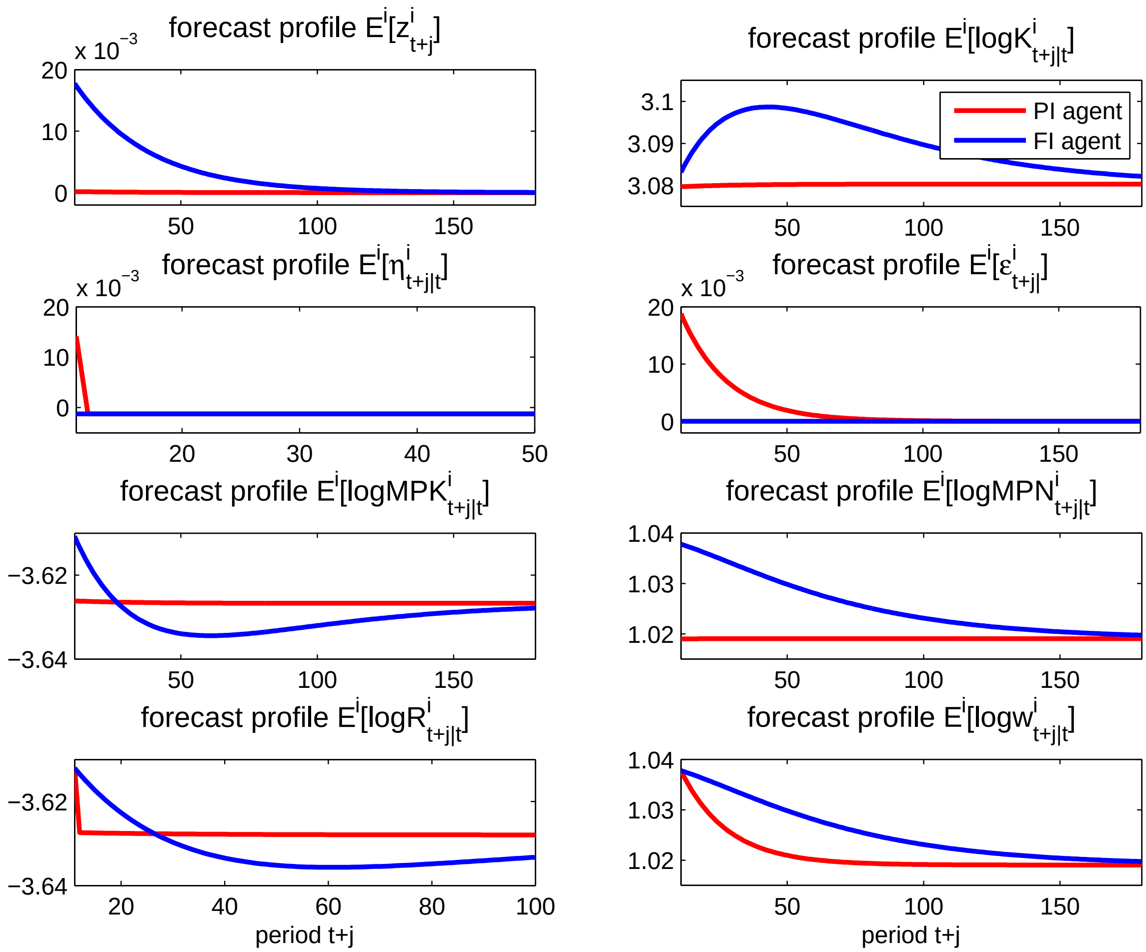

Figure 11: Aggregate capital

