Adoption Curves and Social Interactions

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1. Introduction

There has been a resurgence of interest in the role of social interactions in determining the rate at which technologies are adopted (Bandiera and Rasul (2006), Burke, Fournier, and Prasad (2007), Conley and Udry (2007), Goolsbee and Klenow (2002), Manski (2004,2006), Munshi (2004), Skinner and Staiger (2005), Young (2007)). Some of these recent efforts represent an attempt at resolution of early debates that emerged when Griliches produced his seminal work on the adoption of hybrid corn varieties in agriculture (1957,1958). Griliches was criticized by rural sociologists Havens and Rogers (1961) among others for ignoring the social determinants of adoption decisions[1]. An important methodological implication of the new social interactions and adoption literature is that it shows that economic incentives and social influences may be synthesized so that any antithesis assumed between economic and social explanations is a false one; private incentives and social incentives are both compatible with the choice-based logic that Griliches developed.

This paper focuses on the properties of a particular rational expectations model of heterogeneous atomistic potential adopters. "Social interactions" in the context of this model constitute positive feedback external spillover effects from the fraction having already adopted to the payoff received by each agent who has adopted; this type of social interaction has been dubbed "endogenous" by Manski (1993) as it involves feedback from the behaviors (as opposed to characteristics) of others onto each individual. Our objective is to identify properties of adoption curves that imply the presence of social interactions under relatively weak assumptions. Of course, any judgment on whether these assumptions are weak enough to be plausible will depend on context. In motivating our analysis in various places, we will return to the hybrid corn example, which continues to be of interest; Skinner and Staiger (2005) and Sutch (2008) are recent studies that revisit Griliches' analysis. The general literature on adoption of new technologies, network effects, learning effects, and the relationship of adoption to general social interactions is very large and had moved far beyond Griliches's early studies; see the survey by Hoppe (2002), but the hybrid corn example remains an exemplar.

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[1] See David (2005) for a discussion of this debate.

We characterize equilibria for rational adopters who maximize intertemporal profits. This characterization is oriented towards uncovering observational implications of social interactions on adoption curves that are robust to various types of observed and unobserved heterogeneity. In doing this, we are particularly concerned to identify observable implications that are robust to the presence of heterogeneity. To our knowledge the integration of social interactions into a rational expectations model of atomistic heterogeneous potential adopters and the study of the observational implications for the adoption curve is new.[2] We recognize that for many questions the adoption curve may be of less interest than other features of patterns of adoption (e.g. delays in adopting superior technologies, patterns of strategic interaction between a small number of major players, etc.), especially for policy making. Independent of any intrinsic interest in adoption curves, our analysis argues these curves may be used to uncover social interaction effects of the type that have been a primary focus of the recent literature, and can do so in a way that requires relatively weak assumptions by a researcher[3]

Our analysis reveals two properties that may assist empirical researchers in interpreting some interesting patterns in the data. First, we demonstrate that social interactions can produce jumps in the fraction of a population who have adopted by a particular date[4]. Second, we demonstrate that social interactions can produce pattern reversals in which agents whose private characteristics suggest they would adopt earlier than others but in fact adopt later. The potential for pattern reversals follows from the requirement that agents with greater ability to profit from a new technology adopt before those with relatively lesser ability. The reason that this requirement empirically

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[2]See Aradillas-Lopez (2007), Brock and Durlauf (2001a,b,2007), Graham (2008), Ioannides and Zabel (2008), Lee (2007) and Manski (1993) for examples of the relevant econometric literature on social interactions.

[3]Geroski (2000 p. 1) argues that the "dominant stylized fact" concerning new technology diffusion is that "the usage of new technologies over time typically follows an *S*-curve." He goes on to organize the diffusion literature according to whether this shape is due to social factors (what he calls an epidemic model) or individual factors (what he calls a probit model). Our analysis combines both explanations into a single model and considers what robust implications follow from the presence of social factors. We discuss the question of *S*-shaped adoption curves in Section 6.

[4]The capacity of rational expectations to produce interesting jumps in equilibrium variables has long been recognized; see Sargent and Wallace (1973) for a classic example.

distinguishes cases with and without social interactions is that under monotonic (in ability) adoption curves, lower ability agents face an environment in which a larger percentage of the population will have adopted than higher ability ones and hence may experience stronger social interaction effects because they adopt at a later date. In order to reconcile the difference in the strength of social interactions with the monotonicity requirement in adoption times and individual ability, discontinuities can occur. As for pattern reversals, the presence of discontinuities within adoption curves with respect to unobservables can break the monotonicity of adoption with respect to observables.

To be clear, neither of these properties is necessary for the presence of social interactions, and each is sufficient only in conjunction with additional assumptions on the adoption process. That said, the relatively weak nature of these assumptions (compared to others that appear in the literature) combined with the fact that these patterns only occur when social interactions are large (in a sense made precise below), provide strategies for empiricists to uncover social influences in adoption.[5] While we certainly do not claim to have established that either discontinuities or pattern reversals constitute a *sine qua non* of observable implications of social interactions in adoption contexts, we do believe they represent useful directions for uncovering social influences on individual behavior. As such, we follow a research strategy in the social interactions literature developed in Brock and Durlauf (2007), Graham (2008) of trying to identify implications of social interactions that hold for an array of assumptions on unobserved heterogeneity and so may be regarded as robust implications[6]. Concerns that unobserved heterogeneity can mask social

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[5]We thank an anonymous referee for this formulation.

[6]The ability of discontinuities and pattern reversals to reveal social interaction effects in adoption curves does not rely on the existence of multiple equilibria, which is the source of the pattern reversal findings in Brock and Durlauf (2007). Rather, it follows from the requirement that agents with greater ability to profit from a new technology adopt before those with relatively lesser ability. The reason this requirement empirically distinguishes cases with and without social interactions is that lower ability agents face an environment in which a larger percentage of the population will have adopted than higher ability ones and hence may experience stronger social interaction effects. In order to reconcile the difference in the strength of social interactions with the monotonicity requirement in adoption times and individual ability, discontinuities and pattern reversals can occur. The uncovering of evidence of social interactions via multiple equilibria would require repeated

influences were formalized at least as far back as Granovetter (1978); our findings suggest one way to constructively proceed.

This approach to uncovering social interactions complements the set of strategies that have been previously proposed to uncover social influences in adoption. For example, Strang and Tuma (1993) propose estimation methods for spatial and temporal heterogeneity in adoption; this is done by making parametric assumptions on the adoption times process and treating unobserved heterogeneity as independent and identically distributed across agents. Similarly, Goolsbee and Klenow (2002) use individual level data to estimate social influences by regressing individual adoption decisions on past aggregate adoption decisions in cities. This type of strategy presupposes a richer microeconomic data set than we do in order to construct individual-specific control variables as well functional form assumptions on the payoff function to adoption. It further relies on instrumental variables strategies for controlling for self-selection and unobserved group effects, the validity of which requires strong assumptions on the nature of the unobserved heterogeneity. Regression approaches have recently been augmented by analyses that use information social networks to help facilitate identification of social effects, e.g. Burke, Fournier, and Prasad (2007) and Conley and Udry (2007); these approaches also rely on functional form assumptions in developing inferences. Another approach to uncovering social interactions in adoption is due to Skinner and Staiger (2005) who employ cross state differences in adoption rates and their correlations with social capital measures to argue that social interactions are present. Analyses of that type require strong exchangeability assumptions about state-level behavior which may be problematic.[7] Our approach of course relies on a range of assumptions as well, hence our emphasis that we provide a complement to other strategies.

In terms of economic structure, two papers are closest to ours. Cabral (1990), in what appears to be a relatively neglected contribution, studies adoption curves when social interactions (in his language, network externalities) are present and shows how

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observations of the same environment, so that differences in the equilibrium outcomes can be used to infer the presence of the multiplicity. No such mechanism is available here.

[7] Brock and Durlauf (2001c) discuss the analogous difficulty in the context of cross-country growth regressions; see Durlauf (2002) for a discussion of the difficulties of using regressions to infer social capital effects.

discontinuities in adoption curves may result.[8] We differ from Cabral first in terms of our analysis of forward looking agents who face dynamic profit flows and second in our analysis of observable implications in the presence of various types of heterogeneity.[9] Our analysis also shares much in common with Young (2007) who compares the properties of adoption models with and without different types of social interactions, and like us focuses on uncovering properties of adoption curves that are robust to heterogeneity. While our goals are similar, our microstructures and analyses are not. We analyze environments in which adopters are forward looking whereas Young focuses on myopic adjustment rules that correspond to different types of social interactions. Heterogeneity in Young's analysis is associated with parameters of these adjustment processes whereas ours concerns observable and unobservable individual productivity. We find that social interactions produce different observable implications from those identified by Young. Unlike Young, we do not differentiate between types of endogenous social interactions.

2. Basic model

We model potential adopters as rational farsighted individuals who choose adoption times. We claim no novelty for the microeconomic specification of this forward looking model with the important exception of the introduction of social interactions. Variations on the model without social interactions appear in several papers in Punzo (2001); Brock (2001) also gives a review of some of this type of literature[10].

