

# Functional linear instrumental regression under second order stationarity.

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## Abstract

We consider the problem of estimating the slope parameter in functional linear instrumental regression, where in the presence of an instrument  $W$ , i.e., an exogenous random function, a scalar response  $Y$  is modeled in dependence of an endogenous random function  $X$ . Assuming second order stationarity jointly for  $X$  and  $W$  a nonparametric estimator of the functional slope parameter and its derivatives is proposed based on an  $n$ -sample of  $(Y, X, W)$ . In this paper the minimax optimal rate of convergence of the estimator is derived assuming that the slope parameter belongs to the well-known Sobolev space of periodic functions. We discuss the cases that the cross-covariance operator associated to the random functions  $X$  and  $W$  is finitely, infinitely or in some general form smoothing.

*Keywords:* Functional linear model, Instrument, Orthogonal series estimation, Spectral cut-off, Optimal rate of convergence, Sobolev space.

*JEL classifications:* Primary C14; secondary C30.

## 1 Introduction

The analysis of functional data is becoming very important in a diverse range of disciplines, including medicine, linguistics, chemometrics as well as econometrics (see for instance Ramsay and Silverman [2005] and Ferraty and Vieu [2006], for several case studies). In particular, there is a wide diversity of applications in economics. Forni and Reichlin [1998] study business cycle dynamics and Preda and Saporta [2005] consider shares at the Paris stock exchange, to name but a few. Roughly speaking, in all these applications the dependence of a response variable  $Y$  on the variation of an explanatory random function  $X$  is modeled by a functional linear regression model, that is,

$$Y = \int_0^1 \beta(t)X(t)dt + \sigma U, \quad \sigma > 0, \quad (1.1)$$

for some error term  $U$ . The important point to note here is that often in economical applications the commonly used hypothesis, that the regressor  $X$  is exogenous, can be rejected

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using, for example, a test proposed by Blundell and Horowitz [2007]. Thus analyzing the influence of endogeneity is of particular interest in econometrics. One objective is then to estimate nonparametrically in the presence of an instrument  $W$  the slope function  $\beta$  or its derivatives based on an  $n$ -sample of  $(Y, X, W)$ .

**Background.** Suppose first the regressor  $X$  is exogenous, i.e.,  $\mathbb{E}[UX(s)] = 0$ ,  $s \in [0, 1]$ . In this case the estimation of the slope function  $\beta$  has been considered, for example, in Cardot et al. [2003], Müller and Stadtmüller [2005], Hall and Horowitz [2007] or Crambes et al. [2008]. Assuming the random function  $X$  to be centered the most popular approach is to multiply both sides in (1.1) by  $X(s)$ . Then taking the expectation leads to

$$\mathbb{E}[YX(s)] = \int_0^1 \beta(t) \text{Cov}(X(t), X(s)) dt, \quad s \in [0, 1]. \quad (1.2)$$

The normal equation (1.2) is the continuous equivalent of a normal equation in a classical linear model. To be more precise, suppose the random function  $X$  and the slope function  $\beta$  to be square integrable, then their generalized Fourier coefficients  $X_i := \int_0^1 X(s)\psi(s)ds$  and  $\beta_i := \int_0^1 \beta(s)\psi(s)ds$ ,  $i \in \mathbb{N}$ , with respect to some orthonormal basis  $\{\psi_i\}$  are well-defined. The functional linear model (1.1) (FLM for short) and hence the normal equation (1.2) can be rewritten as

$$Y = \sum_{i=1}^{\infty} \beta_i X_i + \sigma U, \quad \text{and} \quad \mathbb{E}[YX_j] = \sum_{i=1}^{\infty} \beta_i \cdot \text{Cov}(X_i, X_j), \quad j = 1, 2, \dots, \quad (1.3)$$