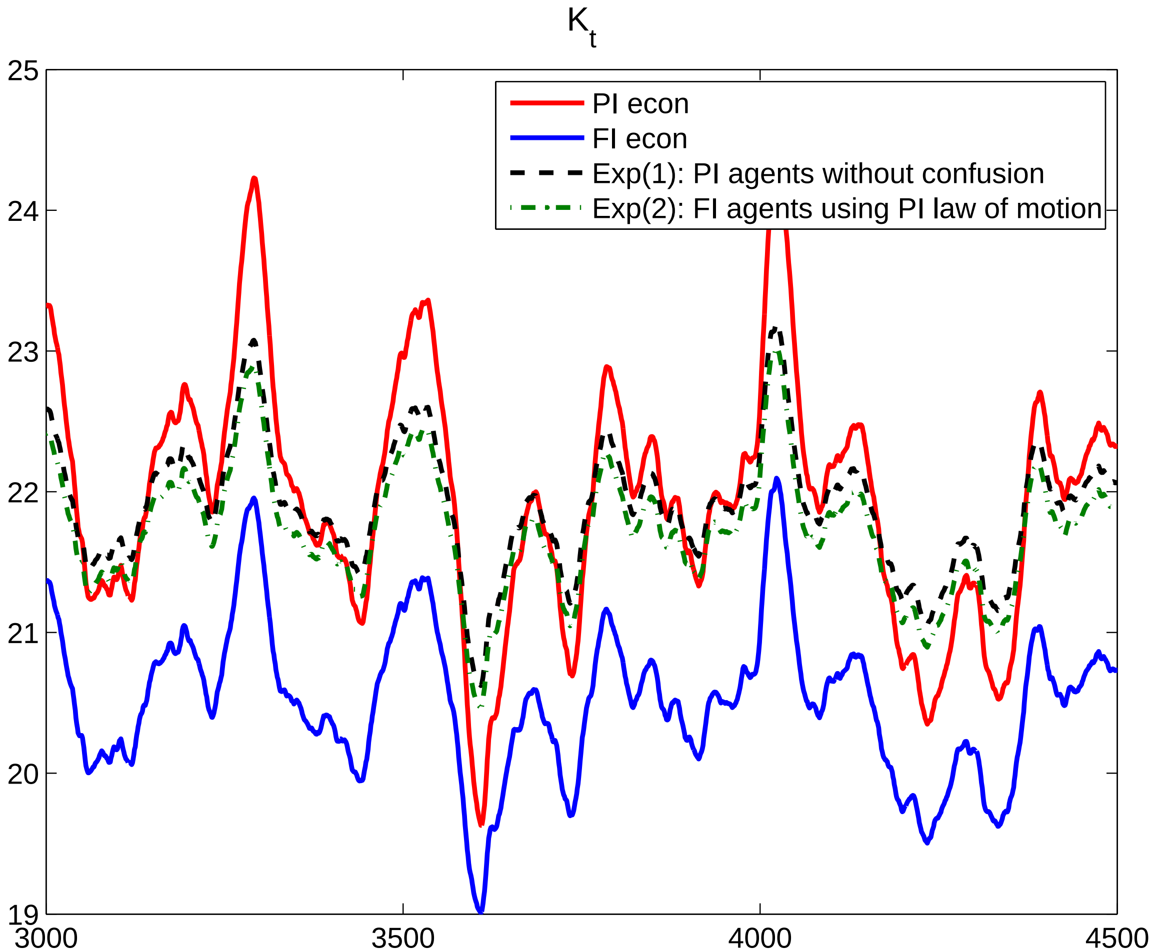

Figure 12: Asymmetry of savings

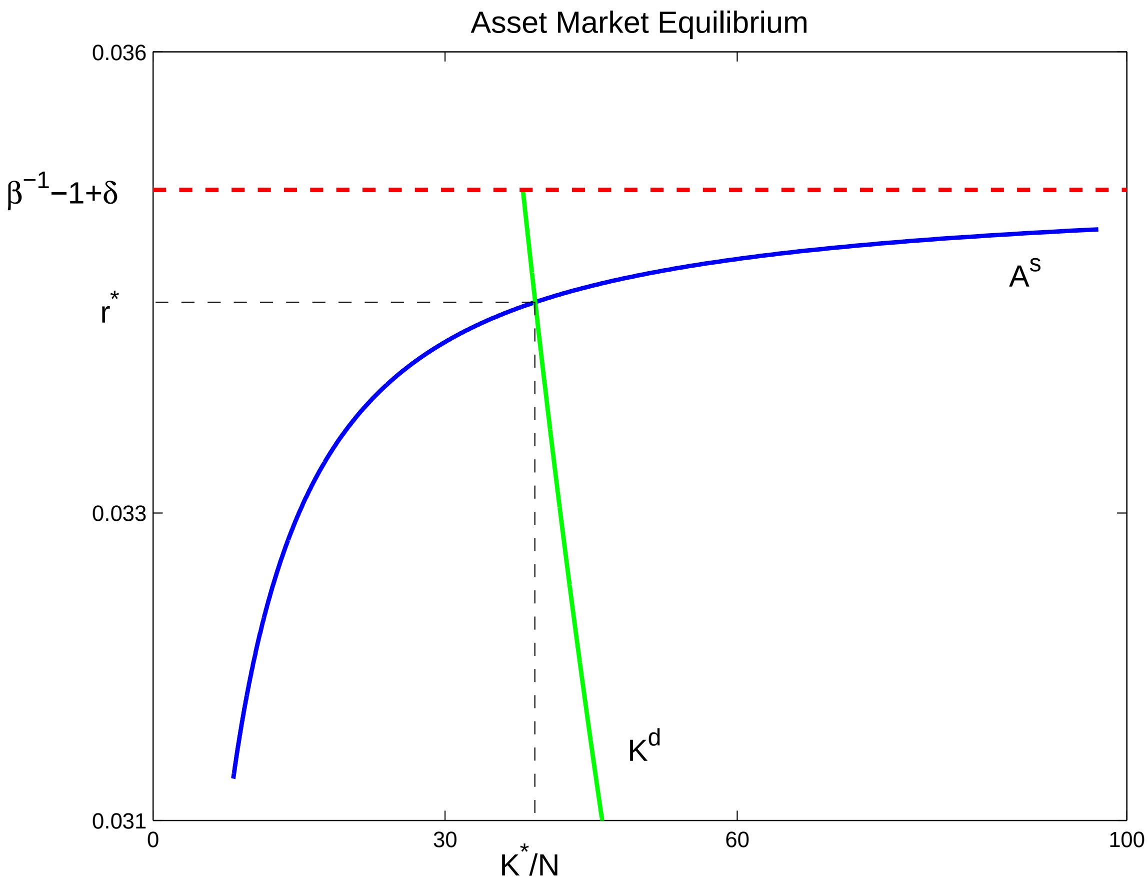

