A Simple Model of Bankruptcy in General Equilibrium
5th Annual CARESS Cowles Conference

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Introduction, i.e. rubbish - or more bluntly, crap - at the beginning of the seminar, shall last no longer than 15 minutes (which is, from my own viewpoint, even too liberal)…If there is not a serious equation, model, result,…(on the blackboard or a slide) by the end of 15 minutes, then anyone in the audience - including me - is perfectly free to simply leave: THE SEMINAR IS FORMALLY OVER!

Mention no more than 5 (or, if there is a very good reason, really stretching it, 10) references, preferably just those which the speaker thinks are the most substantively important (not the most politically correct or self-serving).
I remove the assumption that households, having made financial commitments today \((s = 0)\), must fulfill these commitments in the realized state \((s > 0)\) tomorrow.

Not considered in this model is defaulting (choosing not to fulfill the financial commitment on an asset-by-asset basis).

Rather, a household considers the value of its entire portfolio and faces a binary decision:
- If the household fulfills its commitment (solvency).
- If not (bankruptcy).
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As a result of bankruptcy, financial payouts to creditors must be appropriately diluted.

I assume assets are pooled (each asset with its own pool) and that all creditors receive the same fraction of the original asset payouts.

For a nonbankrupt household $h$, the payout on asset $j$ is

$$\rho_j(s) r_j(s) \quad \text{when } z_j^h \geq 0 \quad \text{and} \quad r_j(s) \quad \text{when } z_j^h < 0$$

$\rho_j(s)$—endogenous dilution factor

$$\rho_j(s) \sum_{\text{all } h} (\text{long positions}) = 1 - \sum_{\text{solvent}} (\text{short positions})$$

$$+ \sum_{\text{bankrupt}} \left( \text{fraction of household’s seized assets paid back to asset pool } j \right) \cdot$$
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$$+ \sum_{\text{bankrupt}} \left( \frac{\text{fraction of household's seized assets paid back to asset pool } j}{\text{asset pool } j} \right).$$
   - Model of household default of unsecured credit
   - Households choose "degree" of default
   - Penalty for default: utility loss
   - Chain reactions of default possible
   - Model with real assets and investment constraints (though constraints could be removed when dealing with numeraire assets)

   - Theorems describing the beneficial normative role that default can play

   - Model of household default of secured credit
   - Collateral requirements exogenously set
   - Investment constraints endogenously depend on collateral requirement and scarcity of collateral
I model chapter 7 bankruptcy

-70% of individual bankruptcies are chapter 7
- with chapter 13, still liable for debt (5 – year repayment)
- more potential for abuse with chapter 7

in the unsecured credit markets

- with secured credit, there is no anonymous interaction between the bankrupt household and its creditors (this is pre-contracted)

without investment constraints

- justification of constraints is as a policy tool to mitigate effects of bankruptcy (need to understand bankruptcy first)
- setting of such constraints is arbitrary

and with assets in unit net supply

- short-hand way to model a diverse asset structure.
The model

2–period model with $S$ states of uncertainty in the second time period households $h \in \mathcal{H}$ (continuum) with finite types $f \in \{1, \ldots, F\}$

$\mathcal{H}_f = \{\text{households of type } f\} \text{ with } \int_{h \in \mathcal{H}_f} dh = \frac{1}{F} \quad \forall f$

- $L$ physical commodities per state ($G = L(S + 1)$)
- consumption $x^h \in \mathbb{R}^G_+$
- endowments $e^f \gg 0$
- $u^f$ is $C^2$, differentiably strictly increasing, differentiably strictly quasi-concave, and satisfies the boundary condition.
- spot prices $p \gg 0$

real

financial

- $J$ numeraire assets in unit net supply
- assets $z^h \in \mathbb{R}^J$
- initial assets $z^f(0) > 0$
- yields matrix $Y = \left[ r^f(s) \right]_{\forall j, \forall s > 0}$ is strictly positive, has full rank, and is in general position.
- asset prices $q$
Under chapter 7, a household forfeits its portfolio $z^h$ (including those assets held in a long position with positive value) and walks away from its debt.

