Will you lend me to short ? The role of the box in leverage and repo fails *

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Abstract

Repo markets have recently attracted a lot of attention as policy makers tried to prevent un-orderly adjustments of leverage in the market. We build a repo market model to understand the role of the physical nature of securities for different scenarios: repo rolls, fails and security market leverage. It is impossible to have a short position in a security without managing to borrow (reverse in) it first. The constraint is captured by the concept of what is called the box in market parlance: the balance of title ownership in each security, which must be non-negative. The level of repo specialness is a function of how binding this constraint is. Leveraging a position becomes possible if engaging in a sequence of repo and security trades in a way that scarce collateral is allocated in agents' boxes. We call this process the repo collateral multiplier.

Key words and phrases: repo, reverse repo, box, short sale, leverage, repo collateral multiplier, fail, specialness.

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1 Introduction

1.1 Motivation

Never as acutely before, has *repo*'s role in the provision of liquidity attracted as much attention from policy-markers, as in the context of the recent credit crisis. Nevertheless, repo and term repos have always been part of the tools at the central bankers disposal. Automatic sterilization when using repo as a monetary tool and the credit protection from the use of a collateral mean that repo has been a tool of choice in the execution of open market operations. Traditionally, in such operations, however, the focus has been the supply of money, mostly using General Collateral (GC) repos on government bonds. Specific security categories have not been targeted with the exception of the innovative repo operations on fails by the Bank of England[1]. Now policy makers try to manage the leverage cycle by intervening in such markets to try to prevent disorderly de-leveraging. This became necessary as mortgage related products became difficult to finance, beyond what credit consideration can justify, and this alone played a central role in the credit crisis that started in the summer 2007, not just because of more expensive borrowing to purchase homes, but also due to propagated effect to the rest of the economy, as financial institutions began to seek out, sometimes under stress, alternate sources of funding. Their action started to affect traditionally calm corners of the financial market. In the end, through second round effects, interbank lending - the backbone of the interest rate derivatives- became affected. Credit of large holders became affected. Credit of big holders of securities became as important as credit of issuers. This implied that one of the policy remedies was now to focus on particular securities funding and targeted repo operations. Repos (see for example the Term Auction Facility (TAF) program introduced by the Federal Reserve[2]) have been one of the main tools to try to re-establish and normalize funding conditions.

While important work has been done on the equilibrium modeling of repos (in the pioneering article by Duffie (1996), in Duffie et al. (2002) and, more recently, in Vayanos and Weill (2008)), one senses that a broad general equilibrium framework is needed to understand domino effects and contagion associated with fluctuating leverage as well as welfare effects of policies associated with leverage through repos. While we do not attempt a full study of the contagion effects and welfare policies, we build a basic incomplete markets general equilibrium framework to model repo and securities markets.

[1] http://www.jdawiseman.com/papers/finmkts/opnot1609.pdf

[2] Official Release: http://www.federalreserve.gov/newsevents/press/monetary/20071212a.htm

1.2 What is a repo trade?

A *repo* trade consists in an asset sale combined with an agreement of future repurchase of the same amount at a predetermined date and price. Repos are associated to securities because prices of securities are not expected to change sign[3]. Cash-flows (e.g., coupon or dividend) received from the securities during the repo trade are passed on to the original owner.[4] What distinguishes repo trades from simple sales and purchases of securities is how the front leg and the back leg of the trade are linked as one trade. The sale price and the future repurchase price correspond to a level of interest rate which is called the *repo* rate. The repo trade is a collateralized loan of cash at the repo rate. The duration of the repo transaction is short and smaller than the time to maturity of the security. The repo rate is a market level. Higher interest rates are an upward pressure on the repo rate. On the other hand, the value associated to desirability of being in possession of the security (e.g., high demand for shorts) and to the credit protection brought by the collateral both push the repo rate lower.

The language used for repo can be tricky at first, but in fact the terminology becomes very natural provided one focuses on the effect of given trades on title balance of given security, called the amount in *the box* in market parlance. In the case of bearer securities for which the title is represented by a physical piece of paper, the box can be literally thought as a box or the vault where one puts such titles. In fact, such record and safe-keeping is most often done electronically or delegated. Nevertheless the humble original bearer form of securities left its institutional mark on how the securities market operates. Getting long a security in the securities market or in the repo market both increase the amount of the security in the box. A security that one lends disappears from one's box, like a book that one lends disappears from one's shelves. Very often (but not always) the positions taken in the securities market and the repo market are in opposite directions. A repo trade involves two parties: the *lender* and the *borrower* of the security. Ms. Arrow, *long* in the securities market, may lend the asset to Mr. Bow and obtain a loan from him to finance the purchase. Mr. Bow is *short* in the

[3]In fact this is a reasonable definition of a security : a financial contract whose price is expected to stay positive, something conveyed by the word "security." This is true of bonds and shares (thanks to the corporate veil). Such definition is actually very close to the Japanese word for security that actually means valuable certificate. The terminology of security is meaningful in the repo context since it is difficult to use as a collateral something that can actually become a liability.

[4]Such proceeds are not passed on in the case of Buy/Sells and Sell/Buys, and this is the main difference to distinguish Repo and Reverse Repo from the corresponding Sell/Buy (corresponding to Repo) and Buy/Sells (corresponding to Reverse Repo)

securities market and extended a loan to Ms. Arrow. In such a transaction Ms. Arrow is long the security and short in repo (and the opposite for Mr. Bow), but because cash received from the borrowed security is passed on to Ms. Arrow by Mr. Bow, it looks like she is borrowing money for the term of the repo to buy the security. She receives cash-flows occurring during the repo transaction.

In the example above Ms. Arrow obtained a *repo* from Mr. Bow and Mr. Bow *reverses in* the security. The *repo* operation can be seen as a loan protected by a collateral that is kept in hands of the party reversing in the security. There are some similarities with mortgages: the collateral securing the loan is the object whose purchase is being financed. The key difference is that in a *repo* trade the collateral's legal title ownership is passed on to the creditor, whereas in the case of mortgages it is kept by the debtor. The net balance of title ownership of the leveraged long in the securities market is zero, Ms. Arrow has no security "in the box.". The consequence of this is that the collateral obtained by a reverse repo can be sold (even by somebody who never was a owner of the security). One can go short in the securities market in a way impossible in real estate.[5] Actually this is the only way to go short in securities: one needs to borrow a security in order to short it.[6] The institutional constraint is that a negative balance/title ownership in a security is not allowed: the box contains non-negative balances of securities title of ownership. This constraint implies that the portfolio set is no longer a linear space: naked negative security positions are not allowed.

1.3 On leverage

Two different types of agents trade in the repo markets: *Real Money Institutions* and *Leveraged Institutions*. Real Money Institutions are typically investors that need to put a given amount of funds at work. Examples are insurance companies, real corporations, individuals, mutual funds, trust banks, sovereign wealth funds, central banks investing their reserves, and depositary institutions investing their deposits. Real Money Institutions own securities on a non-leveraged basis. They trade and re-balance their portfolios, but this is typically without the use of leverage (this is the reason of their name *"real money"*). Such Real Money Institutions typically do not go short (sometimes

[5] There is funding market in the securities world where the asset is pledged but the title not transferred: the asset back commercial paper (ABCP) market.

[6] Our paper is about shorting securities (stocks and bonds), which implies physical delivery. We do not attempt to model other types of shorting, in particular, derivatives (e.g. interest rate swaps, although our analysis is also relevant for this case as dealers tend to hedge this derivative with Treasuries).

a legal constraint). In essence, they do not need to enter the repo market - but will do with the correct incentives. Real money institutions will lend their securities for a fee, or engage in equivalent repo transactions. They will be the final source of securities for the repo market. The initial lending of securities by real money institutions is an important step for the development of leverage in any security market. Repo dealers typically run a *matched book* (net zero balance for securities in the box). They ultimately have to find a source for the bonds in demand with real money institutions. On the other hand, *leveraged accounts*, such as security market makers (dealers) or hedge funds, will need the repo market pretty much for everything they do. They can go short a security and the repo market is the tool to do it. But even when they buy a security, leveraged accounts do not directly invest funds like the capital of their firm or funds under management. In fact, they can have positions on their balance-sheet typically involving a multiple of such a capital or funds under management. They build the leverage in the securities market using the repo market (or in the derivatives market where the derivative market marker uses repo market for hedging). A large long position in a given security can be built by engaging in a sequence of asset purchases and repos. This leverage mechanism for the leveraged institutions is introduced in Section 2 of the paper under the name of the *repo collateral multiplier.*

1.4 Equilibrium and the short sales constraints

Equilibrium analysis is particularly important in a repo context since, as Duffie (1996) remarks, it is possible to bound *repo* rates from above by arbitrage, but there is no obvious arbitrage argument to find a lower bound and these rates may even become negative. In the case of treasuries and next-day repurchases, the upper bound is at or near to the overnight interest rate in the market for federal funds. At a lower repo rate, the owner of the specific security could lend the security and invest the associated cash loan at a higher interest rate, but the scarcity of the security prevents an unlimited arbitrage. Securities in relative scarcity trade *on special,* that is, below that upper bound called the general collateral rate.

Duffie's (1996) leading paper on repo markets, first introduced *repo specialness* in the field of study. Subsequent empirical work was done by Jordan and Jordan (1997). Duffie et al. (2002) modeled search in the repo market and show that it generates a positive lending fee. Vayanos and Weill (2008) build a search model, in the spirit of Diamond (1982), and explain price differentials among otherwise identical assets. Our paper brings new insights into the *specialness* phenomenon by linking it to the shadow prices of binding *box constraints* that agents face when short selling a security.

Pricing in repo markets can not dispense with an equilibrium approach. Repo trades are not only made on bonds but also on stocks. A classical problem for the existence of equilibrium, in incomplete markets of securities with price dependant returns, such as stocks, is the well known Hart's (1975) discontinuity. Hart's counterexample to the existence of an equilibrium is based on a collapse in the span of intertemporal transfers that can be achieved by freely trading any linear combinations of assets (and this is what we challenge here). Such a collapse occurs on an exceptional set of "bad" spot prices where one asset becomes redundant. Since then there have been many attempts to resurrect the existence of equilibria[7]. All these attempts owe a debt to Radner (1972), who demonstrated existence of equilibrium in a general multi-period model under bounded short sales. However, as Radner (1989, p.314) himself reckons there is no obvious argument to justify the imposed bound on short sales. In the present paper we provide with a natural bound on the short sales, as collateral and initial wealth is consumed to collateralize leverage. In order to short sell a security the agent must borrow the security through a *reverse repo*. Technically, the obligation to reverse in securities before shorting them (non-negative "title balance" in the *box*) re-establishes the upper hemi-continuity of the budget correspondence. This is because as prices approach a "Hart change" in span, agents try to compensate for the approaching loss in trading ability by a higher use of leverage.[8] In the presence of haircut, this very activity will be prevented by the scarcity of collateral: in the end the budget set remains compact and regular.[9]

[7]See for example Duffie and Shafer (1985), Geanakoplos and Mas Colell (1985), Geanakoplos and Polemarchakis (1986), and Bottazzi (1995, 2002b).

[8]Traditional approaches have been to find financial instruments structures that avoid such behavior, at least for most initial good endowments. This can be done if around Hart prices where the span drops, financial structure are sufficiently dependent on price (see Bottazzi (2002b) for the concept of Transverse Financial structures) - this is precisely how Ku and Polemarchakis (1990) construct a counter example : they use the fact that out of the money options have no local dependance on the underlying's price. Bad prices cease to be exceptional and so are initial endowments leading to them. Securities market do no longer have no existence problem but derivatives market need a different approach. Hart points should not be dismissed just because existence problem is solved. Hart points in the model where funding is not explicitly modeled are likely to be in regions where funding constraints are binding and of high relevance.

[9]Zero-haircuts are sometimes achieved at some point in time or if health of a given counterparty warrants it, but this would be changed if the transaction in which such a counterparty engages changes the actual credit perception. For exemple, LTCM lost such arrangements in 1998, and one can think they would disappear as one approaches a "Hart point."

1.5 Fails vs. default

We do not contemplate the introduction of any legal requirements for the leveraged institutions to keep the collateral security in their *box* along the duration of the repo transaction: Such a requirement would anyhow be counterfactual. The borrower of collateral can lend the collateral further, keep it or sell it in the securities markets. When the delivery of the security becomes due, the long in repo is required to honor his promise of delivering back the collateral by the totality he received in the previous date. Normally these longs get back the security by engaging in another *reverse repo* (rolling the reverse-repo). It is possible for the borrower of the security to be forced into the securities markets if a *short squeeze*[10] on a given security occurs. It may occur that the amount delivered back by the borrower of the security is below the one promised. This is called a *fail*. Such a fail may happen with nobody being at fault (e.g., in a loop of fails, of no economical consequence). We want to understand fails, not try to come with a normative theory to prevent them.

The agent who borrowed cash to buy a security may also find himself unable to give back the cash he promised when he purchased the asset. But it better be a short term glitch as this is much more serious than a repo fail and may lead to *default*, triggering a full bankruptcy.[11] In practice, however, as long as things are resolved quickly it rarely comes to this. The default notice is only sent by the counterparty if his view is that the counterparty is in serious trouble and rolling the funding would not work anyhow.[12] In a standard situation the long side is much more likely to face difficulties getting hold on collateral than the short side since a security is more likely to be scarce and become *special*[13] at any given point in time than the cash itself, even though such scarcity of cash happens when the financial system

[10] A *short squeeze* is a situation where borrowers of securities feel such discomfort from the anticipation or the reality of not being able to get securities back, that they cover their securities positions. For example the discomfort can come from a high level of specialness (and punitive repo rates), the potential penalties or the fear of unexpected difficulties to get a hold of the collateral. A short squeeze is a funding transaction inducing a security transaction.

