Rational agents are the quickest*

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This version: January 2009

Abstract

We consider agents who make decisions by proceeding sequentially through a checklist of criteria: for any pair of alternatives the first criterion that ranks the pair determines the agent's decision. An agent with complete and transitive preferences can use a 'quick' checklist where the number of criteria is a small proportion of the number of the agent's indifference classes. An irrational agent on the other hand can never use fewer criteria than a rational agent uses and on some domains must use more. Moreover, as the size of the domain increases, the proportion of irrational agents that can use a quick checklist goes to 0 at a super-exponential rate. In contrast to the view that utility-maximization is computationally demanding, it is irrational agents who face the more difficult procedural task.

JEL codes: D01

Keywords: Checklists, rationality, computational efficiency, lexicographic orders.

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*I am grateful to Jose Apesteguia, Mark Machina, Paola Manzini, Marco Mariotti, and particularly Claire Blackman for helpful discussions and advice.

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1 Introduction

Consider an agent who makes decisions by proceeding through a fixed sequence of criteria. For any pair of alternatives, the agent checks whether the first criterion recommends one of the alternatives over the other; if not, the agent proceeds to the second criterion; and so on. If no criterion ranks the alternatives, the agent selects both (declares either alternative to be acceptable). For example, the field of alternatives might be a set of possible meals, the first criterion might categorize meals by their inclusion of meat (vegetarian, no red meat, some red meat, all other cases), the second by country of origin (French, Italian, Spanish, all other countries), the third by flavor (predominantly salty, sweet, or bitter, or none of the above). Criteria need not be transitive – e.g., they might cycle – and they need not be complete. We call the sequence of criteria a *checklist*.

A checklist C implicitly maximizes a preference relation $\succ$ in which case we say C is a *checklist for* $\succ$. Conversely, given any preference $\succ$, there will under mild restrictions be a nonempty set of checklists for $\succ$. Our main question is: for which preferences $\succ$ is there a *quick* checklist for $\succ$, that is, a checklist that employs the smallest theoretically possible number of criteria? To ensure the contest is fair, we compare preference relations that have the same number of indifference classes and endow criteria with the same capacity to discriminate among alternatives. We show that if $\succ$ is complete and transitive – $\succ$ is *rational* – then there is always a checklist for $\succ$ that uses the theoretical minimum number of criteria; in fact, the ratio of the number of criteria in the quickest checklist for a rational $\succ$ to the number of indifference classes in $\succ$ converges to 0 rapidly. In contrast, *any* nonrational preference relation will fail to have a checklist that is as quick as those used by rational agents on some subset of the domain of alternatives and for some bound on the discriminatory power of the admissible criteria. Even a failure of completeness, which might seem to require fewer decision-making discriminations, will lead to a slow checklist on some domains.

We also calculate the proportion of irrational preferences on a given domain that have quick checklists, and show that as the size of the domain increases this proportion shrinks to 0 at a super-exponential rate. And finally we show that rational preferences can use checklists that in expectation proceed through the smallest number of criteria.

This study pursues an agenda that Mandler, Manzini, and Mariotti (2008) (henceforth MMM) lays out but could not fully address. MMM considered checklists with binary criteria that partition alternatives into just two categories. The preferences that have checklists that use binary criteria are rational and conversely a rational preference always has a quick checklist that uses binary criteria. But because binary criteria do not allow an irrational preference relation to have any checklist at all, MMM could not stage a horse race between rational and irrational agents. Notice that each criterion in our opening 'meals' example divides meals into four types, not two, and that example can therefore admit irrational preferences.

Our results contrast with the views of Herbert Simon (1976, 1990). Consider the following sample of Simon's thinking, also discussed in MMM:

> The assumption of a utility function postulates a consistency of human choice that is not always evidenced in reality. The assumption of maximization may also place a heavy (often unbearable) computational burden on the decision maker (Simon (1990, p. 16)).

But if we measure computational burden by the number of checklist criteria, we come to a different conclusion: it is the irrational agents who bear the heaviest burden. Simon also seems to take utility to exist independently of an agent's choices rather than being a mere representation of choice behavior. If instead an agent's checklist is primitive then rational preferences (which have a utility representation) could simply be the outcome of criteria that are themselves complete and transitive. Agents might not use such well-behaved criteria but there is nothing computationally difficult about doing so. Indeed, the complete and transitive criteria have a capacity to make preference discriminations that outstrips any of the alternatives.

Tversky and Simonson (1993) adopt a view somewhere between Simon's and ours; they recognize the dexterity of rational decision-making and are thus forced to find it paradoxical that agents often depart from rationality when they attempt to reduce decision-making complexity.

Several other recent papers have considered sets or sequences of 'criteria' to represent choice behavior: Kalai, Rubinstein, and Spiegler (2002) (henceforth KRS), Apesteguia and Ballester (2005, 2008a, 2008b) (henceforth AB), and Manzini and Mariotti (2007). Besides MMM (2008), KRS and AB (2008b) are closest to the present paper in that they assess a set of criteria by the number of criteria in the set (what we call 'length'). But KRS and AB pursue a different agenda; they seek the most concise explanations of an agent's decisions, not the minimum speed with which an agent comes to a decision (on the latter see Salant (2003)). The number of equivalence classes used in criteria is therefore irrelevant, whereas for our aim of comparing the computational efficiency of different agents it is imperative to give every agent access to criteria of equivalent discriminatory power. Also, agents in our model do not use an unordered set of criteria; they proceed lexicographically through a fixed sequence of criteria, where it is the first criterion that ranks x and y that determines the checklist's overall ranking of x and y, as in Manzini and Mariotti (2007), AB (2008a), MMM (2008), and much earlier the lexicographic utility theory of Chipman (1960, 1971) (see also Fishburn (1974)). Chipman's framework in fact bears close resemblance to ours, but because Chipman was pursuing a representation for preferences that do not have a classical utility function, he allowed criteria to make a continuum of discriminations; our measure of computational efficiency therefore could not arise.

2 Criteria and Checklists

An agent will be represented by an asymmetric binary relation $\succ$, henceforth a *preference relation*, on a set of alternatives X, called the *domain of* $\succ$.

The agent makes decisions by proceeding through a list of criteria $C_1, ..., C_T$. Each criterion C_i is also an asymmetric relation on X but criteria will typically divide X into a smaller number of equivalence classes than preference relations do. For a variation on the example in the introduction, X could be a set of hundreds or thousands of cars, while one criterion might categorize cars by the numbers of doors (four doors, two doors, hatchbacks, minivans, all others), a second criterion might categorize by country of origin (Japanese, German, Korean, American, all others), and so on. Criteria need not be transitive and a criterion C_i need not rank every pair of the equivalence classes that C_i implicitly defines.

Definition 1 *The sequence $C = (C_1, ..., C_T)$ is a* checklist for $\succ$ *if and only if*

(1) each C_i is an asymmetric relation on X, and

(2) $x \succ y \Longleftrightarrow \exists i$ with $1 \leq i \leq T$ such that $x\, C_i\, y$ and not $y\, C_j\, x$ for all $j < i$.

The integer T is the length of C.

We will also say that a preference $\succ$ *has the checklist C* if C is a checklist for $\succ$. Definition 1 amounts to a mild generalization of Chipman's (1960) lexicographic definition of utility representation, and of kindred concepts in set theory, e.g., Cuesta Dutari (1943, 1947).

There is no need to take $\succ$ as primitive in Definition 1. We may instead directly identify an agent with a checklist C; an agent with C need not think about the preferences he or she is implicitly maximizing. Of course we may still deduce $\succ$ from C via condition (2) above.

Definition 1 can evidently be seen as a decision procedure for choice sets A that consist of two elements. Given $A = \{x, y\}$, the agent proceeds through the checklist until he comes to the first criterion C_i that ranks x and y and then eliminates the C_i-inferior item; if no criterion ranks x and y the agent selects both alternatives.

Checklists can also be applied to larger choice sets. A checklist $C = (C_1, ..., C_T)$ can be applied to an arbitrary choice set A by eliminating in each round i any element of A that has survived so far and that C_i ranks as inferior to some other survivor. Formally, we may define 'survivor sets' $S_i(A)$ recursively by

$$S_0(A) = A$$

$$S_i(A) = \{x \in S_{i-1}(A) : \nexists y \in S_{i-1}(A) \text{ with } y\, C_i\, x\} \text{ for } i = 1, ..., T,$$

(as in Manzini and Mariotti (2007), Apesteguia and Ballester (2008a), Mandler, Manzini, Mariotti (2008)). Suppose c is a choice function on a domain of nonempty choice sets, say the finite sets: $c(A)$ is a nonempty subset of any finite A. So an agent with the choice function c uses the checklist C if $S_T(A) = c(A)$ for any finite A. While this is the principal interpretation we have in mind for checklists, the extra generality of choice sets with more than two elements would be pointless for our purposes. As we will see, agents with rational preference relations can use the shortest possible checklists when they choose from two-element sets; and it is easy to see that they can use exactly the same checklists when they choose from finite sets. Hence rational agents will continue to have the shortest possible checklists on the larger domain. While our conclusion that only rational preferences are 'uniformly quick' will technically be a statement about choice from two-element sets, any agent who both uses a checklist and who chooses from two-element sets according to a rational preference must also choose rationally from any finite set. Our conclusion that only rational agents are uniformly quick therefore extends to agents who choose from arbitrary choice sets.

Whether we take the preference $\succ$ or checklist C in Definition 1 as primitive, a checklist with a small number of criteria relative to the number of indifference classes in $\succ$ means that the checklist C is making choice discriminations quickly. We now ask which $\succ$ have this efficiency feature.

3 Checklists Quick and Slow

We measure the speed or efficiency of a checklist by the number of criteria it uses, that is, by the checklist's length. The minimum length of a checklist for a $\succ$ is determined in part by the number of indifference classes in $\succ$ and the number of discriminations that checklist criteria make. It is important on both scores to have an accurate count of indifference or equivalence classes.

Definition 2 *Given an asymmetric relation $>$ on X, the binary relation $\approx$ on X is defined by*

$$x \approx y \Longleftrightarrow \{z \in X : z > x\} = \{z \in X : z > y\} \text{ and } \{z \in X : x > z\} = \{z \in X : y > z\}$$

and is called the equivalence (or indifference) relation *of $>$.*

It is easy to confirm that $\approx$ is in fact an equivalence relation and thus furnishes an appropriate definition of indifference for preferences. For a textbook discussion of $\approx$, specialized to the case where $\succ$ is transitive, see Fishburn (1970).

Definition 3 *Given an asymmetric relation $>$ and its equivalence relation $\approx$, a $>$-equivalence class is a nonempty $I \subset X$ such that (1) $x, y \in I \Longrightarrow x \approx y$ and (2) ($x \in I$ and $x \approx y$) $\Longrightarrow y \in I$.*

Given an asymmetric relation $>$ on X, the $>$-equivalence classes form a partition of X.

When we apply Definitions 2 and 3 to a preference relation $\succ$ we use $\sim$ to denote the equivalence relation of $\succ$ and call it an *indifference relation*. To keep our meaning clear, we reserve *equivalence* for equivalence relations of criteria. We assume throughout the remainder of the paper, and without further remark, that any preference $\succ$ has finitely many indifference classes.