In principle, this system legislates a "fresh start" with no wage garnishments and no confiscation of current period endowments.

However...
Incorporating bankruptcy

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However...
The court appoints a case trustee to oversee the liquidation of a bankrupt household’s assets to repay the creditors.

The case trustee is endowed with "avoiding powers"; trustee can void any transfer of assets that occurred 90 days prior to the bankruptcy filing.

For bankrupt households with short positions in all assets, there is no role for "avoiding powers" as such households would not have any assets to shelter from the state.

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- The case trustee is endowed with "avoiding powers"; trustee can void any transfer of assets that occurred 90 days prior to the bankruptcy filing.
- For bankrupt households with short positions in all assets, there is no role for "avoiding powers" as such households would not have any assets to shelter from the state.

But...
Incorporating bankruptcy (cont.)

If a bankrupt household currently holds long positions in assets $\left( z_j^h \right)^+$,

- then its sale of assets in the last 90 days will likely be voided.
- having consumed the benefits from this asset sale in previous periods, there must be a corresponding cost (in terms of lost endowments) in the current period.

Cost of bankruptcy (in current period)

$$\sum_j \alpha_j r_j(s)(z_j^h)^+$$

Household wealth (in current period)

$$p(s)e^h(s) - \sum_j \alpha_j r_j(s)(z_j^h)^+$$

Key first assumption: $\alpha >> 0$
The Bankruptcy Abuse Prevention and Consumer Protection Act of 2005 sought to do as its name indicates.

- Any household applying for chapter 7 bankruptcy must pass a "means test". Simply put, if a household’s average monthly income over the past 6 months exceeds the state median, then it cannot declare bankruptcy.

Key second assumption: $\forall f \in \mathcal{F}, \exists s > 0 \text{ s.t. } e^f_i(s) > \frac{1}{|\mathcal{F}|} \sum_{\bar{f} \in \mathcal{F}} e^\bar{f}_i(s) \ \forall i$.

- By making this assumption, then by law, $\forall h, \exists s > 0 \text{ s.t. } h$ cannot declare bankruptcy.
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- By making this assumption, then by law, $\forall h, \exists s > 0 \text{ s.t. } h$ cannot declare bankruptcy.
Recall $r(s) \gg 0$, so no Arrow securities.

Suppose a household declares bankruptcy in state $s = 1$.

"This household can use its portfolio to replicate an Arrow security. The following portfolio replicates the desired Arrow security:

$$z' = \left[ \begin{array}{ccc} r_1(1) & \ldots & r_J(1) \\ \vdots & \ddots & \vdots \\ r_1(J) & \ldots & r_J(J) \end{array} \right]^{-1} \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$}

The household could then short this replicated portfolio to $\infty$ and obtain an arbitrage profit."

To bound assets and obtain existence, such a "story" cannot be possible (notice that $z'$ is a linear combination of both long and short asset positions and recall that $\alpha \gg 0$).
Lemma

With $\alpha \gg 0$ and $r(s) \gg 0 \ \forall s > 0$, then the assets $z^h$ of bankrupt households are bounded.

Proof:

Household wealth in a bankrupt state

$$p(s)e^h(s) - \sum_j \alpha_j r_j(s) \left( z^h_j \right)^+$$

Household wealth in a nonbankrupt state (at least one such state exists)

$$p(s)e^h(s) + \sum_j \left\{ r_j(s) \left( z^h_j \right)^- + \rho_j(s)r_j(s) \left( z^h_j \right)^+ \right\}$$

- Suppose $\exists z^ν_j \to +\infty$, then consumption is outside the consumption set in bankrupt state ($\Rightarrow \Leftarrow$).
- Suppose $\exists z^ν_j \to -\infty$, then with $z^ν$ bounded above, consumption would be outside the consumption set in nonbankrupt state ($\Rightarrow \Leftarrow$).
Definition of equilibrium terms