[11] This is why the new operation of the Fed, that is non-recourse is innovative. See http://www.federalreserve.gov/newsevents/press/monetary/20081125a.htm

[12] The August 1998 Russian default created such situation for Leveraged buyer of GKO who had to re-buy the securities at a price much higher than the post Russian domestic default value.

[13] Specific repo rates are typically compared with the *general collateral rate,* the repo rate quoted for instruments that are not *on special.* The *repo specialness* of a given instrument is the difference between the *general collateral rate* and the *specific collateral rate* for that instrument.

as a whole is under stress, and funding overall becomes hard.[14] In normal circumstances, however, such scarcity is almost surely linked to the financial difficulties of a given counterparty: somebody having genuine problem raising needed cash is fighting for financial survival. In comparison, failing to return a security in time is much more benign, and in the case of the US, the explicit penalty for that is loosing the interest on the cash.[15]

The risk of default is usually mitigated by requiring a *haircut*: the value of the collateral exceeds the value of the loan, at the time the agreement is made. It is the system of *margin calls* that enables institutions to ask for a collateral top up in order to minimize the loss caused by default.[16] Different longs in the repo market, doing *reverse repos* on the same asset may require different *haircuts* depending on counterparties and assets, and, therefore, the prices for these *repo* will be different.[17] [18]

In this paper we will focus on fails, and leave the impact of default for further work.[19] This is because fails are a phenomenon specific to repo mar-

[14] In 2007 and 2008 dollars have been hard to buy, forcing intervention by central banks supplying dollars.

[15] Note that in the mortgages case, only the debtor could default on loan repayments and in this event the creditor would confiscate the collateral from the debtor, also, unlike repo, non-recourse clauses of many mortgage markets mean the borrower may not face full personal bankruptcy.

[16] Note that a *haircut* is also present in the case of mortgages in the form of a minimum down-payment.

[17] In contrast, the literature on one-sided default considers a pool of consumers, and takes their average expected default rate for market clearing purposes (Dubey, Geanakoplos and Shubik (2005) and Araujo, Fajardo and Páscoa (2005) consider different default risks, resulting from different loan/collateral ratios, captured by a spread).

[18] In an interesting unpublised work on the 2007/2008 liquidity crisis, that came to our attention very recently, Ewerhart and Tapking (2008) construct a partial equilibrium repo model with two banks that may fail or default, addressing haircut and collateral choices, for given default probabilities and given repayments in case of default. However, as fallacy of composition is central in what we currently observe, we believe that a partial equilibrium two-agents analysis is not enough to understand the liquidity problem: when looking at pricing, leverage and fails for each bilateral repo trade, one should depart from the usual ceteris paribus condition on the rest of the repo and securities markets.

[19] This paper does not model default, but here is how the potential default would potentially affect the repos rate. In the case of default, positions that are not in the box of the defaulting party disappear. The repo counterparty will then have the security in its box but for the back leg where it is selling back the security, the transaction is now with a bankrupted counterparty, and realistically has to be liquidated in the market. The repo counterparty becomes subject to market risk. This means that, in anticipation, when a counterparty is under financial stress, the securities it possess become difficult to finance (as repo counterparties want to be compensated for the liquidation cost), both the rate goes higher and the haircut is potentially increased, also the securities borrowed become more expensive to borrow.

kets of securities, while default and collateral are associated in other markets as well.[20] Fails happen because the borrower of a security may find it difficult to deliver a security back to its original owner. Agents may have different preferences and endowments and, therefore, different marginal rates of inter-temporal substitution, hence choosing differently whether to honor commitments or fail and suffer future penalties. On a timeline, repos are typically short term transactions used to fund a leveraged asset position or to short an asset. As such, the funding aspect needs to be rolled periodically. When funding is rolled in the form of new repo transactions, preferences and endowments and, therefore, different marginal rates of inter-temporal substitution come into play to produce a new adjusted repo rate. We model such repo funding and rolling in the standard GEI equilibrium first. To study fails asymmetry of information is introduced. This forces us to model repo contracts as bilateral contracts (which is according with the real word). In a context of asymmetric information the agents form beliefs about the possible deliveries of the collateral asset by their counter-parties. This leads us to the concept of a temporary equilibrium.

The rest of this paper is as follows. Section 2 starts with a leverage process through the *repo* collateral multiplier. This serves as a warm-up example to understand *repo* markets, but this concept is also relevant throughout the paper. Section 3 establishes a general equilibrium model with bilateral trading in the *repo* markets and where rolling repos is possible for the initial repo positions. There we address the issue of existence of an equilibrium for an economy with fully honored repo promises. In Section 4 fails on the promised repo deliveries are allowed and we examine different equilibrium concepts for this economy. Section 5 discusses *repo specialness*. The Appendix is devoted to proofs.

2 Leverage

Bank deposits consist of a certain amount of cash deposited by the agents of the economy. Banks invest part of these deposits in further loans or in the financial markets; thereby re-injecting the deposits into the economy. Just a fraction of the initial deposits are kept by the bank as cash reserves in order to respond appropriately to the withdrawals of the individuals at any moment in time. A minimum ratio for reserves requirement is often required by law for depositary institutions, and as such, reserves are one the first dams of protection against potential bank-runs. The amplification effect of bank

[20] See Geanakoplos and Zame (2002) and Araujo et al. (2002).

deposits is known in the literature as the *money multiplier*. By analogy, we want to introduce what we call *repo collateral multiplier* to designate how the interplay between the repo and the securities markets can lead to both short and long positions outstripping the total supply of a security in the economy, yet the net position is represented by the amount of security in the economy.

In this section we present an example about securities and repos trades between two agents. One stands with cash, the other with and endowment of the security. At any moment in time the amount of the security in each person's box has to be non-negative, that is, the amount that an agent short sells or lends cannot be greater than what he has in his box. Whenever someone borrows a security (reverses in), he gives the cash loan to his counterparty. The value of the loan is the haircutted value of the collateral (which is the security kept in the hands of the borrower of the security).

Let us illustrate the mechanism at work with the repo collateral multiplier. Assume there can be countably many transactions for each date. Each transaction occurs in what we call a *moment*. Thus in one date we allow for countably many moments to carry out transactions. Consider one security, available in an amount C and priced in the securities markets at q. We write $c = qC$. The *haircut* is exogenous[21] and denoted by h, where $h \in [0, 1]$. The repo collateral multiplier takes place between two agents, Ms. A and Mr. B. The security in Ms. Arrow and Mr. Bow' *box* at each moment n are obtained by adding the position in the repo market with the position in the securities market, which correspond to our second and third columns respectively in the tables below.

The following example shows that Ms. Arrow can leverage her cash balance $\frac{1}{1-h}$ times at most by engaging in a sequence of purchases and repo operations. The reader familiar with the impact of reserve requirement of depositary institutions on the money multiplier argument will understand the analogy. More precisely, she could have ordered a purchase of the amount $\frac{C}{1-h}$ at the preceding date, knowing that she would be able to pay it at the end of the current date even though she only started with a cash position of c. Let the initial positions be

Moment 0	Cash Deposit	Repo Position	Security Position	Box Position
Ms. Arrow	c	0	0	0
Mr. Bow	0	0	C	C

[21] The typical haircut to a hedge fund is between 0% and 2% on repo, nothing on reverses.

Now let Ms. Arrow buy the security from Mr. Bow with her cash. Note that Mr. Bow can sell the security to Ms. Arrow because he already has it as endowment (and hence the security is in his box). The positions become

Step 1, Moment 1	Cash Deposit	Repo Pos.	Security Pos.	Box Pos.
Ms. Arrow	0	0	C	C
Mr. Bow	c	0	0	0

Next, Ms. Arrow, who has the balance C in her box, lends C of the security to Mr. Bow and uses this to collateralize a loan (repo), so Ms. Arrow can borrow the haircutted amount hc in cash. So the positions become

Step 1, Moment 2	Cash Deposit	Repo Pos.	Security Pos.	Box Pos.
Ms. Arrow	hc	$-C$	C	0
Mr. Bow	$(1-h)c$	C	0	C

Step 2 starts and agents replicate Step 1. This is moment 3. Now Ms. Arrow can use her cash deposit to buy the security she just lent before, which left her box empty. Mr. Bow sells hC, a portion of the security C he received as collateral from Ms. Arrow in moment 2. Mr. Bow is entitled to sell this amount of the security because he has it in his box. Also, observe that at moment 3 Ms. Arrow cannot afford a larger purchase of the security because of Mr. Bow's moment 2 haircut. The positions become

Step 2, Moments 3	Cash Deposit	Repo Pos.	Security Pos.	Box Pos.
Ms. Arrow	0	$-C$	$(1+h)C$	hC
Mr. Bow	c	C	$-hC$	$(1-h)C$

At this point hC of the security in in Ms. Arrow's box, she posts her collateral in a repo with Mr. Bow and borrows a further h^2c amount of cash. The positions become

Step 2, Moment 4	Cash Deposit	Repo Pos.	Security Pos.	Box Pos.
Ms. Arrow	h^2c	$-(1+h)C$	$(1+h)C$	0
Mr. Bow	$(1-h^2)c$	$(1+h)C$	$-hC$	C

Now Step 3 of the leverage build up starts. At moment 4 Ms. A uses her cash position to buy more of the security. Then in moment 5 Mister B short sells $h(hC)$ of the security he received as collateral from Ms. A in moment 4. The positions become

Step 3, Moments 5	Cash	Repo Pos.	Security Pos.	Box Pos.
Ms. Arrow	0	$-(1+h)C$	$(1+h+h^2)C$	h^2C
Mr. Bow	c	$(1+h)C$	$-(h+h^2)C$	$(1-h^2)C$

Repeating all the steps, after the n^{th} iteration of repo operations followed by cash market operations, we get

Step n	Cash	Repo Pos.	Security Pos.	Box Pos.
Ms. Arrow	0	$-(1+h+...+h^{n-1})C$	$(1+h+...+h^n)C$	h^nC
Mr. Bow	c	$(1+h+...+h^{n-1})C$	$-(h+...+h^n)C$	$(1-h^n)C$

In the limit the positions are

Step ∞	Cash	Repo Pos.	Security Pos.	Box Pos.
Ms. Arrow	0	$-\frac{C}{1-h}$	$\frac{C}{1-h}$	0
Mr. Bow	c	$\frac{C}{1-h}$	$-\frac{hC}{1-h}$	C

In the limit the amount of the security in Ms. Arrow and Mr. Bow's box are 0 and C, respectively, which coincide with the initial positions in moment 0. Net Ms. Arrow has managed to leverage her cash $\frac{1}{1-h}$ times to build a security long position.

A *repo* position in moment n is denoted by $z_n < 0$, while $z_n > 0$ refers to a *reverse repo*. We refer to $y_n < 0$ as a securities market short sale and $y_n > 0$ to a purchase long position. The security in Ms. Arrow and Mr. Bow' *box* at each moment n are denoted by e_n^A and e_n^B, respectively, and are obtained by adding the position in the repo market with the position in the securities market. The positions in the box evolve from repo and securities trading as follows:

$$e^i_{j,n} = e^i_{j,n^-} + y^i_{j,n} + z^i_{j,n},\ \forall(\xi, i, j) \tag{E}$$

The box constraint (Box) (non-negative title balance in the box) is:

$$e^i_{j,n} \geq 0,\ \forall(\xi, i, j) \tag{Box}$$

As we will show in the next section, the box constraint can be disentangled in two constraints. The first says that one can only sell securities one has in the box (nobody can sell a security without getting their hands on the title):

$$e^i_{j,n^-} + y^i_{j,n} < 0 \;\Rightarrow\; e^i_{j,n^-} + y_{j,n} \geq -z^i_{j,n} \tag{L}$$

The second says that one can only lend securities one has in the box:

$$z^i_{j,n} < 0 \Longrightarrow e^i_{j,n^-} + y^i_{j,n} \geq -z^i_{j,n} \tag{S}$$

Duffie (1996) had already imposed constraint S and a variant of constraint L requiring $e^i_{j,\xi^-} + y^i_{j,\xi} = -z^i_{j,\xi}$ when $z^i_{j,\xi} < 0$ (see the next section on a comparison and implications for existence of equilibrium).

The above condition E (or L and S) was written in incremental form for each moment, that is, referring to the trades at each moment, but could have also been written for the date and in accumulated form, that is, using the accumulated trades of the date (see the next sections where this form is used).[22]

We may conclude that at every node (date), where repo markets are open countably many times, the repo transactions can be looped without any uncertainty being resolved[23]. We list now some of the causes that diminish the possible leveraged position in the security used as collateral, and lower the level of the corresponding repo collateral multiplier.[24] First, the velocity of circulation of the collateral asset may be diminished as a consequence of a high risk of a failure of the security in the repo markets, likewise the worry about the credit quality of the collateral or the counterparty. Second, and sometimes for the previous reason, the repo rate on the repurchase of the collateral asset may be excessively *low,* so that the cost of borrowing the security becomes prohibitive and only counterparties who really cannot dispense with the possession of the collateral borrow it. Third, in the face of greater uncertainty as in the previous two points, an intermediary may increase the difference between the price of selling and buying the asset, both in the cash and repo market. This will compensate the market-maker for greater risk but also increase the cost of leverage. Fourth, harsh penalties in case of a fail on the delivery of the collateral asset may prevent this asset from circulating appropriately. If the fail penalties are too harsh when one cannot give back the collateral asset(s), then the agent may refuse to engage

[22] See Luque (2009) for the role that these two constraints play along the repo collateral multiplier construction. Also there we indicate the relation between the accumulated and incremental positions under these two constraints.

[23] Duffie (1996), p. 507, had already remarked that someone who borrowed a security yesterday can engage today in sequencial borrowings of a small security amount (using short proceeds to reverse in again), in order to meet a large delivery requirement today, leaving that same requirement to be met tomorrow. How small that quantity can be depens on the how many times repo markets clear in one day. This idea (where haircuts are not used) should not be confused with leverage.