A word of explanation on why we begin with a strict preference $\succ$ and define indifference classes from $\succ$ rather than simply beginning with a weak preference $\succsim$ and taking the symmetric part of $\succsim$ as our definition of indifference. The difficulty with the latter tactic is that a $\succsim$ might generate two or more indifference classes, unranked by $\succsim$, that have identical upper and lower contour sets; that is, we might have x and y such that neither $x \succsim y$ nor $y \succsim x$ holds but where

$$\{z \in X : z > x\} = \{z \in X : z > y\} \text{ and } \{z \in X : x > z\} = \{z \in X : y > z\}$$

where $>$ denotes the strict part of $\succsim$. Assuming that an agent with $\succsim$ chooses all $>$-undominated elements from any choice set, x and y are fully interchangeable from a behavioral point of view. Hence no criterion in any checklist for $\succsim$ would need to distinguish between x and y. So if we were to label x and y as being in separate indifference classes, we would give a checklist for $\succsim$ unwarranted credit for tackling a $\succsim$ with more indifference classes than show up in the agent's behavior. Conversely, $\succsim$ might classify two elements as indifferent when their upper and lower contour sets do differ and hence are behaviorally distinguishable.

Definition 4 *Given a positive integer p, $C = (C_1, ..., C_T)$* is a p-checklist for $\succ$ *if and only if C is a checklist for $\succ$ and each C_i has p or fewer C_i-equivalence classes.*

For convenience, we have assumed that each criterion C_i in a p-checklist has the same upper bound p on C_i-equivalence classes. See the discussion following Theorem 2 for the minor differences that arise when p varies by criterion.

We use a definition of completeness that applies to strict preference relations.

Definition 5 *A preference relation* $\succ$ *on the domain* X *is* complete *if and only if for every* $x, y \in X$ *either* $x\,(\succ \cup \sim)\,y$ *or* $y\,(\succ \cup \sim)\,x$ *or both.*

We identify 'rationality' with a $\succ$ that is complete and transitive. It is easy to confirm that an asymmetric relation $\succ$ is complete and transitive if and only if it is negatively transitive (i.e., where $x \not\succ y \not\succ z$ implies $x \not\succ z$), the traditional definition of rationality for asymmetric relations.

One way that a rational $\succ$ can arise is when each criterion C_i in a p-checklist $C = (C_1, ..., C_T)$ is itself rational (complete and transitive) on X. To see this, for any $x \in X$ and criterion i, let x_i be the number of C_i-equivalence classes J for which $x\,C_i\,J$,[1] and then identify x with the integer whose expansion in base p has x_i as its ith digit: call this integer $\text{Int}(x)$. (The smallest of these integers will be 0 and the largest must be $\leq p^{\text{length of } C} - 1$.) If C is a checklist for $\succ$, then $x \succ y$ if and only if the first criterion C_i that ranks x and y has $x\,C_i\,y$, i.e., the first digit i where $\text{Int}(x)$ and $\text{Int}(y)$ differ in base p (reading from the left) has $x_i > y_i$ and so $\text{Int}(x) > \text{Int}(y)$. It is also easy to confirm that $x \sim y$ if and only if $\text{Int}(x) = \text{Int}(y)$. Since the indifference classes of $\succ$ are order isomorphic to a subset of the integers (as ordered by $>$), $\succ$ must be complete and transitive:[2]

Observation 1 *If each criterion in a checklist* C *is complete and transitive and* C *is a checklist for* $\succ$*, then* $\succ$ *is complete and transitive.*

Notice that all criteria in a 2-checklist (the model studied in Mandler, Manzini, Mariotti (2008)) must be complete and transitive; so the preferences that have 2-checklists must be rational. To compare the checklist length of rational and irrational agents we must therefore go to higher values of p. The following observation shows that we need only go to $p \geq 3$.

[1]For any binary relation R on X, any subsets $A, B \subset X$, and any $c \in X$, we use $A\,R\,B$ to mean $a\,R\,b$ for all $a \in A$, $b \in B$ and use $c\,R\,A$ to mean $c\,R\,a$ for all $a \in A$.

[2]Two partially ordered sets, $(A, \geq_A)$ and $(B, \geq_B)$, are order-isomorphic if and only if there is a bijection $f : A \longrightarrow B$ such that, for all $x, y \in A$, $x \geq_A y \Longleftrightarrow f(x) \geq_B f(y)$.

Observation 2 *Any preference $\succ$ has a p-checklist if $p \geq 3$.*

The simplest way to build a p-checklist for $\succ$, if $p \geq 3$, is just to 'list' the preference rankings in $\succ$: for any two $\succ$-indifference classes I and J with $I \succ J$, define a criterion $C_{I,J}$ by $x\, C_{I,J}\, y$ if and only if $x \in I$ and $y \in J$, and then form a checklist C that consists for all such criteria, arranged in any order, and no others. Such a C is evidently a checklist for $\succ$ and it is a p-checklist for any $p \geq 3$ since each $C_{I,J}$ has the three equivalence classes, namely I, J, and $X \backslash (I \cup J)$. An alternative and usually shorter construction is to define for each $\succ$-indifference class I a criterion C_I by $y\, C_I\, x$ if and only if $x \in I$ and $y \succ x$, and again let C consist of all such criteria, arranged in any order. We again get a p-checklist for $\succ$ if $p \geq 3$, this time with a number of criteria equal to the number of $\succ$-indifference classes n. If $\succ$ is complete, our first construction in contrast uses $\frac{n(n-1)}{2}$ criteria. The worst case – where the quickest checklist has length n – can in fact occur (see Example 2).

We turn to the lower bound on the length of the quickest checklist. For any real number a, let $\lceil a \rceil$ denote the smallest integer b such that $b \geq a$.

Theorem 1 *A preference relation $\succ$ with n indifference classes has no p-checklist with length strictly less than $\lceil \log_p n \rceil$.*

Proof. If $(C_1, ..., C_T)$ is a p-checklist for $\succ$, we may define the p-checklist $(C'_1, ..., C'_T)$ for $\succ$ by

$$x' C'_i y' \Longleftrightarrow (\exists x, y \in X \text{ such that } x \sim x', y \sim y', \text{ and } x C_i y)$$

for $i = 1, ..., T$. Then, for any $a, b \in X$ with $a \sim b$ and any $i \in \{1, ..., T\}$, a and b are always in the same C'_i-equivalence class. Therefore, if we let $[x]$ denote the $\succ$-indifference class that contains x, the preference relation $\rhd$ on $\{I \subset X : I = [x] \text{ for some } x \in X\}$ defined by $[a] \rhd [b]$ if and only if $a \succ b$ must also have a p-checklist of length T. It is therefore without loss of generality to assume that each $\succ$-indifference class is a singleton.

Given a p-checklist C of length k, define a *C-undecided set* to be a $A \subset X$ such that $x, y \in A \Longrightarrow$ (for each $i = 1, ..., k$, $\exists$ a C_i-equivalence class I such that $\{x, y\} \in I$), define a *maximal C-undecided set* to be a C-undecided set A such there there does not exist a

C-undecided set B with $|B| > |A|$, and finally define the *maximal C-undecided cardinality* to equal $|A|$ where A is a maximal C-undecided set.

We show, by induction on k, that if the p-checklist C has length k then the maximal C-undecided cardinality is greater than or equal to $p^{-k}n$. This claim holds for $k = 1$ since if every C_i-equivalence class has strictly fewer than $p^{-k}n$ elements we would have $|X| < n$. Assuming then that the claim holds for arbitrary k, let $C = (C_1, ..., C_k, C_{k+1})$ be an arbitrary p-checklist of length $k+1$. So there is a $(C_1, ..., C_k)$-undecided set A with $|A| \geq p^{-k}n$. Since the p C_{k+1}-equivalence classes, $I_1, ..., I_p$, form a partition of X, the sets $I_1 \cap A, ..., I_p \cap A$ partition A; hence one of these sets, say $I_j \cap A$, must have at least $p^{-(k+1)}n$ elements. And since $I_j \cap A$ is a $(C_1, ..., C_k, C_{k+1})$-undecided set, the maximal $(C_1, ..., C_k, C_{k+1})$-undecided cardinality is at least $p^{-(k+1)}n$.

If C is a p-checklist for $\succ$ of length k, then the maximal C-undecided cardinality must equal 1. If it did not then there would be distinct $x, y \in X$ and a C_i-equivalence class I_i for each $i = 1, ..., k$ such that $x, y \in I_i$ for $i = 1, ..., k$; and then for all $z \in X$ there would be a $R \in \{\succ, \prec\}$ such that $xRz \iff yRz$, i.e., x and y are in the same $\succ$-indifference class (here and subsequently $\prec$ denotes $\{(a, b) \in X \times X : b \succ a\}$). Hence, the claim of the previous paragraph gives $p^{-k}n \leq 1$ or, equivalently, $\log_p n \leq k$. Since k is an integer, $k \geq \lceil \log_p n \rceil$. ■

Theorem 1 leads to the following definition of a quick checklist.

Definition 6 *Given a preference relation $\succ$ with n indifference classes, call a p-checklist for $\succ$ with length $\lceil \log_p n \rceil$* quick.

The label 'quick' is well-deserved when Definition 6 applies. If we fix p, then, as the number of $\succ$-indifference classes n increases, $\log_p n$ increases slowly – at a less-than-polynomial rate.

For any complete and transitive $\succ$ and any $p \geq 2$, $\succ$ has a quick p-checklist. One way to build a quick checklist is to reverse engineer the construction used for Observation 1: identify any x with the expansion in base p of the number of $\succ$-indifference classes that x is $\succ$-better than, say $(x_1 x_2 \cdots x_{\lceil \log_p n \rceil})_p$, and then, for $i = 1, ..., \lceil \log_p n \rceil$, set $x\, C_i\, y$ if and only if $x_i > y_i$. We lay out this argument in more detail in the proof of Theorem 2 (which also covers the case $p = 2$). The following example illustrates a slightly different way to construct a quick checklist.

Example 1 For $X = \{1, 2, 3, 4, 5, 6, 7, 8, 9\}$, let $\succ$ be the usual ordering of the integers. The unique quick 3-checklist is then the (C_1, C_2) given by

$$\{9,8,7\}\, C_1\, \{6,5,4\}\, C_1\, \{3,2,1\} \text{ and } \{9,8,7\}\, C_1\, \{3,2,1\}$$
$$\{9,6,3\}\, C_2\, \{8,5,2\}\, C_2\, \{7,4,1\} \text{ and } \{9,6,3\}\, C_2\, \{7,4,1\}.$$

Criterion C_1 divides X into a top third, a middle third, and a bottom third, and then ranks these thirds as $\succ$ does, while C_2 groups together the top items from the three C_1-equivalence classes, the middle items from the three C_1-equivalence classes, and the bottom three items of the three C_1-equivalence classes, and ranks these three groups so that the $\succ$-ordering of each C_1-equivalence class is preserved. ■

Example 1 provides a model for how to build a quick checklist for any complete and transitive $\succ$. One begins by partitioning X into p C_1-equivalence classes that are ordered by $\succ$. Then at each subsequent stage i for each $A \subset X$ that has not yet been divided into separate criterion equivalence classes partition A into p cells so that these cells are ordered by $\succ$ and group together all the cells across A's that have the same rank according to $\succ$ to form the C_i-equivalence classes. When the number of $\succ$-indifference classes n is an exact power of p, we must make sure that the cells at each stage contain the same number of $\succ$-indifference classes; in all other cases, there will be some latitude in the construction, but one may always proceed by requiring the cells of each A above to have sizes that differ by at most one $\succ$-indifference class. See the proof of Theorem 5 for more details.

