Limit Theory for Panel Data Models with Cross Sectional Dependence and Sequential Exogeneity

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Introduction

- The starting point of our analysis is the observation that in many cross-sectional data-sets random sampling is not feasible. Examples include large scale macro time series, trade and foreign exchange, studies of economic growth, industrial organization.
- In these settings the assumption of iid data, even conditional on common factors, is difficult to defend.
- We derive a general Central Limit Theorem and asymptotic distributions for moment conditions related to panel data and spatial models with large n.
- Regressors can be sequentially (rather than just strictly) exogenous.
- The data can be cross sectionally dependent conditional on common factors.
- Cross sectional dependence can come from the presence of common factors, which leads to the need for random norming.

Introduction (cont.)

- The limit theorem for sample moments is derived by showing that the moment conditions can be recast such that a martingale difference array central limit theorem can be applied.
- Cross-sectional conditional moment restrictions hold for a class of games with common and private information.
- Analyze GMM estimators of a fixed effect panel model without imposing i.i.d. or strict exogeneity conditions.

Models

- We develop a central limit theory (CLT) for data-sets that are generated by models of the form

$$y_{it} = \psi_{it}\left(\mathbf{x}_t, \mathbf{z}_t, \mu_i, u_{-i,t}; \theta_0\right) + u_{it} \text{ for } i = 1, ..., n; \; t = 1, ..., T \quad (1)$$

- ψ_{it} are known functions,
- $y_{it}, \mathbf{x}_t = (x_{1t}, ..., x_{nt})$ and $\mathbf{z}_t = (z_{1t}, ..., z_{nt})$ are observed in the sample and denote, respectively, the dependent variable, the sequentially exogenous and strictly exogenous explanatory variables (conditional on the unobserved components),

Models (cont)

- θ_0 is an unknown parameter vector of *fixed and finite* dimension,
- μ_i is an individual specific effect not observed by the econometrician
- u_{it} is an unobserved error term with $u_{-i,t} = (u_{1t}, \ldots, u_{i-1,t}, u_{i+1,t}, \ldots u_{nt})$.
- Specification (1) includes linear dynamic spatial models as well as dynamic game theoretic models. A special case of (1) arises when $y_{it} = \psi_{it}(x_{it}, z_{it}, \mu_i; \theta_0) + u_{it}$ which is the case for conventional panel models.
- n tends to infinity and T fixed.

Moment Conditions

- The central limit theorem is stated for averages over the cross-section of random variables $x_i = (x_{i1} \ldots, x_{iT})$, $z_i = (z_{i1}, \ldots, z_{iT})$, and $u_i = (u_{i1}, \ldots, u_{iT})$ for $i = 1, ..., n$.
- Analogous to Andrews (2005), we allow in each period t for the possibility of common shocks across observations that are captured by a sigma field $\mathcal{C}_t \subset \mathcal{F}$.
- Let $x_{it}^o = (x_{i1}, \ldots, x_{it})$, $u_{it}^o = (u_{i1}, \ldots, u_{it})$ and $\mathcal{C}_t^o = \mathcal{C}_1 \vee \ldots \vee \mathcal{C}_t$ where $\vee$ denotes the sigma field generated by the union of two sigma fields. For simplicity write $\mathcal{C}$ for $\mathcal{C}_T^o$ in the following.
- Define a vector of sample moments of the form

$$\begin{aligned} m_{(n)} &= n^{-1/2} \sum_{i=1}^{n} m_i, \qquad (2) \\ m_i &= \left(h_{i1} u_{i1}^{+}, ..., h_{T^+i} u_{T^+i}^{+}\right)', \end{aligned}$$

with $T^+ \leq T$,

Moment Conditions (cont.)

- $h_{it} = (x_{it}^{o}, z_i)$ denotes a vector of instruments corresponding to t
- $u_i^{+} = \left(u_{i1}^{+}, ..., u_{iT^{+}}^{+}\right)$ denotes a vector of transformed disturbances with $u_{it}^{+} = \sum_{s=t}^{T} \pi_{ts} u_{is}$ for some nonstochastic constants π_{ts}.
- The class of transformations includes first differences, $u_{it}^{+} = u_{i,t+1} - u_{it}$, as well as the Helmert transformation, $u_{it}^{+} = \alpha_t \left[u_{it} - \left(u_{i,t+1} + \ldots + u_{iT}\right) / (T - t)\right]$, $\alpha_t^2 = (T - t)/(T - t + 1)$, for $t = 1, \ldots, T$.