Define the household specific asset payouts for nonbankrupt households $r_j^h(s)$ as:

$$
\begin{align*}
    r_j^h(s) &= \rho_j(s) r_j(s) & \text{if } z_j^h \geq 0 \\
    r_j^h(s) &= r_j(s) & \text{if } z_j^h < 0.
\end{align*}
$$

Define $S_f^* = \{ s > 0 : p(s) e^f(s) > \frac{1}{F} \sum_{\tilde{f} \in F} p(s) e^{\tilde{f}}(s) \}$. Then a household will declare bankruptcy in state $s > 0$ if

$$
  s \notin S_f^* \text{ for } h \in \mathcal{H}_f \\
  \sum_j r_j^h(s) z_j^h < - \sum_j \alpha_j r_j(s) (z_j^h)^+.
$$

Define the set of bankrupt households in state $s > 0$:

$$
\mathcal{H}'_s = \{ h \in \mathcal{H} : (1) \text{ holds} \}.
$$
Household budget set

The budget set for each household is given by:

\[ B^h(p, q, \rho) = \{(x, z) \in X^h \times \mathbb{R}^J : p(0)(e^f(0) - x(0)) + qz^f(0) - qz \geq 0, \]

\[ p(s)(e^f(s) - x(s)) + \max \left\{ \sum_j r_j^h(s)z_j, - \sum_j \alpha_j r_j(s)(z_j)^+ \right\} \geq 0 \ \forall s \notin S_f^*, \]

\[ p(s)(e^f(s) - x(s)) + \sum_j r_j^h(s)z_j \geq 0 \ \forall s \in S_f^* \}

for \( h \in \mathcal{H}_f \).
Bankrupt households forfeit the value of their long positions
\[ \sum_j \rho_j(s) r_j(s) \left( z_j^h \right)^+ \] and must pay the bankruptcy cost
\[ \sum_j \alpha_j r_j(s) \left( z_j^h \right)^+ . \]

I assume that these total funds \( \sum_j \left( \rho_j(s) + \alpha_j \right) r_j(s) \left( z_j^h \right)^+ \) are divided among the vector of asset pools with the fraction
\[ \frac{r_k(s) \left( z_k^h \right)^-}{\sum_k r_k(s) \left( z_k^h \right)^-} \] to each asset pool \( k \).

Defining
\[ \delta^h(s) = \frac{-\sum_j \left( \rho_j(s) + \alpha_j \right) r_j(s) \left( z_j^h \right)^+}{\sum_k r_k(s) \left( z_k^h \right)^-} , \]

then \(-\delta^h(s) r_k(s) \left( z_k^h \right)^-\) is returned to asset pool \( k \).

By the definition of \( h \in \mathcal{H}'_s \), the variable \( 0 \leq \delta^h(s) < 1 \).
A bankruptcy equilibrium is \( ((x^h, z^h)_{h \in \mathcal{H}}, p, q, \rho) \) s.t.

\[
\forall h \in \mathcal{H}, \text{ given } (p, q, \rho) \quad (x^h, z^h) \in \arg \max_{x} u^h(x). \tag{H}
\]

\[
\text{subj to } (x, z) \in B^h(p, q, \rho)
\]

\[
\int_{h \in \mathcal{H}} z_j^h = 1 \quad \forall j.
\]

\[
\int_{h \in \mathcal{H}} x_j^h(s) = \frac{1}{F} \sum_{f \in \mathcal{F}} \epsilon_f^h(s) \quad \forall (l, s) \notin \{(L, 1), \ldots, (L, S)\}. \tag{M}
\]

\[
\int_{h \in \mathcal{H}} x_L^h(s) = \frac{1}{F} \sum_{f \in \mathcal{F}} \epsilon_L^h(s) + \sum_j r_j(s) \quad \forall s > 0.
\]

\[
\rho_j(s) \int_{h \in \mathcal{H}} (z_j^h)^+ + \int_{h \notin \mathcal{H}_s'} (z_j^h)^- + \int_{h \in \mathcal{H}_s'} \delta^h(s) (z_j^h)^- = 1 \quad \forall j, s > 0. \tag{AC}
\]
Implications of bankruptcy

Consider the aggregate consistency (AC) condition for some \( j, s \):

\[
\rho_j(s) \int_{h \in \mathcal{H}} (z_j^h)^+ + \int_{h \notin \mathcal{H}'_s} (z_j^h)^- + \int_{h \in \mathcal{H}'_s} \delta^h(s) (z_j^h)^- = 1.
\]

The dilution factor \( 0 < \rho_j(s) \leq 1 \).