respectively. Therefore, the FLM (1.1) extends the linear model (LM for short)  $Y = \sum_{i=1}^k \beta_i X_i + \sigma U$ ,  $k \in \mathbb{N}$ , to an infinite number of regressors. Since in analogy to the estimation in the LM recovering from (1.3) the coefficients  $(\beta_i)_{i \in \mathbb{N}}$  necessitates the inversion of the infinite dimensional covariance matrix  $\Sigma_{\infty} := (\text{Cov}(X_i, X_j))_{i,j \in \mathbb{N}}$ , the estimation of  $\beta$  is called an inverse problem. It is well-known that in both, the linear and the functional linear model identification as well as the accuracy of any estimator depends strongly on the properties of the covariance matrix  $\Sigma_k := (\text{Cov}(X_i, X_j))_{i,j=1}^k$  and  $\Sigma_{\infty}$  respectively. That is, in both cases the coefficients can be identified as long as the covariance matrix  $\Sigma_k$  and  $\Sigma_{\infty}$  respectively, is not singular. Moreover, in the LM a high degree of multicollinearity between the regressors  $X_1, \dots, X_k$ , that is, the smallest eigenvalue of  $\Sigma_k$  is close to zero, produces unacceptable uncertainty in the coefficient estimates. However, as long as the covariance matrix  $\Sigma_k$  is not singular an ordinary least squares estimator (LSE for short) will be consistent and leads under fairly weak assumptions to a minimal asymptotic variance. In general the situation in the FLM is different. Since under very mild assumptions zero is an accumulation point of the eigenvalues of  $\Sigma_{\infty}$  we always have to face a classical multicollinearity problem in the presence of many regressors. Therefore, although the covariance matrix  $\Sigma_{\infty}$  is not singular the LSE will not longer be consistent. This corresponds to the setup of ill-posed inverse problems.

In the literature several approaches are proposed in order to circumvent in the FLM the instability issue. Essentially, all of them replace the covariance matrix  $\Sigma_{\infty}$  in equation (1.3) by a regularized version. A popular example is based on the functional principal

components regression (c.f. Bosq [2000], Müller and Stadtmüller [2005] or Cardot et al. [2007]), which corresponds to a method called spectral cut-off in the literature of numerical analysis (c.f. Tautenhahn [1996]). Another example is the Tikhonov regularization (c.f. Hall and Horowitz [2007]), where the regularized solution  $\beta_\alpha$  is defined as unique minimizer of the Tikhonov functional  $F_\alpha(\beta) = \sum_{j=1}^{\infty} \{\mathbb{E}[YX_j] - \sum_{i=1}^{\infty} \beta_i \cdot \text{Cov}(X_i, X_j)\}^2 + \alpha \sum_{j=1}^{\infty} \beta_j^2$  for some strictly positive  $\alpha$ . Regularization using a penalized least squares approach after projection onto some basis (such as splines) is also considered in Ramsay and Dalzell [1991], Eilers and Marx [1996] or Cardot et al. [2003]. The common aspect of all these regularization schemes is the introduction of an additional regularization parameter  $\alpha$  (for example, the parameter determining the weight of the penalty in the Tikhonov functional). The risk of the resulting regularized estimator can then be decomposed, roughly speaking, into a function of the risk of the estimators of  $\mathbb{E}[YX_j]$  and  $\text{Cov}(X_i, X_j)$ ,  $i, j \in \mathbb{N}$ , plus an additional bias term which is a function of the regularization parameter  $\alpha$ . The optimal value of  $\alpha$  is then obtained by balancing these two terms. However, in order to obtain a rate of convergence additional regularity assumptions on the slope function  $\beta$  and the infinite dimensional covariance matrix  $\Sigma_\infty := (\text{Cov}(X_i, X_j))_{i,j \in \mathbb{N}}$  are necessary (a detailed discussion in the context of inverse problems in econometrics can be found in Carrasco et al. [2006] or Johannes et al. [2008]).

The objective of this paper is to study the estimation of the slope function  $\beta$  when the regressor  $X$  is endogenous, which to the best of our knowledge has not yet been considered in the literature. In the following the approach of this paper is described in more details.

**Methodology.** To treat the endogeneity problem, we assume that an instrument  $W$ , i.e., an exogenous random function, is given. Assuming the random function  $W$  to be centered and square integrable, we consider its generalized Fourier coefficients  $W_i := \int W(s)\psi'_i(s)ds$ ,  $i \in \mathbb{N}$ , with respect to some orthonormal basis  $\{\psi'_i\}$  not necessarily the same as  $\{\psi_i\}$  used above in the decomposition of  $X$  and  $\beta$ . Then multiplying the equation (1.1) by  $W_j$  and taking the expectation leads to the normal equation

$$\mathbb{E}[YW_j] = \sum_{i=1}^{\infty} \beta_i \cdot \text{Cov}(X_i, W_j), \quad j = 1, 2, \dots, \quad (1.4)$$