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[8]We became aware of Cabral's paper after writing the first draft of this one. We wish to emphasize Cabral's priority in uncovering the discontinuity property for adoption, despite differences in our microeconomic specifications.

[9]Other papers share important similarities with our analysis and Cabral's. de Paula (2007) studies a model of synchronization in the presence of social interactions. This analysis produces interdependences in hazard functions for individuals and in turn is related to Brock and Durlauf (2001b) and especially Sirakaya (2006) each of which directly embeds social interactions in a duration framework. Adsera and Ray (1998) consider a dynamic migration model with spillovers and show that lags in the effects of social interactions can eliminate multiple equilibria.

[10]Reinganum's pioneering work (1981a,1981b) is an especially important predecessor in as it provides adoption settings in which decisions are interdependent because of the

Formally, we consider a population of individuals each of whom chooses an adoption time t in order to maximize the present discounted value of current and future profits. We allow for heterogeneity across the individuals, by associating each individual with a scalar x which may be interpreted as individual ability. Ability should be understood as indexing individual productivity; we take no stance on its determinants as it is treated as exogenous to the model. This heterogeneity is drawn from some continuously differentiable distribution function $F_X(x)$ with associated density $f_X(x)$. The support of x is assumed to be an interval $I=[0,\hat{x}]$, where $\hat{x}$ may be infinite.

Individual actors choose adoption times t in order to maximize the objective function $J(t,x)$ defined as

$$J(t,x)=\left(-\exp(-\rho t)C+\int_t^\infty \exp((a-\rho)s)\pi(x,q^e(s))ds\right) \tag{1}$$

In this expression ρ is the discount rate, C is the cost of adoption, a is the rate of technical progress, $\pi(x,q^e(s))$ is expected profit flow at date s (which depends upon the type x), where $q^e(s)$ is the expected fraction of adopters in the population who have adopted by date s. The rate of technical progress is designed to proxy for quality improvements that augment the value of the adopted technology, such as improved software for computers. We assume that all agents have the same expectations. The dependence of profits on this fraction constitutes what we mean by social interactions. Relative to standard adoption models, the only innovation is the presence of $q^e(s)$ in the profit function. Our objective is to understand whether observed adoption behavior can reveal the role of social interactions in individual decisions.

We make the following assumptions on various elements of this decision problem.

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competitive structure. Reinganum shows that this force alone (with no heterogeneity across individual actors) is enough to produce many Nash equilibria with heterogeneous adoption times. In her analysis, early adoption by one firm hurts the profitability of adoption by others, which is the opposite of social interaction effects we study.

Assumption A.1. The discount rate exceeds the rate of technical progress, i.e. $\rho - a > 0$.

Assumption A.2. The expected level of adoptions at time t, $q^e(t)$, is a piecewise differentiable, monotone nondecreasing function with a finite number of points of discontinuity. For all t, conditional on $q^e(t)$ right derivatives are well defined and left limits exist.

Assumption A.3. The profit function $\pi(x,q)$ is strictly increasing in x, weakly increasing in q and is twice continuously differentiable in (x,q).

Assumption A.4. Each potential adopter of type x does not take into account the impact of his choice of adoption time t upon the adoption time choices of others.

Assumption A.1 is necessary to make sure that each individual's maximization problem is well defined; it simply ensures that the present discounted value of profits is bounded. Assumption A.2 restricts the set of admissible beliefs by imposing a monotonicity requirement. This is substantively restrictive. We make the assumption in order to render the problem of characterization of the set of rational expectations equilibria tractable.[11] We conjecture that monotone beliefs may be plausible in microfounded models where adopters are learning about their new technology by newsletters, user groups, and other mechanisms of information transfer where the usefulness of the information transferred is increasing in the fraction that have already adopted the particular technology under scrutiny. The assumption is also plausible in contexts where the new technology is generally understood to be one that will eventually become widespread; in the case of hybrid corn the US Department of Agriculture systematically proselytized on its

[11]The assumption is especially useful in finding rational expectations equilibria in that we follow the standard strategy in macroeconomics models of in essence conjecturing that agents form beliefs that have certain properties and then showing that these beliefs are justified in the equilibrium law of motion for the system.

behalf (Sutch (2008)), so it seems reasonable to assume that farmers expected the use of hybrid corn to monotonically rise. Similarly, one could imagine such beliefs for advances such as the Pentium chip that are recognized as state-of-the-art and certain to eventually supplant previous technologies. Assumption A.3 restricts the payoff function and is therefore substantive in its impositions of monotonicity, but is of course far weaker than assuming a particular functional form. The requirement that the heterogeneity scalar x has a monotonic effect on profits limits what sorts of interpretation may be placed on it. The assumption that the profit function is increasing in q means that we are focusing on strictly positive interactions. As such, the assumption is consistent with claims about network goods; Gandal (2008) surveys evidence of these effects for a very wide of contexts. Our analysis does not distinguish between mechanisms for network effects, i.e. a direct benefit from the adoption of a technology by others as occurs with a word processing program or information externalities by which the usage of a technology by others communicates relative payoffs to an individual or even factors such as increasing support structure for a technology based on market size[12]. Assumption A.4 allows us to ignore strategic interactions, i.e. we treat each agent as atomistic. The assumption is sensible when an individual adopter is small relative to the overall group.

All propositions stated in the paper assume at least some of A.1-A.4; for ease of exposition we assume all of them for each of our lemmas and theorems. Further, we always assume that there is a finite date in the past when the new technology first appears and the first adoption occurs after that date; without loss of generality we designate the date at which the innovation first appears as 0.[13] Finally, we conceptualize each agent as

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[12]Our blackbox approach to social interactions is common in the social interactions literature and indeed a limitation of the literature.

[13]Sutch (2008) shows that this is precisely what happened in the case of hybrid corn. Hybrid corn was invented by Donald F. Jones in 1917-1918, was developed and introduced on a trial basis by Henry A. Wallace in 1924, was first sold commercially in 1925, competitors began sales in 1928, and widespread commercial adoption began in 1932. In 1933 .1 percent of the nation's corn acreage was planted in hybrid corn and by 1960 hybrid corn was planted on 96.3 percent of the nation's corn acreage. 1917 corresponds to our $t = 0$.

choosing an adoption time from $(-\infty, T)$, this avoids problems of corner solutions in the optimization problem.

The first order necessary condition (FONC) and secondary order necessary condition (SONC) for the optimal adoption time t^* by type x are given by

$$t^*(x) = \frac{1}{a}\left(\ln(\rho C) - \ln\left(\pi\left(x, q^e(t)\right)\right)\right) \tag{2}$$

and

$$-\rho^2 C \leq \exp(at)\left(\left(a-\rho\right)\pi\left(x, q^e(t)\right) + \frac{\partial \pi\left(x, q^e(t)\right)}{\partial q^e(t)} \frac{dq^e(t)}{dt}\right) \tag{3}$$

respectively for dates t, if $q^e(t)$ is differentiable at t. However, as will become clear, we do not wish to assume differentiability at all dates. If $q^e(t)$ is not differentiable at date t, observe that by A.3 $q^e(t^-) < q^e(t^+)$ where "$-$" denotes the operation of taking the left limit and "+" denotes the operation of taking the right limit. A local maximum is characterized (in our case) by the left limit being greater than the right derivative with a zero between these two values. From (1), the first and second order necessary conditions for a local maximum at t may be expressed as

$$\begin{gathered} J'(t^-, x) = e^{-\rho t}\left(\rho C - e^{at}\pi\left(x, q^e(t^-)\right)\right) \geq \\ 0 \geq \\ J'(t^+, x) = e^{-\rho t}\left(\rho C - e^{at}\pi\left(x, q^e(t^+)\right)\right) \end{gathered} \tag{4}$$

since this inequality can be decomposed in terms of the usual FONC and SONC, $J'(t,x)=0, J''(t,x)\leq 0$ when $q^e(t)$ is differentiable at t.[14]

In the case where $q^e(t)$ is differentiable at date t, the FONC is quite intuitive as it amounts to equating the marginal benefit to adoption at a given time with the associated marginal cost. To understand the SONC, rewrite (2) as

$$\rho C = \exp(at)\pi\left(x, q^e(t)\right). \tag{5}$$

Substituting into (3), the second order condition holds if and only if

$$0 \leq a\pi\left(x, q^e(t)\right) + \frac{\partial \pi\left(x, q^e(t)\right)}{\partial q^e(t)} \frac{dq^e(t)}{dt}. \tag{6}$$

Equation (6) is satisfied provided that $\pi\left(x, q^e(t)\right) \geq 0$ $\frac{\partial \pi\left(x, q^e(t)\right)}{\partial q^e(t)} \geq 0$ and $\frac{dq^e(t)}{dt} \geq 0$. The first inequality is immediate from the FONC, since the cost of adoption is positive. The second inequality holds by Assumption A.3 above. The third inequality cannot be assessed without specification of the expectations formation process. In our subsequent analysis $\frac{dq^e(t)}{dt} \geq 0$ will be shown to hold in equilibrium under rational expectations. In the case where $q^e(t)$ is not differentiable at t we will show that $q^e(t^-) \leq q^e(t^+)$ holds in equilibrium. Hence it will be the case that the SONC holds for interior critical points t for (2). If $\rho C < \pi\left(x, q^e(0)\right)$, this is a signal that optimal t is some negative number for type x.