So one can build a quick checklist for a complete and transitive $\succ$ from a sequence of complete and transitive criteria; conversely, Observation 1 reported that any sequence of complete and transitive criteria will generate a $\succ$ that is complete and transitive. If we think of the checklist rather than $\succ$ as primitive, it is not 'difficult' for an agent to have rational preferences, as Simon suggests in his work; rational preferences can merely indicate that an agent uses a checklist with complete and transitive criteria. Moreover, we will now see that a checklist user that implicitly maximizes a rational $\succ$ can use a checklist that makes choice discriminations more efficiently that any other checklist user who shares the same p.

Consider the following $\succ$ that has no quick checklist (and hence is not rational).

Example 2 For $X = \{a, b, c, d\}$, let $\succ$ be defined by $a \succ b \succ c \succ d$ (so $\succ$ is both incomplete and intransitive). Suppose $p = 3$. If there were a quick 3-checklist, it must have length $2 = \lceil \log_3 4 \rceil$. So then either C_1 makes two out of the three $\succ$-rankings – that is, C_1 has $a\, C_1\, b\, C_1\, c$ or $b\, C_1\, c\, C_1\, d$ or ($a\, C_1\, b$ and $c\, C_1\, d$) – or C_1 makes one $\succ$-ranking and C_2 makes the other two. If the criterion C_i that makes two rankings has $a\, C_i\, b\, C_i\, c$ then the omitted item d must be placed in the same C_i-equivalence class as one of $\{a, b, c\}$ and thus be ranked by C_i vis-à-vis b if joined with a or c and be ranked vis-à-vis a if joined with b, contrary to the rankings given by $\succ$. The C_i that makes two rankings similarly cannot have $b\, C_i\, c\, C_i\, d$. And finally if C_i has ($a\, C_i\, b$ and $c\, C_i\, d$) then (since $p = 3$) either $\{a, c\}\, C_i\, \{b, d\}$ or $a\, C_i\, \{b, c\}\, C_i\, \{d\}$, which again imposes rankings not given by $\succ$.

The shortest-length 3-checklist for $\succ$ has length 3. Here are two samples:

$$C_1 = \{(a, b)\},\ C_2 = \{(b, c)\},\ C_3 = \{(c, d)\}$$
$$C_1' = \{(a, b)\},\ C_2' = \{(b, c)\},\ C_3' = \{(c, d), (c, b)\}.$$

For C_3', the three C_3'-equivalence classes are $\{a\}$,$\{b, d\}$, and $\{c\}$.

The above $\succ$ is not quite the worst case. The $\succ$'s that come in last in the checklist-efficiency horse race are the $\succ$'s with n indifference classes that require a 3-checklist of length n. The smallest n where such a $\succ$ arises is 5: let $\succ$ on $\{a, b, c, d, e\}$ be defined by $a \succ b \succ c \succ d \succ e \succ a$. The argument that there is no 3-checklist shorter than length 5 is similar to the case above and we leave the details to the reader. ■

One moral of Example 2 is that while it may seem that incomplete preferences should have a speed advantage (your criteria need to make fewer rankings), incompleteness forces an agent to ensure that any $\succ$-unranked pair is unranked by every criterion. This restriction burns up criterion equivalence classes and hence slows down the quickest checklist.

In some carefully constructed cases, an irrational agent – an agent with a $\succ$ that fails to be complete and transitive – can have a quick checklist.

Example 3 Let $X = \{a, b, c, d, e, f, g, h, i\}$ and define $\succ$ by

$$\{a, b, c\} \succ \{d, e, f\} \succ \{g, h, i\} \text{ and } \{a, b, c\} \succ \{g, h, i\}$$
$$a \succ b \succ c \succ a \text{ and } d \succ e \succ f \succ d \text{ and } g \succ h \succ i \succ g.$$

Thus there are three triples that $\succ$ orders linearly but within each triple $\succ$ orders the items as a cycle. The following 3-checklist (C_1, C_2) for $\succ$ is quick:

$$\{a, b, c\}\, C_1\, \{d, e, f\}\, C_1\, \{g, h, i\} \text{ and } \{a, b, c\}\, C_1\, \{g, h, i\}$$
$$\{a, d, g\}\, C_2\, \{b, e, h\}\, C_2\, \{c, f, i\}\, C_2\, \{a, d, g\}.$$

Outside of the labeling the only difference between the present example and Example *1* is the C_2 is a cycle rather a line. In both cases the C_1-equivalence classes, as ordered by $\succ$ are order isomorphic. ■

We can mimic Example 3 to generate all of the $\succ$'s with nine indifference classes that have quick 3-checklists. For any nine-indifference-class $\succ$ with a quick 3-checklist (C_1, C_2), each C_i must have three C_i-equivalence classes and each C_i-equivalence class must consist of three $\succ$-indifference classes. Moreover, we must put the three $\succ$-indifference classes in each C_1-equivalence class into separate C_2-equivalence classes. A 3-checklist that does not meet these requirements would have to put some pair x, y such that not $x \sim y$ in *both* the same C_1-equivalence class and the same C_2-equivalence class, and then since x and y would be ranked the same way relative to every other $z \in X$ we would have $x \sim y$ (the proof of Theorem 1 makes this point in more detail). Now up to an order isomorphism there are only four asymmetric orderings of three equivalence classes. Labeling the equivalence classes a, b, and c, these are: (1) $\{(a, b), (b, c), (a, c)\}$, (2) $\{(a, b), (b, c), (c, a)\}$, (3) $\{(a, b), (b, c)\}$, and (4) $\{(a, b)\}$.[3] By letting C_1 order the C_1-equivalence classes according to one of the orderings (1) through (4) and letting C_2 order the items in each C_1-equivalence class according to one of the orderings (1) through (4) (the same ordering for every C_1-equivalence class), we can generate, up to an order isomorphism, all of the preferences with nine indifference classes (on a domain with nine items) that have quick 3-checklists. Evidently there are 16 such checklists: Example 3 is one (C_1 is the ordering (1) and C_2 is the ordering (2)) and Example 1 is another (both C_1 and C_2 are the ordering

[3] The orderings $\{(a, b), (c, b)\}$, $\{(a, b), (a, c)\}$, and $\varnothing$ do not qualify since they have 2, 2, and 1 indifference class respectively.

(1)). Now each of these 16 checklists is a checklist for one preference $\succ$ (up to an order isomorphism) and 15 of these $\succ$ are not rational. Since up to an order isomorphism there are approximately 400 billion distinct irrational $\succ$'s with nine indifference classes (when the domain contains nine items) it is apparently an exceedingly unusual event for an irrational $\succ$ with nine indifference classes to have a quick 3-checklist.

We further pursue the question of how many irrational $\succ$'s have quick checklists in section 4. Here we take a different line of attack and argue that for any irrational $\succ$ there is always a subdomain of X on which $\succ$ does not have a quick p-checklist for some p. Notice the delicacy of the checklist in Example 3: the items in each of the three C_1-equivalence classes must be ordered by $\succ$ in the same way (i.e., be order isomorphic). Consequently, if we were to swap one of the items in X in the example for a tenth item j then in order for the new domain to have a quick 3-checklist the item j would have to be $\succ$-ordered relatively to the eight remaining items in the same way as the item that j replaces. For one particular swap, we may be lucky – j may have the desired fit – but if we in turn swap j for each of the original nine we will necessarily come to a case where j will fail to share the $\succ$-orderings of the omitted item. This would not be true if $\succ$ were complete and transitive since then we could build new checklist criteria according to the model laid out in Example 1. But the above swapping argument does apply, a little surprisingly, if the source of the irrationality of $\succ$ is merely incompleteness. Although the details are long and messy, this is a rough guide to how the proof of our main theorem (Theorem 2) proceeds. There is one proviso however: the arguments we have given for Theorem 2 do not apply if the largest possible domain for $\succ$ contains just nine or fewer $\succ$-indifference classes. If this largest domain has, say, exactly nine indifference classes, then there may well be a quick 3-checklist C for $\succ$ and it is easy to see that on any subdomain the checklist formed by dropping from C the ranking of elements that are missing in the subdomain will remain quick. So the smallest domain where it might be the case that an irrational $\succ$ will fail to have a quick checklist on some subdomain must have at least 10 indifference classes. This is in fact the domain restriction we impose in Theorem 2; the irrational agents must fail to have a quick checklist on some subdomain at the first point where this failure is conceivable.

We lay out the definitions that formalize the above sketch.

Definition 7 *Let $\succ$ be a preference relation with domain X. The binary relation $\succ'$ is a* subrelation of $\succ$ *if and only if there is a subdomain $X' \subset X$ such that $\succ' = \succ \cap (X' \times X')$.*

We may think of a subdomain X' in two ways. First, it could be that every so often – on the first of every month – the agent discovers that only alternatives in X' will be available in the coming month. So during the month the agent need only use a checklist for the subrelation of $\succ$ that has domain X'. Our question is then 'which agents can use a quick checklist every month?' Second, the agent or the world might set the domain X' on which the checklist operates once and for all but we as outside analysts do not know what this X' is. We then ask 'which agents necessarily have the flexibility to use a quick checklist no matter how X' is set?'

Just as the domain might change periodically or we are unaware of the domain, the number of discriminations p that criteria can make might also change periodically or we do not know the p that the agent will use (or be allowed to use). We therefore formally define what it means for an agent to be capable of using a quick checklist under all circumstances as follows.

Definition 8 *The preference $\succ$ has* uniformly quick checklists *if and only if for all subrelations $\succ'$ of $\succ$ and all $p \geq 3$ there is a quick p-checklist for $\succ'$.*

Theorem 2 *A preference with $n \geq 10$ indifference classes has uniformly quick checklists if and only if it is complete and transitive.*

We note one easy extension to Theorem 2. So far we have required each criterion C_i in a checklist to have the same upper bound on C_i-equivalence classes. We could instead specify a sequence of upper bounds $\overrightarrow{p} = (p_1, p_2, ...)$ where each $p_i \geq 3$. Suppose $\succ$ has n indifference classes and let $q(n)$ denote the smallest integer such that $\prod_{i=1}^{q(n)} p_i \geq n$. We can then call a $\overrightarrow{p}$-checklist for $\succ$ that has $q(n)$ criteria *quick* and define a $\succ$ to have *uniformly quick checklists* if and only if for all subrelations $\succ'$ of $\succ$ and all $\overrightarrow{p}$, there is a quick $\overrightarrow{p}$-checklist for $\succ'$. The proof of the 'if' half of Theorem 2 extends easily to these new definitions while the harder 'only if' half of the Theorem extends vacuously given that our previous definitions are a special case of the new definitions.

We give the proof of the 'if' half of Theorem 2 here and relegate the tedious proof of the 'only if' half to an appendix.

Half of proof of Theorem 2. Suppose $\succ$ is complete and transitive. Fix a p and a subrelation $\succ'$ of $\succ$ with domain X' and with n indifference classes. For any $x \in X'$, let $[x]$ be the $\succ'$-indifference class that contains x. For any $\succ'$-indifference class $I \subset X'$, define

$$r(I) = |\{J \subset X : J \text{ is a } \succ' \text{-indifference class such that } I \succ' J\}|,$$

which can be viewed as the common utility number of the elements in I. Next, identify each integer r such that $0 \leq r \leq n-1$ with its expansion in base p. That is, for each r there is a unique ordered sequence of $T = \lceil \log_p n \rceil$ integers, $(k(r,1), ..., k(r,T))$, where $0 \leq k(r,j) \leq p$ for $j = 1, ..., T$, such that $r = \sum_{j=1}^{T} k(r,j) p^{T-j}$. For $j = 1, ..., T$, define C_j by $x \, C_j \, y$ if and only if $k(r([x]), j) > k(r([y]), j)$. For any $x, y \in X'$, suppose there is a smallest integer i with $1 \leq i \leq T$ such that $k(r([x]), i) \neq k(r([y]), i)$. If say $k(r([x]), i) > k(r([y]), i)$ then $r[x] > r[y]$ and so $x \succ' y$. Moreover, $x \, C_i \, y$ and not $y \, C_k \, x$ for all integers $k < i$, as desired. If on the other hand $k(r([x]), i) = k(r([y]), i)$ for all integers i with $1 \leq i \leq T$, then $r([x]) = r([y])$ and so $x \sim' y$. And then not $x \, C_i \, y$ and not $y \, C_i \, x$ for all i with $1 \leq i \leq T$. ■

4 The proportion of irrational preferences with quick checklists

We fix $p \geq 3$ and consider the proportion of preferences that have quick p-checklists as the size of the domain increases. We first consider the proportion of all preferences with quick checklists – it is a more straightforward calculation – and then turn to our real concern, irrational preferences.