Moment Conditions (cont.)

- Let $u_i^{+\prime} = \Pi u_i'$ where Π is a $T^+ \times T$ matrix with t, s-th element π_{ts}.
- Observe that the lower diagonal elements of Π are zero.
- Furthermore, let $H_i = diag_{t=1}^{T^+}(h_{it})$, then we can express the moment vectors as

$$m_i = H_i' u_i^{+\prime} = H_i' \Pi u_i' = \sum_{t=1}^{T} H_i' \pi_t u_{it}, \tag{3}$$

where π_t denotes the t-th column of Π.

Information Structure

- Define the following sub-σ-fields of $\mathcal{F}$:

$$\mathcal{B}_{n,i,t} = \sigma\left\{\left(x^o_{tj}, z_j, u^o_{t-1j}, \mu_j\right)_{j=1}^{n}, u_{t,-i}\right\},$$

and $(i = 1, \ldots, n)$

$$\begin{aligned}
&\mathcal{F}_{n,0} = \mathcal{C}, \\
&\mathcal{F}_{n,i} = \sigma\left\{\left(x^o_{j1}, z_j, \mu_j\right)_{j=1}^{n}, (u_{j1})_{j=1}^{i-1}\right\} \vee \mathcal{C}, \\
&\mathcal{F}_{n,n+i} = \sigma\left\{\left(x^o_{j2}, z_j, u^o_{j1}, \mu_j\right)_{j=1}^{n}, (u_{j2})_{j=1}^{i-1}\right\} \vee \mathcal{C}, \\
&\vdots \\
&\mathcal{F}_{n,(T-1)n+i} = \sigma\left\{\left(x^o_{jT}, z_j, u^o_{j,T-1}, \mu_j\right)_{j=1}^{n}, (u_{jT})_{j=1}^{i-1}\right\} \vee \mathcal{C}.
\end{aligned} \tag{4}$$

Assumptions

- **Assumption 1:** For some $\delta > 0$ and some finite constant K the following conditions hold for all $t = 1, \ldots, T$, $i = 1, \ldots, n$, $n \geq 1$:
(a) The $2+\delta$ absolute moments of x_{it}, z_{it}, u_{it}, and μ_i exist, and are uniformly bounded by K. In addition,
$$E\left[\left|u_{it}\right|^{2+\delta} \mid \mathcal{B}_{n,i,t} \vee \mathcal{C}\right] \leq K. \tag{5}$$
(b) The following conditional moment restrictions hold:
$$E\left[u_{it} \mid \mathcal{B}_{n,i,t} \vee \mathcal{C}_t^o\right] = 0. \tag{6}$$
(c) Let $\tilde{V}_{(n)} = \sum_{t=1}^{T} \tilde{V}_{t,n}$ with
$$\tilde{V}_{t,n} = n^{-1} \sum_{i=1}^{n} E\left[u_{it}^2 \mid \mathcal{F}_{n,(t-1)n+i}\right] H_i' \pi_t \pi_t' H_i.$$
There exists a matrix $V = \sum_{t=1}^{T} V_t$, where V has finite elements and is positive definite *a.s.*, V_t is $\mathcal{C}$ measurable, and $\tilde{V}_{t,n} - V_t \xrightarrow{p} 0$ as $n \longrightarrow \infty$.

Alternative Moment Conditions

- **Assumption 2:** The following conditional moment restrictions hold:

$$E\left[u_{it} \mid \mathcal{B}_{n,i,t} \vee \mathcal{C}\right] = 0. \tag{7}$$

- Condition (7) implies (6) because $\mathcal{B}_{n,i,t} \vee \mathcal{C}_t^o \subset \mathcal{B}_{n,i,t} \vee \mathcal{C}$.
- The moment condition (7) is satisfied in models where the common factors are strictly exogenous.
- Our analysis includes the important case where no common factors are present by allowing $\mathcal{C}_t = \{\emptyset, \Omega\}$. In this case conditions (6) and (7) are identical, and Assumption 2 is automatically implied by Assumption 1.