The following lemma provides a sufficient condition for uniqueness of an individual adoption time.

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[14] Throughout, we write $t^- = t = t^+$ for the parts of functions that are continuous in t.

Lemma 1. Uniqueness of optimal adoption time

Given, A.1-A.5, the optimal time to adopt for each agent type exists and is unique.

For comparative purposes, we note the case where there are no social interactions, i.e. $\pi\left(x,q^{e}\left(t\right)\right)=\pi\left(x\right)$. In this case the optimal adoption times follow

$$t^{*}\left(x\right)=\frac{1}{a}\left(\ln\left(\rho C\right)-\ln\pi\left(x\right)\right) \text{ if } x<\bar{x}, \qquad t^{*}\left(x\right)<0 \text{ if } x\geq\bar{x} \tag{7}$$

where $\bar{x}$ is defined by $t^{*}\left(\bar{x}\right)=0$ (if such an $\bar{x}$ exists). Observe that $t^{*}\left(x\right)$ is decreasing in x by Assumption A.3; this simply means that higher productivity types adopt earlier as the profit incentives are higher.

3. Adoption curves and rational expectations equilibria

We now consider equilibria under social interactions by imposing a rationality requirement on beliefs about adoption levels. This assumption may naturally be criticized, especially in the context of a technological innovation. On the other hand, to the extent that our analysis is designed to compare environments with and without social interactions, expectations only matter when social effects are absent, hence deviations from rationality are not relevant under the "null hypothesis" that social effects are absent. An interesting exercise, beyond the scope of this paper, is to understand whether plausible alternative expectations formation mechanisms can restrict adoption patterns in robust ways analogous to those we find.

To develop a rational expectations equilibrium, observe first that, for any expectations process, one can construct the actual adoption curve $q\left(t\right)$ that describes the percentage of the population which has adopted by date t. We first note a lemma.

Lemma 2. Monotonicity of adoption with respect to ability

Given A.1-A.4, suppose $t < t_1$ and $x < x_1$. If x_1 adopts at time t_1, then for all $x < x_1$, the adoption time t of type x satisfies $t \geq t_1$.

Monotonicity means that the adoption rate may be calculated using the formula

$$q(t) = \int_{a(t)}^{\infty} f_X(z)\,dz = 1 - F_X(a(t)) \tag{8}$$

where the lower integral limit $a(t)$ is implicitly defined by

$$t^*(a(t)) = t. \tag{9}$$

Eq. (8) means that there is a tight link between the adoption curve and the distribution function of the type-specific heterogeneity. To understand eq. (9), recall that we have shown that for each x there exists a unique adoption time $t^*(x)$, a decreasing function which may be discontinuous, i.e. $t^*(x^-) > t^*(x^+)$ may occur. At continuity points of $t^*(x)$, the associated inverse function is well defined and one can solve for the value $x = a(t)$ that satisfies (9) at date t. At discontinuity points of $t^*(x)$ we use the right limit x^+ as x decreases towards the point of discontinuity. This choice rule is consistent with Assumption A.2; this ensures that $q^e(t)$ is increasing in t.

Since the profit functions are assumed to depend on the expected adoption curves of each agent, rational expectations equilibria thus are straightforward to define as they require that the beliefs about adoption rate coincide with the actual adoption rates along an equilibrium path.

Definition: rational expectations equilibrium (REE).

A *rational expectations equilibrium* is a pair of functions $t^*(x)$ and $q^*(t)$ such that

i. individual adoption times are optimal,

and

ii. the aggregate adoption curve is consistent with these individually optimal choices.

The requirements of a rational expectations equilibrium, implicitly characterizes the optimal adoption times. At points of differentiability, the optimal times must fulfill

$$t^*(x) = \frac{1}{a}\left(\ln(\rho C) - \ln\left(\pi\left(x, q^*(t)\right)\right)\right) \text{ if } t^*(x) > 0 \tag{10}$$

and the equilibrium fraction who have adopted by date t, $q(t)$, must fulfill

$$q^*(t) = \Pr\left(x \middle| \rho C \exp(-at) \le \pi\left(x^*(t), q^*(t)\right)\right) = 1 - F_X\left(x^*(t)\right). \tag{11}$$

Substituting (11) into (10)

$$t^*(x) = \frac{1}{a}\left(\ln(\rho C) - \ln \pi\left(x^*(t), 1 - F_X\left(x^*(t)\right)\right)\right) \tag{12}$$

In contrast to this function, one can consider a best response function for each type in which the population fraction adopting corresponds to the distribution of types:

$$S(x) = \frac{1}{a}\left(\ln(\rho C) - \ln \pi\left(x, 1 - F_X(x)\right)\right). \tag{13}$$

This $S(x)$ function differs from $t^*(x)$ as it is the best response function for an individual without self-consistency imposed between the adoption time and adoption curves as an equilibrium condition.

4. Restrictions on the shape of the adoption curve

In this section, we consider how social interactions manifest themselves in observed adoptions decisions. We follow the tradition started with Griliches (1957) in placing primary emphasis on the shape of the equilibrium adoption curve $q^*(t)$. We consider how social interactions, in the sense that the profit function $\pi(x,q)$ increases in both x and q restricts this curve. Our goal is to understand how restrictions emerge in light of heterogeneity as characterized by $F_X(x)$ and the unobservability of $\pi(x,q)$. In other words, we are interested in uncovering empirical implications of social interactions that are robust to unobserved heterogeneity in terms of individual types and the associated profit functions that characterize agents.

In order to develop restrictions on the shape of the adoption curve which are generateed by social interactions, we focus on eq. (13). When social interactions are absent, $\pi(x,q)=\pi(x)$ and so is monotonically increasing in x by Assumption A.3; this implies that $S(x)$ is monotonically decreasing as well. On the other hand, when social interactions are present the $\pi(x,1-F_X(x))$ in (13) is not necessarily increasing in x despite Assumption A.3 because $1-F_X(x)$ is decreasing in x; by (13) nonmonotonicity may be transferred to $S(x)$. Intuitively, this means that, as a best response, higher ability agents may not find it as profitable to adopt as lower ability agents because of the absence of others who adopt at the same or earlier times. The breakdown, in the presence of social interactions, of the monotonicity of the profit function with respect to x and associated nonmonotonicity of $S(x)$ has important implications for the behavior for the observable, $q^*(t)$.

We start by concentrating on a form of $S(x)$ such that 1) $S(0)=\infty$, when x is small, and 2) the function has a unique minimum $\underline{x}$. Let $x_1(t)$ denote the smallest and $x_2(t)$ the largest solutions to $t=S(x)$. We wish to demonstrate that $x_1(t)$ can be part of a rational expectations equilibrium, but that $x_2(t)$ cannot. Suppose that $x_2(t)$ is part of an REE. At $x_2(t)$, higher t^* values are associated with larger x values. This solution is not one that will be produced by agents choosing optimally conditional on their commonly shared belief that $x_2(t)$ is an equilibrium; this contradiction follows from Assumption A.3 that the partial derivative of the profit function with respect to x is positive. Hence only $x_1(t)$ has the potential to be part of an REE solution. This type of argument generalizes to produce Lemma 3; the proof is trivial given our previous results and is therefore omitted.

Lemma 3. Monotonicity along equilibrium adoption paths

Given A.1-A.4, along an equilibrium adoption path $q^*(t)=1-F_X\left(x^*(t)\right)$,

A. the process for optimal adoption times $t^*(x)$ must obey $\frac{dx}{dt}<0$ for points of differentiability.

B. At jumps, larger x types adopt earlier than smaller x types.

How does the possibility of nonmonotonicity of $S(x)$ interact with the requirement that $\frac{dx}{dt}<0$ holds in equilibrium? The reconciliation of these requirements places restrictions on $q^*(t)$. Intuitively, in order to ensure that $\frac{dx}{dt}<0$, at differentiable times it is necessary that for higher x's, a sufficient number of adopters are present to ensure that adoption is monotonic in type. In order for this to always hold, it may be necessary for the $q^*(t)$ function to exhibit discontinuities. This can produce discontinuities of the following form: as t increases from slightly below t^* to slightly above t^*, a mass point of size $F_X(x^*) - F_X(x') > 0$ adopts where $x' < x^*$ is defined by

$$S(x') = S(x^*). \tag{14}$$

This jump in the number of adopters means that the $t^*(x)$ is never increasing in x. The adoption of a mass point at date t^* causes a jump of size

$$q(t^+) - q(t^-) = F_X(x^*) - F_X(x') > 0, \tag{15}$$

to occur at time $t = t^*$. Hence an observable implication of social interactions, when the interactions are strong enough to induce a nonmonotonicity in the function $S(x)$ in x, is that there may exist at least one jump in $q^*(t)$.

As our objective is to explicitly link discontinuities in the REE function $q^*(t)$ to the identification of social interactions, we formulate a theorem for the particular case of $S(x)$ in which the function has a single local minimum $x_{\min}$ and a single finite local maximum $x_{\max}$; Figure 1 illustrates the qualitative shape of $S(x)$ that is assumed in the theorem The structure of the theorem makes it evident that other formulations are possible. While we conjecture that a more general theorem may be produced which directly maps the nonmonotonicity of $S(x)$ to discontinuities in $q^*(t)$, we have yet to show this; the difficulty is that in dynamic models, one needs to evaluate the best reply function against the full dynamic path of the associated adoption curve. A parametric example of the theorem is provided in the Appendix.