For any n, let X_n denote a generic domain of n objects, $\{1, ..., n\}$, let $\succ(n)$ denote the set of all preferences on X_n,

$$\succ(n) = \{\succ \subset X_n \times X_n : \ \succ \text{ is asymmetric}\},$$

and let $|\succ(n)|$ denote the cardinality of $\succ(n)$. As for quick checklists, let $q(n)$ denote the set of preferences on X_n that have quick checklists,

$$q(n) = \{\succ \in \succ(n) : \succ \text{ has a quick } p\text{-checklist}\},$$

and let $|q(n)|$ denote the cardinality of $q(n)$.

Theorem 3 *The proportion of preferences on X_n that have quick p-checklists converges to 0 at a super-exponential rate as $n \to \infty$. That is, for any $k > 0$, $e^{kn}\frac{|q(n)|}{|\succ(n)|}$ converges to 0.*

Proof. Since there are $\frac{n(n-1)}{2}$ unordered pairs of elements in X_n and since for any $x, y \in X_n$ an asymmetric relation $\succ$ must specify one of three possibilities ($x \succ y$, $y \succ x$, or (not $x \succ y$ and not $y \succ x$)), $|\succ(n)| = 3^{\frac{n(n-1)}{2}}$.

The number of ways that X_n can be partitioned into p cells is given by the Stirling number of the second kind $S(n, p)$ (see Goldberg et al. (1972) and Knuth (1997) and for the formula for $S(n, p)$ we use below). The number of asymmetric binary relations on the domain $\{1, ..., p\}$ is $3^{\frac{p(p-1)}{2}}$, a total that includes all of the asymmetric binary relations with m equivalence classes for any $m \leq p$. Hence $S(n, k)3^{\frac{p(p-1)}{2}}$ provides an upper bound on the number of asymmetric binary relations C on X_n with p or fewer C-equivalence classes. Finally, since a preference on X_n has at most n indifference classes, a quick p-checklist has no more than $\lceil \log_p n \rceil$ criteria (Theorem 1). Hence an upper bound $U(n)$ on $|q(n)|$ is given by

$$U(n) = \left(S(n, k)3^{\frac{p(p-1)}{2}}\right)^{\lceil \log_p n \rceil}.$$

Since (Goldberg et al. (1972)) we may write $S(n, p)$ as

$$S(n, p) = \frac{1}{p!}\sum_{i=0}^{p}(-1)^{p-i}\binom{p}{i}i^n,$$

we have

$$S(n, p) \leq \frac{1}{p!}\sum_{i=0}^{p}\binom{p}{i}p^n.$$

Hence there is a $r > 0$ such that $S(n, k)3^{\frac{p(p-1)}{2}}$ is bounded above by r^n. Hence

$$U(n) \leq (r^n)^{1+\log_p n} = e^{(1+\log_p n)n \ln r}.$$

So, for any fixed $k > 0$, $e^{kn} |q(n)|$ is bounded above by $e^{(1+\log_p n)n \ln r + kn}$. Since $|\succ(n)| = 3^{\frac{n(n-1)}{2}} = e^{\frac{n(n-1)}{2} \ln 3}$ and since

$$\lim_{n \longrightarrow \infty} \left(\frac{n(n-1)}{2} \ln 3 - \left((1 + \log_p n) n \ln r + kn \right) \right) = \infty, \qquad (1)$$

$e^{kn} \frac{|q(n)|}{|\succ(n)|} \longrightarrow 0$ as $n \longrightarrow \infty$. ■

For all but the smallest values of n, the set of possible preferences on X_n is vast. Only a comparative handful of these are rational; when n is reasonably large, most preferences on X_n have n indifference classes, and yet there are only $n!$ preference relations on X_n with n indifference classes that are complete and transitive (and up to an order isomorphism there is in fact only one rational preference on X_n with n indifference classes). It is therefore no surprise that the conclusion of Theorem 3 continues to hold if we narrow our focus to irrational preferences.

Let $\succ_{\text{ir}}(n)$, $|\succ_{\text{ir}}(n)|$, $q_{\text{ir}}(n)$, and $|q_{\text{ir}}(n)|$ denote the restriction of $\succ(n)$, $|\succ(n)|$, $q(n)$, and $|q(n)|$ to preferences that are not complete and transitive (e.g., $\succ_{\text{ir}}(n) = \{\succ \in \succ(n) :$ $\succ$ is not complete and transitive$\}$ and $q_{\text{ir}}(n) = \{\succ \in \succ_{\text{ir}}(n) : \succ$ has a quick p-checklist$\}$).

Theorem 4 *The proportion of irrational preferences on X_n that have quick p-checklists converges to 0 at a super-exponential rate as $n \to \infty$. That is, for any $k > 0$, $e^{kn} \frac{|q_{\text{ir}}(n)|}{|\succ_{\text{ir}}(n)|}$ converges to 0.*

Proof. Only one adjustment is needed to the previous proof. For each asymmetric binary relation $\succ_{n-1}$ on X_{n-1} we may select a binary relation R on X_n that only ranks n relative to the elements of X_{n-1} (each ordered pair in R has n as one of its coordinates) such that $\succ_{n-1} \cup R$ is in $\succ_{\text{ir}}(n)$. Hence $3^{\frac{(n-1)(n-2)}{2}}$ can serve as a lower bound for $|\succ_{\text{ir}}(n)|$. Since $3^{\frac{(n-1)(n-2)}{2}} = e^{\frac{(n-1)(n-2)}{2} \ln 3}$ and since

$$\lim_{n \longrightarrow \infty} \left(\frac{(n-1)(n-2)}{2} \ln 3 - \left((1 + \log_p n) n \ln r + kn \right) \right) = \infty, \qquad (1)$$

we conclude that $e^{kn} \frac{|q_{\text{ir}}(n)|}{|\succ_{ir}(n)|} \longrightarrow 0$ as $n \longrightarrow \infty$. ■

The size of the domain n need not be at all large for the proportion of irrational preferences with quick checklists to be extremely small. To illustrate, the following table reports for $3 \leq n \leq 9$ the number of preference relations on X_n that fail to be complete and transitive, the number of those preferences that do not have quick 3-checklists, and the ratio of these two numbers, i.e., the proportion of irrational preferences on X_n with a quick 3-checklist. To keep the count under control, we give only the number of preferences – both for the total and for preferences with quick checklists – that are unique up to an order isomorphism.[4]

n	# irrational preferences	# with quick 3-checklists	ratio
3	3	3	1
4	34	31	.91176471
5	566	226	.39929329
6	$21,448$	$1,084$	.05054084
7	$2,142,224$	$3,847$	.00179580
8	$575,016,091$	$11,143$	.00001938
9	$415,939,242,776$	$217,902$	.00000052

5 Minimum expected run time

While rational agents can always use checklists with the shortest possible length, the possibility remains that compared to their irrational peers rational agents may have to use checklists that in expectation proceed through more criteria before coming to a decision. We show here that subject to one proviso this is not the case; rational agents not only can use the shortest checklists but can use checklists that on average execute the most quickly.

Suppose a preference $\succ$ on a domain X with n elements has the checklist $C = (C_1, ..., C_T)$. For any pair $\{x, y\} \subset X$, let $t_C(x, y) = t_C(y, x)$ be the index of the crite-

[4]Since there are only five relevant C_i orderings of C_i-indifference classes when $p = 3$, it is feasible to enumerate the irrational preferences on X_n that have quick 3-checklists and then weed out the preferences that are order isomorphic. I am indebted to Claire Blackman for writing code that performs this enumeration (it is available on request). The total number of preferences on X_n up to order isomorphism is given in Harary (1969); for the number of irrational preferences we subtract the number of rational preferences, which equals 2^{n-1}, from the Harary total.

rion that determines the agent's decision between x and y: $t_C(x, y)$ is the largest integer $i \in \{1, ..., T\}$ such that (not $x\,C_j\,y$ and not $y\,C_j\,x$ for $j \in \{1, ..., i-1\}$). Notice that if x and y are not ranked by $\succ$ then $t_C(x, y) = T$. We define the *expected run time* of C to be $Et_C = \sum_{\{x,y\} \subset X} t_C(x, y)(1/2^{n(n-1)/2})$. Since $2^{n(n-1)/2}$ is the number of unordered pairs in X, Et_C is the expectation of the number of criteria the agent must examine, assuming each pair is equally likely. We could apply any strictly increasing transformation to t without affecting the conclusion of the theorem below.

A preference $\succ$ on X has *minimum expected run time* if, for all $p \geq 2$, $\succ$ has a p-checklist C such that, for all preferences $\succ'$ on X and any p-checklist C' for $\succ'$, $Et_C \leq Et_{C'}$. Also, we say that $\succ$ *has strictly smaller expected run time than* $\succ'$ if, for all $p \geq 3$, $\succ$ has a p-checklist C such that, for all checklists C' for $\succ'$, $Et_C \leq Et_{C'}$ and if strictly inequality holds for some $p \geq 3$, that is, for some $p \geq 3$, $\succ$ has a p-checklist C such that, for all checklists C' for $\succ'$, $Et_C < Et_{C'}$.

A preference $\succ$ has *singleton indifference classes* if each $\succ$-indifference class contains a single element of X. Given our maintained assumption that any $\succ$ has finitely many indifference classes, the domain of a $\succ$ with single indifference classes must be finite.

Theorem 5 *Any rational preference $\succ$ with singleton indifference classes has minimum expected run time. If $\succ'$ is defined on the same domain as $\succ$ and lacks a quick p-checklist then $\succ$ has strictly smaller expected run time than $\succ'$.*

Proof. Let $\succ$ on the domain X be a rational preference with n singleton indifference classes and fix some $p \geq 2$. We recursively define a p-checklist $\widehat{C} = (\widehat{C}_1, ..., \widehat{C}_{\lceil \log_p n \rceil})$ for $\succ$ as follows.