which provides a natural extension of the linear instrumental regression (LIR for short)  $Y = \sum_{i=1}^k \beta_i X_i + \sigma U$  with  $\mathbb{E}[W_j U] = 0$ ,  $j = 1, \dots, q$ , to an infinite number of regressors and instruments. Therefore, in the presence of an instrument  $W$  we call (1.1) functional linear instrumental regression (FLIR for short). The estimation of the coefficients in both, linear and functional linear instrumental regression is then again an inverse problem, since it involves now the inversion of the cross-covariance matrix  $\Sigma_{kq} := (\text{Cov}(X_i, W_j))_{i,j=1}^{k,q}$  and  $\Sigma_{\infty\infty} := (\text{Cov}(X_i, W_j))_{i,j \in \mathbb{N}}$  respectively. Furthermore, in both cases the coefficients are identifiable as long as  $\Sigma_{kq}$  and  $\Sigma_{\infty\infty}$  respectively, is not singular and moreover, the obtainable accuracy of any estimator depends now on the properties  $\Sigma_{kq}$  and  $\Sigma_{\infty\infty}$  respectively. It is worth to pointing out that the FLIR parallels recent developments in econometric theory such as nonparametric instrumental regression (c.f. Darolles et al. [2002], Newey and Powell [2003] or Hall and Horowitz [2005]), nonparametric instrumental quantile re-

gression of Horowitz and Lee [2007] or semi-nonparametric estimation of Engel curve with shape-invariant specification of Blundell et al. [2007].

The estimator of the slope function in FLIR considered in this paper is based on a two stage least squares approach. To be more precise, consider first the LIR. Then as long as the cross-covariance matrix  $\Sigma_{kq}$  is not singular a two stage least squares procedure (2SLS for short) will lead to a consistent estimator. That is, in a first step a linear regression of the endogenous vector  $X = (X_1, \dots, X_k)^t$  onto the vector of instruments  $W = (W_1, \dots, W_q)^t$  is performed, resulting into an estimator  $\widehat{W}$  of the optimal linear instrument  $\widetilde{W}$ , i.e., the best linear predictor  $\widetilde{W} := \Sigma_{kq} \Sigma_q^{-1} W$  of  $X$  with  $\Sigma_q := (\text{Cov}(W_i, W_j))_{i,j=1}^q$ . Note that the optimal linear instrument is well-defined as long as the covariance matrix  $\Sigma_q$  of  $W$  has full rank. Then in the second step an estimator of the  $k$ -vector of coefficients  $(\beta_j)$  is obtained considering a linear regression of  $Y$  onto  $\widehat{W}$ . Applying a 2SLS approach in FLIR we have to face additional technical difficulties given through the facts that the optimal linear instrument, i.e., the best linear predictor of the random function  $X$  given the random function  $W$ , is not always well-defined and that both stages of the estimation procedure necessitate the solution of an ill-posed inverse problem (see the discussion above in case of an exogenous regressor). Therefore, assuming the optimal linear instrument is well-defined, we apply in each stage a regularization scheme in order to circumvent the instability issue. Although the estimation in the first step has to be stabilized, it has only a minor influence on the obtainable accuracy of the final estimator. Particularly, the proposed estimator of  $\widetilde{W}$  will in general not be optimal. The main complexity of the estimation problem is contained in the second stage. To be more precise, if the optimal linear instrument  $\widetilde{W}$  is given, then the second step in fact only consists of the estimation in a FLM (1.1) given now with exogenous regressor  $\widetilde{W}$ . Thereby, the relationship between the regularity assumption on the slope function  $\beta$  and the infinite dimensional covariance matrix  $\widetilde{\Sigma}_\infty := (\text{Cov}(\widetilde{W}_i, \widetilde{W}_j))_{i,j \in \mathbb{N}}$  associated to the instrument  $\widetilde{W}$  determines the obtainable accuracy of any estimator of  $\beta$  (see also the discussion above in case of an exogenous regressor). Nevertheless, the instrument  $\widetilde{W}$  is not given and thus has to be estimated. However, the estimation in the first step is possible without changing the optimal rate of the estimator of  $\beta$ , where only higher moment conditions are the price to pay.