Theorem 1. Discontinuities in the adoption curve as evidence of social interactions

Given A.1-A.4, suppose that

i. $S(x)$ is continuously differentiable in x, $S(0)=\infty$,

ii. $S(x)$ is initially strictly decreasing in x, until it takes a positive local minimum at $x_{\min}$,

iii. $S(x)$ increases in the open interval $(x_{\min}, x_{\max})$,

iv. $S(x)$ takes a unique local maximum at $x_{\max} > x_{\min}$, and then strictly decreases in x for all $x > x_{\max}$ until some point $\tilde{x} > x_{\max}$ is reached

v. For all $x > \tilde{x}$, $S(x) < 0$.

Define $x_* < x_{\min} < x_{\max}$ via $t^* = S(x_{\max}) = S(x_*)$. Then,

A. There exists an REE, $q^*(t)$ defined by, $q^*(t) = 1 - F_X(x^*(t))$, for which $x^*(t)$ satisfies the equation,

$$t = S(x^*(t)) \tag{16}$$

where $x^*(t)$ is defined to be the largest solution to the equation $t = S(x)$, for $t \le t^*$ and $x^*(t)$ is defined to be the unique solution to the equation $t = S(x)$, for $t > t^*$ where $t^* = S(x_{\max})$.

B. There exists one jump point of positive size in the equilibrium adoption curve $q^*(t)$ which occurs at $t^* = S(x_{\max})$,

C. There is a mass point, $1-F_X(\tilde{x})$, at $t=0$.

This discontinuity property represents a variant of a partial identification argument in econometrics, cf. Manski (2003). By this, we mean that the presence of social interactions can, for certain magnitudes of the interactions, place restrictions on observable data, but that these restrictions neither identify the exact magnitude of the interactions (in this case the effects of q on $\pi(x,q)$ nor are the restrictions necessary for the interactions to be present[15]. The presence of a discontinuity means that in comparing the adoption model with and without social interactions, i.e. comparing the implications of the dependence of the profit function on q, a discontinuity may be interpreted as evidence of social interactions. However, the absence of a discontinuity does not imply that social interactions are absent. The discontinuity property requires that the strength of the influence of social interactions is large. Given the absence of any restrictions on $\pi(x,q)$ beyond Assumption A.2, one cannot be more precise about what is meant by larger; if a functional form were chosen in which a parameter measured the strength of social interactions, then one could interpret the presence of discontinuities as a statement about the parameter's magnitude. "Large" is model-specific.

Of course, discontinuities in $F_X(x)$ can produce discontinuities in $q^*(t)$ even when social interactions are absent. If individual adoption times and associated values of x are observable (so that $F_X(x)$ is observable), this alternative explanation can be assessed as any discontinuity at t_d would have to align with a discontinuity at x_d so that $t^*(x_d)=t_d$. An example in the hybrid corn context is the date 1936 when the news that hybrid corn produced much more then open pollinated varieties in that drought year spread like wildfire through the farming community and caused the demand for hybrid seed to explode (Sutch (2008, page 18)). This would look like a jump in the data but would not be

[15]Our form of partial identification differs from that typically found in Manski's work as his approach produces bounds on unknown objects of interest whereas ours uncovers the sign of an object of interest (the partial derivative of the profit function with respect to adoption) and information about the magnitude as it affects the equilibrium behavior of the adoption curve.

due to social interactions in the sense that the jump is not caused by $\frac{\partial \pi}{\partial q} > 0$. Of course if one conditions on relative yield information in 1936, separation of causes of jumps in adoption rates due to $\frac{\partial \pi}{\partial q} > 0$ in contrast to an exogenous driver of jumps (such as information on the relative productivity of hybrid corn in drought conditions) should be possible.[16]

Once one introduces individual heterogeneity, it is possible to develop empirical implications of social interactions which do not rely on the presence of jumps. For example, if one observes the adoption under two different distributions of observed heterogeneity, one of which stochastically dominates the other, the absence of social interactions places strong restrictions on the associated adoption curves.

Theorem 2. Stochastic dominance and the absence of social interactions

Suppose that the distribution function $F_{X,1}(\cdot)$ (with associated adoption curve $q_1(t)$) is stochastically dominated by the distribution function $F_{X,2}(\cdot)$ (with associated adoption curve $q_2(t)$. If there are no social interactions, i.e. $\pi(x,q) = \pi(x)$, then it must be the case that $q_1(t) \leq q_2(t)$ for all *t*.

The application of this Theorem would require that a researcher is able to identify distinct cases of adoption under different $F_{X,i}(\cdot)$'s.

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[16]One might argue that in the hybrid corn historical record our model is most relevant to the period after the 1936 jump. Following Sutch (2008), this is so because of the combination of the key initial role of Henry Agard Wallace in promoting the use of hybrid corn and the fact that it did not dramatically expand until prompted by the news of the very high relative yields in the drought year, 1936. Sutch discusses the continued improvement of varieties of hybrid corn, the role of demonstration plots, the sharing of information about tailoring varieties to specific land types and the role of agricultural extension offices in the diffusion process of hybrid corn.

5. Unobservable heterogeneity

In this section we introduce unobservable heterogeneity and consider how empirical implications are affected. Interestingly, unobservable heterogeneity can produce a different route to uncovering social interactions that is not present when the heterogeneity is absent. We focus on the case of one observable and one unobservable. Before doing so, we first illustrate how this case can capture key features of the general case of multiple observables and multiple unobservables, so long as π is monotonically increasing in all variables. An elementary result in functional separability theory as applied to general utility theory (Varian (1992, p. 150)) is that for a vector x and scalar q, if 1) $\pi(x,q)$ is strictly increasing in both arguments and 2) $\pi(x,q)\geq\pi(x',q)$ if and only if $\pi(x,q')\geq\pi(x',q')$ for all x, x', q and q', then there exists a "subprofit" function $u(x)$ (mapping x to a scalar) and an aggregator function $V(u,q)$ such that

$$\pi(x,q)=V(u(x),q). \tag{17}$$

If $V(u,q)$ is strictly increasing in (u,q) one may simply repeat our earlier analysis by replacing x in the above analysis with u. Hence in an REE, it must be the case that $q=1-F_U(u)$. In an REE, when $V(u,q)$ is independent of q, individuals with higher levels of u adopt first. When $V(u,q)$ varies in q, the best response adoption times for individuals with level u is given by

$$S(u)=\frac{1}{a}\left(\ln\rho C-\ln V\left(u,1-F_U(u)\right)\right) \tag{18}$$

In parallel to our earlier discussion, the unconditional equilibrium adoption curve is given by

$$q^*(t) = 1 - F_U(t) \qquad (19)$$

where

$$\begin{gathered} t^*(u) = \frac{1}{a}\left(\ln(\rho C) - \ln \pi\left(u^*(t), 1 - F_U\left(u^*(t)\right)\right)\right) \text{ if } t^*(u) > 0 \\ t^*(u) < 0 \text{ if } \ln\left(\pi\left(u, q^*(0)\right)\right) \geq \ln(\rho C). \end{gathered} \qquad (20)$$

(Note that since we are focusing only on positive adoption times, we use the convention that we just state $t^*(u)$ for negative adoption times even though the equality still holds for negative adoption times.)

Functional separability provides a general strategy for introducing unobservables. Suppose that productivity is a vector rather than a scalar attribute. Partition the vector as $x = (x_1, x_{-1})$ where x_1 is observable and x_{-1} is not. We may now repeat our earlier analysis to conditional adoption curves $q^*(t|x_1)$. These curves must obey

$$q^*(t|x_1) = 1 - F_{U|X_1}\left(u(t \mid x_1)\right). \qquad (21)$$

This indicates how the scalar observable/ scalar unobservable case we consider is in fact quite general.

We now focus on $x = (x_1, x_2)$, where each component is one dimensional. Notice that if one had infinite data on each "cell" x_1, i.e. one observes the conditional adoption curve $q(t \mid x_1)$ for every x_1, then this would be enough to "reveal" the one dimensional unobservable x_2 except at those dates when nonmonotonicity of the conditional best response time $S(x_2 \mid x_1)$ occurs. To put it another way if the function $S(x_2 \mid x_1)$ is one-to-one for all x_1 then knowledge of the conditional adoption curves at *t* and the observable covariate x_1 reveals the unobservable x_2. On the other hand when the function $S(x_2 \mid x_1)$ is not monotone in x_2 for some x_1 then our results derived above on the characterization of REE's apply, i.e. there will typically be a discontinuous burst of adoption as time t passes through a critical value where the function is not monotone for this particular value of x_1.