Set $\widehat{C}_0 = \varnothing$. Given the binary relation $\widehat{C}_i$ on X, let $\widehat{\mathcal{I}}_i$ denote the partition of X whose cells are the $\widehat{C}_i$-equivalence classes. So in particular $\widehat{\mathcal{I}}_0 = \{X\}$. Given the partitions $\widehat{\mathcal{I}}_0, ..., \widehat{\mathcal{I}}_r$ of X, we let $\widehat{\mathcal{J}}_r = \bigvee_{j=0,...,r} \widehat{\mathcal{I}}_j$ denote the coarsest common refinement of $\widehat{\mathcal{I}}_0, ..., \widehat{\mathcal{I}}_r$. For the recursion, take $\widehat{C}_0, ..., \widehat{C}_{i-1}$ as given and define for each $J \in \widehat{\mathcal{J}}_{i-1}$ a partition $\{J(1), ..., J(q)\}$ of J, where $q = \min[p, |J|]$, such that (1) each cell contains either $\left\lfloor \frac{|J|}{q} \right\rfloor$ or $\left\lceil \frac{|J|}{q} \right\rceil$ elements (where $\lfloor a \rfloor$ is the largest integer $\leq a$) and (2) $J(r) \succ J(s)$ if $r < s$. If $|J| < p$, set $J(m) = \varnothing$ for $|J| < m \leq p$. For any $k = 1, ..., p$ such that $\bigcup_{J \in \widehat{\mathcal{J}}_{i-1}} J(k) \neq \varnothing$, define a $\widehat{C}_i$-equivalence class by $\widehat{I}_i(k) = \bigcup_{J \in \widehat{\mathcal{J}}_{i-1}} J(k)$ and then set $\widehat{C}_i$

by $I_i(r)\,\widehat{C}_i\,I_i(s)$ if and only if $r < s$ and both $\widehat{I}_i(r)$ and $\widehat{I}_i(s)$ are nonempty. It is easy to check that each $J \in \widehat{\mathcal{J}}_{\lceil \log_p n \rceil}$ is a singleton and that $\widehat{C} = (\widehat{C}_1, ..., \widehat{C}_{\lceil \log_p n \rceil})$ is a p-checklist for $\succ$.

Using the fact that $\left\lfloor \frac{\lfloor n/p \rfloor}{p} \right\rfloor = \left\lfloor \frac{n}{p^2} \right\rfloor$ and $\left\lceil \frac{\lceil n/p \rceil}{p} \right\rceil = \left\lceil \frac{n}{p^2} \right\rceil$, it follows that for any $i = 1, ..., \lceil \log_p n \rceil$ and any $J, K \in \widehat{\mathcal{J}}_i$, $||J| - |K|| \leq 1$. It will be useful for the final paragraph to note that if a partition $\mathcal{L}$ of X contains only cells with cardinalities that differ by at most 1 then $\mathcal{L}$ contains only sets with either $\left\lceil \frac{n}{|\mathcal{L}|} \right\rceil$ or $\left\lfloor \frac{n}{|\mathcal{L}|} \right\rfloor$ elements.

Let $\succ'$ be an arbitrary preference on X and let $C' = (C'_1, ..., C'_T)$ be a p-checklist for $\succ'$. It is sufficient to show that for any $\tau \leq T$ the first τ criteria in C' make no more discriminations than do the first $\min[\tau, \lceil \log_p n \rceil]$ criteria in $\widehat{C}$. Let $\tau_{\min}$ henceforth denote $\min[\tau, \lceil \log_p n \rceil]$. That is, if we define, for an arbitrary checklist C,

$$D_C(\tau) = \{\{x, y\} \subset X : \exists j \text{ with } 1 \leq j \leq \tau \text{ such that either } x\,C_j\,y \text{ or } y\,C_j\,x\},$$

it is sufficient to show that $\left|D_{\widehat{C}}(\tau_{\min})\right| \geq |D_{C'}(\tau)|$ for any $\tau \leq T$.

Before proceeding, define for any partition $\mathcal{L}$ of X the set

$$D_{\mathcal{L}} = \{\{x, y\} \subset X : \exists I, J \in \mathcal{L} \text{ such that } x \in I, y \in J, \text{ and } I \neq J\}.$$

Given criteria $(C_1, ..., C_t)$ the equivalence classes of each criterion C_j define a partition $\mathcal{I}_j$ of X, and therefore $\mathcal{J}_t = \bigvee_{j=1,...,t} \mathcal{I}_j$ and $D_{\mathcal{J}_t}$ are also well-defined. Since x and y are in different cells of $\mathcal{J}_t$ only if $x\,C_j\,y$ for some $j \in \{1, ..., t\}$, $|D_{\mathcal{J}_t}| \geq |D_C(t)|$. Since inequality can occur only if some C_j is incomplete, for any initial segment of the criteria in the $\widehat{C}$ we defined for $\succ$, say $\widehat{C}_1, ..., \widehat{C}_i$, we have $|D_{\widehat{\mathcal{J}}_i}| = \left|D_{\widehat{C}}(i)\right|$. Notice also that if each C_j in $(C_1, ..., C_t)$ has at most p equivalence classes then $\mathcal{J}_t$ can have at most $\min[p^t, n]$ cells but $\widehat{\mathcal{J}}_i$ has exactly $\min[p^i, n]$ cells.

To get our result, fix some $\tau \leq T$. For $i = 1, ..., \tau$, let $\mathcal{I}'_i$ denote the partition formed by the equivalence classes of C'_i. Beginning with $\mathcal{J}'_\tau = \bigvee_{j=1,...,\tau} \mathcal{I}'_j$, we define a new partition by taking any pair $I, J \in \mathcal{J}'_\tau$ of maximal and minimal cardinality and replacing them with two sets that partition $I \cup J$ into sets with cardinalities $\left\lceil \frac{|I \cup J|}{2} \right\rceil$ and $\left\lfloor \frac{|I \cup J|}{2} \right\rfloor$. By iterating this operation, we arrive at a partition $\mathcal{K}$ such that if $K, J \in \mathcal{K}$ then $||K| - |J|| \leq 1$ and where $|\mathcal{K}| = |\mathcal{J}'_\tau|$. It is easy to confirm that when this operation is applied to $\mathcal{L}_1$ to generate $\mathcal{L}_2$, we have $|D_{\mathcal{L}_2}| \geq |D_{\mathcal{L}_1}|$. So $|D_{\mathcal{K}}| \geq |D_{\mathcal{J}'_\tau}|$. As noted earlier, each $K \in \mathcal{K}$ has $\left\lceil \frac{n}{|\mathcal{K}|} \right\rceil$ or $\left\lfloor \frac{n}{|\mathcal{K}|} \right\rfloor$ elements and each $J \in \widehat{\mathcal{J}}_{\tau_{\min}}$ has $\left\lceil \frac{n}{|\widehat{\mathcal{J}}_{\tau_{\min}}|} \right\rceil$ or $\left\lfloor \frac{n}{|\widehat{\mathcal{J}}_{\tau_{\min}}|} \right\rfloor$ elements. Hence, since $|\mathcal{K}| \leq \min[p^\tau, n]$ and $|\widehat{\mathcal{J}}_{\tau_{\min}}| = \min[p^\tau, n]$, we have $|K| \geq |J|$ for any $K \in \mathcal{K}$ and $J \in \widehat{\mathcal{J}}_{\tau_{\min}}$. It follows that $\left|D_{\widehat{C}}(\tau)\right| = |D_{\widehat{\mathcal{J}}_{\tau_{\min}}}| \geq |D_{\mathcal{K}}|$. Since $|D_{\mathcal{J}'_\tau}| \geq |D_{C'}(\tau)|$, we have $\left|D_{\widehat{C}}(\tau_{\min})\right| \geq |D_{C'}(\tau)|$ as desired.

As for the second sentence of the theorem, suppose $\succ'$ on X lacks a quick p-checklist. Then $t_{C'}(x, y) = T$ for some $\{x, y\} \subset X$ but $t_C(x, y) < T$ for all $\{x, y\} \subset X$. Combined with the fact that $\left|D_{\widehat{C}}(\tau_{\min})\right| \geq |D_{C'}(\tau)|$ for any $\tau < T$, $\succ$ must have strictly smaller expected run time than $\succ'$. ■

By gluing Theorem 5 to Theorem 2 we conclude that for any irrational $\succ$ there will be a subdomain on which $\succ$ must have strictly smaller expected run time than a rational preference with singleton indifference classes restricted to the same subdomain.

A $\succ$ with minimum expected run time in fact terminates very quickly. One may easily show if C is a checklist for $\succ$ that always minimizes its expected run time then $Et_C < 2$.

Why the restriction to singleton indifference classes? If C is a checklist for $\succ$ and $\succ$ has multiple-element indifference classes then any pair of indifferent elements will be decided only by the final criterion in C. This fact can allow an irrational $\succ$ that does have singleton indifference classes to run more quickly on average.

6 Conclusion

Being rational may not be as difficult a task as Herbert Simon supposed. If we identify a preference relation with a decision procedure and not with an agent who must uncover from his mind a preexisting utility function, then there need be no intrinsic challenge to choosing rationally. In the case of checklists, rational preferences will emerge if the constituent building blocks of a checklist – the criteria – are themselves rational. Irrationality is still possible, of course, but there is no computational argument in its favor in the present setting. Indeed, it is the rational agents who hold the computational advantage. The agents who always use rational criteria in their checklists can make choice discriminations on some subdomains with strictly greater efficiency than the agents who use irrational criteria and they will never be less efficient on any subdomain.

7 Appendix: Remainder of Proof of Theorem 2

We prove the remaining 'only if' direction of Theorem 2. Suppose to the contrary that $\succ$ fails to be complete and transitive but has uniformly quick checklists. In Part I, we assume that $\succ$ has a subrelation $\succ'$ on a domain X' such that X' has 10 $\succ'$-indifference classes and such that $\succ'$ is complete but not transitive, while in Part II we consider the remaining cases. In both parts, we proceed by finding a subrelation $\succ''$ of $\succ$ with domain $X'' \subset X'$ that has 9 $\succ''$-indifference classes and that fails to have a quick 3-checklist, i.e., a checklist of lenght 2. The following Lemmas are useful in both parts.

Lemma 1. If $\succ$ has 9 indifference classes and (C_1, C_2) is a 3-checklist for $\succ$, then (1) for any pair of distinct $\succ$-indifference classes I and J there does not exist a C_1-equivalence class K_1, a C_2-equivalence class K_2, a $x \in I$, and a $y \in J$, such that $\{x, y\} \subset (K_1 \cap K_2)$, (2) each C_i-equivalence class, $i = 1, 2$, equals the union of three $\succ$-indifference classes, and (3) if $\succ$ is complete, then C_i is complete for $i = 1, 2$.[5]

Proof. The proof of Theorem 1 gives conclusion (1). Given (1), there can be no C_1-equivalence class that contains a four element subset Y such that any $x, y \in Y$ are in distinct $\succ$-indifference classes. Since the C_1-equivalence classes form a partition of X, there cannot be a C_1-equivalence class that intersects only two or fewer $\succ$-indifference classes; if there were, there would then be a C_1-equivalence class that intersects four or more $\succ$-indifference classes. Hence each C_1-equivalence class equals the union of three $\succ$-indifference classes. If $\succ$ is complete and there were two C_1-equivalence classes I_1 and I_1' such that neither $I_1 C_1 I_1'$ nor $I_1' C_1 I_1$ holds, then each pair of the six $\succ$-indifference class in $I_1 \cup I_1'$ would have be in distinct C_2-equivalence classes, contradicting the fact that (C_1, C_2) is a 3-checklist.

Moving to C_2, (1) implies that any two of the three $\succ$-indifference classes in a C_1-equivalence class must be in distinct C_2-equivalence classes. Since this conclusion applies

[5] Following Definition 5, the completeness of C_i means that, for every $x, y \in X$, either $x\,(C_i \cup \approx)\,y$ or $y\,(C_i \cup \approx)\,x$, where $\approx$ is the equivalence relation of C_i.

to each of the three C_1-equivalence classes and there are only three C_2-equivalence classes, each C_2-equivalence class must equal the union of three $\succ$-indifference classes, which completes the proof of (2). Since any two of the three $\succ$-indifference classes in a C_1-equivalence classes are in distinct C_2-equivalence classes, for any pair of C_2-equivalence classes I_2 and I_2' there are two distinct $\succ$-indifference classes I and J in a single C_1-equivalence class such that $I \subset I_2$ and $J \subset I_2'$. So if there were two C_2-equivalence classes I_2 and I_2' such that neither $I_2 C_2 I_2'$ nor $I_2' C_2 I_2$ holds, there must be two distinct $\succ$-indifference classes I and J in a single C_1-equivalence class such that neither $I C_2 J$ nor $J C_2 I$ holds; so $\succ$ could not be complete. ■

Lemma 2. Suppose (i) (C_1, C_2) is a checklist for $\succ$, (ii) either $I C_1 J$ or $J C_1 I$ for any two distinct C_1-equivalence classes I and J, (iii) K is a C_1-equivalence class with $x, z \in K$, and (iv) $x \succ y \succ z$. Then $y \in K$.