Suppose the slope function  $\beta$  belongs to the Sobolev space of periodic functions  $\mathcal{W}_p$  (defined below). Given an  $n$ -sample of  $(Y, X, W)$  our objective is not only the estimation of the slope function  $\beta$  itself but also of its derivatives. We show that the relationship between the Sobolev spaces and the covariance matrix  $\widetilde{\Sigma}_\infty$  associated to the optimal linear instrument  $\widetilde{W}$ , i.e., the “smoothing” property of  $\widetilde{\Sigma}_\infty$ , is essentially determining the optimal rate of convergence of any estimator. We now describe two examples. First consider the covariance matrix  $\widetilde{\Sigma}_\infty$  to be finitely smoothing, that is, the range of  $\widetilde{\Sigma}_\infty$  equals  $\mathcal{W}_a$  for some  $a > 0$ . Then the optimal rate is a polynomial of the sample size  $n$ . It is worth to note that all published results in the FLM with exogenous regressor consider only this case (c.f. Hall and Horowitz [2007] and Crambes et al. [2008]). However, assuming  $\widetilde{\Sigma}_\infty$  to be finitely smoothing excludes several interesting situations, such as our second example. Suppose  $\widetilde{\Sigma}_\infty$  to be infinitely smoothing, that is, the range of  $|\log(\widetilde{\Sigma}_\infty)|^{-1}$  equals  $\mathcal{W}_a$  for some  $a > 0$ . Then the optimal rate is a logarithm of the sample size  $n$ . The important point to note

here is the theory behind these cases can be generalized by using an index function  $\kappa$  (c.f. Nair et al. [2005]), which ‘links’ the range of  $\tilde{\Sigma}_\infty$  and the Sobolev spaces. Then  $\tilde{\Sigma}_\infty$  is called in some general form smoothing and moreover the index function  $\kappa$  determines the functional form of the optimal rate of convergence. A similar approach in the context of nonparametric instrumental regression, where the conditional expectation plays the same role as the covariance matrix  $\tilde{\Sigma}_\infty$ , can be found in Chen and Reiß [2008] or Johannes et al. [2008].

In this paper we deal with the estimation of the slope function when the regressor  $X$  and the instrument  $W$  are jointly second order stationary (defined below). We derive a lower bound of the rate of convergence for any estimator of  $\beta$  or its derivatives assuming some general form of smoothing of  $\tilde{\Sigma}_\infty$ . Assuming second order stationarity we propose an orthogonal series estimator of  $\beta$  and its derivatives based on a spectral cut-off (thresholding in the Frequency domain). Then we show that the rate of the lower bound provides also an upper bound for the risk of the orthogonal series estimator. Therefore, the rate is optimal and hence the proposed estimator is minimax-optimal. The results for general smoothing  $\tilde{\Sigma}_\infty$  imply then as propositions the minimax optimal rate of convergence in estimating  $\beta$  and its derivatives respectively in case of finitely as well as infinitely smoothing  $\tilde{\Sigma}_\infty$ .

**Organization of the paper.** We summarize in Section 2 the model assumptions and define the estimator of  $\beta$  and its derivatives. In Section 3 we provide minimal conditions to ensure consistency of the estimator. Furthermore, we derive a lower and an upper bound for the risk in the Sobolev norm when  $\tilde{\Sigma}_\infty$  is in some general form smoothing. This results are illustrated in Section 4 assuming  $\tilde{\Sigma}_\infty$  to be finitely or infinitely smoothing and Section 5 concludes. All proofs can be found in the Appendix.

## 2 Formalization of the model and definition of the estimator

**Model.** The setting of this paper can be summarized through the model

$$Y = \int_0^1 \beta(t)X(t)dt + \sigma U, \quad \sigma > 0, \quad (2.1a)$$

where  $Y \in \mathbb{R}$  is a response variable, the endogenous random function  $X$  is defined on the interval  $[0, 1]$  and  $U$  is a centered error term with variance one such that

$$\mathbb{E}[UW(t)] = 0, \quad t \in [0, 1] \quad (2.1b)$$

for some instrument  $W$ , i.e., an exogenous random function defined also on  $[0, 1]$ . The objective is the nonparametric estimation of the slope function  $\beta$  and its derivatives based on a  $n$ -sample of  $(Y, X, W)$ . We assume throughout the paper that the random functions  $X$  and  $W$  are defined on the interval  $[0, 1]$  that (technically) simplifies the notations. Of course, it does not touch the applicability of the model and suggested estimator in a general setting when  $X$  and  $W$  are defined on some compact intervals  $I_1$  and  $I_2$ , respectively. Moreover, we suppose that the random functions  $X$  and  $W$  have a finite second moment, i.e.,  $\int_0^1 \mathbb{E}|X(t)|^2 dt < \infty$  and  $\int_0^1 \mathbb{E}|W(t)|^2 dt < \infty$ . In order to simplify the presentation we