Using the market clearing condition \( \int_{h \in \mathcal{H}} z_j^h = 1 \), (AC) is equivalent to:

\[
\rho_j(s) = 1 + \frac{\int_{h \in \mathcal{H}'_s} \left(1 - \delta^h(s)\right) (z_j^h)^-}{\int_{h \in \mathcal{H}} (z_j^h)^+}
\]

Recalling \( 0 \leq \delta^h(s) < 1 \),

1. If \( z_j^h \geq 0 \ \forall h \in \mathcal{H}'_s \) (or \( \mathcal{H}'_s = \emptyset \)), then \( \rho_j(s) = 1 \).
2. If \( z_j^h < 0 \) for some \( h \in \mathcal{H}'_s \), then \( \rho_j(s) < 1 \).
Existence

**Theorem**

*Under the smooth assumptions listed, with strictly positive \( Y \), and with the two key assumptions on how bankruptcy is implemented, then a bankruptcy equilibrium \( ((x^h, z^h)_{h \in \mathcal{H}}, p, q, \rho) \) exists.*
Boundedness of Assets (sketch)

- For nonbankrupt households \((h \notin \mathcal{H}_s')\), financial payout \(\hat{Y}^h z^h\) is bounded where \(\hat{Y}^h\) is the strictly positive matrix:

\[
\hat{Y}^h = \begin{bmatrix}
  r_1^h(1) & \ldots & r_j^h(1) \\
  \vdots & \ddots & \vdots \\
  r_1^h(S) & \ldots & r_j^h(S)
\end{bmatrix}.
\]

- Impose an artificial investment constraint \(z_j^h \geq -K_j^h\). Then existence is guaranteed, markets clear, and the \((AC)\) conditions are met.

- Consider a sequence of economies as \(K \to \infty\). Then, either
  1. the investment constraints cease to bind (assets bounded)
  2. the constraints continue to bind
     1. the payout matrix \(\hat{Y}^h \to Y^-,\) a rank deficient payout matrix
     2. the payout matrix \(\hat{Y}^h \to Y,\) the original full rank payout matrix
Features of the equilibrium

- Chain reaction of bankruptcy is possible (aka contagion in finance). Suppose one household type \( h \in \mathcal{H}_f \) declares bankruptcy. Then \( \rho_j(s) < 1 \) for all \( j \) s.t. \( z_j^h < 0 \). The value of all household portfolios are now

\[
\sum_j \rho_j(s) r_j(s) \left( z_j^h \right)^+ + \sum_j r_j(s) \left( z_j^h \right)^- < r(s) \cdot z^h.
\]

This may lead to further bankruptcy declarations. Particularly vexing is that for bankrupt households with both short and long positions, if \( \rho(s) << 1 \), then the liquidated long positions will have less value and this reduces the dilution factor further.

- No distinction between complete markets and incomplete markets. With bankruptcy, the first basic welfare theorem fails in both cases. I need to characterize equilibria as resulting from a dynamic trading environment over both commodity and financial markets.
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Extension I: Real costs of bankruptcy

- Bankrupt households turn over \( \sum_j \left( \rho_j(s) + \alpha_j \right) r_j(s) \left( z_j^h \right)^+ \).

- Suppose during the liquidation process, only \( \sum_j \beta \rho_j(s) r_j(s) \left( z_j^h \right)^+ \) is made available to creditors, with \( \beta \in [0, 1] \).