[24] See also Bottazzi (2002a).

in another repo, since the risks of not returning the collateral asset(s) are too high. Finally, and sometimes for related reasons, some market participants may hold excess precautionary balances of a given security in their box, exacerbating the very scarcity which they try to protect themselves against. All the previous factors above may explain how the market spontaneously adjusts the amount of leverage in the securities market.

3 A model with Repo Rolls

The economy is represented by four dates, $t \in \{0, 1, 2, 3\}$.[25] In each date there are countably many transactions. The budget constraint of each date accounts for all the transactions carried out until the end of the date. The leveraging process of the *repo collateral multiplier* may be thought as occurring within the same date.

Agents can trade commodities and securities at date 0, and also engage in bilateral repos. Along the paper we assume that promises in the securities market are honored. Instead, we will focus on several scenarios regarding repo promises. In this section, we allow the agents to roll repos at date 1 but all repo promises must eventually be honored at date 2. At date 1, commodities, securities and repo markets are open. The set of states of nature at date 1 is $\mathbf{S} = \{1, ..., s, ..., S\}$. At date 2 there are no repo markets. The set of states of nature at date 2 and 3 are $\mathbf{S'} = \{1', ..., s', ..., S'\}$ and $\mathbf{S''} = \{1'', ..., s'', ..., S''\}$, respectively. The only market opened in the last date is the market for goods. More generally, we shall denote a date-state node in the economy by $\xi = 0, (s, s', s'')_{s \in \mathbf{S}, s' \in \mathbf{S'}, s'' \in \mathbf{S''}}$.

The set $\mathbf{J} = \{1, ..., j, ..., J\}$ represents the securities available in the economy. Securities are real[26] and live up to date 3. The set of commodities is $\mathbf{L} = \{1, ..., l, ..., L\}$. There is a finite set $\mathbf{I} = \{1, ..., i, ..., I\}$ of individuals (or agents). Agent $i \in \mathbf{I}$ is endowed with a vector $\omega^i_\xi \in \mathbb{R}^L_{++}$ of physical commodities at date-state ξ.

The only way to short sell a security is by borrowing it through a *reverse repo* and then selling it. So ultimately the whole supply of securities in the initial moment of date 0 comes from the real money institutions, which are typically the ones who have significant positive initial securities endow-

[25] We could also have considered an initial date $t = -1$ that represents the beginning of the economy where the transactions in the securities markets are negotiated by phone. These transactions take place in the next date $t = 0$, together with the usual transactions in repo and commodity markets.

[26] i.e. securities pay in commodities or a numeraire.

ments.[27] The securities endowments at the initial date are $e_0^i \in \mathbb{R}_{++}^J$ if i is a *real money institution*, and $e_0^i = \mathbf{0}$ if i is a *leveraged institution.*[28]

We model repos on real assets.[29] The proceeds of security j at node $\xi = 1, (s, s')_{s \in \mathbf{S}, s' \in \mathbf{S}'}$ are exogenously given by $B_{j\xi}$, where $B_{j\xi} \in \mathbb{R}_+^L$ are the real proceeds of asset j in terms of the L physical commodities. Given spot prices $p_\xi \in \mathbb{R}_+^L$, the nominal return on security j is $a_{j\xi}(p_\xi, B_{j\xi}) \equiv p_\xi B_{j\xi}$. By taking into account the security proceeds, we have that the total endowment of physical commodities at state s of date 1 is given by $\sum_i \tilde{\omega}_s^i = \sum_i \omega_s^i + \sum_{j \in J} B_{js} \sum_i \left(y_{j0}^i + e_{j0}^i\right)$, while the total endowments of physical commodities at date 2 with a history (s, s') are $\sum_i \tilde{\omega}_{ss'}^i = \sum_i \omega_{ss'}^i + \sum_{j \in \tilde{J}} B_{jss'} \sum_i (y_{js}^i + y_{j0}^i + e_{j0}^i)$.

We study *repos* that are settled in a bilateral trade system. A repo contract $\left((i,k), \left(\pi_{j\xi}^{ik} z_{j\xi}^{ik}\right), r_{j\xi}^{ik}\right)$ initiated at node ξ consists on a pair of individuals (i,k), the loan dealt, $\pi_{j\xi}^{ik} z_{j\xi}^{ik}$, where $z_{j\xi}^{ik}$ represents the amount of security j engaged in the repo and $\pi_{j\xi}^{ik} = h_{j\xi}^{ik} q_{j\xi}$ is the haircutted price of the collateralized loan, with the haircut $h_{j\xi}^{ik} \in [0,1]$ exogenously given. The interest rate on this loan is called the *repo rate,* denoted by $\hat{r}_{j\xi}^{ik}$. In order to facilitate the formulas we use the term $r_{j\xi}^{ik} = 1 + \hat{r}_{j\xi}^{ik}$. From the point of view of agent i, we say that agent i has a *reverse in* position with agent k if $z_{j\xi}^{ik} > 0$. In this case, agent i, *long in repo*, awards to k with a loan of $\pi_{j\xi}^{ik} z_{j\xi}^{ik}$ and receives $z_{j\xi}^{ik}$ as collateral from k. Conversely, we say that agent i obtains a *repo* from k if $z_{j\xi}^{ik} < 0$. In this case, i, *short in repo*, receives the loan $\pi_{j\xi}^{ik} z_{j\xi}^{ik}$ from k in exchange of giving $z_{j\xi}^{ik}$ units of security j to agent k as collateral.

3.1 Date 0

Date 0 represents the initial date when transactions take place in commodities, cash and repo markets. The budget constraint for agent i at the initial date is

[27] Real money institutions are often gauged against a long (thus their index), they are natural shorts in the securities.

[28] It may happen that the leveraged institutions cannot cover their position in the professional market (understand dealers and the leveraged community). Then dealers may go to their real money accounts and borrow the security missing. Typically the security is trading *special* when this happens so from the point of view of the real money account this is just a return enhancement: the money borrowed through the operation is a good rate and can be reinvested in the money market or in short dated securities at a higher rate.

[29] See Luque (2009) for an extention of the model to nominal securities, i.e., securities that pay an amount of units of account known ex-ante for each date-state.

$$p_0(x_0^i - \omega_0^i) + q_0 y_0^i + \sum_{(k,j)} \pi_{j0}^{ik} z_{j0}^{ik} \leq 0 \tag{BC.0}$$

Recall that the net trade in asset j, denoted by y_{j0}^i, accounts for the gross trades net of the asset endowments, that is, $y_{j0}^i = \varphi_{j0}^i - e_{j0}^i$ where φ_{j0}^i denotes the asset gross trades and e_{j0}^i the asset endowments.[30] Hence, an agent is only initially net short in securities if $e_{j0}^i + y_{j0}^i < 0$ and stays net long in securities if $e_{j0}^i + y_{j0}^i > 0$. The security cash market transactions at the initial date take place at a price denoted by q_{j0}.

Let us next impose the box constraint for date 0 (Box.0), which dictates that for this date Mr. i's box contains non-negative balances of securities title of ownership, when repo and security positions are added (that is, when to quantities purchased or borrowed we subtract quantities sold or lent):

$$y_{j0}^i + e_{j0}^i + \sum_{k \neq i} z_{j0}^{ik} \geq 0 \tag{Box.0}$$

As it has been pointed out in Section 2, the box constraint can be disentangled into two constraints: (L) and (S). The first is that if an agent has not enough endowments to cover a *sale* (decided at the previous date), then he must borrow it through *reverse repo*. This constraint for Mr. i is as follows.

$$\left(y_{j0}^i + e_{j0}^i\right) < 0 \implies y_{j0}^i + e_{j0}^i \geq -\sum_{k \neq i} z_{j0}^{ik} \tag{L.0}$$

Observe the interesting interaction between constraints (BC.0) and (L.0). A long in repo with net position $\sum_{k \neq i} z_{j0}^{ik} > 0$ faces the cost $\sum_{k \neq i} \pi_{j0}^{ik} z_{j0}^{ik}$ in his budget constraint (BC.0) for awarding the loans, but is able to short sell at most $\sum_{k \neq i} z_{j0}^{ik}$ from constraint (L.0), and this income enters in his budget constraint at the security market price q_{j0}.

The second portraits the fact that only those with specific collateral j may use collateralized borrowing (repo).

$$\sum_{k \neq i} z_{j0}^{ik} < 0 \implies y_{j0}^i + e_{j0}^i \geq -\sum_{k \neq i} z_{j0}^{ik} \tag{S.0}$$

In fact, we can show that imposing constraints (L.0) and (S.0) is equivalent to impose the box constraint (Box.0). That constraint (Box.0) implies

[30] This is, if φ_{j0}^i denotes the asset gross trades and e_{j0}^i the asset endowments, the net position is such $y_{j0}^i = \varphi_{j0}^i - e_{j0}^i$. Hence, a short sale happens when $e_{0j}^i + y_{j0}^i = \varphi_{j0}^i < 0$ and a purhcase when $e_{j0}^i + y_{j0}^i = \varphi_{j0}^i > 0$.

(L.0) and (S.0) is direct. In the other way, let constraints (L.0) and (S.0) for security j be written as

$$y_{j0}^{i} + e_{j0}^{i} \geq 0 \quad \vee \quad y_{j0}^{i} + e_{j0}^{i} \geq -\sum_{k \neq i} z_{j0}^{ik}$$

$$-\sum_{k \neq i} z_{j0}^{ik} \leq 0 \quad \vee \quad y_{j0}^{i} + e_{j0}^{i} \geq -\sum_{k \neq i} z_{j0}^{ik}$$

Now we have that $[(L.0) \wedge (S.0)]$ occurs if and only if

$$\left[y_{j0}^{i} + e_{j0}^{i} \geq 0 \ \wedge -\sum_{k \neq i} z_{j0}^{ik} \leq 0 \right] \vee \left[y_{j0}^{i} + e_{j0}^{i} \geq -\sum_{k \neq i} z_{j0}^{ik} \right]$$

which is equivalent to (Box.0).

Apart from the box constraint, we also impose another constraint that limits the total supply of a security in the economy. The resulting position of all the repo shorts will outstrip the initial supply of the security by the leveraging process. This is because the total supply of a security in the economy can be put in a sequence of repos and purchases (as the repo collateral multiplier illustrates). The leverage constraint at date 0 says that the quantity of each security j that can be put on loan (and borrowed) in a repo contract is bounded by the total endowments of the security leveraged by one minus the maximum haircut (maximum percentage retained in each repo operation), that is,

$$\left| z_{j0}^{ik} \right| \leq \frac{\sum_i e_{j0}^{i}}{1 - \bar{h}_{j0}^{i}} \qquad \text{(Lev.0)}$$

where $\bar{h}_{j0}^{i}$ equals $h_{j0}^{ik} \in [0,1]$ if i is a leverage institution, and 0 if i is a real money institution. Our separation of the institutions in two groups (real money and leveraged institutions) is more for descriptive purposes. In truth all the shades of behavior towards leverage can found.[31] The conventional wisdom is that real money institutions are stronger hands than leveraged institutions - as long only portfolios are less likely to put a firm at risk.

The next proposition establishes that the short sales in the securities market are bounded. The bound comes from the non-negativity constraints associated with the box of every agents. This a fundamental feature in our model since it kills the Hart's (1975) discontinuity points for securities markets. The explicit relationship between shorting and reverse repos is at the

[31] The arguments in our results do not depend on the inability of real money institutions to go short or use leverage. The proofs would be the same if all agents had access to leverage.

heart of the result, which re-establishes the upper hemmicontinuity of the budget correspondence by making the budget set compact.

Proposition 1: *The intersection of constraints (Box.0) and (Lev.0) implies that short sales in the securities market have the following bound*

$$y^i_{j0} + e^i_{j0} \geq -(I-1)\frac{\sum_i e^i_{j0}}{1-\bar{h}^i_{j0}} \tag{BSS.0}$$

Proof. From the box constraint we have that $e^i_{j0} + y^i_{j0} \geq -\sum_{k\neq i} z^{ik}_{j0}$. By constraint (Lev.0) we have $-z^{ik}_j \geq -(\sum_i e^i_{j0}/1-\bar{h}^i_{j0})$, which implies $-\sum_{k\neq i} z^{ik}_{j0} \geq -(I-1)\sum_i e^i_{j0}/(1-\bar{h}^i_{j0})$. The desired result follows. ■

Remark 1: The intersection of (Box.0) and (Lev.0) is convex. [32] The intersection of these two constraints is illustrated in Picture 1. The shadow area represents the intersection of the constraints, which is a convex cone. The convexity result is specially important in order to apply Debreu's (1952) existence theorem. As we had remarked before, Duffie (1996) already had constraints (S.0) and (L.0), but the latter was written in equality terms, that is, if the agent short sells a security j then he must have in his box exactly that quantity being shorted. In (L.0) we allow for inequality, that is, the amount shorted cannot be greater than the amount in the box. This constraint *a la Duffie* is as follows:

$$y^i_{j0} + e^i_{j0} < 0 \implies y^i_{j0} + e^i_{j0} = -\sum_{k\neq i} z^{ik}_{j0}$$

In the Appendix we show that under our specification, the budget set is convex, whereas under Duffie's (1996) it is not.

3.2 Date 1 : Repo rolls

As explained in the introduction, the agents may have different marginal utilities and income levels and therefore different marginal rates of intertemporal substitution. Moreover, the income differences are exacerbated by difference in leverage. These differences may lead to choose differently whether to honor or roll their commitments in a repurchase agreement. In this second date, an agent is allowed to roll part or all of a *reverse repo* made at the first date 0. Each agent chooses how much of the collateral security

[32] The intersection is also non empty. The reason is that the budget set contains the no trading point.

received in a reverse repo he wants to deliver back and how much of the collateral asset engaged in repo he wants to repurchase.[33] The part of the initial repo that is renegotiated is called *repo roll*.