assume that the mean function of  $X$  and  $W$  are zero. Then multiplying both sides in (2.1a) by  $W(s)$ ,  $s \in [0, 1]$ , and taking the expectation leads to

$$\mathbb{E}[YW(s)] = \int_0^1 \beta(t) \text{Cov}[X(t), W(s)] dt =: [T_{WX}\beta](s), \quad s \in [0, 1], \quad (2.2)$$

where the function  $\mathbb{E}[YW(\cdot)]$  is square integrable and  $T_{WX}$  denotes the cross-covariance operator associated to the random functions  $X$  and  $W$ . Note that the cross-covariance matrix  $\Sigma_{\infty\infty}$  considered in the introduction satisfies  $\Sigma_{\infty\infty} = (\int_0^1 [T_{WX}\psi_i](s)\psi_j'(s)ds)_{i,j \in \mathbb{N}}$ . Estimation of  $\beta$  is thus linked with the inversion of the cross-covariance operator  $T_{WX}$  of  $(X, W)$ , and, hence called an inverse problem. Throughout the paper we require the following assumption, which provides a necessary and sufficient condition for the existence of a unique solution of equation (2.2).

**ASSUMPTION 2.1.** *The cross-covariance operator  $T_{WX}$  associated to the random functions  $X$  and  $W$  is injective and the function  $\mathbb{E}[YW(\cdot)]$  belongs to the range  $\mathcal{R}(T_{WX})$  of  $T_{WX}$ .*

In case a solution of the normal equation (2.2) does not exist all the results below can also straightforward be obtained for the unique least-square solution with minimal norm, which exists if only if  $\mathbb{E}[YW(\cdot)]$  is contained in the direct sum of  $\mathcal{R}(T_{WX})$  and its orthogonal complement  $\mathcal{R}(T_{WX})^\perp$  (for a definition and detailed discussion in the context of inverse problems c.f. Engl et al. [2000] or Carrasco et al. [2006]).

**Notations and basic assumptions.** In this paper we suppose that the random function  $(X, W)$  is second order stationary and, hence there exists a function  $c_{WX} : [-1, 1] \rightarrow \mathbb{R}$  such that  $\text{Cov}[X(t), W(s)] = c_{WX}(s - t)$ ,  $t, s \in [0, 1]$ . Notice that due to the finite second moment of  $X$  and  $W$  the cross-covariance function  $c_{WX}(\cdot)$  is square integrable. Therefore, its Fourier coefficients with respect to the Fourier complex exponentials, i.e.,

$$c_k := \int_{-1}^1 c_{WX}(t) \exp(-2\pi kit) dt, \quad \text{for all } k \in \mathbb{Z}, \quad (2.3)$$

are well-defined and by applying the well-known convolution theorem we have

$$T_{WX}\phi_k = c_k \cdot \phi_k \quad \text{with} \quad \phi_k(t) := \exp(2\pi kit), \quad t \in [0, 1], \quad \text{for all } k \in \mathbb{Z}. \quad (2.4)$$

Thereby, it is convenient to consider the real-valued random functions  $X$  and  $W$  as elements of the Hilbert space  $L^2[0, 1]$  of square integrable complex valued functions defined on  $[0, 1]$ , which is endowed with inner product  $\langle f, g \rangle = \int_0^1 f(t)\overline{g(t)}dt$  and associated norm  $\|f\| = \langle f, f \rangle^{1/2}$ ,  $f, g \in L^2[0, 1]$ . Here and subsequently,  $\overline{g(t)}$  denotes the complex conjugate of  $g(t)$ . Furthermore, the cross-covariance operator  $T_{WX}$  is a well-defined mapping from  $L^2[0, 1]$  into itself. Consider the centered complex valued random variables  $\langle \phi_k, X \rangle$  and  $\langle W, \phi_k \rangle$ ,  $k \in \mathbb{Z}$ , which due to the identity (2.4) satisfy

$$c_k = \mathbb{E}[\langle \phi_k, X \rangle \langle W, \phi_k \rangle] \quad \text{and} \quad 0 = \mathbb{E}[\langle \phi_k, X \rangle \langle W, \phi_j \rangle] \quad \text{for all } j \neq k. \quad (2.5)$$