- Define how the remaining funds are distributed:

\[
\delta_{RC}^h(s) = \frac{-\sum_j \beta \rho_j(s) r_j(s) \left( z_j^h \right)^+}{\sum_k r_k(s) \left( z_k^h \right)^-}.
\]

- Then the (AC) equation yields \( \forall j, \forall s > 0 \):

\[
\rho_{j,RC}(s) = 1 + \frac{\int_{h \in H_s'} \left(1 - \delta_{RC}^h(s) \right) \left( z_j^h \right)^-}{\int_{h \in H} \left( z_j^h \right)^+}.
\]

- \( \rho_{j,RC}(s) \leq \rho_j(s) \), with < if \( \left( z_j^h \right)^- \cdot \sum_k \left( z_k^h \right)^+ < 0 \) for some \( h \in H_s' \).
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- Then the $(AC)$ equation yields $\forall j, \forall s > 0$:

$$\rho_{j,RC}(s) = 1 + \int_{h \in \mathcal{H}'_s} \left(1 - \delta_{RC}^h(s) \right) \left(z_j^h\right)^- \int_{h \in \mathcal{H}} \left(z_j^h\right)^+.$$ 

- $\rho_{j,RC}(s) \leq \rho_j(s)$, with $<$ if $\left(z_j^h\right)^- \cdot \sum_k \left(z_k^h\right)^+ < 0$ for some $h \in \mathcal{H}'_s$. 

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- Define how the remaining funds are distributed:

$$\delta^h_{RC}(s) = \frac{- \sum_j \beta \rho_j(s) r_j(s) \left( z^h_j \right)^+}{\sum_k r_k(s) \left( z^h_k \right)^-}.$$ 

- Then the (AC) equation yields $\forall j, \forall s > 0$:

$$\rho_{j, RC}(s) = 1 + \int_{H'_s} \left( 1 - \delta^h_{RC}(s) \right) \left( z^h_j \right)^- \int_{H} \left( z^h_j \right)^+ .$$

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Then the \( (AC) \) equation yields \( \forall j, \forall s > 0 \):

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\( \rho_{j,RC}(s) \leq \rho_j(s) \), with \( < \) if \( \left( z_j^h \right)^- \cdot \sum_k \left( z_k^h \right)^+ < 0 \) for some \( h \in \mathcal{H}'_s \).
Extension II: Firm bankruptcy

Introduce firms $i \in I = \{1, \ldots, I\}$. Firms use the same set of $J$ numeraire assets to finance production investment.

If a firm has negative profit $\pi(s)$ in some state, then it must declare bankruptcy and will only pay back the fraction $\delta^i(s) \sum_j r_j(s) \left( z^i_j \right)^-$ of its financial debt such that $\pi(s) = 0$ (limited liability of shareholders).

If a firm has positive profit, then obviously $\delta^i(s) = 1$.

The condition $(AC)$ for this production model is the same as for the pure exchange model with the second line added:

$$
\rho_j(s) \int_{h \in \mathcal{H}} \left( z^h_j \right)^+ + \int_{h \notin \mathcal{H}_s} \left( z^h_j \right)^- + \int_{h \in \mathcal{H}_s} \delta^h(s) \left( z^h_j \right)^- \\
+ \rho_j(s) \sum_{i \in I} \left( z^i_j \right)^+ + \sum_{i \in I} \delta^i(s) \left( z^i_j \right)^- = 0 \ \forall j, s > 0.
$$
Extension III: Infinite time horizon

2—period assumptions

- There is a cost of bankruptcy parameterized by $\alpha$.
- Households must fail the "means test" in some state.
- Nonbankrupt households must repay their debt entirely.
- All households face the same asset prices when borrowing and these prices are independent of the size of the loan.

∞—time replacements

- Bankruptcy is harmful by lowering the credit score.
- Bankrupt households may not declare again for 8 years.
- Nonbankrupt households choose how much debt to hold.
- All households with the same credit score (as determined by bankruptcy history and debt held) and requesting the same size loan face the same asset prices.