A counterparty having done a *reverse repo* will be able to adjust or roll his balance at the second date where the repo roll takes place. Then, the long must *reverse in* at least the amount of collateral that is needed to fulfill the promise. This situation is normal in a framework of next-day delivery securities markets and same-day settlement of repo markets. Such borrowing of securities and the potential scrambling to find the security back with closing the position is how leveraged institutions short a given security in the cash market.

Let agent i's delivery of collateral j to agent k be d_{js}^{ik}, is a fraction of the amount promised z_{j0}^{ik} and hence belongs to $\left[0, z_{j0}^{ik}\right] > 0$. If the full amount of security borrowed by the long is not returned at date 1, the long has to cover at date 1 the missing amount of undelivered security through a new reverse repo or by buying back the security. Then, the totality of the repo promise initiated at date 0 appears in the agents' budget constraints as fulfilled, but at the cost of rolling the part of the promised deliveries to a *new repo* z_{js}^{ik}.[34]

The new *repos* at date 1 follow the same mechanism than those *repos* initiated at date 0, except that it will not be possible any more to cover at date 2 the new repo promises with repo rolls. The price of a *repo* between agents i and k initiated at state s of date 1 is denoted by π_{js}^{ik}. The associated *repo rate* on the amount $\pi_{js}^{ik} z_{js}^{ik}$ is $r_{js}^{ik} - 1$.[35] The budget constraint at state s of date 1 is the following.

$$p_s(x_s^i - \omega_s^i) + q_s y_s^i + \sum_{(k,j)} \left(\pi_{js}^{ik} z_{js}^{ik} - r_{j0}^{ik} \pi_{j0}^{ik} z_{j0}^{ik}\right) \leq a_s \left(y_0^i + e_0^i\right) \qquad \text{(BC.s)}$$

Note that in a *repo* we just focus on the cash flows of the loan (the repo rate). This is because if a long *reversed in* a security and it lies in his box, the long just passes-through all the cash flow he received to his repo counterpart. The security is only playing the role of collateral. Net

[33]The intended magnitudes for the repo roll may be different between the short and long parts. This difficulty is solved in equilibrium through the *repo rate*, which makes the two parties to agree in the repo contract.

[34]Observe that z_{js}^{ik} may be larger than the repo roll $z_{j0}^{ik} - d_{js}^{ik}$. We could have modeled the specific case in which the new repos exactly match the repo roll, $z_{j0}^{ik} - d_{js}^{ik} = z_{js}^{ik}$. However, we think that allowing for the more general case $z_{js}^{ik} \geq z_{j0}^{ik} - d_{js}^{ik}$ is more realistic, since an agent does not know whether his counterparty is doing a reverse repo in order to short sell a security or to cover a previous reverse repo promise.

[35]Observe that the *new repo rate* $r_{js}^{ik} - 1$ may differ from the repo rate $r_{j0}^{ik} - 1$ corresponding to the initial repo contract at date 0.

the *borrower of securities* only has the cash flows of the cash he lent. The payment back to the short is known as *manufactured dividend.*

Next we impose the box constraint for date 1 and state s, which assure the non-negativity of final balance of security j in agent i's box:

$$y^i_{js} + y^i_{j0} + e^i_{j0} + \sum_{k \neq i} (z^{ik}_{js} - z^{ik}_{j0}) \geq 0 \qquad \text{(Box.s)}$$

Finally, we impose the constraint that requires that the new repo position (long or short) of a security j at state s has to be smaller or equal than the total initial supply of the security leveraged up by the repo collateral multiplier:

$$\left| z^{ik}_{js} \right| \leq \frac{\sum_i e^i_{j0}}{\left(1 - \bar{h}^i_{j0}\right)\left(1 - \bar{h}^i_{js}\right)} \qquad \text{(Lev.s)}$$

where $\bar{h}^i_{js}$ equals $h^{ik}_{js} \in [0, 1]$ if i is a leveraged institution, and 0 if i is a real money institution.

Proposition 2: *The intersection of constraints (Box.s) and (Lev.s) implies that short sales in the securities market at date 1 and state s are bounded as follows*

$$y^i_{js} + y^i_{j0} + e^i_{j0} \geq -\frac{(2 - \bar{h}^i_{js})(I - 1) \sum_i e^i_{j0}}{(1 - \bar{h}_{j0})\left(1 - \bar{h}_{js}\right)} \qquad \text{(BSS.s)}$$

Proof. (Box.s) and (Lev.0) gives us $y^i_{js} + y^i_{j0} + e^i_{j0} \geq -\sum_{k \neq i} z^{ik}_{js} - ((I - 1)\sum_i e^i_{j0})/(1 - \bar{h}^i_{j0})$. Then, by using constraint (Lev.s), $-z^{ik}_{js} \geq -\sum_i e^i_{j0}/((1 - \bar{h}^i_{j0})(1 - \bar{h}^i_{js}))$, we obtain our desired result. ■

Note that constraint (BSS.s) is different than constraint (BSS.0) since in the second date we must account for the leverage $\frac{1}{1 - \bar{h}_{j0}}$ that was already done by the leveraged institutions in date 1 by using the repo collateral multiplier, but also for the corresponding leverage of the second date.

3.3 Date 2 with fully honored promises

We assume that at date 2 agents are not allowed to roll on the repo promises made at date 1. For a *repo* signed at state s the short i must pay the *repo rate* $r^{ik}_{js} - 1$ on the loan $\pi^{ik}_{js} z^{ik}_{js}$ in order to repurchase the security. The long side then receives the repo interests and gives back the collateral. The budget constraint at date 1 for state $s \in S$ is as follows.

$$p_{s'}(x^i_{s'} - \omega^i_{s'}) + q_{s'} y^i_{s'} \leq a_{s'} \left(y^i_s + y^i_0 + e^i_0\right) + \sum_{(k,j)} r^{ik}_{js} \pi^{ik}_{js} z^{ik}_{js} \qquad (BC.s')$$

Since there is no possibility to engage in reverse repos at date 2 in order to short sale a security, the only possible security trades $y_{s'}$ are those that satisfy the following box constraint

$$y^i_{js'} + y^i_{js} + y^i_{j0} + e^i_{j0} - \sum_{k \neq i} z^{ik}_{js} \geq 0 \qquad (\text{Box}.s')$$

Observe that (Box.s') is a plain *no failing condition*, since repo promises made at date 1 and state s appear fully honored in the box of date 2. Notice that, at this date, no one can short sell a security by borrowing it through reverse repo at date 2.

3.4 The last date

Since it is only in the interest of an agent to repurchase a security if it pays in the next date, it is necessary to consider one more date, $t = 3$, where securities have returns. To avoid having to deal with infinite horizon issues we assume that assets die at date 3. In the last date the securities pay dividends, but have no value and, therefore, there are no repo contracts and security trades. Only the market for commodities is open. The corresponding budget constraint in state s'' is the following

$$p_{s''}(x_{s''} - \omega_{s''}) \leq a_{s''}(y^i_{s'} + y^i_s + y^i_0 + e^i_0) \qquad (BC.s'')$$

3.5 Equilibrium

Let us consider the economy constructed above, where rolling the *initial repos* is allowed. The consumer i's problem is to choose a vector $(\bar{x}^i, \bar{y}^i, \bar{z}^i) \in \mathbb{R}^{(1+S+S')L}_+ \times \mathbb{R}^{J(1+S)}_+ \times \mathbb{R}^{(I-1)J(1+S)}_+$ that maximizes his utility $u^i(x)$ subject to his budget and portfolio constraints (BC.ξ), (Box.ξ) and (Lev.ξ) at each node $\xi = 0, (s, s', s'')_{s \in \mathbf{S}, s' \in \mathbf{S}', s'' \in \mathbf{S}''}$, given prices $(\bar{p}, \bar{q}, \bar{r})$.

Definition 1: An equilibrium for this economy is an allocation of commodity bundles, net security positions and repo positions $(\bar{x}, \bar{y}, \bar{z})$ together with a price vector $(\bar{p}, \bar{q}, \bar{r})$ such that: (1.i) $\forall i \in I$, $(\bar{x}^i, \bar{y}^i, \bar{z}^i)$ solves the consumer i's problem given $(\bar{p}, \bar{q}, \bar{r})$; (1.ii) Commodity markets clear: $\sum_{i \in \mathbf{I}} (\bar{x}^i_\xi - \tilde{\omega}^i_\xi) = 0$, at all nodes ξ; (1.iii) Securities markets clear: $\sum_{i \in \mathbf{I}} \bar{y}^i_\xi = 0$, $\xi = 0, s, s'$; and (1.iv) Repo markets clear: $\bar{z}^{ik}_{j0} + \bar{z}^{ki}_{j0} = 0$ and $\bar{z}^{ik}_{js} + \bar{z}^{ki}_{js} = 0$, $\forall i, k, j, s$.

Assume that: (A1) Utility functions $u_i : \mathbb{R}^{(1+S+S')L} \to \mathbb{R}$ are continuous, strictly quasi-concave and strictly increasing in x_0, x_s and $x_{s'}$ and bounded on $X_0 \times X_s \times X_{s'}$. (A2) The endowments of commodities are positive, i.e.

$\omega_\xi^i \in \mathbb{R}_{++}^L$, $\forall i$, ξ; and the initial asset security endowments are positive for the real money institutions, that is, $e_0^i = \mathbb{R}_{++}^J$ for all agents i with real money accounts. (A3) The real returns matrix B is such that $B > 0$.

Proposition 3: *Let assumptions A1-A3 hold. Then there exists an equilibrium for this economy with repo rolls and fully honored repo promises in the penultimate date.*

4 Modelling Fails

4.1 A Failing Rule

In the present section we no longer have agents covering their collateral shortfalls with new repos (repo rolls), but fails are allowed if one's counterparty is failing on him. We assume that agents always repurchase what their long repo counterparties wish to deliver, that is, repo *shorts* do not default but repo *longs* may fail. Thus we consider a three dates economy, $t = 0, 1, 2$. At the initial date 0 the repos are initiated and these repos have to be settled at the next date ($t = 1$). The last date ($t = 2$) just serves for guaranteeing that securities retain a value at the date of the repo settlement and, therefore, to simplify, we assume that no uncertainty is to be resolved between dates 1 and 2 (that is, each node s at date 1 has just one successor s^+).

Since long and short repo positions will be treated asymmetrically, we need to use two different non-negative variables to denote an agent's short and long repo positions. We denote agent i's *reverse in* position with agent k by $\theta_{j0}^{ik} \geq 0$. In this case, agent i, *long in repo*, awards to k a loan $\pi_{j0}^{ik}\theta_{j0}^{ik}$ and receives θ_{j0}^{ik} as collateral from k. Conversely, we denote agent i's *repo* position with agent k by $\varphi_{j0}^{ik} \geq 0$. The total net repo position of agent i is then $z_{j0}^{ik} = \theta_{j0}^{ik} - \varphi_{j0}^{ik}$, so that $z_{j0}^{ik} < 0$ means that agent i is net short in repo and $z_{j0}^{ik} > 0$ means that agent i is net long in repo. Date 0 budget constraint is now the following

$$p_0(x_0^i - \omega_0^i) + q_0 y_0^i + \sum_{(k,j)} \pi_{j0}^{ik}(\theta_{j0}^{ik} - \varphi_{j0}^{ik}) \leq 0 \qquad (\widehat{BC}.0)$$

We allow an agent involved in a reverse repo to *fail* if he is sold back in repo to other counterparties and the amounts they delivered were below their commitment. In particular, a borrower i of security j (that is, an agent i for whom $z_{j0}^{ik} > 0$ for some k) *fails* at state $s \in S$ of date 1 if the deliveries

$\left(d_{js}^{ni}\right)_{n\neq i}$ of his long counterparts[36] together with securities that are otherwise in his box $y_{js}^{i}+y_{j0}^{i}+e_{j0}^{i}$ are insufficient to cover his own commitment to deliver $\sum_{n\neq i}\theta_{j0}^{in}$, that is,

$$y_{js}^{i}+y_{j0}^{i}+e_{j0}^{i}<\sum_{n\neq i}\left(\theta_{j0}^{in}-d_{js}^{ni}\right)$$

where $d_{js}^{ni}\in\left[0,\varphi_{j0}^{in}\right]$.