Now an equivalent formulation of Assumption 2.1 is given by

$$|c_k|^2 > 0, \quad \text{for all } k \in \mathbb{Z} \quad \text{and} \quad \sum_{k \in \mathbb{Z}} \frac{|\mathbb{E}[Y\langle W, \phi_k \rangle]|^2}{|c_k|^2} < \infty. \quad (2.6)$$

**Optimal linear instrument.** Let  $x_k := \text{Var}\langle X, \phi_k \rangle$ ,  $w_k := \text{Var}\langle W, \phi_k \rangle$  and define  $\lambda_k := c_k^2/w_k \leq x_k$ ,  $k \in \mathbb{Z}$ , where due to the finite second moment of  $X$ , i.e.,  $\mathbb{E}\|X\|^2 = \sum_{k \in \mathbb{Z}} x_k$ , the sequences  $(\lambda_k)_{k \in \mathbb{Z}}$  is summable. If we further assume that  $\sup_{k \in \mathbb{Z}} |\lambda_k/w_k| < \infty$ , then the complex valued random function

$$\widetilde{W} := \ell(W) := \sum_{k \in \mathbb{Z}} \frac{\bar{c}_k}{w_k} \cdot \langle W, \phi_k \rangle \cdot \phi_k \quad (2.7)$$

is well-defined, i.e.,  $\|\widetilde{W}\| < \infty$ . Note that  $\ell$  is a linear operator mapping  $L^2[0, 1]$  into itself. If in addition  $\sum_{k \in \mathbb{Z}} \lambda_k/w_k < \infty$ , then  $\ell$  is a Hilbert-Schmidt operator and  $\widetilde{W} = \ell(W)$  is the best linear predictor of  $X$  based on  $W$ . That is,  $\ell$  minimizes the mean prediction error  $\mathbb{E}\|X - \ell'(W)\|^2$  over all Hilbert-Schmidt operator  $\ell'$  (c.f. Bosq [2000]). Therefore, we call  $\widetilde{W}$  optimal linear instrument. Throughout the paper we suppose the linear predictor  $\widetilde{W}$  is well-defined, i.e.,  $\sup_{k \in \mathbb{Z}} |\lambda_k/w_k| < \infty$ , which implies an additional restriction on the behavior of the sequences  $(\lambda_k)_{k \in \mathbb{Z}}$  and  $(w_k)_{k \in \mathbb{Z}}$  as  $|k| \rightarrow \infty$ . In particular the sequence of variances  $(w_k)_{k \in \mathbb{Z}}$  associated to the instrument  $W$  has to tend slower to zero than the sequence  $(c_k)_{k \in \mathbb{Z}}$  of cross-covariances associated to  $X$  and  $W$ . Note that although we suppose the optimal linear instrument  $\widetilde{W}$  exist, in general it is not known to the econometrician.

**Moment assumptions.** The results derived below involve additional conditions on the moments of the random functions  $X$  and  $W$  and the error term  $U$ , which we formalize now. Let  $\mathcal{F}$  be the set of all centered second order stationary random functions  $(X, W)$ . Here and subsequently,  $\mathcal{F}_{\eta, \tau}^m$ ,  $m \in \mathbb{N}$ ,  $\eta, \tau \geq 1$ , denotes the subset of  $\mathcal{F}$  containing all random functions  $(X, W)$  such that the  $m$ -th moment of the corresponding random variables  $\{\langle X, \phi_k \rangle / \sqrt{x_k}\}$  and  $\{\langle W, \phi_k \rangle / \sqrt{w_k}\}$  are uniformly bounded and such that the linear predictor of  $X$  based on  $W$  is well-defined, that is

$$\mathcal{F}_{\eta, \tau}^m := \left\{ (X, W) \in \mathcal{F} \text{ with } \sup_{k \in \mathbb{Z}} \mathbb{E} \left| \frac{\langle X, \phi_k \rangle}{\sqrt{x_k}} \right|^m \leq \eta \text{ and } \sup_{k \in \mathbb{Z}} \mathbb{E} \left| \frac{\langle W, \phi_k \rangle}{\sqrt{w_k}} \right|^m \leq \eta \right. \\ \left. \text{and associated values } (\lambda_k)_{k \in \mathbb{Z}} \text{ such that } 1 \vee \sup_{k \in \mathbb{Z}} |\lambda_k/w_k| \leq \tau \right\}. \quad (2.8)$$

In what follows,  $\mathcal{E}_\eta^m$  stands for the set of all centered error terms  $U$  with variance one and finite  $m$ -th moment, i.e.,  $\mathbb{E}|U|^m \leq \eta$ .