Proof. If $y \notin K$ then y must be in a C_1-equivalence class $L \neq K$ and $L C_1 J$ would contradict $y \succ z$ while $L C_1 K$ would contradict $x \succ y$. ■

Part I.

We assume that $\succ$ has a subrelation $\succ'$ on a domain X' with 10 $\succ'$-indifference classes such that $\succ'$ is complete but not transitive. Without loss of generality, we suppose that $|X'| = 10$.

Since $\succ'$ is complete, for any $X'' \subset X'$ each element of X'' is both a $\succ'$- and a $\succ''$-indifference class, where $\succ''$ is the subrelation of $\succ$ that has domain X''. Fix a $X'' \subset X'$ with 9 elements and hence 9 $\succ''$-indifference classes. Since $\succ$ has uniformly quick checklists, there is a (C_1'', C_2'') that is a 3-checklist for $\succ''$. By Lemma 1, there are three C_1''-equivalence classes, say I_α'', I_β'', and I_γ''. Since the three elements in any of these C_1''-equivalence classes must be in distinct C_2''-equivalence classes, for any $\{i, j\} \subset \{\alpha, \beta, \gamma\}$, $(I_i'', \succ)$ and $(I_j'', \succ)$ must be order-isomorphic. Also by Lemma 1, C_2'' is complete. Let ω denote the sole element of $X' \backslash X''$.

(1) Suppose that each $(I_i'', \succ)$ is order-isomorphic to the cycle, that is, to $(\{a, b, c\}, \{(a, b), (b, c), (c, a)\})$. For any I_i'', say I_γ'' for concreteness, it is easy to confirm that there is a two-element subset $\{x, y\} \subset I_\gamma''$ such that $(\{x, y, \omega\}, \succ)$ is order-isomorphic to the line $(\{a, b, c\}, \{(a, b), (b, c), (a, c)\})$. Set $\widehat{X}'' = I_\alpha'' \cup I_\beta'' \cup \{x, y, \omega\}$.

By assumption, there is 3-checklist $(\widehat{C}_1'', \widehat{C}_2'')$ for the subrelation of $\succ$ that has domain $\widehat{X}''$. We show that I_α'' and I_β'' are $\widehat{C}_1''$-equivalence classes. Case I. Some $\widehat{C}_1''$-equivalence class $\widehat{I}''$ contains at least two elements of I_i'' for either $i = \alpha$ or $i = \beta$. Lemma 2 then implies that $\widehat{I}''$ contains the third element of I_i''. Some other $\widehat{C}_1''$-equivalence class $\widehat{J}''$ therefore contains at least two element of I_j'', $j = \{\alpha, \beta\} \backslash \{i\}$. Hence both I_α'' and I_β'' are $\widehat{C}_1''$-equivalence classes. Case II. Each $\widehat{C}_1''$-equivalence class contains at most one element of I_i'' for both $i = \alpha$ and $i = \beta$. It follows that each of the three $\widehat{C}_1''$-equivalence classes must contain exactly one element of I_i'', $i = \alpha, \beta$. Since (C_1'', C_2'') is a checklist for $\succ''$, either $I_\alpha'' \succ I_\beta''$ or $I_\beta'' \succ I_\alpha''$. Choose indices so that $I_\alpha'' \succ I_\beta''$. Let $\widehat{I}''$ be one of the $\widehat{C}_1''$-equivalence classes and let $x, y \in \widehat{I}''$ be such that $x \in I_\alpha''$, $y \in I_\beta''$ and hence $x \succ y$. Since $(I_\alpha'', \succ)$ is order-isomorphic to the cycle there is a $z \in I_\alpha''$ such that $x \succ z$. Since $I_\alpha'' \succ I_\beta''$, we have $z \succ y$, and so Lemma 2 implies that $z \in \widehat{I}''$, returning us to case I.

The three $\widehat{C}_1''$-equivalence classes are therefore I_α'', I_β'', and $\{x, y, \omega\}$. But since $(\{x, y, \omega\}, \succ)$ is not order-isomorphic to $(I_\alpha'', \succ)$ (or $(I_\beta'', \succ)$), we rule out the possibility that each $(I_i'', \succ)$ is order-isomorphic to the cycle.

(2) Since $\succ$ on I_i'' is complete, the only other possibility is that each $(I_i'', \succ)$ is order-isomorphic to the line $(\{a, b, c\}, \{(a, b), (b, c), (a, c)\})$. Since $\succ''$ is not transitive, C_1'' cannot be complete and transitive. Since, by Lemma 1, C_1'' is complete, the C_1'' ordering of C_1''-equivalence classes must be a cycle.

Let I_α'', I_β'', and I_γ'' continue to denote the C_1''-equivalence classes; assign indices so that $I_\alpha'' \succ I_\beta'' \succ I_\gamma'' \succ I_\alpha''$. We define three domains $\widehat{X}_k''$, $k = \alpha, \beta, \gamma$, by omitting an arbitrary element of $x_k \in I_k''$ from X'' and replacing it ω. Let $\widehat{I}_k''$ denote $(I_k'' \cup \{\omega\}) \backslash \{x_k\}$.

Fix $k \in \{\alpha, \beta, \gamma\}$. For the 3-checklist $(\widehat{C}_1'', \widehat{C}_2'')$ that we assume exists for the subrelation of $\succ$ that has domain $\widehat{X}_k''$, we show that for each $\widehat{C}_1''$-equivalence class $\widehat{I}''$ there is a $I'' \in \{I_\alpha'', I_\beta'', I_\gamma''\}$ such that $\widehat{I}'' \backslash \{\omega\} \subset I''$. So the three $\widehat{C}_1''$-equivalence classes must then be $\widehat{I}_k''$, I_i'', and I_j'', where $\{i, j\} = \{\alpha, \beta, \gamma\} \backslash \{k\}$. To accomplish this, we exclude the possibilities (a) and (b) below. Note first that since the C_1'' ordering of C_1''-equivalence classes is a cycle, the subrelation of $\succ$ that has domain $\widehat{X}_k''$ is not transitive.

(2a) Each $\widehat{C}_1''$-equivalence class consists of a $\succ$-cycle. For either assignment of indices i and j so that $\{i, j\} = \{\alpha, \beta, \gamma\} \backslash \{k\}$ and for any $\widehat{C}_1''$-equivalence class $\widehat{I}''$, $\widehat{I}''$ obviously cannot equal either I''_i or I''_j since I''_i and I''_j are $\succ$-lines rather than $\succ$-cycles. Furthermore $\widehat{I}''$ cannot contain two elements of I''_i and one element of I''_j since then $\widehat{I}''$ would again be a $\succ$-line; hence each $\widehat{C}''_1$-equivalence class contains one element of I''_i and one element of I''_j. But then for any pair of $\widehat{C}''_1$-equivalence classes, $\widehat{I}''$ and $\widehat{J}''$, we cannot have either $\widehat{I}'' \succ \widehat{J}''$ or $\widehat{J}'' \succ \widehat{I}''$ (as we must since $\widehat{C}''_1$ is complete and $(\widehat{C}''_1, \widehat{C}''_2)$ is a checklist for $\succ$ on the domain $\widehat{X}''_k$).

(2b) Each $\widehat{C}''_1$-equivalence class consists of a $\succ$-line but there is a $\widehat{C}''_1$-equivalence class $\widehat{I}''$ such that there is no $I'' \in \{I''_\alpha, I''_\beta, I''_\gamma\}$ with $\widehat{I}''\backslash\{\omega\} \subset I''$. Assigning the indices i and j so that $\{i,j\} = \{\alpha,\beta,\gamma\}\backslash\{k\}$, $\succ$ is a linear order on $I''_i \cup I''_j$. So if, for each $\widehat{C}''_1$-equivalence class $\widehat{I}''$, $\widehat{I}'' \cap (I''_i \cup I''_j) \neq \varnothing$ then the $\succ$ ordering of $\widehat{C}''_1$-equivalence classes must be linear, contradicting the fact that the subrelation of $\succ$ that has domain $\widehat{X}''_k$ is not transitive. Hence only two $\widehat{C}''_1$-equivalence classes intersect $I''_i \cup I''_j$. Since $\succ$ is linear on $I''_i \cup I''_j$ and either $I''_i \succ I''_j$ or $I''_j \succ I''_i$, these two $\widehat{C}''_1$-equivalence classes equal I''_i and I''_j.

Since for each $\widehat{C}''_1$-equivalence class $\widehat{I}''$ there is a $I'' \in \{I''_\alpha, I''_\beta, I''_\gamma\}$ such that $\widehat{I}''\backslash\{\omega\} \subset I''$, we can label the $\widehat{C}''_1$-equivalence classes $\widehat{I}''_\alpha(k)$, $\widehat{I}''_\beta(k)$, and $\widehat{I}''_\gamma(k)$, where, for $i \in \{\alpha,\beta,\gamma\}$, $\widehat{I}''_i(k)\backslash\{\omega\} \subset I''_i$. Since $I''_\alpha \succ I''_\beta \succ I''_\gamma$, $\widehat{I}''_\alpha(k) \succ \widehat{I}''_\beta(k) \succ \widehat{I}''_\gamma(k)$ for each $k \in \{\alpha,\beta,\gamma\}$. So, setting $k = \alpha$, $\omega \succ \widehat{I}''_\beta(\alpha) = I''_\beta$ and, setting $k = \gamma$, $I''_\beta = \widehat{I}''_\beta(\gamma) \succ \omega$, a contradiction.

Part II.

We turn to the remaining possibility where there is no $X' \subset X$ such that $\succ'$, the restriction of $\succ$ to X', defines 10 $\succ'$-indifference classes and is complete. We first show that in this case there is at least a $X' \subset X$ such that $\succ'$ defines 10 $\succ'$-indifference classes.

Lemma 3. Let $\succ$ be an asymmetric relation on a domain Y with $n \geq 3$ indifference classes and, for $Y' \subset Y$, let $\succ_{Y'}$ denote the restriction of $\succ$ to Y'. If $\succ_Y$ is incomplete then for any m with $3 \leq m \leq n$ there is a $Y' \subset Y$ such that (1) $|Y'| = m$, (2) Y' has m $\succ_{Y'}$-indifference classes, and (3) $\succ_{Y'}$ is incomplete.

Proof. Since the status of a $I \subset Y$ as an indifference class can vary according to the domain on which a subrelation of $\succ$ is defined, in this proof we will always denote explicitly, by use of subscripts for $\succ$ and for $\sim$, the relation that defines any indifference class. Note that for $Y'' \subset Y' \subset Y$ if x and y are in distinct $\succ_{Y''}$-indifference classes then

x and y are in distinct $\succ_{Y'}$-indifference classes. So if Y'' has j $\succ_{Y''}$-indifference classes then Y' has at least j $\succ_{Y'}$-indifference classes.