Although we allow for fails triggered by others' fails, we require that the agents' net trades in securities, at date 1, y_{s}^{i}, be such that what is available for delivery to short counterparties, given the effective deliveries by long counterparties, cannot be negative:

$$y_{js}^{i}+y_{j0}^{i}+e_{j0}^{i}+\sum_{n\neq i}d_{js}^{ni}\geq 0 \qquad \text{(Box1.}s\text{)}$$

In the case of fail, we consider a concave delivery rule known as the *constrained equal award* reimbursement rule (R) that makes awards as equal as possible, subject to the condition that no counterparty receives more than his claim (see Herrero-Villar (1999)). That is, the long i delivers back to k the following amount

$$R_{js}^{ik}\left((\theta_{j0}^{in},d_{js}^{ni})_{n\neq i},y_{js}^{i},y_{j0}^{i}\right)=\begin{cases}\theta_{j0}^{ik} & \text{if } y_{js}^{i}+y_{j0}^{i}+e_{j0}^{i}+\sum_{n\neq i}d_{js}^{ni}\geq\sum_{n\neq i}\theta_{j0}^{in}\\ \min\left\{\theta_{j0}^{ik},\alpha_{js}^{i}(y_{js}^{i}+y_{j0}^{i}+e_{j0}^{i}+\sum_{n\neq i}d_{js}^{ni})\right\}, & \text{otherwise}\end{cases}$$

where $\alpha_{js}^{i}\in[0,1]$ is a parameter to be determined in equilibrium so that the sum of reimbursements matches what is available for reimbursement. Let $H_{js}^{i}\equiv\min\left\{y_{js}^{i}+y_{j0}^{i}+e_{j0}^{i}+\sum_{n\neq i}d_{js}^{ni},\sum_{n\neq i}\theta_{j0}^{in}\right\}$ be agent i's total amount available for reimbursement and suppose α_{js}^{i} is such that, in equilibrium, (ι) $H_{js}^{i}=\sum_{n}\min\{\theta_{j0}^{in},\alpha_{js}^{i}H_{js}^{i}\}$. We can actually have

$$R_{js}^{ik}\left((\theta_{j0}^{in},d_{js}^{ni})_{n\neq i},y_{js}^{i},y_{j0}^{i}\right)=\min\left\{\theta_{j0}^{ik},\alpha_{js}^{i}H_{js}^{i}\right\} \qquad \text{(CEA)}$$

in both cases, provided that $\min\{\theta_{j0}^{in},\alpha_{js}^{i}\sum_{n\neq i}\theta_{j0}^{in}\}=\theta_{j0}^{in}$, which is equivalent to ($\iota\iota$) $\alpha_{js}^{i}\geq\theta_{j0}^{in}/\sum_{n\neq i}\theta_{j0}^{in}$ for every $n\neq i$. That is, deliveries are made

[36] Recall that we defined the agent i's promise of delivering back collateral j to agent k by d_{js}^{ik}, which belongs to $\left[0,z_{j0}^{ik}\right]$ for $z_{j0}^{ik}>0$. Then, d_{js}^{ni} refers to the delivery of agent n to i for $z_{j0}^{in}<0$. Also note that the agent n's delivery to i can be thought by agent i as a repurchase d_{js}^{ni} since he is not allowed to default on promised repurchases.

according to the *constrained equal award* delivery rule (R) as described by (CEA), where $\alpha^i_{js} \in [0,1]$ is taken as given by agent i and will, in equilibrium, satisfy conditions (ι) and ($\iota\iota$) above.[37] See Picture 3 for a graphical representation of the CEA rule.

We could have considered other concave delivery rules, so that the convexity of the consumer's choice set is preserved, and our argument does not actually depend on the precise allocation rule.

Notice that, for each pair of agents (i,k), a repo contract between them might not be symmetric, in the sense that the effective returns depend on who is in each side of the contract, as delivery beliefs and effective awards given by rule (R) (taking into account the α^i_{js} and H^i_{js} specific to the long) may be affected by this. Hence, for each security j and each pair (i,k), we have now two possible repo contracts, one where i is the long and k the short, with repo rate r^{ik}_{j0}, and another where i is the short and k is the long, with repo rate r^{ki}_{j0}. To simplify, as we are not allowing for default (i.e., shorts' not honoring the promise to repurchase), we assume the same haircut for both contracts and, hence, the same loan price π^{ik}_{j0}.

Under the possibility of fails and the given rule (R), the budget constraint at state s should be written as follows:

$$p_s(x^i_s - \omega^i_s) + q_s y^i_s \leq a_s(y^i_{j0} + e^i_{j0}) + \sum_{j\in J, k\neq i} (r^{ik}_{j0}\pi^{ik}_{j0}R^{ik}_{js}\left((\theta^{in}_{j0}, d^{ni}_{js})_{n\neq i}, y^i_{js}, y^i_{j0}\right) - r^{ki}_{j0}\pi^{ik}_{j0}d^{ki}_{js}) \qquad (\widehat{BC}.s)$$

Also we impose the constraint that prevents agent i from failing when his long counterparts are not failing,[38]

$$y^i_{js} + y^i_{j0} + e^i_{j0} \geq \sum_{k\neq i}\left(\theta^{ik}_{j0} - \varphi^{ik}_{j0}\right), \ \forall s, j, k \qquad (\text{Box2}.s)$$

The budget constraint of the last date, $t = 2$, is the following.

$$p_{s^+}(x^i_{s^+} - \omega_{s^+}) \leq a_{s^+}(y^i_s + y^i_0 + e^i_0) \qquad (\widehat{BC}.s^+)$$

4.2 Perfect Foresight

Let us look for different equilibrium concepts allowing for fails according to the rule (R) and where the agents form *beliefs* on the deliveries by others.

[37] Observe that the delivery rule is equivalent to a box constraint, since it dictates how much the long should deliver given the non-negativity constraint of the tittle of a security (which is assured by the (SC1.s) constraint).

[38] We could have dispensed with this no-intended fails condition (Box2), but in this case it might be appropriate to introduce fail penalties. The discussion of the form and impact on repo trades of such penalties is beyond the purpose of this paper.

We start with the simplest case to model, the *Perfect Foresight Equilibrium* (PFE). Here the beliefs about future deliveries and repo positions are formed simultaneously by the agents. Moreover, we require the beliefs to be *degenerate* (i.e. put all the probability in one point) and *self-fulfilling* (i.e., the believed deliveries are consistent with prices and plans in the sense that all equilibrium conditions hold). Notice that agent i only forms beliefs about the deliveries of his counterparts and no one else in the economy needs to form beliefs about these variables. That is, the set of delivery variables for which each agent forms beliefs is disjoint from the corresponding set of any other agent. In this respect, a notion of perfect foresight in deliveries differs from the usual perfect foresight equilibrium in price variables, which requires *common* beliefs about future prices.[39]

Consumers form beliefs about the delivery rates δ_{sj}^{ik} of their long counterparties and choose consumption x^i, security y^i and repo positions $(\theta_0^i, \varphi_0^i)$ so that the above budget and portfolio constraints hold for counterparties' deliveries $d_{js}^{ki} = \delta_{js}^{ki}\varphi_{j0}^{ik}$.

The leverage constraint that places an upper bound on the quantity of each security that can be put on loan in each repo contract is now written as follows:

$$\theta_{j0}^{ik}, \varphi_{j0}^{ik} \leq \frac{\sum_i e_{j0}^i}{1 - \bar{h}_{j0}^i} \qquad (\widehat{Lev}.0)$$

More formally, let us define the *consumer i's problem*: Consumer i, having formed beliefs on his counterparties delivery rates [40] $\delta_{sj}^{ki} \in [0,1]$, $\forall\ (s,j)$, $k \neq i$, chooses a vector $\left(x^i, y^i, \theta_0^i, \varphi_0^i\right)$ in order to maximize his utility u^i subject to the budget constraints $(\widehat{BC}.0)$, $(\widehat{BC}.s)$, $(\widehat{BC}.s^+)$ and the portfolio constraints (Box.0), $(\widehat{Lev}.0)$, (Box1.s), (Box2.s), for $d_{js}^{ki} = \delta_{js}^{ki}\varphi_{j0}^{ik}$, $\forall\ (s,j)$, $k \neq i$, given prices $(\bar{p}, \bar{q}, \bar{r})$.

Definition 2: A *Perfect Foresight Equilibrium* consists of a vector of prices $(\bar{p}_0, \bar{q}_0, \bar{r}_0)$, an allocation $\left(\bar{x}_0, \bar{y}_0, \bar{\theta}_0, \bar{\varphi}_0\right)$ of net asset positions and repo contracts at the initial date, a family of beliefs (degenerate) about future repo delivery rates $\bar{\delta}_{sj}^{ki}$ and future prices $(\bar{p}_s, \bar{p}_{s'}, \bar{q}_s)$, and consumption plans $(\bar{x}_s, \bar{x}_{s^+})$ such that:

(2.i) For every i, $\left(\bar{x}^i, \bar{y}^i, \bar{\theta}_0^i, \bar{\varphi}_0^i\right)$ solves consumer i's problem at believed delivery rates $\bar{\delta}_{sj}^{ki}$ and given prices $(\bar{p}, \bar{q}, \bar{r})$

[39] Observe that here, as usual, future prices $(p_s, p_{s'}, q_s)$ should be seen as common beliefs about the equilibrium values.

[40] As usual in models with perfect foresight where price beliefs and price variables are made to coincide, we use the same notation for deliveries and delivery beliefs at date 1.

(2.ii) Securities and commodities markets clear at every node ξ, that is, $\sum_{i\in I}\bar{y}^i_\xi = 0$ and $\sum_{i\in I}(\bar{x}^i_\xi - \omega^i_\xi) = 0$.

(2.iii) Repo markets clear: $\bar{\theta}^{ik}_{j0} - \bar{\varphi}^{ki}_{j0} = 0$, $\forall\,(i,j,k)$

(2.iv) Delivery beliefs are self-fulfilling *(consistency):* for $d^{ki}_{js} = \delta^{ki}_{js}\varphi^{ik}_{j0}$ we have

$$\bar{d}_s : R^{ik}_{js}\left((\theta^{in}_{j0}, d^{ni}_{js})_{n\neq i}, y^i_{js}, y^i_{j0}\right) - d^{ik}_{js} = 0, \forall j, i, k.$$

Condition (2.iv) is requiring that agent k's belief about what his long counterparty i will deliver (d^{ik}_{js}) should coincide with what agent i decides to deliver to k according to rule (R). Notice that, in condition (2.iv), we have a system of $I(I-1)$ equations and $I(I-1)$ unknowns, that is, the vector $\left(\bar{d}^{ik}_{js}\right)_{i,k\neq i}$ should be a fixed point of the function R_{js}.

Observe that the above equilibrium notion is compatible with different delivery scenarios depending on whether agents' beliefs are more or less pessimistic. The following two equilibria are perfect foresight examples. One is the no fail equilibrium that we previously analyzed. The other is one where nobody expects any collateral back and hence nobody lends, which implies the no-short sale at equilibrium. Since repo transactions are bilateral this sets the scene for the growth of the security lending markets by slowly adding bilateral relationships of counterparties trusting each others to return borrowed collateral.

Proposition 4: *Under assumptions A1-A3, there exists a perfect foresight equilibrium (PFE) satisfying the above conditions.*[41]

Remark 2: When $r^{ik}_{j0} = r^{ki}_{j0}$ and agents i and k have beliefs $\bar{\delta}^{nm}_{sj} = 1$,for $m = i, k$ and for any repo long counterparty n of these two agents in security j, the two repo contracts between agents i and k become formally the same (as date 1 payoffs of these repo contracts become linear and described as in Section 3, due to (Box2.s)). One may argue that, in this case, only the net position $\theta^{ik}_{j0} - \varphi^{ik}_{j0}$ should be bounded (by (Lev.0), as in Section 3). However, an equilibrium under ($\widehat{Lev}$.0) induces an equilibrium under the formalization with bounded net repo positions (given by (Lev.0)) for this specific case. In fact, take an equilibrium under constraint ($\widehat{Lev}$.0) for which this case holds. Now, suppose that there is for agent i or agent k, say for the former, a better plan $(x^i, y^i, \theta^i, \varphi^i)$ in the choice set, with repo positions $(\theta^{ik}_0, \varphi^{ik}_0)_{k\neq i}$ that might not satisfy ($\widehat{Lev}$.0) for security j. Let $(\tilde{\theta}^{ik}_{j0}, \tilde{\varphi}^{ik}_{j0})$ be such that $\tilde{\theta}^{ik}_{j0} - \tilde{\varphi}^{ik}_{j0} = \theta^{ik}_{j0} - \varphi^{ik}_{j0}$ and $\tilde{\theta}^{ik}_{j0}\tilde{\varphi}^{ik}_{j0} = 0$. The redefined plan $(x^i, y^i, \tilde{\theta}^i, \tilde{\varphi}^i)$ with

[41] Actually, within a large class of collateral allocation rules, satisfying concavity and continuity properties, there is a perfect foresight equilibrium compatible with that rule and those beliefs.

$(\theta_{j0}^{ik}, \varphi_{j0}^{ik})_{k\neq i}$ replaced by $(\widetilde{\theta}_{j0}^{ik}, \tilde{\varphi}_{j0}^{ik})_{k\neq i}$ (and $(\widetilde{\theta}_{j'0}^{ik}, \tilde{\varphi}_{j'0}^{ik})_{k\neq i} = (\theta_{j'0}^{ik}, \varphi_{j'0}^{ik})_{k\neq i}$ for any other security j') satisfies $(\widehat{Lev}.0)$, together with all the other constraints of agent i's problem, and is still better than the equilibrium plan for agent i, contradicting individual optimality.

The perfect foresight requirement is certainly demanding since agents must forecast accurately the future deliveries by others in every state. We would like to allow for non-degenerate and not self-fulfilling beliefs, being formed using signals on date 0 variables. However, if delivery beliefs are not self-fulfilling can we still have market clearing at dates 1 and 2 as equilibrium requirements? Mistakes in forecasting what counterparties will deliver may preclude the occurrence of Walras law at these dates. To see this, let us aggregate date 1 budget constraints of state s across agents:

$$p_s \sum_i (x_s^i - \omega_s^i - a_s \sum_i \varphi_{j0}^i) + q_s \sum_i y_s^i - \sum_{(i,k)} \sum_{j\in J} r_{j0}^{ik} \pi_{j0}^{ik} (R_s^{ik} - d_{js}^{ik}) = 0 \quad \text{(C.1)}$$

The value of aggregate excess demand in commodity and security markets will be zero when the third term above is zero. In the case of no fails it is zero because effective deliveries are the same as promises. In the case of perfect foresight this term is also zero because deliveries are a fixed point of the rule. However, errors in forecasting fails (i.e., $d_{js}^{ik} \neq R_{js}^{ik}((\theta_{j0}^{in}, d_{js}^{ni})_{n\neq i}, y_{js}^i, y_{j0}^i)$, when k anticipates a deliver by i different from what i delivers according to the rule) may prevent Walras law and, therefore, also market clearing at date 1.