**Estimation of  $\beta$  as an ill-posed inverse problem.** Consider the optimal linear instrument  $\widetilde{W}$  defined in (2.7), then due to Assumption 2.1 the normal equation (2.2) implies

$$\beta = \sum_{k \in \mathbb{Z}} \frac{g_k}{\lambda_k} \cdot \phi_k \quad \text{with } g_k := \langle g, \phi_k \rangle, \quad k \in \mathbb{Z}, \quad \text{and } g := \mathbb{E}[Y \widetilde{W}(\cdot)]. \quad (2.9)$$

Moreover,  $\lambda_k = \mathbb{E}|\langle \widetilde{W}, \phi_k \rangle|^2$ ,  $k \in \mathbb{Z}$ , are the eigenvalues of the covariance operator  $T_{\widetilde{W}}$  associated to  $\widetilde{W}$ . In other words, the estimation of  $\beta$  necessitates the inversion of the covariance operator  $T_{\widetilde{W}}$ . Accordingly, replacing in (2.9) the unknown function  $g$  by a consistent estimator  $\widehat{g}$  does in general not lead to a consistent estimator of  $\beta$  even in case of known values  $\{\lambda_k\}$ . To be more precise, since the sequence  $(\lambda_k)_{k \in \mathbb{Z}}$  tends to zero as  $|k| \rightarrow \infty$ ,

$\mathbb{E}\|\widehat{g}-g\|^2 = o(1)$  does generally not imply  $\sum_{k \in \mathbb{Z}}^\infty |\lambda_k|^{-2} \cdot \mathbb{E}|\langle \widehat{g}-g, \phi_k \rangle|^2 = o(1)$ . Consequently, the estimation in FLIR is called ill-posed and additional regularity assumptions on the slope function  $\beta$  are necessary in order to obtain a uniform rate of convergence (c.f. Engl et al. [2000]).

In this paper we assume that the slope function  $\beta$  belongs to the well-known Sobolev space  $\mathcal{W}_\nu$ ,  $\nu > 0$ , of periodic functions, which can be defined for  $\nu \in \mathbb{R}$  by

$$\mathcal{W}_\nu := \left\{ f \in L^2[0, 1] : \|f\|_\nu^2 := \sum_{k \in \mathbb{Z}} \gamma_k^\nu |\langle f, \phi_k \rangle|^2 < \infty \right\}, \quad (2.10)$$

where  $\{\phi_k\}$  are the complex exponentials given in (2.4) and the weights  $\{\gamma_k\}$  satisfy

$$\gamma_k = 1 + |2\pi k|^2, \quad k \in \mathbb{Z}. \quad (2.11)$$

Let  $\mathcal{W}_\nu^\rho := \{f \in \mathcal{W}_\nu : \|f\|_\nu^2 \leq \rho\}$  for  $\rho > 0$ . Notice that for integer  $\nu \in \mathbb{N}$  the Sobolev space of periodic functions  $\mathcal{W}_\nu$  is equivalently given by

$$\mathcal{W}_\nu = \left\{ f \in H_\nu : f^{(j)}(0) = f^{(j)}(1), \quad j = 0, 1, \dots, \nu - 1 \right\},$$

where  $H_\nu := \{f \in L^2[0, 1] : f^{(\nu-1)} \text{ absolutely continuous, } f^{(\nu)} \in L^2[0, 1]\}$  is a Sobolev space (c.f. Neubauer [1988a,b], Mair and Ruymgaart [1996] or Tsybakov [2004]).

In the literature several approaches are proposed in order to circumvent an instability issue due to the inversion of the covariance operator (for a detailed discussion in the context of inverse problems in econometrics we refer e.g. to Carrasco et al. [2006] and Johannes et al. [2008]). Essentially, all of them replace equation (2.9) by a regularized version which avoids that the denominator becomes too small. For example, Hall and Horowitz [2007] use in a functional linear model with exogenous regressor a Tikhonov regularization. There is a large number of alternative regularization schemes in the numerical analysis literature available like the iterative Tikhonov regularization, Landweber iteration or the  $\nu$ -Method to name but a few (c.f. Engl et al. [2000]). However, in this paper we regularize equation (2.9) by introducing a threshold  $\alpha > 0$  and weights  $\{\gamma_k\}$  defined in (2.11). For  $\nu \geq 0$  we consider the regularized version  $\beta_\nu$  given by

$$\beta_\nu := \sum_{k \in \mathbb{Z}} \frac{g_k}{\lambda_k} \cdot \mathbb{1}\{\lambda_k / \gamma_k^\nu \geq \alpha\} \cdot \phi_k, \quad (2.12)$$

which obviously belongs to the Sobolev space  $\mathcal{W}_\nu$ . Thresholding in the Fourier domain in this situation is new, however has been used, for example, in a deconvolution problem in Mair and Ruymgaart [1996], Neumann [1997] or Johannes [2007] and coincides with an approach called spectral cut-off in the numerical analysis literature (c.f. Tautenhahn [1996]).