Figure 13: Wealth concentration

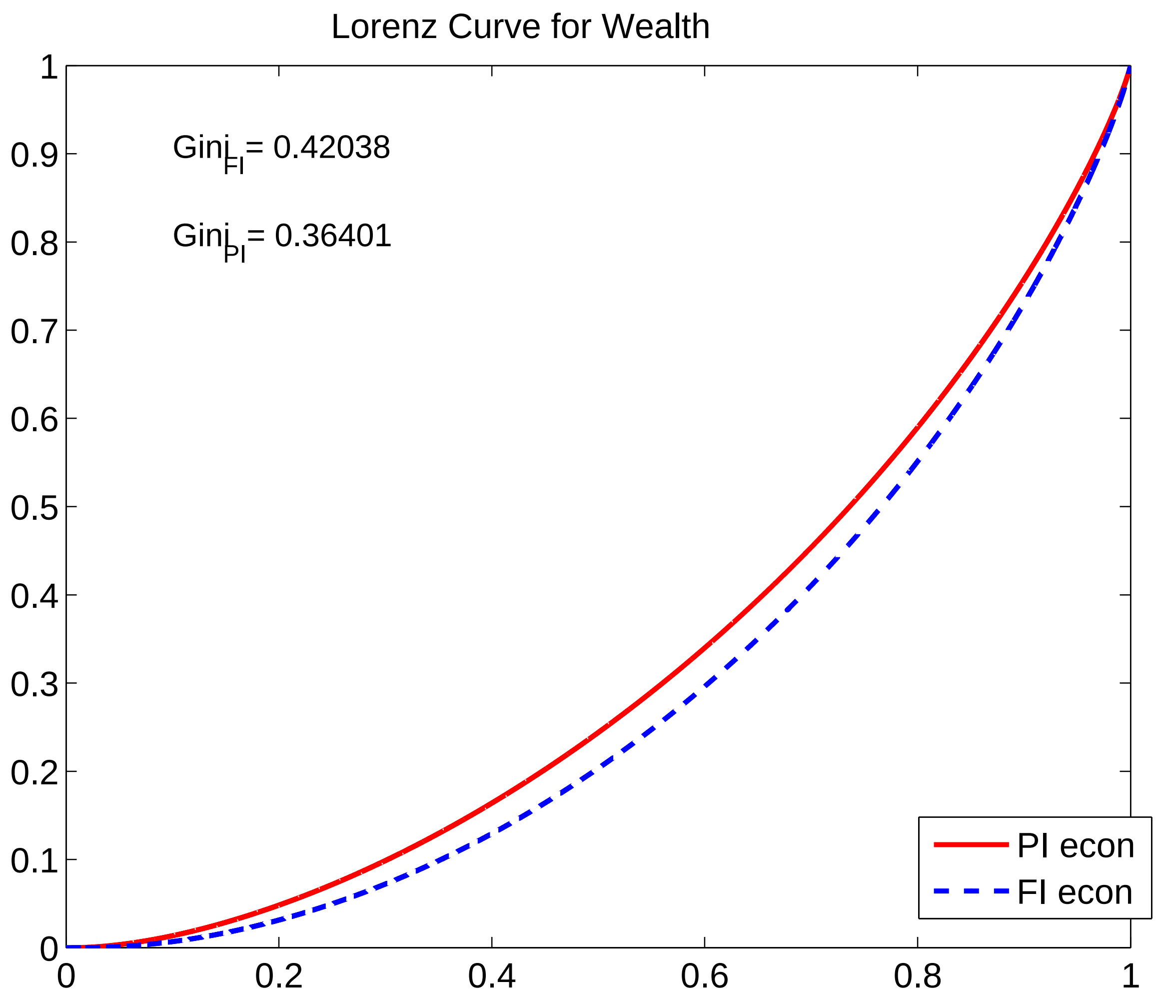