Hence, our following step will be to relax our equilibrium notion to a *Temporary Equilibrium* (TE), where market clearing at dates 1 and 2 does not necessarily hold. We do this in the next subsection. Having done this, we will see that, for some classes of non-perfect foresight delivery beliefs, we still have market clearing properties at dates 1 and 2, for price forecasts that are degenerate and common across agents.

4.3 A temporary equilibrium approach

A well known problem of temporary equilibrium with price beliefs is that it may fail to exist if the agents have very heterogeneous expectations (Hart 1974) leading to non compact budget sets. In our context, we do not need to assume that price expectations overlap (as Green (1973, 1974), Hart (1974) and Hammond (1977, 1983) did) or to impose bounded short-sales (as Radner (1972) and Milne (1980) did) since short sales are already endogenously bounded by the box constraint (see Proposition 1). With regard to the delivery beliefs this problem is not relevant since the delivery variables are disjoint from agent to agent. So, the construction of a temporary equilibrium seems easier than in usual contexts.

A temporary equilibrium guarantees market clearing at date 0 assuming that agents maximize their expected utility with respect to beliefs on states, deliveries and future prices, and in such a way that the budget sets at the following dates are non-empty almost surely. Let us start by defining agents' beliefs and constructing their expected utilities with respect to these beliefs.

Each agent i assigns probabilities to the vector consisting of a state s occurring at date 1, the deliveries $\left(d_{js}^{ki}\right)_{k\neq i,j\in J}$ of securities he lent out at the initial date and future prices $(p_s, q_s, p_{s^+}) \in P_{-0}$. Let us introduce the eligible variables that can be used as a *signal* to form such beliefs : $\tau = (p_0, q_0, r, y_0, \theta, \varphi)$. Notice that repos are only traded at date 0, so that z can be part of the signal. Moreover, even though repo interest rates are paid at date 1 they are actually date 0 prices (see the Appendix, where positions are rewritten in terms of final proceeds, so that r appears on date 0 budget constraints instead). The agent may without loss of generality elect to restrict the information he uses to any subset of signal variables (e.g., prices and counterparties promised deliveries). One may be tempted to reduce the relevant signals used by agents to some reasonable subset of variables, for example to the promised deliveries of one's counterparties. Our framework allows such focus, but this is not acceptable in all situations. For example, if a third party aggressively acquires the security in an attempt to control the availability of such a security without lending it out or selling it. Ignoring such an activity to assess my counterparty's ability to return the security to me is naive. One need not know exactly the activity of that third party and use it as a signal: the repo rate would reveal the scarcity of the security as the security would trade on special.

We introduce an assumption on beliefs. Let $\mathbf{D}_s^i \subseteq \mathbb{R}_+^{J(I-1)}$ be the set of deliveries $\left(d_{js}^{ki}\right)_{k\neq i,j\in J}$ of securities agent i lent out at the initial date and let $P_{-0} \equiv (\triangle^{L+J-1})^S \times (\triangle^{L-1})^S$ be the set of normalized future prices (p_s, q_s, p_{s^+}). Denote by $\mathbf{M}\left(S \times P_{-0} \times \mathbf{D}_s^i\right)$ the set of probability measures on $S \times P_{-0} \times \mathbf{D}_s^i$.

Assumption (A4): every agent i's joint probability measure on state, future prices and deliveries is a function of τ, given by $\mu^i : \tau \longmapsto \mu^i(\tau) \in \mathbf{M}\left(S \times P_{-0} \times \mathbf{D}_s^i\right)$, assumed to be norm continuous[42] and such that, for each $\tau = (p_0, q_0, r, y_0, \theta, \varphi)$, we have (i) supp $\mu^i(\tau)$ is closed and (ii) $d_{js}^{ki} \leq \varphi_{js}^{ik}$ for every d in supp$\mu(\tau)$.

The equilibrium concept that we will be using requires market clearing

[42]That is, if $\tau_n \to \tau$ then $\sup \sum_{i=1}^{n} |\mu(\tau_n)(F_i) - \mu(\tau)(F_i)| \to 0$ where the supremum is taken over all finite sequences $\{F_i\}$ of disjoint Borel subsets of $S \times P_{-0} \times \mathbf{D}_s^i$.

only on date 0 variables and these variables should be individually optimal in the sense that each agent i maximizes a naturally constructed expected utility function E^i whose explanatory variables are just date 0 choices, current prices and beliefs.

First, let us find the maximal utility that agent i can achieve at state s, when deliveries are $d(s)$, future prices are (p_s, p_{s^+}) and the signal τ gives information on date 0 choices and prices. This maximal utility, denoted by $\tilde{U}^i(\tau, s, p_s, p_{s^+}, q_s, d(s))$, is given as follows

$$\max\{u_s^i(x_0, x_s, x_{s'}) : \{x_s, x_{s'}, y_s\} \text{ satisfy } (\widehat{BC}.\xi)_{\xi=s,s^+}, (Box1.s), (Box2.s) \\ \text{for } \tau, p_s, p_{s^+}, q_s, d(s)\}$$

We are now ready to compute the *expected indirect utility* E^i, for beliefs $\mu^i(\tau)$ and current choices $(x_0^i, y_0^i, \theta^i, \varphi^i)$ (given through τ):

$$E^i(\tau) = \int_{S\times P_{-0}\times \mathbf{D}_s^i} \tilde{U}^i(\tau, s, p_s, p_{s^+}, q_s, d(s)) d\mu^i(\tau)$$

We are ready to say how agent i should choose current variables in order to maximize the above expected indirect utility[43].

At the initial date agent i's demand, g^i, for current consumption, current security and repo positions under the beliefs $\mu^i(\tau^i)$ is the result of the maximization of the expected indirect utility function E^i subject to budget feasibility at the initial date and the condition that the budget set for the following dates is *non-empty* for every realization in the support of $\mu^i(\tau)$. That is, given $(p_0, q_0, r, x_0^{-i}, \theta^{-i}, \varphi^{-i}, y_0^{-i})$,

$$g^i(p_0, q_0, r, x_0^{-i}, \theta^{-i}, \varphi^{-i}, y_0^{-i}) = \arg \max_{(x_0^i, y_0^i, \theta^i, \varphi^i)} E^i(\cdot, p_0, q_0, r, x_0^{-i}, \theta^{-i}, \varphi^{-i}, y_0^{-i}) \tag{G.i}$$

such that (I) (x_0^i, y_0^i, z^i) satisfy $(\widehat{BC}.0), (Box.0), (\widehat{Lev}.0)$, (II) $(\widehat{BC}.s)$ hold for some (x_s, y_s) satisfying (Box1.s), (Box2.s), $x_s \geq 0$, $\forall (s, p_s, q_s, d_s) \in$supp $\mu^i(\tau)$, and (III) $(\widehat{BC}.s^+)$ holds for some $x_{s^+} \geq 0$, $\forall (s, p_{s^+}) \in$supp $\mu^i(\tau)$.

Definition 3: A family of beliefs $\bar{\mu}(\tau)$ on the set of states, future prices and deliveries at date 1 is a *Temporary Equilibrium* if there exists an allocation of net asset positions and repo contracts $(\bar{y}_0, \bar{z}_0)$ together with a price vector $(\bar{p}, \bar{q}, \bar{r})$ such that:

[43] Observe that E^i is well defined since $\tilde{U}^i$ is a measurable function on $S \times P_{-0} \times \mathbf{D}_s^i$ (see Lemma III.39 in Castaing and Valadier (1975)). Moreover, E^i is continuous and bounded due to assumptions (A1) and (A4) (see Grandmont (1982, Proposition 1); actually, for this purpose, weak* continuity would suffice).

(3.i) $\left\{\bar{x}_0^i, \bar{y}_0^i, (\bar{\theta}^{ik}, \bar{\varphi}^{ik})_{k\neq i}\right\} \in g^i(\bar{p}_0, \bar{q}_0, \bar{r}, \bar{x}_0^{-i}, \bar{y}_0^{-i}, (\bar{\theta}^{ki}, \bar{\varphi}^{ki})_{k\neq i})$
(3.ii) Securities market at date 0 clear: $\sum_i \bar{y}_0^i = 0$.
(3.iii) Repo markets clear: $\bar{\theta}_{j0}^{ik} - \bar{\varphi}_{j0}^{ki} = 0, \forall i, k, j$.
(3.iv) Goods market at date 0 clear: $\sum_i \bar{x}_0^i = 0$.

Proposition 5: *Under assumptions A1 to A4, there exists a Temporary Equilibrium.*

Remark 3: One may wonder whether it is possible to have a model that is easily comparable with the perfect foresight model, so that the temporary equilibrium aspect is focused on delivery variables only. This means that the beliefs on futures prices should be degenerate and consensual. In this hybrid model we would have a market clearing requirement to pin down those consensual price beliefs. The problem is to marry beliefs on deliveries with some notion of market clearing for date 1 and 2. Walras law will be a necessary condition for any kind of market clearing concept that we can come up with (see equation C1 and the subsequent discussion). There are three possible conditions on beliefs under which we can address date 1 market clearing.

(1) The first is to require that the beliefs' support is such that the forecasting errors (i.e., $d_{js}^{ik} \neq R_{js}^{ik}$) on delivery values are canceled out across agents and across assets, that is,

$$\sum_{(i,k)} \sum_{j\in J} r_{j0}^{ik} \pi_{j0}^{ik} (R_s^{ik}((\theta_{j0}^{in}, d_{js}^{ni})_{n\neq i}, y_{js}^i, y_{j0}^i) - d_{js}^{ik}) = 0 \ , \ \forall d_s^{ik} \in \text{sup p } \mu^k(\tau), \forall k$$

Under this hypothesis, there exists a temporary equilibrium with common and degenerate future price forecasts vector (p_s, p_{s^+}, q_s) consistent with market clearing of commodities and securities in every state of date 1, for each realization of deliveries.

(2) The second possibility is to require that, for each bilateral repo contract and each state, *the mean forecasting error on deliveries is zero, that is,*

$$\int_{\mathbf{D}_s} (R_s^{ik}((\theta_{j0}^{in}, d_{js}^{ni})_{n\neq i}, y_{js}^i, y_{j0}^i) - d_{js}^{ik}) d\mu_\tau(s|.) = 0 \text{ for every pair } (i,k)$$

Under this hypothesis, there exist a temporary equilibrium with degenerate and common future price beliefs consistent in mean with market-clearing at each state s, that is,

$$\sum_i \left(\int_{\mathbf{D}_s} (\tilde{x}_s, \tilde{x}_{s^+}, \tilde{y}_s)^i(\tau, s, p_s, p_{s^+}, q_s, d(s)) d\mu_\tau(s|.) - (\omega_s^i, \omega_{s^+}, 0) \right) = 0$$

(3) The third possibility is an hybrid of the previous two and the same result as in case (2) holds. It says that at each state of date 1 the disagreements on *expected* deliveries values cancel out across agents and assets, that is,

$$\sum_{(i,k)} \sum_{j \in J} \int_{\mathbf{D}_s} r_{j0}^{ik} \pi_{j0}^{ik} (R_s^{ik}((\theta_{j0}^{in}, d_{js}^{ni})_{n \neq i}, y_{js}^i, y_{j0}^i) - d_{js}^{ik}) d\mu_\tau(s|.) = 0$$

5 A Note on Repo Specialness

The analysis of existence of equilibrium and asset valuation in equilibrium is particularly important in a repo context since, as Duffie (1996) remarks, it is possible to bound *repo* rates from above by arbitrage, but there is no obvious arbitrage argument to find a lower bound and these rates may even become negative. In fact, one should think of such repo rate as a market clearing price, influenced by funding need, the rent associated with holding a specific security and the potential of default of the party involved. The highest of such repo rate for a given term and securities issuer is called the general collateral (GC) rate. This is the repo rate associated to securities that are not scarce, and where the principal benefit for the borrower comes from the credit protection offered by the collateral. In the case of treasuries and next-day repurchases, the upper bound is at or near to the overnight interest rate in the market for federal funds (assuming readily access to Fed fund funding). When the repo rate of a specific security is below this GC rate, there is an incentive for the owner of the specific security to lend it in the repo market (and borrow funds at a favorable rate to reinvest the cash at a higher rate, for example by borrowing another security and investing at GC rate). Such opportunities, however, are not scalable and are limited by the very scarcity of the security available at the date *repo* agreements are made. The higher cost of borrowing the security reflects its relative scarcity, in which case the security trades *on special* in the repo market.

In this section we obtain by arbitrage a pricing formula for the repo rate of each repo contract. From this expression we can find both the repo rate upper bound (CG) and lower bound (LB).

In our model the haircut h_j^{ik} is exogenously fixed (in market practice the haircut is around 1% or 2% and charged on leveraged institutions). Thus, all changes in the demand and supply for the security are driven through the repo rate. We consider the model with repo rolls proposed in Section 3. The constraints and the associated Lagrange multipliers are (BC.0)-β_0^i and (Box.0)-μ_0^i for date 0,[44] (BC.s)-β_s^i and (Box.s)-μ_s^i for date 1, (BC.s')-$\beta_{s'}^i$, (Box.s')-$\mu_{s'}^i$ for date 2, and (BC.s'')-$\beta_{s''}^i$ for date 2. Then, one can find that in the absence of arbitrage opportunities the repo rate $r_{j0}^{ik} - 1$ for a bilateral contract between agents i and k on security j must satisfy the following equation[45]

$$r_{j0}^{ik} = \frac{\beta_0^i}{\beta_1^i} - \frac{\mu_0^i - \mu_s^i}{\beta_1^i \pi_{j0}^{ik}} \tag{RS}$$

where $\beta_1^i = \sum_s \beta_s^i$.