Without any restrictions on the unobservables, one cannot identify any observable implications for social interaction effects on adoption curves. Any adoption pattern that is generated with social interactions can be replicated with them by suitable choice of the process describing the unobservable variable. We proceed by using a "weak" assumption on the unobservable:

A.5. $F_{X_2|X_1}$ is stochastically increasing in X_1 in the sense that $x_1' > x_1''$ implies

$$F_{X_2|X_1}\left(x_2 \middle| x_1'\right) \le F_{X_2|X_1}\left(x_2 \middle| x_1''\right). \tag{22}$$

Stochastic dominance is an example of a shape restriction. Shape restrictions are not only weaker than functional form restrictions, they are often interpretable in economic terms in ways that functional forms are not. For example, if one thinks of individual farmers as characterized by an unobservable ability level and an observable education level, then stochastic dominance says nothing more than higher education levels imply that the density of abilities is shifted to the right. Brock and Durlauf (2007) show how stochastic dominance can facilitate identification of social interactions, using partial identification arguments. This work, in turn, draws from Manski (1997) and Manski and Pepper (2000) who emphasize the constructive role of assumptions such as shape restrictions.

Theorem 3. Monotonicity of adoption curves with respect to observables.

Given A.1-A.5, suppose that both the observables x_1 and the unobservables x_2 are one dimensional.

A. If there are no social interactions, i.e. $\frac{\partial \pi(x_1, x_2, q)}{\partial q} = 0$

$$x_1' > x_1'' \text{ implies } q(t|x_1') > q(t|x_1'') \ \forall\ t \geq 0. \tag{23}$$

B. If

$$x_1' > x_1'' \text{ but } q(t|x_1'') > q(t|x_1'), \tag{24}$$

then it must be the case that $\frac{\partial \pi(x_1, x_2, q)}{\partial q} \neq 0$.

The theorem is intuitive. One expects agents with higher values of observable characteristics x_1 to have better values of unobservable characteristics on average and, hence, to have higher profits on average which cause more of them to adopt at any given observed date. This argument in turns leads to the idea of a pattern reversal as evidence of social interactions. A pattern reversal occurs when differences in characteristics between two populations of agents suggest one relationship between the associated group behaviors whereas the opposite is observed. In our context, a pattern reversal occurs if the one group is adopting more rapidly than another, whereas the private incentives experienced by members of each group predict the opposite pattern. Corollary 1 formalizes this idea and follows immediately from Theorem 3.

Corollary 1. Pattern reversals and order reversals

Assume A.1-A.5. Suppose there exists a fixed time t_0 such that for $x_1' > x_1''$

$$q\left(t_0 \mid x_1'\right) < q\left(t_0 \mid x_1''\right), \tag{25}$$

i.e. a greater percentage of low type agents have adopted by t_0 than high types. Then either $F_{X_2|X_1}\left(x_2 \mid x_1'\right) > F_{X_2|X_1}\left(x_2 \mid x_1''\right)$ or $\frac{\partial \pi\left(x_1, x_2, q\right)}{\partial q} \neq 0$.

As a special case of the Theorem and Corollary, if x_1 and x_2 are independent, then a pattern reversal is interpretable as evidence of social interactions.

Our earlier remarks on functional separability indicate that Theorem 3 and Corollary 1 generalize beyond the scalar observable/scalar unobservable case. Following our earlier discussion, let the profit function equal $\pi\left(x_1, x_{-1}, q\right)$ and assume that the profit function is increasing in all arguments. Assume there exist monotone increasing scalar valued functions V and u such that $\pi\left(x_1, x_{-1}, q\right) = V\left(x_1, u\left(x_{-1}\right), q\right)$. One can then replace the scalar x_{-1} in the argument leading to Theorem 3 and Corollary 1 with the scalar $u\left(x_{-1}\right)$ and obtain generalizations to an arbitrary number of unobservables. Of course, the functional separability of payoffs with respect to a vector of unobservables may be more difficult to justify based on heuristic economic reasoning.

6. Comparisons to other approaches to uncovering social interactions in adoption

In this section, we consider some previous approaches which have been proposed for uncovering social interactions via adoption curves.

A. logistic functions and adoption curves

It is often claimed that logistic adoption curves are evidence of social interactions; Schelling (1997) provides an overview of interpretations of logistic curves as evidence of social effects; see also Geroski (2000). This type of argument typically derives from a view of social interactions as a type of infection phenomena in which the fraction of those who have not adopted, $1-q(t)$, changes via

$$\frac{dq(t)}{dt} = Rq(t)\left(1-q(t)\right). \tag{26}$$

See Daley and Gani (2001) for discussion of this equation in epidemiology and Berndt, Pindyck and Azoulay (2003) for an example of an analysis that employs this type of equation to empirically model social effects on diffusion[17].

For our model, a logistic curve does not represent evidence of social effects. In our model, the derivative of the adoption curve is

$$\frac{dq(t)}{dt} = -f_X(x)\left(\frac{dS(x)}{dx}\right)^{-1} = R\left(1-F_X(x)\right)\left(F_X(x)\right), \tag{27}$$

and can be constructed by solving the ordinary differential equation,

$$S'(x) = -\frac{f_X(x)}{R\left(1-F_X(x)\right)F_X(x)}. \tag{28}$$

But it follows from the definition of $S(x)$ that

$$S'(x) = -\left(\frac{1}{a}\right)\frac{\pi'(x)}{\pi(x)}. \tag{29}$$

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[17]Interestingly, Griliches (1962) defended himself against the claim by Havens and Rogers (1961) that he ignored social factors by arguing that his use of logistic curves captured "interaction."

Hence, given $f_X(x)$ (and the associated distribution function $F_X(x)$), in order to have a logistic dynamic of the form (26) generated by our model, all one must do is to construct a $\pi(x)$ function that satisfies the equation,

$$\ln(\pi(x)) - \ln(\pi(x_0)) = \frac{a}{R}\int_{x_0}^{x} \frac{f_X(x)}{((1-F_X(x))F_X(x))}dx, \tag{30}$$

for some lower bound x_0. This indicates how logistic dynamics can result from the shape of the profit function $\pi(x)$ with no implication about social interactions in the adoption process. Since the profit function is not observable, one cannot use the shape of the adoption curve to infer anything about social interactions. Put differently, $\pi(x)$ represents a latent variable that varies across types and is therefore interpretable as a form of unobserved heterogeneity; eq. (30) shows that the mapping of a logistic curve to social interactions is not robust when this type of heterogeneity is present.

One can develop a parallel analysis to demonstrate that, in the context of our model, logistic adoption curves may be generated in absence of social interactions. Suppose that the profit function does not embody social interactions and has the form

$$\pi(x,q) = Ax^{\alpha},\ 0<\alpha<1 \tag{31}$$

Mimicking our earlier arguments, profit maximization implies that equilibrium adoption times are implicitly defined by

$$x(t) = \left(\frac{\rho C}{A}\right)^{1/\alpha} e^{-(a/\alpha)t} \tag{32}$$

Previous arguments establish that $\frac{dq}{dt} = -f_X(x)\frac{dx}{dt}$ therefore (32) implies $\frac{dx}{dt} = -\frac{ax}{\alpha}$. Further, (26) indicates the necessary condition for a logistic adoption curve. Combining these yields the following differential equation for $F_X(x)$.

$$\frac{a}{\alpha}\frac{dF_X(x)}{F_X(x)\left(1-F_X(x)\right)} = r\frac{dx}{x} \tag{33}$$

Integrating both sides of (33) (using partial fractions for the LHS) one obtains the solution

$$F_X(x) = \frac{c_0 x^{r\alpha/a}}{1+c_0 x^{r\alpha/a}},\ c_0 > 0 \tag{34}$$

and

$$q(t) = \frac{d_0}{1+d_0 e^{-rt}} \tag{35}$$

where the constant $d_0 = c_0\left(\frac{\rho C}{A}\right)^{r/a}$. This is a logistic function.

Our demonstration that a logistic shape may arise from a variant of our model without interactions derives from a particular distribution function $F_X(x)$. One can therefore object that this is a knife edge case. However, the derivation of a logistic curve itself depends on special functional form assumptions. Further, as originally noted by Feller (1940) and further argued in Brock (1999) and Dinardo and Winfree (2007), it is very difficult to distinguish logistic functions from other *S*-shaped functions. This has led to some authors arguing that social versus individual explanations of adoptions are distinguished by accelerating versus nonaccelerating adoption curve shapes[18]. We note

[18]Reader (2004, p. 90) argues that this view is especially true in social learning models: "In general, social learning processes are argued to result in accelerating diffusion curves, such

here that our argument on the inability of the logistic to differentiate presence or absence of social interactions immediately adapts to this case[19].

B. relative acceleration rates

Young (2007) is a recent effort that employs adoption curve shapes to uncover social interactions. His analysis is based on a function he calls the relative acceleration rate, which measures the rate of change of the adoption curve when different fractions of adoptions have occurred. Formally, letting r denote the fraction of the population that as adopted, the relative acceleration rate equals

$$g(r) = \frac{d^2 q(t)}{dt^2} \Bigg/ \frac{dq(t)}{dt} \text{ evaluated at } t_r \text{ where } q(t_r) = r. \tag{36}$$

Young argues that for one type of social interactions model, a social learning process (in which each agent updates beliefs about the relative payoffs between adoption and nonadoption based on observed choices of others), the relative acceleration rate may be increasing whereas in another type of social interactions model, a contagion process (in which each agent adopts when he comes into contact with someone else who has adopted), the relative acceleration rate can never increase.