Moment Conditions - Additional Observations

- When (6) holds but not (7) several cases leading to different limiting distributions for the central limit theorem below can be distinguished.
- It is important to note that unless V_t is a constant, a martingale central limit theorem for $m_{(n)}$ can not be established for a martingale defined on the σ-fields $\widetilde{\mathcal{B}}_{n,i,t} \vee \mathcal{C}_t^o$ where

$$\widetilde{\mathcal{B}}_{n,i,t} = \sigma \left\{ \left(x_{j2}^o, z_j, u_{j1}^o, \mu_j \right)_{j=1}^n, (u_{j2})_{j=1}^{i-1} \right\}.$$

- This is despite the fact that $E\left[h_{it} u_{it}^+ \mid \widetilde{\mathcal{B}}_{n,i,t} \vee \mathcal{C}_t^o \right] = 0$ and the CLT holds for the marginal $n^{-1/2} \sum_{i=1}^n h_{it} u_{it}^+$.
- However, joint convergence of the elements in $m_{(n)}$ only holds on the enlarged σ-fields $\mathcal{F}_{n,(t-1)n+i}$.
- With only (6) holding, $E\left[h_{it} u_{it}^+ \mid \mathcal{F}_{n,(t-1)n+i} \right]$ may not be zero and needs to be appropriately handled.

Stable Convergence

The presence of stochastic norming factors $\tilde{V}_{(n)}$ in the CLT requires a stronger form of convergence:

Definition

Let $(\Omega, \mathcal{F}, P)$ be a probability space and let $\mathcal{B}(\mathbb{R}^p)$ denote the Borel σ-field on $\mathbb{R}^p$. If $\{Z_n : n = 1, 2, \ldots\}$ and Z are $\mathbb{R}^p$-valued random vectors on $(\Omega, \mathcal{F}, P)$, and $\mathcal{F}_0$ is a σ-field such that $\mathcal{F}_0 \subset \mathcal{F}$, then

$$Z_n \xrightarrow{d} Z \ (\mathcal{F}_0\text{-stably})$$

if for all $U \in \mathcal{F}_0$ and all $A \in \mathcal{B}(\mathbb{R}^p)$ with $P(Z \in \partial A) = 0$,

$$P(\{Z_n \in A\} \cap U) \rightarrow P(\{Z \in A\} \cap U)$$

as $n \longrightarrow \infty$.

Stable Convergence - Comments

- In the following we will apply Definition 1 to establish stable convergence for $Z_n = S_{nk_n}$. The definition generalizes the definition of Hall and Heyde (1980) to allow for stable convergence on a sub σ-field $\mathcal{F}_0$ rather than on $\mathcal{F}$.
- Restricting stable convergence to $\mathcal{F}_0$ is important in our setting because the σ-fields $\mathcal{F}_{ni}$ are not nested in the sense that

$$\mathcal{F}_{ni} \subset \mathcal{F}_{n+1i} \text{ for all } i \leq k_n, n \geq 1,$$

a condition maintained by the central limit theorem of Hall and Heyde (1980).

Martingale Central Limit Theorem

Theorem

Let $\{S_{ni}, \mathcal{F}_{ni}, 1 \leq i \leq k_n, n \geq 1\}$ *be a zero mean, square integrable martingale array with differences* X_{ni}. *Let* $\mathcal{F}_0 = \cap_{n=1}^{\infty} \mathcal{F}_{n0}$ *with* $\mathcal{F}_{n0} \subseteq \mathcal{F}_{n1}$ *for each* n *and* $E[X_{n1} | \mathcal{F}_{n0}] = 0$ *and let* η^2 *be an a.s. finite random variable measurable w.r.t.* $\mathcal{F}_0$. *If*

$$\max_i |X_{ni}| \xrightarrow{p} 0, \quad \sum_{i=1}^{k_n} X_{ni}^2 \xrightarrow{p} \eta^2, \quad E\left(\max_i X_{ni}^2\right) \text{ is bounded in } n$$

then

$$S_{nk_n} = \sum_{i=1}^{k_n} X_{ni} \xrightarrow{d} Z \ (\mathcal{F}_0\text{-stably})$$

where the random variable Z *has characteristic function* $E\left[\exp\left(-\frac{1}{2}\eta^2 t^2\right)\right]$. *In particular,* $S_{nk_n} \xrightarrow{d} \eta\xi$ *(*$\mathcal{F}_0$*-stably) where* $\xi \sim N(0,1)$ *is independent of* η *(possibly after redefining all variables on an extended probability space).*