The shadow prices of the box constraints at the dates when the repo contract is initiated and settled have opposite effects. The harder to get a security at date 0 through reverse repo (in order to meet a short sale) the higher will be the shadow price (μ_0^i) of the box constraint at date 0 and, therefore, the lower the repo rate that the lender of cash is willing to accept. On the other hand, the more severe is the box constraint at date 1 (the higher μ_s^i), the more (less) one wants to lend (borrow) of a security at date 0, in order to have deliveries at date 1 that accommodate the security trades at that date. In other words, the higher is shadow price μ_s^i of (Box.s), the higher will be the repo rate that the lender of cash is willing to accept.

Expression (RS) gives us the lower bound for the repo rate of security j in the economy, which is, $r_j^{LB} - 1 = \min_{(i,k)}(r_{j0}^{ik} - 1)$. To find the upper bound, first notice that, if there were a risk free bond with prices q_0 and q_1 at dates 0 and 1, respectively, we would have $(\beta_0^i \backslash \beta_1^i) = (q_1 \backslash q_0) \equiv 1 + i$, where i is the risk-free interest rate. The ratio $(\beta_0^i \backslash \beta_1^i)$ would be the same for all agents in the economy. Even when constraints (Box.s) are binding, by an arbitrage argument, pointed out by Duffie (1996), one can infer that the repo rate must be bounded from above by this risk-free interest rate i. Hence, $\mu_0^i \geq \mu_s^i$ and i can be thought as the General Collateral rate.

Expression (RS) is interesting since it relates the repo specialness with the illiquidity of a security. Expression (RS) says that the larger the shadow price of the box constraint at date 0, μ_0^i, the more the repo rate will be on special. This shadow price portraits how strong the demand for borrowing a

[44] We suppose that the leveraged upper bound on repo positions given by (Lev.0) is not hit by any individual agent.

[45] We used the non-arbitrage deflators given by the ratios of the equilibrium Lagrange multipliers of the budget constraints at different dates. We could have used more general non-arbitrage deflators.

given security through reverse repo is.

Moreover, as we show in the next expression, prices of securities are also affected by the multipliers μ_0^i and μ_s^i corresponding to date 0 and date 1 (state s) box constraints.

$$q_{j0} = \frac{1}{\beta_0^i}(\mu_0^i + \sum_{s\in S}\left(\beta_s^i a_{js} + \mu_s^i\right) + \sum_{s'\in S'}(\beta_{s'}^i a_{js'} + \mu_{s'}^i)) \tag{SP}$$

The tighter is the box constraint (the higher is μ_0^i), the higher will be the price of the security that serves as collateral in repo. This result adds to Jarrow (1980) and Jones and Lamont (2002), who explain security price differentials due to short sale restrictions.

6 Appendix

6.1 Convexity

Under our specification (constraints L and S), the intersection of the constraints is convex, whereas under Duffie's (1996) it is not. To see this, let us write constraints (L.0) and (S.0), with the former *a la Duffie,* as follows:

$$y_{j0}^i + e_{j0}^i \geq 0 \quad \vee \quad y_{j0}^i + e_{j0}^i = -\sum_{k\neq i} z_{j0}^{ik}$$

$$-\sum_{k\neq i} z_{j0}^{ik} \leq 0 \quad \vee \quad e_{j0}^i + y_{j0}^i \geq -\sum_{k\neq i} z_{j0}^{ik},\ \forall j \in J,$$

Picture 2 portraits the intersection between the constraint (L) *a la Duffie* with constraints (S) and (Lev), which is non-covex. In Picture 1 we portrait the intersection of our constraint (L.0) with constraints (S.0) and (Lev.0), which is convex.

6.2 Proof of Proposition 3

First, note that the returns matrix depends on prices. In particular, it depends on $\pi_{j\xi}^{ik} = h_{j\xi}^{ik} q_{j\xi}$. As Hart's (1975) counterexample shows, when the returns matrix depends on the asset prices, upper semicontinuity of the budget correspondences fails and in order to guarantee existence of equilibrium we must require the asset short sales to be bounded from below. This is satisfied by the expression (BSS.0) of Proposition 1 and expression (BSS.s) of Proposition 2.[46] Positions in repos are bounded by constraints (Box.0),

[46] Observe that we can use the BSS expression for an equilibrium result since in our economy we have a finite number of nodes.

(Lev.0) at date 0, and (Box.s) and (Lev.s) at date 1 and state s. Attainable consumption bundles are bounded by $x^i_\xi \in \left[0, \sum_i \tilde{\omega}^i_\xi\right]$ at each node ξ.

Let us construct a generalized game where allocated consumption and security vectors have an upper bound that exceed the attainability upper bound by an arbitrary small amount and repo positions are bounded by (Lev.ξ). Denote $\mathbf{X}\times\mathbf{Y}\times\mathbf{Z}$ the bounded set of allocated vectors in commodities, positions in the securities markets, positions in the repo markets and fails.

By adding consumers budget constraints, we have that the values of the aggregate excess demand at dates 0, 1 and 2 are

$$p_0 \sum_{i\in I}(x^i_0 - \omega^i_0) + q_0 \sum_{i\in I} y^i_0 + \sum_{(i,k)\in I\times I} \pi^{ik}_0 \left(z^{ik}_0 + z^{ki}_0\right) \leq 0 \qquad \text{(B.0)}$$

$$\begin{aligned} p_s \sum_{i\in I}(x^i_s - \omega^i_s) + q_s \sum_{i\in I} y^i_s - a_s \sum_{i\in I}\left(y^i_0 + e^i_0\right) + \\ + \sum_{(i,k)\in I\times I}\left(\pi^{ik}_s\left(z^{ik}_s + z^{ki}_s\right) - r^{ik}_0 \pi^{ik}_0\left(z^{ik}_0 + z^{ki}_0\right)\right) \leq 0 \quad \text{(B.1.s)}\end{aligned}$$

and

$$p_{s'} \sum_{i\in I}(x^i_{s'} - \omega^i_{s'}) - a_{s'} \sum_{i\in I}\left(y^i_s + y^i_0 + e^i_0\right) - \sum_{(i,k)\in I\times I} r^{ik}_s \pi^{ik}_s \left(z^{ik}_s + z^{ki}_s\right) \leq 0 \qquad \text{(B.2.}s'\text{)}$$

Observe that the haircut is assumed fixed. The repo rate is actually decided in the initial date for the repos when the repos are negotiated. In order to model these facts, we rewrite constraints (B.0), (B.1.s) and (B.2.s') by making the following change of variables. For the repos initiated at $t = 0$ let us define $\tilde{z}^{ik}_0 + \tilde{z}^{ki}_0 = r^{ik}_s\left(\bar{z}^{ik}_0 + \bar{z}^{ki}_0\right)$, and for the repos initiated at $t = 1$, let $\tilde{z}^{ik}_s + \tilde{z}^{ki}_s = r^{ik}_s\left(\bar{z}^{ik}_s + \bar{z}^{ki}_s\right)$. Then, after the change in variables, the aggregated budget constraints are such:

At the initial date ($t = 0$) an auctioneer chooses (p_0, q_0) and $\left(\frac{1}{r^{ik}_0}\right)_{(i,k)\in I\times I}$, $\forall\,(i,k) \in I \times I$, $i \neq k$, in order to maximize

$$p_0 \sum_{i\in I}(\bar{x}^i_0 - \omega^i_0) + q_0 \sum_{i\in I} \bar{y}^i_0 + \sum_{(i,k)\in I\times I} \frac{1}{r^{ik}_0}\pi^{ik}_0 \left(\tilde{z}^{ik}_0 + \tilde{z}^{ki}_0\right) \leq 0 \qquad (\tilde{\text{B}}.0)$$

At date 1 there is an auctioneer for each state s who chooses p_s, q_s and $\left(\frac{1}{r_s^{ik}}\right)_{(i,k)\in I\times I}$, $\forall (i,k) \in I \times I$, $i \neq k$, in order to maximize

$$p_s \sum_{i\in I}(\bar{x}_s^i - \omega_s^i) + q_s \sum_{i\in I} \bar{y}_s^i - p_s B_s \sum_{i\in I} \left(\bar{y}_0^i + e_0^i\right) + \sum_{(i,k)\in I\times I} \left(\frac{1}{r_s^{ik}} \pi_s^{ik} \left(\tilde{z}_s^{ik} + \tilde{z}_s^{ki}\right) - \pi_0^{ik} \left(\tilde{z}_0^{ik} + \tilde{z}_0^{ki}\right)\right) \leq 0 \quad (\tilde{\text{B}}.1.\text{s})$$

At date 2 there is an auctioneer in each state s' who chooses $p_{s'}$ in order to maximize

$$p_{s'} \sum_{i\in I}(x_{s'}^i - \omega_{s'}^i) - p_{s'} B_{s'} \sum_{i\in I} \left(y_s^i + y_0^i + e_0^i\right) - \sum_{(i,k)\in I\times I} \pi_s^{ik} \left(\tilde{z}_s^{ik} + \tilde{z}_s^{ki}\right) \leq 0 \quad (\tilde{\text{B}}.2.s')$$

The price vector $\left(\frac{1}{r_\xi}, p, q\right)$ is positive for every component. Observe that $r_\xi = 1 + \hat{r}_\xi$ verifies that $1 + \hat{r}_\xi > 0$, where $\hat{r}_\xi$ is the repo interest rate. Thus $\frac{1}{r_\xi} > 0$. Then we can normalize the price vector so that each normalized price belongs to the simplex.

Each individual chooses $(\bar{x}^i, \bar{y}^i, \bar{z}^i)$ to solve his maximization problem on $\mathbf{X} \times \mathbf{Y} \times \mathbf{Z}$.

Now observe that lower semicontinuity of the budget constraints (BC.0) and (BC.s) follows from the assumption of positive initial endowments of goods so that the interior of both constraints is nonempty. The intersection of the constraint correspondences is nonempty and convex (by Remark 1), and compact (by Proposition 1). It is easy to see that this intersection of correspondences is continuous. Then we have that the generalized game satisfies all the assumptions of Debreu's (1952) theorem[47], and hence existence of an equilibrium[48] for the generalized game is guaranteed. Moreover, we can show that this equilibrium is an equilibrium in the sense of Definition 1. For the first date, it is enough to check that at $(\bar{x}, \bar{y}, \bar{z}, \bar{p}, \bar{q}, \bar{r})$ we have the following claims.

[47] **Debreu's (1952) theorem:** If action sets are compact and convex, the payoff functions are quasiconcave on its own action and continuous on the action space, and the constraint correspondences have non-empty compact convex values and are continuous, then the generalized game has an equilibrium for the generalized game.

[48] An equilibrium for this generalized game is a vector $(\bar{x}, \bar{y}, \bar{z}, \bar{p}, \bar{q}, \bar{r})$, such that, for each player, the respective action solves his optimization problem parametrized by the other players' actions.

Claim 1: $\sum_{i\in I}(\bar{x}_0^i - \omega_0^i) = 0$
Proof. Suppose $\sum_{i\in I}(\bar{x}_{l0}^i - \omega_{l0}^i) > 0$. Then the auctioneer chooses $p_{l0} = 1$, but then the whole function ($\tilde{\text{B}}$.0) becomes positive, a contradiction with the aggregation of budget constraints. Then it must be $\sum_{i\in I}(\bar{x}_{l0}^i - \omega_{l0}^i) \leq 0$. We have that it actually holds as equality. Otherwise the respective price would have to be zero, $p_{l0} = 0$, which would make $\bar{x}_{l0}^i$ hit the bound of $\mathbf{X}_l$, for every individual, and so $\sum_{i\in I}(\bar{x}_{l0}^i - \omega_{l0}^i) > 0$, a contradiction. Therefore, $\sum_{i\in I}(\bar{x}_{l0}^i - \omega_{l0}^i) = 0, \forall l \in L$, as desired. ■

Claim 2: $\sum_{i\in I}\bar{y}_{j0}^i = 0$, $\forall j \in J$
Proof. Suppose $\sum_{i\in I}\bar{y}_{j0}^i > 0$. Then the auctioneer chooses $q_{j0} = 1$, but then the whole function ($\tilde{\text{B}}$.0) becomes positive, a contradiction with the aggregation of budget constraints. Then it must be $\sum_{i\in I}\bar{y}_{j0}^i \leq 0$. We have that it actually holds as equality. Otherwise the respective price would have to be zero, $q_{j0} = 0$, making $\bar{y}_{j0}^i$ hit the bound of $\mathbf{Y}_j$ for every individual, this is, $\bar{y}_{j0}^i = \mathbf{Y}_j, \forall i \in I$, but then $\sum_{i\in I}\bar{y}_{j0}^i > 0$, a contradiction. ■

Claim 3: $\tilde{z}_0^{ik} + \tilde{z}_0^{ki} = 0$
Proof. Suppose $\tilde{z}_{j0}^{ik} + \tilde{z}_{j0}^{ki} > 0$, so that agents i and k have a reverse repo net position. Then the auctioneer chooses $\frac{1}{r_{j0}^{ik}} = 1$, but then the whole function ($\tilde{\text{B}}$.0) becomes positive, a contradiction with the aggregation of budget constraints. Then it must be $\tilde{z}_{j0}^{ik} + \tilde{z}_{j0}^{ki} \leq 0$. We have that it actually holds as equality. Otherwise agents i and k have a repo net position $\tilde{z}_{j0}^{ik} + \tilde{z}_{j0}^{ki} < 0$ and the respective price is $\frac{1}{r_{j0}^{ik}} = 0$. But then the agents will lend through reverse repo at zero price[49] to increase their consumption in date $t = 1$. The reverse repo net position will hit the bound of $\mathbf{Z}_j$, so $\tilde{z}_{j0}^{ik} + \tilde{z}_{j0}^{ki} > 0$, a contradiction. Therefore, $\tilde{z}_{j0}^{ik} + \tilde{z}_{j0}^{ki} = 0$ as desired. ■