To understand the behavior of the relative acceleration rate in our model, algebraic manipulation reveals that, even when social interactions are absent, the relative acceleration rate for our model is

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as the logistic, exponential or the hyperbolic sine…Cumulative distributions characterized by nonaccelerating functions (e.g. linear or logarithmic) are thought to be compatible with asocial (individual) learning models." Reader notes that this holds for prominent studies such as Boyd and Richerson (1985).

[19]See Manuelli and Seshadri (2008) for a different framework in which *S*-shaped adoption curves are produced without social interactions.

$$g(r)=\frac{df_X\left(x_r^*(t)\right)}{dx}\left(f_X\left(x_r^*(t)\right)\frac{dt^*\left(x_r^*(t)\right)}{dx}\right)^{-1}-\left(\frac{d^2t^*\left(x_r^*(t)\right)}{dx^2}\right)\left(\frac{dt^*\left(x_r^*(t)\right)}{dx}\right)^{-2} \quad (37)$$

where the quantities are evaluated at $x^*\left(t_r\right)$. Additional algebraic manipulation produces

$$g(r)=$$
$$a\pi\left(x_r^*(t)\right)\left(\frac{d\pi\left(x_r^*(t)\right)}{dx}\right)^{-1}\left(\frac{df_X\left(x^*(t)\right)}{dx}\right)\times \quad (38)$$
$$\left(\left(\frac{df_X\left(x^*(t)\right)}{dx}\right)f_X\left(x^*(t)\right)+\pi\left(x_r^*(t)\right)\left(\frac{d\pi\left(x_r^*(t)\right)}{dx}\right)^{-1}\left(\frac{d^2\pi\left(x_r^*(t)\right)}{dx_2}-\left(\frac{d\pi\left(x_r^*(t)\right)}{dx}\right)^2\right)\right).$$

For our purposes, what is important about this expression is that one can manufacture a wide variety of shapes of $g\left(r\right)$ by varying the values of $\frac{df_X\left(x\right)}{dx}$ and $\frac{d\pi^2\left(x\right)}{dx^2}$. Economic theory of course does not restrict these functions. Hence, the relative acceleration rate is not restricted by our forward looking model. By implication, one cannot distinguish types of social interactions, i.e. contagion versus social learning. Again, $\frac{df_X\left(x\right)}{dx}$ and $\frac{d\pi^2\left(x\right)}{dx^2}$ constitute forms of unobserved heterogeneity that break any logical link between the shape of the adoption curve and social interactions.

7. Conclusions

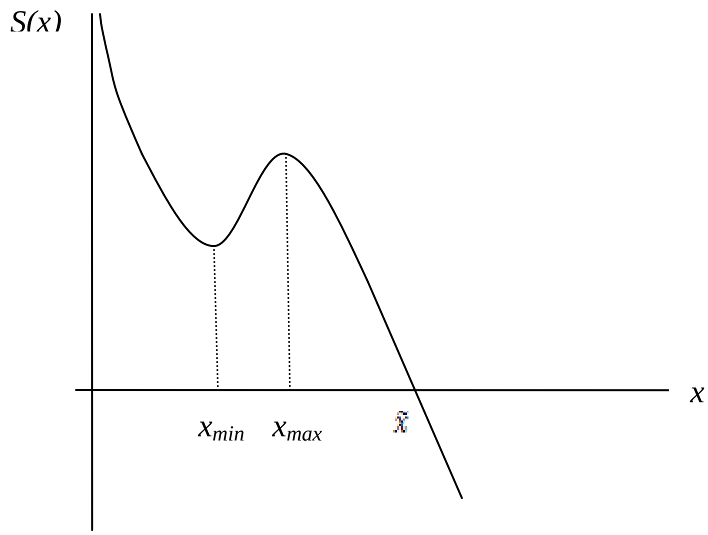
In this paper we have analyzed a model of adoption decisions in which social interactions are present. Our analysis indicates that even in the presence of observable and unobservable heterogeneity, it is possible to uncover properties of adoption curves that observationally differentiate environments in which social interactions matter from those that do not.

While we have not translated these observational differences into econometric analogs, their presence provides a basis for constructing formal econometric tests. For jumps in the adoption curve $q(t)$ in continuous time data sets, there exist a number of methods from the time series and finance literatures that allow for the identification of jumps in stochastic processes - Ait-Sahalia (2004), Ait-Sahalia and Jacod (2008) and Barndoff-Neilson and Shephard (2006) are major recent contributions. For cases where data are measured in discrete time, these formal methods will not be applicable. Nevertheless, the jumps we find in discrete time translate into large changes in a discretely sampled adoption curve. If one finds that adoption increases slowly, suddenly increases rapidly at one time increment, and then increases slowly again, this is suggestive of a jump in the underlying continuous adoption process. Notice that this type of finding is much sharper than the finding of an *S*-shaped adoption curve, which we in fact have argued is not a robust implication of social interactions models. While assessing whether a particular change in an adoption curve is too large to be plausibly associated with fundamentals requires judgment, this does not invalidate its utility.

Pattern reversals are also in principle estimable. For this case, the empirical objects of interest are pairs of conditional adoption curves $q(t|x')$ and $q(t|x'')$, where $x' > x''$ and one can assume that private incentives to adopt are increasing in x. Recalling Theorem 3, the absence of a pattern reversal requires that $q(t|x) \geq q(t|x') \ \forall t$. This requirement in turn is a form of stochastic dominance requirement (on adoption curves) and may be assessed using methods developed in Barrett and Donald (2003) and Linton, Maasoumi, and Whang (2005). Unlike the jump case, pattern reversal claims are not sensitive to whether continuous time or discrete time data are available.

Figure 1

Shape of $S(x)$ in Theorem 1



Appendix 1: Proofs of Theorems

Proof of Lemma 1.

Assumption A.1 implies that $\pi\left(x, q^e(t)\right) \leq \pi(x,1) \quad \forall t$. Suppose by way of contradiction that there exist two optimal times $t_1 < t_2$ such that. $J(t_1, x) = J(t_2, x)$ when each is evaluated for a given set of beliefs $q^e(t)$. There are four cases to consider (i) $q^e(t)$ is differentiable at both t's, (ii) $q^e(t)$ is differentiable at t_1 but not t_2, (iii) $q^e(t)$ is differentiable at t_2 but not t_1, (iv) $q^e(t)$ is not differentiable at either t. For each of the cases, $J'\left(t_i^-, x\right) \geq 0 \geq J'\left(t_I^+, x\right)$ by eq. (4). Using the representation in (4) and cancelling off $e^{-\rho t}$, it must be the case that

$$\rho C - e^{at_i}\pi\left(x, q^e\left(t_i^-\right)\right) \geq 0 \geq \rho C - e^{at_i}\pi\left(x, q^e\left(t_i^+\right)\right) \tag{A.1}$$

By Assumption A.2 any jump in $q^e(t)$ must be positive. Thus any jump in $\rho C - e^{at_i}\pi\left(x, q^e\left(t_i^-\right)\right)$ can only jump down if it jumps at all. Since $q^e\left(t_1^+\right) \leq q^e\left(t_2^-\right)$ by A.2, we have a contradiction to eq. (A.1). To see this, consider the RHS of eq. (A.1) for t_1:

$$0 \geq \rho C - e^{at_1}\pi\left(x, q^e\left(t_1^+\right)\right) > \rho C - e^{at_2}\pi\left(x, q^e\left(t_1^+\right)\right) > \rho C - e^{at_2}\pi\left(x, q^e\left(t_2^-\right)\right) \geq 0 \tag{A.2}$$

The second inequality between the 0's follows because $t_1 < t_2$ and the third inequality follows from monotonicity of $\pi(\cdot,\cdot)$ and $q^e\left(t_1^+\right) \leq q^e\left(t_2^-\right)$. We thus arrive at a contradiction. This argument addresses all four cases and ends the proof.

Proof of Lemma 2.

Suppose by way of contradiction that $t < t_1$. Since t_1 is optimal,

$$0 \le g\left(x_1, t_1^-\right) = \rho C - \exp\left(at_1\right)\pi\left(x_1, q^e\left(t_1^-\right)\right) < \rho C - \exp\left(at\right)\pi\left(x, q^e\left(t^+\right)\right) \le 0. \quad \text{(A.3)}$$

This chain of inequalities follows from optimality of t_1, the assumptions $t < t_1$ and $x < x_1$, and the implication of Assumption A.2 that $q^e\left(t_1^-\right) \ge q^e\left(t^+\right)$. As eq. (A.3) is internally contradictory, it must be the case that $t \ge t_1$ □.

Proof of Theorem 1.

We first prove that the candidate REE above *is* an REE. It will help to follow the argument if one considers the graph of $S(x)$ with x on the horizontal axis and S on the vertical axis.