Since $\succ_Y$ has at least three $\succ_Y$-indifference classes and is incomplete, there exist $a, b \in Y$ such that $a \perp b$. (We use $\perp$ to indicate that two alternatives are unranked by $\succ$: $x \perp y \iff$ (not $x \succ y$, not $y \succ x$, and not $x \sim y$). And, as in the proof of Theorem 1, $\prec$ denotes $\{(x, y) \in X \times X : y \succ x\}$.) Since a and b are in separate $\succ_Y$-indifference classes, there exists some c such that, for at least one of the labelings ($s = a$ and $t = b$) and ($s = b$ and $t = a$), we have ($c \succ s$ and not $c \succ t$) or ($s \succ c$ and not $t \succ c$). So, defining $Y_3 = \{a, b, c\}$ and letting $\succ_3$ denote the restriction of $\succ_Y$ to Y_3, $\succ_3$ is incomplete and Y_3 has three $\succ_3$-indifference classes. To argue by induction, suppose for any integer k between 3 and $m - 1$ that there is a $Y_k \subset Y$ with $|Y_k| = k$ such that, $\succ_k$, the restriction of $\succ$ to Y_k, is incomplete and such that Y_k has k $\succ_k$-indifference classes. We show that there is a $Y_{k+1} \subset Y$ with $|Y_{k+1}| = k + 1$ such that $\succ_{k+1}$ is incomplete and such that Y_{k+1} has $k + 1$ $\succ_{k+1}$-indifference classes. Let $x \in Y \backslash Y_k$ be such that, for all $w \in Y_k$, it is not the case that $x \sim_Y w$. If $Y_k \cup \{x\}$ cannot serve as the desired Y_{k+1}, then for the restriction of $\succ_Y$ to $Y_k \cup \{x\}$, say $\succ_{k,x}$, we must have $x \sim_{k,x} y$ for some $y \in Y_k$. Given the induction premise that Y_k has k $\succ_k$-indifference classes, there can be at most one such y. Then, however, since not $x \sim_Y w$ for all $w \in Y_k$, there must be some $z \in Y \backslash (Y_k \cup \{x\})$ such that

$$(z \succ s \text{ and not } z \succ t) \text{ or } (s \succ z \text{ and not } t \succ z). \tag{i}$$

for at least one of the labelings ($s = x$ and $t = y$), ($s = y$ and $t = x$). Since x and y are in distinct $\succ_{Y_k \cup \{x,z\}}$-indifference classes, and each pair in Y_k are in distinct $\succ_{Y_k}$-indifference classes, $Y_k \cup \{x, z\}$ must have either $k + 1$ or $k + 2$ $\succ_{Y_k \cup \{x,z\}}$-indifference classes (see the first paragraph of the proof). In the former case, $Y_k \cup \{x, z\}$ can serve as Y_{k+1}. So suppose $Y_k \cup \{x, z\}$ has $k + 2$ $\succ_{Y_k \cup \{x,z\}}$-indifference classes. We show that, for either $Z = Y_k \cup \{z\}$ or $Z = (Y_k \cup \{x, z\}) \backslash \{y\}$, Z has $k + 1$ $\succ_Z$-indifference classes. Given that for either selection of Z, we have either $\{a, b, c\} \subset Z$ or, if $y \in \{a, b, c\}$, $\{a, b, c, x\} \backslash \{y\} \subset Z$, the Lemma is then proved. Since both Y_k and $(Y_k \cup \{x\}) \backslash \{y\}$ have k indifference classes, Z can fail to have $k + 1$ $\succ_Z$-indifference classes only if Z has k $\succ_Z$-indifference classes and $z \sim_Z v$ for some $v \in Z \backslash \{z\}$. Suppose $Y_k \cup \{z\}$ in fact has $k \succ_{Y_k \cup \{z\}}$-indifference classes. Then for the $v \neq z$ such that $z \sim_{Y_k \cup \{z\}} v$ either $v \in Y_k \backslash \{y\}$ or $v = y$. If the first case, $v \in Y_k \backslash \{y\}$, obtains then $zRy \iff vRy$ for both $R = \succ$ and $R = \prec$ and so, by (i) and the fact $x \sim_{Y_k \cup \{x\}} y$, it is not the case that $zRx \iff vRx$ for both $R = \succ$ and $R = \prec$; hence not $z \sim_{(Y_k \cup \{x,z\}) \backslash \{y\}} v$. Furthermore we cannot have either $z \sim_{(Y_k \cup \{x,z\}) \backslash \{y\}} x$ or $z \sim_{(Y_k \cup \{x,z\}) \backslash \{y\}} v'$ for $v' \in Y_k \backslash \{v\}$ since then, given that $x \sim_{Y_k \cup \{x\}} y$, we would have, respectively, $y \sim_{Y_k} v$ or $v' \sim_{Y_k} v$, contradicting the induction premise that Y_k has $k \succ_{Y_k}$-indifference classes. Thus $(Y_k \cup \{x, z\}) \backslash \{y\}$ has $k+1 \succ_{(Y_k \cup \{x,z\}) \backslash \{y\}}$-indifference classes and may serve as Y_{k+1}. The second case, $v = y$, is similar. Then (i) implies not $z \sim_{(Y_k \cup \{x,z\}) \backslash \{y\}} x$. It furthermore cannot be the case that $z \sim_{(Y_k \cup \{x,z\}) \backslash \{y\}} v'$ for $v' \in Y_k \backslash \{y\}$ since then $y \sim_{Y_k} v'$, again violating the assumption that Y_k has $k \succ_{Y_k}$-indifference classes. So again $(Y_k \cup \{x, z\}) \backslash \{y\}$ may serve as Y_{k+1}. ■

For subrelations $\succ_*$ of $\succ$ on $X_* \subset X$, we now drop the cumbersome clarification needed in the proof of Lemma 3 between $\succ_*$-indifference classes and $\succ$-indifference classes. We have no further need to consider indifference classes defined from $\succ$ on the entire domain X; so instead we will always name the subdomain $X_* \subset X$ explicitly and refer to a $\succ_*$-indifference class simply as a $\succ$-indifference class.

Lemma 3 allows us to assume that $\succ$ on X' has 10 $\succ$-indifference classes and is incomplete. We again suppose, without loss of generality, that $|X'| = 10$. By setting $Y = X'$, Lemma 3 also implies that there is a nine-element $X'' \subset X'$ such that $\succ$ on X'' has nine $\succ$-indifference classes and such that $\succ$ on X'' is incomplete. Since the proof is done if $\succ$ on X'' fails to have a 3-checklist of length 2, suppose it does have such a checklist. Since $\succ$ on X'' is incomplete, it follows from Lemma 1 that for either $i = 1$ or 2 (or both) C_i must be incomplete. Since Lemma 1 also implies that each C_i defines three C_i-equivalence classes, the C_i ordering of C_i-equivalence classes must be order-isomorphic to one of two incomplete orderings: letting $\{a, b, c\}$ denote the C_i-equivalence classes, these two orderings are $\{(a, b)\}$, which we label Inc(1), and $\{(a, b), (b, c)\}$, which we label Inc(2). Let Com denote either one of the two complete orderings of $\{a, b, c\}$ (the line and cycle). A 3-checklist of length 2 must therefore take the following form: either C_1-C_2 or C_2-C_1 must be of type Inc(1)-Inc(1), Inc(2)-Inc(2), Inc(1)-Inc(2), Inc(1)-Com, or Inc(2)-Com, where in each case 'C_i-C_j is of type A-B' means that the C_i (resp. C_j) ordering of

C_i (resp. C_j)-equivalence classes is order-isomorphic to A (resp. B). Since the cases are broadly similar, we consider only one case – where C_1-C_2 is of type Inc(2)-Com – in fine detail.

Before proceeding to Inc(2)-Com, we list, for each of the possibilities given in the previous paragraph, the number of elements in X'' that are ranked relative to each $x \in X''$, which in the terminology of graph theory is called the *degree of* x. Given a set Y with 9 elements and a 3-checklist (C', C'') that is quick for some asymmetric relation $\succ_{(C',C'')}$ that has domain Y and that has nine indifference classes, we define the degree of x, $d(x)$, to be $\left|\{y \in Y : y \succ_{(C',C'')} x \text{ or } x \succ_{(C',C'')} y\}\right|$ for each of the nine $x \in Y$. When the nine possible $d(x)$, one for each $x \in Y$, are arranged in decreasing order, they are called a *degree sequence*. If we instead began with the 3-checklist (C'', C') that is quick for some $\succ_{(C'',C')}$ with the same domain Y and also with nine indifference classes, we would arrive at the same degree sequence. There are therefore five possible degree sequences, each simple to calculate, which we list in the following table:

Table of degree sequences

Inc(1)-Inc(1)	5,5,5,5,3,3,3,3,0
Inc(2)-Inc(2)	8,7,7,7,7,5,5,5,5
Inc(1)-Inc(2) or Inc(2)-Inc(1)	7,7,6,5,5,5,5,3,3
Inc(1)-Com or Com-Inc(1)	7,7,7,7,7,7,6,6,6
Inc(2)-Com or Inc(2)-Com	8,8,8,7,7,7,7,7,7

.

Inc(2)-Com. We assume there is a $X'' \subset X'$ with $|X''| = 9$ that has a 3-checklist (C_1, C_2) of length 2 where the C_1 ordering of C_1-equivalence classes is of type Inc(2) and the C_2 ordering of C_2-equivalence classes is of type Com. Applying Lemma 1, without loss of generality we can assign labels a, b, c, d, e, f, g, h, and i to the elements of X'' so that $\{a, b, c\}C_1\{d, e, f\}C_1\{g, h, i\}$ and the three C_2-equivalence classes are $\{a, d, g\}$, $\{b, e, h\}$, and $\{c, f, i\}$. For later use, we note the $\succ$-unranked pairs for the $\succ$ for which (C_1, C_2) is a checklist: $a \perp g$, $b \perp h$, and $c \perp i$. Consider the domain of 9 elements $\widehat{X}''(z)$ formed by replacing one element $z \in \{a, b, c, d, e, f, g, h, i\}$ with the tenth element of X' which we label ω. We first show that if $\widehat{X}''(z)$ has a 3-checklist of length 2 then it is either of type Inc(2)-Com or Com-Inc(2). Suppose to begin that $\widehat{X}''(z)$ has nine

$\succ$-indifference classes. Since the degree of any element in both X'' and $\widehat{X}''(z)$ can differ by at most one, the presence in the degree sequence table of numbers 5 or smaller for the checklists Inc(1)-Inc(1), Inc(2)-Inc(2), Inc(1)-Inc(2), and Inc(2)-Inc(1) rules out these possibilities as quick checklists when the domain is $\widehat{X}''(z)$. The remaining possibilities besides Inc(2)-Com and Com-Inc(2) are Inc(1)-Com and Com-Inc(1). But in order for $\widehat{X}''(z)$ to have a Inc(1)-Com or Com-Inc(1) checklist, it must be that ω is not ranked against d, e, or f if they are in $\widehat{X}''(z)$ (these are the elements of X'' with degree eight) and is also not ranked against three elements of a, b, c, g, h, and i if they are in $\widehat{X}''(z)$ (these are the elements of X'' with degree seven). Given that one element of X'' is not an element of $\widehat{X}''(z)$, we conclude that ω could have a degree of at most four in $\widehat{X}''(z)$, which is not consistent with the Inc(1)-Com degree sequence. Now let $\widehat{X}''(z)$ have fewer than nine $\succ$-indifference classes. It is easy to confirm that $\{a, b, c, d, e, f, g, h, i\}\backslash\{z\}$ must have eight $\succ$-indifference classes; so the only case to consider is where $\widehat{X}''(z)$ has eight $\succ$-indifference classes. If a checklist is Inc(1)-Com or Com-Inc(1) and has eight indifference classes then one can easily confirm that the degree sequence is either $(6, 6, 6, 6, 6, 6, 5, 5)$ or $(7, 7, 6, 6, 6, 6, 5, 5)$ while if a checklist is Inc(2)-Com or Com-Inc(2) then the degree sequence is $(7, 7, 6, 6, 6, 6, 6, 6)$ or $(7, 7, 7, 7, 6, 6, 6, 6)$. Therefore it could not be that a $\widehat{X}''(z)$ with eight indifference classes has a Inc(1)-Com or Com-Inc(1) checklist.