Figure 14: Consumer sentiment

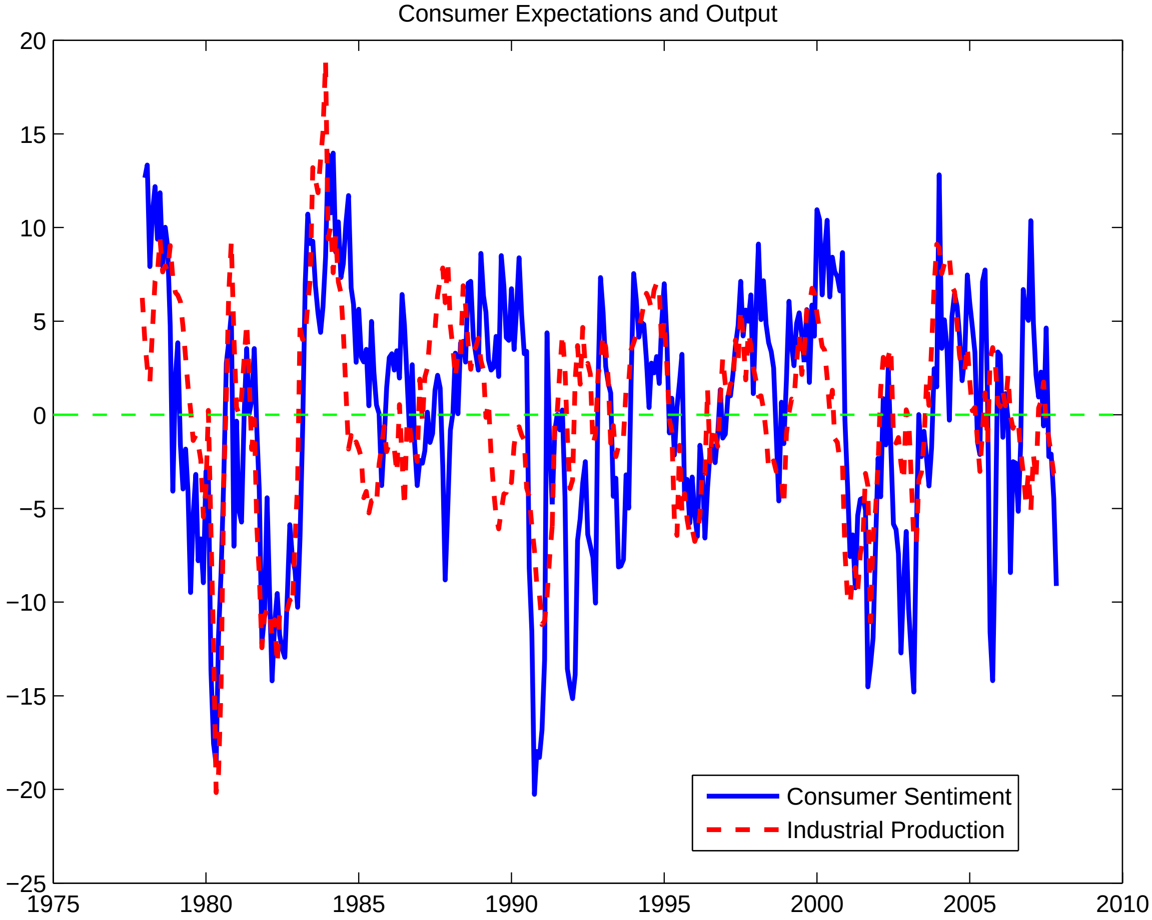

Figure 15: Consumer sentiment