Let $x_i(t)$, $i = 1,2,3$ denote the smallest, middle and largest solutions of the equation, $t = S(x)$; when the solution is unique, these solutions simply coincide. Note that for $t_* = S(x_{\min})$, $t^* = S(x_{\max})$ there are two distinct values of x, x_* and x^* such that $t_* = S(x^*)$ and $t^* = S(x_*)$. Note that $x_* < x_{\min} < x_{\max} < x^*$. We will use the notation $x_i(t) = S_i^{-1}(t)$ when it is useful. For $t < t_*$, the large x solution, $x_3(t)$ is the only solution of the equation $t = S(x)$, so intuitively it must be an REE, if an REE exists at all. For $t > t^*$ the small x solution $x_1(t)$, is the only solution of the equation $t = S(x)$ so intuitively it must be an REE, if an REE exists at all. The first step in the proof of the theorem is to formally demonstrate that the large x solution is part of an REE. The same type argument will also imply that the small x solution is part of an REE. In other words, for each x, we need to prove that the optimal t^* is the part of $S(x)$ described in the theorem statement. Since we showed in Lemma 1 that, for each fixed x, there can be at most one solution to the necessary conditions for a local maximum of $J(t,x)$ all we need to do is show that the relevant parts of $S(x)$ satisfy the set of necessary conditions for each x.

Consider first $x > x_{\max}$. By the implicit function theorem, the implicit function $S(z) - \tau = 0$ has a well defined solution $z(\tau) = S_3^{-1}(\tau)$ in an open neighborhood of the point (t,x) (given that $S'(x) < 0$). We claim that

$$J'(\tau,x) = e^{-\rho\tau}\left(\rho C - e^{\alpha\tau}\pi\left(x, 1 - F_X(S_3^{-1}(\tau))\right)\right) \tag{A.4}$$

equals 0 at $\tau = t = S(x)$ and is decreasing for all τ in an open neighborhood of t, i.e. t is a local maximum of $J(\tau,x)$. Substituting $\tau = S(x)$ into $J(\tau,x)$ and using the definition of $S(x)$ and the identity $S_3^{-1}(S(x)) = x$ it is evident that $J'(\tau,x) = 0$ at $\tau = t = S(x)$. Since $\pi(x.q)$ and $F_X(x)$ are, by assumption, continuously differentiable, and since by the implicit function theorem, $S_3^{-1}(z)$ is differentiable at $z = x$, it must be the case that $J(\tau,x)$ is differentiable at $\tau = t$. Since $S_3^{-1}(z)$ is strictly decreasing, we find by direct computation that $J''(\tau,x) < 0$ at $\tau = t$ by using $J'(\tau,x) = 0$ at $\tau = t$. Hence the best reply property for an REE is satisfied for $x > x_{\max}$. The argument extends to $x = x_{\max}$ by taking right hand limits. Define $x_* < x_{\min} < x_{\max}$ to be the smaller value of the two values of x that satisfy the equation $S(x_{\max}) = S(x_*)$. The same argument as above shows that the best reply property is satisfied for $x < x_*$.

We next verify that the optimal t for $x \in [x_*, x_{\max}]$ is $t^* = S(x_*) = S(x_{\max})$. We do $x_{\max}$ first. If one inserts $x = x_{\max}$, into J' evaluated at t^*, one sees immediately that

$J'(t^*, x_{\max}) = 0$. If one replaces t^* by a slightly smaller value of t, J' will be negative at $x = x_{\max}$ (on the branch $x_3(t) = S_3^{-1}(t)$).

We next consider x_*. This requires us to show that the derivative

$$J'(t^*, x) = e^{-\rho t^*}\left(\rho C - e^{-at^*}\pi\left(x, q^*\left(t^*\right)\right)\right) \tag{A.5}$$

is positive (negative) for t slightly less (greater) than t^* for $t^* = \frac{1}{a}\ln\frac{\rho C}{\pi\left(x_{\max}, 1 - F\left(x_{\max}\right)\right)}$.

Now put $x = x_*$, but evaluate $J'(t^*, x)$ in an interval around t^*, i.e. compute right limits and left limits at t^*. (Note that for the left limit we will be on the branch $q^*(s) = 1 - F_X\left(S_3^{-1}(s)\right)$, $s < t^*$ and for the right limit we will be on the branch $q^*(s) = 1 - F_X\left(S_1^{-1}(s)\right)$, $s > t^*$). The left limit as t approaches t^* from below of $J'(t^*, x)$ is easily seen to be given by

$$J'(t^{*-}, x_*) = e^{-\rho t^*}\rho C\left(1 - \frac{\pi\left(x_*, 1 - F_X\left(x_{\max}\right)\right)}{\pi\left(x_{\max}, 1 - F_X\left(x_{\max}\right)\right)}\right) > 0. \tag{A.6}$$

This same inequality holds for all $x \in [x_*, x_{\max})$ since π is assumed to be strictly increasing in x. Notice that $J'(t^{*-}, x)$ above is the marginal gain to waiting an extra day to adopt before the jump in q^*.

Next, compute the right hand limit of $J'(t, x)$ as t approaches t^* from above. The atom $F_X(x_{\max}) - F_X(x_*)$ has now adopted. Compute $J'(t^*, x)$ at t^* to obtain

$$J'(t^*, x_*) = e^{-\rho t^*}\rho C\left(1 - \frac{\pi\left(x_*, 1 - F_X\left(x_*\right)\right)}{\pi\left(x_{\max}, 1 - F_X\left(x_{\max}\right)\right)}\right) = 0, \tag{A.7}$$

because, by the definition of x_*, $S(x_*) = S(x_{\max})$ implies that $\pi(x_*, 1 - F_X(x_*)) = \pi(x_{\max}, 1 - F_X(x_{\max}))$ and $S(x) = \frac{1}{a} \ln \frac{\rho C}{\pi(x, 1 - F_X(x))}$. We will be finished once we show that $J'(t, x, q^*)$ is negative for t slightly bigger than t^*. We have

$$J'(t^* + \varepsilon, x) = e^{-\rho(t^* + \varepsilon)} \rho C \left(1 - \frac{\pi(x, q^*(t^* + \varepsilon))}{\pi(x_{\max}, 1 - F_X(x_{\max}))} \right). \tag{A.8}$$

Define $h(\varepsilon) = 1 - \frac{\pi(x, q^*(t^* + \varepsilon))}{\pi(x_{\max}, 1 - F_X(x_{\max}))}$. It is clear from (24) that the sign of $J'(t^* + \varepsilon, x)$ must be the same as $h(\varepsilon)$. Since $h(\varepsilon)$ is continuously differentiable at $\varepsilon = 0$, it has a Taylor expansion, $h(\varepsilon) = h(0) + h'(0)\varepsilon + o(\varepsilon)$. We first show that $h(0) < 0$ Recall that $\lim_{\varepsilon -> 0} q^*(t^* + \varepsilon) = 1 - F_X(x_*)$ because under q^* the atom $F_X(x_{\max}) - F_X(x_*)$ is anticipated to have already adopted for dates t slightly greater than t^*. From this, we obtain

$$h(0) = 1 - \frac{\pi(x, 1 - F_X(x_*))}{\pi(x_{\max}, 1 - F_X(x_{\max}))} < 0. \tag{A.9}$$

We next show that $h'(0) < 0$ This derivative equals

$$h'(0) = -a + \frac{\pi_q(x_*, 1 - F_X(x_*)) f_X(x_*)}{S'(x_*) \pi(x_*, 1 - F_X(x_*))} < 0, \tag{A.10}$$

which is negative because $S'(x_*) < 0$. This completes the proof for $x = x_*$.

The proof for each $x \in (x_*, x_{\max})$ proceeds as follows. We have already shown that the left hand limit of $J'(t^*, x)$ is positive for each such x. We must now show that the right hand limit of $J'(t^*, x)$ is negative for each such x. We have

$$J'(t^* + \varepsilon, x) = e^{-\rho(t^*+\varepsilon)} \rho C \left(1 - \frac{\pi(x, q^*(t^* + \varepsilon))}{\pi(x_{\max}, 1 - F_X(x_{\max}))} \right). \tag{A.11}$$

Define $h(\varepsilon) = 1 - \dfrac{\pi(x, q^*(t^* + \varepsilon))}{\pi(x_{\max}, 1 - F_X(x_{\max}))}$. We show that $h(0) < 0$. Recall that $\lim_{\varepsilon -> 0} q^*(t^* + \varepsilon) = 1 - F_X(x_*)$ because under q^* the atom $F_X(x_{\max}) - F_X(x_*)$ is anticipated to have already adopted for dates t slightly greater than t^*. From this, we obtain

$$h(0) = 1 - \frac{\pi(x, 1 - F_X(x_*))}{\pi(x_{\max}, 1 - F_X(x_{\max}))} < 0 \tag{A.12}$$

because $\pi(x_{\max}, 1 - F_X(x_{\max})) < \pi(x, 1 - F_X(x)) < \pi(x, 1 - F_X(x_*))$ for $x \in (x_*, x_{\max})$. This completes the proof.

Proof of Theorem 2.

If there are no social interactions, $\pi(x, q) = \pi(x)$. Hence the equation $t = S(x) = a^{-1} \ln\left(\dfrac{\rho C}{\pi(x)}\right)$ has a unique solution for each t. Denote $q_i(t) = 1 - F_{X,i}(S^{-1}(t))$. Therefore, $q_1(t) = 1 - F_{X,1}(S^{-1}(t)) \leq q_2(t) = 1 - F_{X,2}(S^{-1}(t))$, by stochastic dominance.

Proof of Theorem 3.