In the second date ($t = 1$), if node is s, we have that the function ($\tilde{\text{B}}$.1.s) becomes

$$p_s \sum_{i\in I}(\bar{x}_s^i - \omega_s^i - B_s \sum_{i\in I} e_0^i) + q_s \sum_{i\in I} \bar{y}_s^i + \sum_{(i,k)\in I\times I} \frac{1}{r_s^{ik}} \pi_s^{ik} \left(\tilde{z}_s^{ik} + \tilde{z}_s^{ki}\right) \leq 0 \quad (\tilde{\text{B}}.1.\text{s})$$

since $\sum_{i\in I}\bar{y}_0^i = 0$ from Claim 2 and $\sum_{(i,k)\in I\times I}\left(\tilde{z}_{j0}^{ik} + \tilde{z}_{j0}^{ki}\right) = 0$ from Claim 3. Then, the market clearing equation for the commodities in state s reduces

[49] This means that the loan that the long in repo awards is zero, for any amount of security borrowed as collateral.

to

$$\sum_{i\in I}(\bar{x}^i_s-\omega^i_s)-\sum_{j\in\tilde{J}}B_{sj}\sum_{i\in I}e^i_{j0}$$

Claim 4: $\sum_{i\in I}(\bar{x}_s-\omega^i_s)-\sum_{j\in\tilde{J}}B_{sj}\sum_{i\in I}e^i_{j0}=0,$ with $R_{sj}\in\mathbb{R}^L_+$
Proof. With a similar proof than Claim 1. ■

Claim 5: $\sum_{i\in I}\bar{y}^i_s=0$
Proof. Similar proof to Claim 2 ■

Claim 6: $\tilde{z}^{ik}_s+\tilde{z}^{ki}_s=0$, $\forall s\in S$
Proof. Suppose $\tilde{z}^{ik}_s+\tilde{z}^{ki}_s>0$, so that agents i and k have a reverse repo net position. Then the auctioneer chooses $\frac{1}{r^{ik}_{js}}=1$, but then the whole function ($\tilde{\text{B}}$.1.s) becomes positive, a contradiction with the aggregation of budget constraints. Then it must be $\tilde{z}^{ik}_s+\tilde{z}^{ki}_s\leq 0$. We have that it actually holds as equality. Otherwise agents i and k have a repo net position $\tilde{z}^{ik}_s+\tilde{z}^{ki}_s<0$ and the respective price is $\frac{\pi^{ik}_{js}}{r^{ik}_{js}}=0$ since $\frac{1}{r^{ik}_{js}}=0$. But then the agents lend through reverse repo at zero price to increase their consumption in date 2. The reverse repo net position will hit the bound of $\mathbf{Z}_j$, so $\tilde{z}^{ik}_s+\tilde{z}^{ki}_s>0$, a contradiction. Therefore, $\tilde{z}^{ik}_s+\tilde{z}^{ki}_s=0$ as desired. ■

Inequality (B.2.s') reduces to

$$p_{s'}\sum_{i\in I}(x^i_{s'}-\omega^i_{s'})\leq 0 \qquad (\tilde{\text{B}}.3)$$

Claim 7: $\sum_{i\in I}(x^i_{s'}-\omega^i_{s'})=0$
Proof. Similar proof to Claim 1. ■

Now $(\bar{x}^i,\bar{y}^i,\bar{z}^i)$ is an optimal choice for consumer i at prices $(\bar{p},\bar{q},\bar{r})$. Suppose it was not, say $(\hat{x}^i,\hat{y}^i,\hat{z}^i)$ is budget feasible at $(\bar{p},\bar{q},\bar{r})$ and $u^i(\hat{x}^i)>u^i(\bar{x}^i)$. By strict quasiconcavity, $u^i(\alpha\hat{x}^i+(1-\alpha)\bar{x}^i)>u^i(\bar{x}^i)$ for $\alpha\in(0,1]$. Actually, when α is small enough, the convex combination lies in $\mathbf{X}\times\mathbf{Y}\times\mathbf{Z}$ and is budget feasible at $(\bar{p},\bar{q},\bar{r})$, a contradiction. *Q.E.D.*

6.3 Proof of Proposition 4

The constructed generalized game for this proof follows along the lines of the proof of Proposition 3. Therefore, we just indicate the main differences here. The portfolio constraints are now (Box.0), ($\widehat{Lev}$.0), (Box1.s) and (Box2.s). Observe that short sales at date 0 are bounded (by an argument analogous to the one in Proposition 1). Similarly, date 1 security positions are also bounded by the leveraged total endowment of the security (see Remark 2). The budget constraints are $(\widehat{BC}.0)$, $(\widehat{BC}.\text{s})$ and $(\widehat{BC}.s^{+})$. Adding the consumers' budget constraints of the first date we obtain

$$p_0 \sum_{i \in I}(x_0^i - \omega_0^i) + q_0 \sum_{i \in I} y_0^i + \sum_{(i,k) \in I \times I} \pi_0^{ik} \left(\theta_0^{ik} - \varphi_0^{ik}\right) \leq 0 \qquad \text{(C.0)}$$

Adding the consumers' budget constraints of the second date we get

$$p_s \sum_i (x_s^i - \omega_s^i - B_s \sum_i (y_0^i + e_0^i)) + q_s \sum_i y_s^i - \sum_{(i,k)} \sum_{j \in J} r_{j0}^{ik} \pi_{j0}^{ik} (R_{js}^{ik} - d_{js}^{ik}) = 0$$
(C.1.s)

In the problem of the date 0 auctioneer (but not in the date 1 auctioneers') we do the following change of variables: $(\tilde{\theta}_0^{ik}, \tilde{\varphi}_0^{ki}) = r_{j0}^{ik}(\theta_0^{ik}, \varphi_0^{ki})$ and modify (C.0) accordingly to have him choosing $(p_0, q_0, (1/r_{j0}^{ik})_{j,i,k})$ in the simplex. The generalized game has now some additional artificial players at date 1 and state s:

(a) One chooses $\alpha_{js}^i \in [0,1], \forall j, s, i$, in order to minimize $\sum_{i,s,j}(H_{js}^i - \sum_n \min\{\theta_{j0}^{in}, \alpha_{js}^i H_{js}^i\})^2$ subject to the constraint $\alpha_{js}^i \geq \theta_{j0}^{in} / \sum_{n \neq i} \theta_{j0}^{in}$, for every $n \neq i$ and every j, s,where $H_{js}^i \equiv \min\left\{y_{js}^i + y_{j0}^i + e_{j0}^i + \sum_{n \neq i} d_{js}^{ni}, \sum_{n \neq i} \theta_{j0}^{in}\right\}$ and is, therefore, a function of other players' strategies.

(b) for every s, one player chooses $\delta_{js}^{ki} \in [0,1]$ in order to minimize $\sum_{(i,k,j)}(d_{js}^{ki} - \delta_{js}^{ki}\varphi_{j0}^{ik})^2$, given other players' choices for d_{js}^{ki} and φ_{j0}^{ik}.

(c) For every s, one additional player chooses $(d_{js}^{ik})_{j,i,k}$ to minimize

$$(\sum_{(i,k)} \sum_{j \in J} r_{j0}^{ik} \pi_{j0}^{ik} (R_{js}^{ik}\left((\theta_{j0}^{in}, d_{js}^{ni})_{n \neq i}, y_{js}^i, y_{j0}^i, \alpha_{js}^i\right) - d_{js}^{ik}))^2$$

subject to $d_{js}^{ki} \in [0, \varphi_{j0}^{ik}], \forall j, i, k$, given $(\alpha_{js}^i)_{i,j,s}$, the consumers' choices $((\theta_{j0}^{ik})_{k \neq i}$, $(\varphi_{j0}^{ik})_{k \neq i}, y_{js}^i, y_{j0}^i)_i$ and also the date 0 auctioneer's choices. Consistency in the deliveries follows then by Brower's fixed point theorem, that is, this additional player chooses a vector d such that $R_{js}^{ik}\left((\theta_{j0}^{in}, d_{js}^{ni})_{n \neq i}, y_{js}^i, y_{j0}^i\right) = d_{js}^{ik}, \forall j, i, k$.

Date 1 and state s auctioneer maximizes the left hand side of (C1.s), by choosing (p_s, q_s) in the simplex, for given choices of the other players. Notice that the last term in the left hand side of (C.1.s) vanishes, when evaluated at optimal choices of the player that chose $(d_{js}^{ik})_{j,i,k}$. Market clearing in commodities, securities and repo markets follow as in Claims 1, 2, 5 and 3, respectively.

6.4 Proof of Proposition 5

Consider a generalized game where consumer i solves problem (G.i) and date 0 auctioneer solves the same problem as in the proof of Proposition 3. Consumer choice variables are constrained to belong to a compact set, as in the proof of Proposition 4. Recall that the expected indirect utility function is continuous on signals (See Grandmont (1982, Proposition 1, which actually requires just the continuity with respect to the weak* topology in the range space of the function in assumption (A4)). We need to show that consumers' current choice correspondences (i.e, the maximizers of E^i subject to (I), (II) and (III)) are non-empty, convex and compact valued, and upper hemi-continuous. It suffices to show that the constraint correspondence is convex, compact valued and continuous.

Let

$$f((y_0^i, \theta^i, \varphi^i, d_s^i), p_s, q_s) = \max_{\substack{x_s^i \geq 0, \\ y_s^i \in M(y_0^i, \theta^i, \varphi^i, d_s^i)}} \sum_{j \in J, k \neq i} r_{j0}^{ik} \pi_{j0}^{ik} R_{js}^{ik} \left((\theta_{j0}^{in}, d_{js}^{ni})_{n \neq i}, y_{js}^i, y_{j0}^i \right) - q_s y_s^i - p_s x_s^i$$

where $M(y_0^i, \theta^i, \varphi^i, d_s^i) = \{y_s^i : \text{(Box1.s) and (Box2.s) hold, given } (y_0^i, \theta^i, \varphi^i, d_s^i)\}$. Condition (II) can be rewritten as follows:

$$\inf_{\text{sup p}\mu^i(\tau)} [\, p_s \omega_s^i + p_s B_s(y_0^i + e^i) - \sum_{j \in J, k \neq i} r_{j0}^{ki} \pi_{j0}^{ki} d_{js}^{ki} + f((y_0^i, \theta^i, \varphi^i, d_s^i), p_s, q_s) \geq 0 \tag{II'}$$

Notice that $f(\cdot, p_s, q_s)$ is concave on $(y_0^i, \theta^i, \varphi^i, d_s^i)$, since the function being maximized is concave on $(y_0^i, \theta^i, \varphi^i, d_s^i, x_s^i, y_s^i)$. Moreover, f is continuous, since M is a continuous correspondence (as $(y_0^i, \theta^i, \varphi^i, d_s^i) = 0$ belongs to its interior) whose values are closed subsets of the compact set of state s choice variables.

The infimum, over all possible realizations, of the concave functions on (θ, y_0) inside square brackets, is still concave (see Rockafellar (1970), Theorem 5.5, p. 54) and, therefore, the constraint correspondence defined by (II') has convex values. This constraint correspondence is also compact valued and upper hemicontinuous, since the infimum in this condition can be replaced by the minimum and the support of μ^i is compact and depends continuously on the signal (by assumption (A4)[50]). This constraint is also lower hemicontinuous since $(y_0^i, \theta^i, \varphi^i, d_s^i) = 0$ satisfies (II') with strict inequality.

[50] Here is where we need the continuity, of the function mapping signals into measures, with respect to the norm topology on measures.

It is much easier to see that the constraint correspondence defined by condition (III) is also nonempty, convex and compact valued and continuous (as this condition is equivalent to $\inf_{\sup p\mu^i(\tau)} p_{s^+}\omega^i_{s^+} + p_{s^+}B_{s^+}(y^i_s + y^i_0 + e^i) \geq 0$). Finally, the intersection of the constraint correspondences defined by (I), (II') and (III) is lower hemicontinuous since each correspondence is convex valued and lower hemicontinuous and their interiors have a common point $(x^i_0, y^i_0, \theta^i, \varphi^i) = 0$, for each signal.

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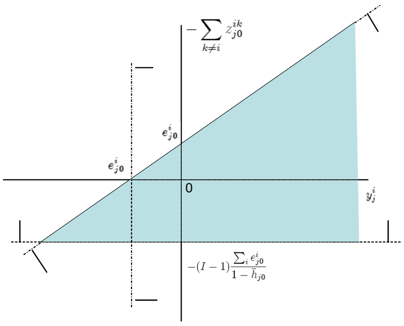
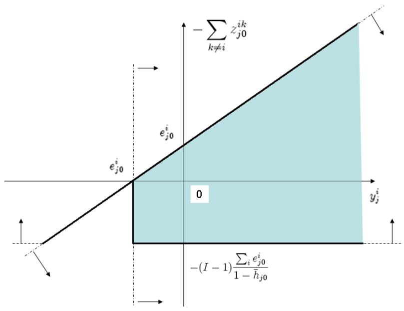
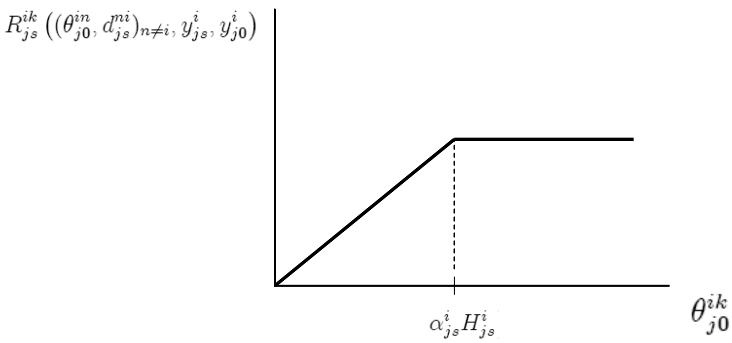
7 Pictures



Picture 1



Picture 2



Picture 3: The Constrained Equal Award Reimbusement rule.