Additional Assumptions

- Assumption 3: Let $\tilde{m}_i = \left(E\left[h_{i1} u_{i1}^{+} \mid \mathcal{F}_{n,i} \right], ..., E\left[h_{T^+ i} u_{T^+ i}^{+} \mid \mathcal{F}_{n,(T^+-1)n+i} \right] \right)'$ and $b_n = n^{-1} \sum_{i=1}^{n} \tilde{m}_i$. One of the following statements holds:
  (a) $b_n \xrightarrow{p} b$ where b is finite a.s. and $\mathcal{C}$ measurable.
  (b) $\sqrt{n} b_n \xrightarrow{p} b$ where b is finite a.s. and $\mathcal{C}$ measurable.
  (c) $\sqrt{n} b_n \xrightarrow{p} 0$.

- Assumption 2 implies that $b_n = 0$, and thus Assumption 2 automatically implies Assumption 3(c). If no common shocks are present, Assumption 3(c) is also automatically implied by Assumption 1.

Main Result

Theorem

(a) Suppose Assumptions 1 and 2 hold. Then

$$m_{(n)} \xrightarrow{d} V^{1/2}\xi \quad (\mathcal{C}\text{-stably}), \tag{8}$$

where $\xi \sim N(0, I_p)$, *and* ξ *and* $\mathcal{C}$ *(and thus* ξ *and* V*) are independent.*
(c) Suppose Assumption 1 and either Assumption 2 or 3(c) hold. Suppose furthermore that

$$n^{-1}\sum_{i=1}^{n} u_{it}u_{is}H_i'\pi_t\pi_s'H_i \xrightarrow{p} 0 \quad \text{for} \quad t \neq s, \tag{9}$$

then $V_{(n)} - V \xrightarrow{p} 0$ *and* $V_{(n)}^{-1/2}m_{(n)} \xrightarrow{d} \xi \sim N(0, I_p)$.

Main Result

Theorem

(b)Let A be some $p_* \times p$ matrix that is $\mathcal{C}$ measurable with finite elements and rank p_* a.s.. Suppose Assumption 1 and either Assumption 2 or 3(c) hold, then

$$Am_{(n)} \xrightarrow{d} (AVA')^{1/2}\xi_*, \tag{10}$$

where $\xi_* \sim N(0, I_{p_*})$, and ξ_* and $\mathcal{C}$ (and thus ξ_* and AVA') are independent. If Assumptions 1 and 3(a) hold, then

$$A\left(m_{(n)} - \sqrt{n}b_n\right) \xrightarrow{d} (AVA')^{1/2}\xi_* \tag{11}$$

and $Am_{(n)}$ diverges. If Assumptions 1 and 3(b) hold then

$$Am_{(n)} \xrightarrow{d} (AVA')^{1/2}\xi_* + Ab \tag{12}$$

Main Result - Intuition

- Let $\lambda = (\lambda_1', \lambda_2', \ldots, \lambda_{T^+}')'$ be some nonstochastic vector, where λ_t is of dimension $(tk_x + k_z) \times 1$ and where $\lambda'\lambda = 1$. Then

$$\lambda' m_{(n)} = n^{-1/2} \sum_{i=1}^{n} c_i' u_i \tag{13}$$

where

$$c_i' = (c_{i1}, \ldots, c_{iT}) = \left(\lambda_1' h_{1i}', ..., \lambda_{T^+}' h_{T^+ i}'\right)$$

- $X_{n,1} = 0$, and for $i = 1, \ldots, n$ define

$$\begin{aligned} X_{n,i+1} &= n^{-1/2} c_{i1} u_{i1}, \\ X_{n,n+i+1} &= n^{-1/2} c_{i2} u_{i2}, \\ &\vdots \\ X_{n,(T-1)n+i+1} &= n^{-1/2} c_{iT} u_{iT}, \end{aligned} \tag{14}$$