The proof of the first part is obtained from a straightforward chain of inequalities that exploits the assumption that $\pi = \pi(x_1, x_2)$ is independent of q and is strictly increasing in both x's as well as Assumption A.5, Formally,

$$\begin{gathered} q(t \mid x_1') = 1 - F_{X_2|X_1}\left(S^{-1}(t \mid x_1') \mid x_1'\right) \geq \\ 1 - F_{X_2|X_1}\left(S^{-1}(t \mid x_1') \mid x_1''\right) > 1 - F_{X_2|X_1}\left(S^{-1}(t \mid x_1'') \mid x_1''\right) = q(t \mid x_1'') \end{gathered}. \tag{A.13}$$

The first inequality follows from eq. (22) and the second follows from $S^{-1}(t \mid x_1') > S^{-1}(t \mid x_1'')$, which follows from $S(x_2 \mid x_1') > S(x_2 \mid x_1'')$, an inequality which follows directly from the assumption that $\pi(x_1, x_2)$ is strictly increasing in the vector (x_1, x_2). Given the proof of the first part of the theorem, the proof of the second part is immediate. This ends the proof.

Appendix 2 : Closed Form Example For Theorem 1.

Here we construct a parametric example of an environment of the type assumed in Theorem 1. First, note that $S(x) = a^{-1} \ln \frac{\rho C}{\pi(x, q(x))}$ falls, then rises, then falls again if and only if $\pi(x, q(x))$, where $q(x) = 1 - F_X(x)$ moves in the opposite way, i.e. rises, then falls, then rises. We therefore construct a $\pi(x, q(x))$ and $q(x) = 1 - F(x)$ that satisfies these last properties. We will choose $A_1 > 0$ and construct two functions with the first function designed so that it is concave, continuously differentiable, and increases then decreases on $(0, A_1)$ and the second function defined on $[A_1, \infty)$ such that they join together to form a continuous function. The parameters of these functions will be designed so they generate an example of Theorem 1. This exercise will prove that Theorem 1 applies to a nontrivial set of examples.

We define the first function by

$$\pi_1(x, q(x)) = x^{\alpha} \left(K + B(1 - F(x))\right)^{\beta} \tag{A.14}$$

where $\alpha > 0, \beta > 0$, $\alpha + \beta = 1$, $K > 0$, $B > 0$ and $F_X(x)$ are chosen so that this function is continuously differentiable, concave, increases, takes a maximum; call it $x_{\min}$ (because it is a local minimum of $S(x)$, on $(0, A_1)$, for appropriately chosen $A_1 > 0$, and decreases to the smaller value $\pi_1(A_1, q(A_1)) < \pi_1(x_{\min}, q(x_{\min}))$). Clearly $\pi(0, q(0)) = 0$. As we will see below, this function will be concave, will increase to a maximum, then take a smaller value at $\pi(A_1, 1 - F_X(A_1))$ for appropriately chosen linear $F_X(x)$ on $(0, A_1)$.

Our second function will be defined by

$$\pi_2(x, 1 - F_X(x)) = \pi_1(A_1, 1 - F_X(A_1)) + L_1(x - A_1) + L_2(F_X(A_1) - F_X(x)) \tag{A.15}$$

where $A_2 > A_1$, $F_X(x)$ is chosen to be linear, and the other parameters are chosen so that $\pi_2(x, 1-F_X(x))$ increases on $[A_1, \infty)$ and $\pi_2(A_2, 1-F_X(A_2)) > \pi_1(x_{\min}, 1-F_X(x_{\min}))$. Note that the rightmost term comes from the term $L_2(q(x)-q(A_1))$ and that the two functions take the same value at $x = A_1$. Therefore the join is continuous at $x = A_1$. We also choose $F_X(x)$ so that it has positive support on $[0, x_S)$, $x_S \ge A_2$

Return to the specification of the first function. Compute

$$\frac{d\pi_1(x, q(x))}{dx} =$$
$$\alpha x^{\alpha-1}(K + B(1-F_X(x))^{\beta} - \beta x^{\alpha}\left(K + B(1-F_X(x))\right)^{\beta-1} Bf_X(x) = \tag{A.16}$$
$$\pi_1(x, q(x))\left(\frac{a}{x} - \frac{\beta Bf_X(x)}{K + B(1-F_X(x))}\right)$$

It is evident that this derivative is positive for small x, is 0 for some $x_{\min} > 0$, and is negative for larger x's iff $\left(\frac{a}{x} - \frac{\beta Bf_X(x)}{K + B(1-F_X(x))}\right)$ has the same properties. We therefore choose $F_X(x)$ to satisfy,

$$F_X(x) = \frac{x}{a_1},\ x \in [0, A_1),$$
$$= \frac{A_1}{a_1} + \frac{x - A_1}{a_2}\ x \in [A_1, A_2), \tag{A.17}$$
$$= 1,\ x \ge A_2$$

where we are free to choose all these parameters to get what we want and satisfy all the explicit and implicit constraints needed to get an example with just one jump. It is easy to check that π_1 rises for small x's, takes a first zero at $x_{\min} = \frac{\alpha}{\alpha+\beta}\frac{a_1(K+B)}{B}$, and is concave on $(0, A_1)$ so our first constraint on our parameter set is that $x_{\min} < A_1$.

Without loss of generality recall that we can assume that $\alpha+\beta=1$ by taking a monotonic transformation. Therefore we know that π_1 is concave on $(0, A_1)$ so we know that we have identified a maximum on $(0, A_1)$. We now wish to construct the "rest" of $F_X(x)$ and the function π_2 to produce a local minimum of π at A_1 and such that for some $A_2 > A_1$, we have $\pi_2(A_2, 1-F_X(A_2)) > \pi_1(x_{\min}, 1-F_X(x_{\min}))$ before the end of the support of $F_X(x)$ is reached. To do this, we identify parameters so that there is a local minimum of π at $x = A_1$. We already know that the left hand derivative of $\pi = \pi_1$ at $x = A_1$ is negative. We must specify the parameters of π_2 so that the right hand derivative of $\pi = \pi_2$ is positive at $x = A_1$. This condition is satisfied if $L_1 - \frac{L_2}{a_2} > 0$.

Finally we require $\frac{A_1}{a_1} < 1$ and $\frac{A_1}{a_1} + \frac{A_2 - A_1}{a_2} \le 1$. Since $S(x) = \frac{1}{a}\ln\left(\frac{\rho C}{\pi(x, q(x))}\right)$, we may choose ρC large enough so that it is above π evaluated at $x_{\min}$ so that $S(x_{\min}) > 0$. We still have quite a bit of freedom to construct the rest of the example to illustrate Theorem 1. Note that we have a nondifferentiability of $S(x)$ at $x = A_1$ whereas Theorem 1 assumes continuous differentiability of $S(x)$. But the proof uses left and right limits so differentiability is not really needed. In any event one can always locally smooth the construction at $x = A_1$.

We still need to choose the parameters so that the local maximum value of π is smaller than $\pi(A_2, 1-F_X(A_2))$ so that we have a positive atom for our jump. This is satisfied provided that

$$\pi_2\left(A_2,1-F_X\left(A_2\right)\right)= \pi_1\left(A_1,1-F_X\left(A_1\right)\right)+L_1\left(A_2-A_1\right)+L_2\left(F_X\left(A_1\right)-F_X\left(A_2\right)\right)>\pi_1\left(x_{\min},1-F_X\left(x_{\min}\right)\right) \tag{A.18}$$

For simplicity, assume $0<a_1=a_2$. The restrictions needed on the various parameters may be summarized as

$$\frac{A_1}{a_1}<1,\ \frac{\alpha a_1\left(K+B\right)}{B}<A_1,\ \frac{A_2}{a_1}\le 1,\ L_1-\frac{L_2}{a_1}>0 \tag{A.19}$$

and

$$L_1\left(A_2-A_1\right)+\frac{L_2\left(A_1-A_2\right)}{a_1}= \left(L_1-\frac{L_2}{a_1}\right)\left(A_2-A_1\right)>\pi_1\left(x_{\min},1-\frac{x_{\min}}{a_1}\right)-\pi_1\left(A_1,\frac{1-A_1}{a_1}\right) \tag{A.20}$$

The approach we have outlined provides a method for the construction of a fairly large class of examples. The construction illustrates that it is straightforward to construct a class of examples that satisfy the conditions of Theorem 1 so that there is an atom jump. For example if one assumes that $K=0$ and $B=1$, it is clear that one can set L_1 large enough and L_2 small enough to satisfy all the above constraints. Define

$$\begin{aligned} S\left(x\right) &= \frac{1}{a}\ln\left(\rho C/\pi_1\left(x,1-\frac{x}{a_1}\right)\right),\ \text{if } x<A_1 \\ &= \frac{1}{a}\ln\left(\rho C/\pi_2\left(x,1-\frac{x}{a_1}\right)\right),\ \text{if } x\ge A_1 \end{aligned} \tag{A.21}$$

Thus, $t^* = S(A_1)$, by continuity there is $x_* < x_{\min} < A_1$, $t^* = S(x_*) = S(A_1)$. The size of the jump atom is $F_X(A_1) - F_X(x_*) = \dfrac{A_1 - x_*}{a_1}$.

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