To exclude Com-Inc(2), observe that if there exists a Com-Inc(2) checklist $(\widehat{C}_1, \widehat{C}_2)$ when the domain is $\widehat{X}''(z)$ then a and g (resp. b and h) if they are elements of $\widehat{X}''(z)$ must be in the same $\widehat{C}_1$-equivalence class since $a \perp g$ (resp. $b \perp h$) and $\widehat{C}_1$ is complete. Now $a \succ \{d, e, f\} \succ g$ and $b \succ \{d, e, f\} \succ h$. Since either $\{a, g\} \subset \widehat{X}''(z)$ or $\{b, h\} \subset \widehat{X}''(z)$ (or both) and $\widehat{C}_1$ is complete, Lemma 2 implies that if $\{d, e, f\} \subset \widehat{X}''(z)$ then d, e, and f must be in the same $\widehat{C}_1$-equivalence class as $\{a, g\}$ or $\{b, h\}$ (whichever pair is a subset of $\widehat{X}''$). Since at least two elements of $\{d, e, f\}$ must be in $\widehat{X}''(z)$, we have a $\widehat{C}_1$-equivalence class with at least four elements, in violation of Lemma 1.

Having established that if $\widehat{X}''(z)$ has a 3-checklist of length 2 the checklist is of type Inc(2)-Com, we proceed to identify a $\widehat{X}''(z)$ that does not have a Inc(2)-Com checklist. Suppose $(\widehat{C}_1, \widehat{C}_2)$ is a 3-checklist for $\widehat{X}''(z)$. Let the three $\widehat{C}_1$-equivalence classes be labeled I, J, and K where $I \, \widehat{C}_1 \, J \, \widehat{C}_1 \, K$. Since $\widehat{C}_2$ is complete and since $a \perp g$, $b \perp h$, and $c \perp i$, any of the pairs (a,g), (b,h), and (c,i) that is a subset of $\widehat{X}''(z)$ must be (i) contained in a single $\widehat{C}_2$-equivalence class and in a single C_2-equivalence class, and (ii) one element of the pair must be I and the other in K. Suppose $z \in \{a,b,c,g,h,i\}$, let y be the unique $y \in \{a,b,c,g,h,i\}$ such that $y \perp z$, and let (s,t) be one of the pairs in $\{(a,g),(b,h),(c,i)\}$ such that $(s,t) \subset \widehat{X}''(z)$. Since (i) $\{s,t\}$ is a subset of a C_2-equivalence class, (ii) $\{s,t,y\} \cap \{d,e,f\} = \varnothing$, and (iii) C_2 is complete, we conclude that either ($y \succ s$ and $y \succ t$) or ($s \succ y$ and $t \succ y$). Since $s \perp t$, we must have $\{s,t\} \subset I \cup K$, and we conclude that $y \notin J$. It follows that $J = \{d,e,f\}$ since otherwise $u \perp v$ for some $u \in \{d,e,f\}$ and some $v \in \{a,b,c,g,h,i\} \backslash \{z\}$. Hence, since $\{a,b,c\} \succ \{d,e,f\} \succ \{g,h,i\}$, $\{a,b,c\} \backslash \{z\} \subset I$ and $\{g,h,i\} \backslash \{z\} \subset K$.

Therefore if $z = c$, $\widehat{X}''(z)$ consists of nine $\succ$-indifference classes we conclude, since $J = \{d,e,f\}$ and $K = \{g,h,i\}$, that $\omega \in I$ and so $\omega \succ \{d,e,f\}$. If for $z = i$, $\widehat{X}''(z)$ also consists of nine $\succ$-indifference classes then $\omega \in K$ and therefore $\{d,e,f\} \succ \omega$, a contradiction.

It remains to dispose of the possibility that for either $z = c$ or $z = i$, $\widehat{X}''(z)$ consists of less than nine $\succ$-indifference classes. We use the following easily confirmed fact: for any domain Y that contains at least 7 elements of X'', any 2 elements of $X'' \cap Y$ are members of distinct $\succ$-indifference classes. So we need only consider the case where, for either $z = c$ or $z = i$, $\widehat{X}''(z)$ consists of eight $\succ$-indifference classes. There must then be a $x \in X'' \backslash \{z\}$ such that (i) if zRx, where $R \in \{\succ, \prec, \perp\}$, then not $zR\omega$ and (ii) for any $w \in (X'' \cup \{\omega\}) \backslash \{z\}$ and $R \in \{\succ, \prec\}$, $wR\omega \iff wRx$. We next show that $\widehat{X}''(v)$ must consist of nine $\succ$-indifference classes for any $v \in X'' \backslash \{z\}$. To do so, fix a $v \in X'' \backslash \{z\}$ and suppose $\widehat{X}''(v)$ has fewer than nine $\succ$-indifference classes. There must then be a $u \in X'' \backslash \{v\}$ such that (i*) if vRu, where $R \in \{\succ, \prec, \perp\}$, then not $vR\omega$ and (ii**) for any $w \in (X'' \cup \{\omega\}) \backslash \{v\}$ and $R \in \{\succ, \prec\}$, $wR\omega \iff wRu$. Since $z \in \widehat{X}''(v)$, and given (i) and (ii*), $u \neq x$. But then for the domain $(X'' \backslash \{z,v\}) \cup \{\omega\}$, ω, x, and u are all elements of the same $\succ$-indifference class, and hence x and u are elements of the same $\succ$-indifference class for the same domain, contradicting the fact mentioned at the beginning of the paragraph. To finish the Inc(2)-Com case, we repeat the argument made in the previous paragraph: since, for $z = a$, $\widehat{X}''(z)$ consists of nine $\succ$-indifference classes, $\omega \in I$ and so $\omega \succ \{d, e, f\}$ and, since, for $z = c$, $\widehat{X}''(z)$ must also consist of nine $\succ$-indifference classes, $\omega \in K$ and therefore $\{d, e, f\} \succ \omega$, a contradiction.

For each of the remaining checklist types, one may use the degree sequence table to argue that for any $z \in X''$ if a quick checklist exists when the domain is $\widehat{X}''(z)$ then it must be of the same type as the checklist when the domain is X''. Below, we provide the key step that there are at least two items $x, y \in X''$ such that the domains $\widehat{X}''(x)$ and $\widehat{X}''(y)$ cannot both have quick checklists, on the assumption that both domains contain 9 $\succ$-equivalence classes. If some $\widehat{X}''(z)$ has 8 $\succ$-equivalence classes, then as in the previous paragraph one may always find an alternative $z' \in X''\backslash\{z\}$ such that $\widehat{X}''(z')$ has no quick checklist.

In each case, let $X' = \{a, b, c, d, e, f, g, h, i, \omega\}$, let $X'' = X'\backslash\{\omega\}$, and let the quick checklist that we assume exists for the domain X'' be denoted (C_1, C_2). The criteria for the quick checklist for $\widehat{X}''(z)$ is denoted $(\widehat{C}_1, \widehat{C}_2)$.

Inc(2)-Inc(1). Choose labels so that $\{a, b, c\}C_1\{d, e, f\}C_1\{g, h, i\}$ and $\{a, d, g\}C_2\{b, e, h\}$. Let I, J, K denote the three $\widehat{C}_1$-equivalence classes, where $I\widehat{C}_1J\widehat{C}_1K$. Consider $\widehat{X}''(z)$ for $z = c$ and $z = i$. Then d and e, which have the largest degree when the domain is X'', will continue to have the largest degree and must therefore be elements of J. The specification of (C_1, C_2) implies $\{a, b, c\} \succ \{d, e, f\} \succ \{g, h, i\}$, $a \succ b$, $d \succ e$, and $g \succ h$. When the domain is $\widehat{X}''(z)$, $\{a, b, c\}\backslash\{z\} \subset I$ since otherwise, given that $\widehat{C}_2$ is Inc(1), the element of $\{a, b, c\}\backslash\{z\}$ not in I could not be $\succ$-superior to both d and e for the $\succ$ that has the checklist $(\widehat{C}_1, \widehat{C}_2)$. Similarly $\{g, h, i\}\backslash\{z\} \subset K$. And since $\{a, b\} \succ \{f\}$, we have $f \in J$ since otherwise (again using the fact that $\widehat{C}_2$ is Inc(1)) f could not be $\succ$-inferior to both a and b for the $\succ$ that has the checklist $(\widehat{C}_1, \widehat{C}_2)$. So, for $z = c$, we have $\omega \in I$ and therefore $\omega \succ \{d, e, f\}$ and, for $z = i$, we have $\omega \in K$ and therefore $\{d, e, f\} \succ \omega$, a contradiction.

Inc(2)-Inc(2). Choose labels so that $\{a, b, c\}C_1\{d, e, f\}C_1\{g, h, i\}$, and let I, J, K denote the three $\widehat{C}_1$-equivalence classes, where $I\widehat{C}_1J\widehat{C}_1K$. For $z = c$ and $z = i$, the domain $\widehat{X}''(z)$ has $J = \{d, e, f\}$. For $z = c$, we have $\omega \in I$ and therefore $\omega \succ \{d, e, f\}$ and, for $z = i$, we have $\omega \in K$ and therefore $\{d, e, f\} \succ \omega$.

Com-Inc(2) and Com-Inc(1). The conclusions for Inc(2)-Inc(2) continue to hold using the same labeling.

Inc(1)-Com and Inc(1)-Inc(2). Choose labels so that $\{a, b, c\}C_1\{d, e, f\}$, $\{a, d, g\}C_2\{b, e, h\}C_2\{c, f, i\}$, and let I, J, K denote the three $\widehat{C}_2$-equivalence classes, where $I\widehat{C}_2J\widehat{C}_2K$. For $z = g$ and $z = i$, the domain $\widehat{X}''(z)$ has $J = \{b, e, h\}$. For $z = g$, we have $\omega \in I$ and therefore $\omega \succ \{b, e, h\}$ and, for $z = i$, we have $\omega \in K$ and therefore $\{b, e, h\} \succ \omega$.

Inc(1)-Inc(1). The easiest argument for the final case is slightly different. Choose labels so that $\{a, b, c\}C_1\{d, e, f\}$ and $\{a, d, g\}C_2\{b, e, h\}$. For $\widehat{X}''(i)$, if ω has $\omega \succ x$ or $x \succ \omega$ for any $x \in \{a, b, c, d, e, f, g, h\}$ then no element of $\widehat{X}''(i)$ is $\succ$-unranked relative to all other elements of $\widehat{X}''(i)$, in violation of the list $(5, 5, 5, 5, 3, 3, 3, 3, 0)$ given by the degree sequence table. If $\omega \perp x$ for all $x \in \{a, b, c, d, e, f, g, h\}$ then $\widehat{X}''(x)$ for any $x \in \{a, b, c, d, e, f, g, h\}$ has a degree sequence that differs from $(5, 5, 5, 5, 3, 3, 3, 3, 0)$.

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