Main Result - Intuition

- We can express $\lambda' m_{(n)}$ as

$$\lambda' m_{(n)} = \sum_{v=1}^{Tn+1} X_{n,v}. \tag{15}$$

- Then $\sum_{v=1}^{Tn+1} X_{n,v}$ is a martingale since
  - $\mathcal{F}_{n,v-1} \subseteq \mathcal{F}_{n,v}$,
  - $X_{n,v}$ is $\mathcal{F}_{n,v}$-measurable,
  - $E\left[X_{n,v} \mid \mathcal{F}_{n,v-1}\right] = 0$ because $\mathcal{F}_{n,(t-1)n+i} \subseteq \mathcal{B}_{n,i,t} \vee \mathcal{C}$.

An Economic Example

- Our framework follows Rust (1994), Aguirregabiria and Mira (2007) and Bajari, Benkard and Levin (2007).
- Players can take actions $a_{it} \in A$ where A is a finite set with elements $a_1, ..., a_J$. Let $\mathbf{a}_t = (a_{1t}, ..., a_{nt})$.
- Information in x_{it} is common knowledge; private signals $\varepsilon_{it}(a_j)$ indexed by actions $a_1, ..., a_J$.
- Let $\varepsilon_{it} = (\varepsilon_{it}(a_1), ..., \varepsilon_{it}(a_J))$, $\boldsymbol{\varepsilon}_t = (\varepsilon_{1t}, ..., \varepsilon_{nt})$ and $\mathbf{x}_t = (x_{1t}, ..., x_{nt})$.
- Players maximize expected discounted utility $\sum_{t=0}^{T} \beta^t E\left[\tilde{U}_i(\mathbf{a}_t, \mathbf{x}_t, \varepsilon_{it}) \middle| \mathbf{x}_0, \varepsilon_{i0}\right]$ with in period utility function

$$\tilde{U}_i(\mathbf{a}_t, \mathbf{x}_t, \varepsilon_{it}) = U_i(\mathbf{a}_t, \mathbf{x}_t) + \varepsilon_{it}(a_{it}).$$

An Economic Example - Distributional Assumptions

- Following Rust (1994) we impose the following additional assumption on the private signals $\boldsymbol{\varepsilon}_t$.
- Assumption 4: The transition probability $\pi(\mathbf{x}_{t+1}, \boldsymbol{\varepsilon}_{t+1} | \mathbf{x}_t, \boldsymbol{\varepsilon}_t, \mathbf{a}_t)$ satisfies
  (a)

$$\pi(\mathbf{x}_{t+1}, \boldsymbol{\varepsilon}_{t+1} | \mathbf{x}_t, \boldsymbol{\varepsilon}_t, \mathbf{a}_t) = \pi_\varepsilon(\boldsymbol{\varepsilon}_{t+1} | \mathbf{x}_{t+1}) \pi_x(\mathbf{x}_{t+1} | \mathbf{x}_t, \mathbf{a}_t). \tag{16}$$

  (b)

$$\pi_\varepsilon(\boldsymbol{\varepsilon}_t | \mathbf{x}_t) = \prod_{i=1}^n \pi_\varepsilon^i(\varepsilon_{it} | \mathbf{x}_t) \tag{17}$$

- It is convenient to define

$$
\begin{aligned}
& V_i\left(\mathbf{x}_t, \varepsilon_{it}, a; \sigma_{-i}\right) \\
= \quad & E\left[U_i\left(a, \sigma_{-i}\left(\mathbf{x}_t, \varepsilon_{-it}\right), \mathbf{x}_t\right) | \mathbf{x}_t, \varepsilon_{it}\right] \\
& + \beta \int \tilde{V}_i\left(\mathbf{x}_{t+1}, \varepsilon_{i(t+1)}; \sigma_{-i}\right) d\pi\left(\mathbf{x}_{t+1}, \varepsilon_{i(t+1)} | \mathbf{x}_t, a\right).
\end{aligned}
$$

- For any x in the support of $\mathbf{x}_t$ and any ε_i in the support of ε_{it} and defining $\varepsilon = (\varepsilon_1, ..., \varepsilon_n)$ a stationary Markov perfect equilibrium (MPE) then is defined as the strategy profile

$$\sigma(x, \varepsilon) = \left(\sigma_1(x, \varepsilon_1), ..., \sigma_n(x, \varepsilon_n)\right)$$

such that

$$\sigma_i(x, \varepsilon_i) = \arg\max_{a \in A} \left\{V_i(x, \varepsilon_i, a; \sigma_{-i}) + \varepsilon_i(a)\right\} \ \forall i, x, \varepsilon_i. \tag{18}$$

An Economic Example - Moment Conditions

- The optimality conditions (18) imply moment conditions that can be exploited for estimation (see Hotz and Miller or Bajari, Benkard and Levin)
- Define the equilibrium choice probabilities

$$\begin{aligned} p_{ij}\left(\mathbf{x}_t\right) &= \Pr\left(\sigma_i\left(\mathbf{x}_t, \varepsilon_{it}\right) = a_j | \mathbf{x}_t\right) \\ &= \int \mathbf{1}\left\{\sigma_i\left(\mathbf{x}_t, \varepsilon_{it}\right) = a_j\right\} d\pi_\varepsilon^i\left(\varepsilon_{it} | \mathbf{x}_t\right) \end{aligned}$$

  and define

$$u_{it}^j = \mathbf{1}\left\{\sigma_i\left(\mathbf{x}_t, \varepsilon_{it}\right) = a_j\right\} - p_{ij}\left(\mathbf{x}_t\right)$$

  and let $u_{it} = \left(u_{it}^1, ..., u_{it}^J\right)'$. It follows by construction that $E\left[u_{it} | \mathbf{x}_t\right] = 0$.
- It also follows from (17) that

$$E\left[u_{it} | \mathbf{x}_t, u_{-i,t}\right] = \int \left(\mathbf{1}\left\{\sigma_i\left(\mathbf{x}_t, \varepsilon_{it}\right) = a_j\right\} - p_{ij}\left(\mathbf{x}_t\right)\right) d\pi_\varepsilon^i\left(\varepsilon_{it} | \mathbf{x}_t\right) = 0$$

An Economic Example - Estimation

- We assume that we observe public information x_{it} for players $i = 1, ..., n$ and at $t = 1, ..., T$ as well as their actions a_{it} and let $y_{it,j} = 1\{a_{it} = A_j\}$.
- As in Rust (1994), assume that θ parametrizes $p_{ij}(x)$.
- Setting $p_i(\mathbf{x}_t, \theta) = (p_{i1}(\mathbf{x}_t), ..., p_{iJ-1}(\mathbf{x}_t))$ the moment conditions $E[u_{it}(\theta) x_{it}^o] = 0$ hold where

$$u_{it}(\theta) = y_{it} - p_i(\mathbf{x}_t, \theta).$$

- With $m_i(\theta) = (x_{i1}^o u_{i1}(\theta), ..., x_{iT}^o u_{Ti}(\theta))'$ the empirical moment condition $m_{(n)}(\theta) = n^{-1/2} \sum_{i=1}^n m_i(\theta)$ can be used for GMM.
- Provide asymptotic theory for $m_n(\theta)$ that does not rely on independent sampling assumptions.

GMM

- Consider the linear dynamic model

$$\begin{aligned} y_{it} &= x_{it}\beta_0 + z_{it}\gamma_0 + \mu_i + u_{it} \\ &= w_{it}\theta_0 + \mu_i + u_{it}, \end{aligned} \tag{19}$$

- We consider moment estimators that are based on first differences of (19) such that

$$E\left[h'_{t-1i}\Delta u_{it}\right] = 0 \text{ for } t = 2, \ldots, T, \tag{20}$$

where Δ is the difference operator. Let $H_i = \text{diag}\left(h_{i1}, ..., h_{i,T-1}\right)$, $\Delta y_i := \left(\Delta y_{i2}, ..., \Delta y_{iT}\right)'$ and $\Delta w_i := \left(\Delta w'_{i2}, ..., \Delta w'_{iT}\right)'$.

- The GMM estimator corresponding to the moment conditions (20) is defined as

$$\tilde{\theta}_n = \left(G'_n \tilde{\Xi}_n G_n\right)^{-1} G_n \tilde{\Xi}_n g_n$$

where $G_n = n^{-1} \sum_{i=1}^n H'_i \Delta w_i$, $g_n = n^{-1} \sum_{i=1}^n H'_i \Delta y_i$, and $\tilde{\Xi}_n$ is some weight matrix.

Theorem

Suppose Assumption 1, and either Assumption 2 or 3(c) hold, and that $G_n \xrightarrow{p} G$, $\tilde{\Xi}_n \xrightarrow{p} \Xi$, *where* G *and* Ξ *are* $\mathcal{C}$*-measurable,* G *and* Ξ *have finite elements and* G *has full column rank and* Ξ *is positive definite a.s.*
(a) Then

$$n^{1/2}(\tilde{\theta}_n - \theta_0) \xrightarrow{d} \Psi^{1/2}\xi, \quad \text{as } n \to \infty,$$

where ξ *is independent of* $\mathcal{C}$ *(and hence of* Ψ*),* $\xi \sim N(0, I_{k_x+k_z})$*, and*

$$\Psi = (G'\Xi G)^{-1} G'\Xi V \Xi G (G'\Xi G)^{-1}.$$

If in addition, $E\left[u_{it}^2 \mid \mathcal{F}_{n,(t-1)n+i}\right] = \sigma^2$ *for a constant* σ^2 *holds, then* $V = \sigma^2 \operatorname{plim}_{n\to\infty}\left(n^{-1}\sum_{i=1}^n H_i' D D' H_i\right)$.

(b) Suppose B *is some* $q \times k_x + k_z$ *matrix that is* $\mathcal{C}$ *measurable with finite elements and rank* q *a.s., then*

GMM Limiting Distribution (cont.)

Theorem

Suppose Assumption 1 holds, and that $G_n \xrightarrow{p} G$, $\tilde{\Xi}_n \xrightarrow{p} \Xi$, *where* G *and* Ξ *are* $\mathcal{C}$*-measurable,* G *and* Ξ *have finite elements and* G *has full column rank and* Ξ *is positive definite a.s.*
(a) If in addition Assumption 3(a) holds then

$$n^{1/2}(\tilde{\theta}_n - \theta_0 - (G'\Xi G)^{-1}G'\Xi b_n) \xrightarrow{d} \Psi^{1/2}\xi$$

and $\tilde{\theta}_n - \theta_0 \xrightarrow{p} (G'\Xi G)^{-1}G'\Xi b$.
(b) If in addition Assumption 3(b) holds then

$$n^{1/2}(\tilde{\theta}_n - \theta_0) \xrightarrow{d} \Psi^{1/2}\xi + (G'\Xi G)^{-1}G'\Xi b.$$

Feasible Implementation

- The limiting variance covariance matrix of $\hat{\theta}_n$ is then given by $\Psi = (G'V^{-1}G)^{-1}$.
- Can be estimated consistently by $\hat{\Psi}_n = \left(G_n' \widetilde{V}_{\Delta(n)}^{-1} G_n \right)^{-1}$ where $\widetilde{V}_{\Delta(n)} = n^{-1} \sum_{i=1}^{n} H_i' \widetilde{\Delta u}_i \widetilde{\Delta u}_i' H_i$.
- Let R be a $q \times (k_x + k_z)$ full row rank matrix and r a $q \times 1$ vector, and consider the Wald statistic

$$T_n = \left\| \left(R \hat{\Psi} R' \right)^{-1/2} \sqrt{n} (R\hat{\theta}_n - r) \right\|^2$$

  to test the null hypothesis $H_0 : R\theta_0 = r$ against the alternative $H_1 : R\theta_0 \neq r$.
- It can be shown that T_n is distributed asymptotically chi-square, even if Ψ is allowed to be random due to the presence of common factors represented by $\mathcal{C}$.

Conclusions

- Prove a general CLT that allows for random norming in the presence of common factors.
- Data can be cross-sectionally dependent of a non-parametric nature, even conditional on common factors.
- Moment conditions are satisfied for dynamic games with common an private information.
- Results are expected to be useful in spatial models and panel models with common factors.