**Definition of the estimator.** Let  $(Y_1, X_1, W_1), \dots, (Y_n, X_n, W_n)$  be an i.i.d. sample of  $(Y, X, W)$ , which we use in a first step to construct an estimator  $\widehat{W}_i$  of the optimal linear instrument  $\widetilde{W}_i$ ,  $i = 1, \dots, n$ , exploiting the identity (2.7). Consider the unbiased estimator of  $c_k = \mathbb{E}\langle \phi_k, X \rangle \langle W, \phi_k \rangle$  and  $w_k = \mathbb{E}|\langle W, \phi_k \rangle|^2$  given by

$$\widehat{c}_k := \frac{1}{n} \sum_{i=1}^n \langle \phi_k, X_i \rangle \langle W_i, \phi_k \rangle \quad \text{and} \quad \widehat{w}_k := \frac{1}{n} \sum_{i=1}^n |\langle W_i, \phi_k \rangle|^2, \quad k \in \mathbb{Z}, \quad (2.13)$$



respectively. Then we define the estimator of  $\widetilde{W}_i$  by regularizing equation (2.7), i.e., by introducing a threshold  $\alpha > 0$ , that is

$$\widehat{W}_i := \sum_{k \in \mathbb{Z}} \frac{\widehat{c}_k}{\widehat{w}_k} \cdot \mathbb{1}\{\widehat{w}_k \geq \alpha\} \cdot \langle W_i, \phi_k \rangle \cdot \phi_k, \quad i = 1, \dots, n, \quad (2.14)$$

where the threshold  $\alpha = \alpha(n)$  has to tend to zero as the sample size  $n$  increases. In a second step we use the estimated optimal linear instrument to construct an estimator of  $\beta$  based on the decomposition (2.9). Consider the identities  $\lambda_k = \mathbb{E}|\langle \widetilde{W}, \phi_k \rangle|^2$  and  $g_k = \mathbb{E}[Y \langle \widetilde{W}, \phi_k \rangle]$ , which motivate the estimators defined by

$$\widehat{\lambda}_k := \frac{1}{n} \sum_{i=1}^n |\langle \widehat{W}_i, \phi_k \rangle|^2 \quad \text{and} \quad \widehat{g}_k := \frac{1}{n} \sum_{i=1}^n Y_i \cdot \langle \widehat{W}_i, \phi_k \rangle, \quad k \in \mathbb{Z}. \quad (2.15)$$

Finally, the estimator  $\widehat{\beta}_\nu$  of  $\beta$  is based on the regularized version (2.12). That is,

$$\widehat{\beta}_\nu := \sum_{k \in \mathbb{Z}} \frac{\widehat{g}_k}{\widehat{\lambda}_k} \cdot \mathbb{1}\{\widehat{\lambda}_k / \gamma_k^\nu \geq \alpha\} \cdot \phi_k, \quad (2.16)$$

which obviously belongs also to the Sobolev space  $\mathcal{W}_\nu$ . It is worth pointing out that due to Parseval's formula  $\sum_{k \in \mathbb{Z}} \widehat{w}_k = \frac{1}{n} \sum_{i=1}^n \|W_i\|^2$  is finite. Thereby, the sum in (2.14) contains only a finite but random number of nonzero summands, and hence only a finite number of values  $\widehat{\lambda}_k$  are nonzero. Consequently, the sum in (2.16) contains only a finite number of nonzero summands. We emphasize that the same threshold is used in the definition of  $\widehat{W}_i$  and  $\widehat{\beta}_\nu$  given in (2.14) and (2.16) respectively. In general this will not lead to an optimal estimator of  $\widetilde{W}_i$ , however as we will see below, it is sufficient to ensure the optimality of  $\widehat{\beta}_\nu$ .

### 3 Optimal estimation of slope function and its derivatives

We shall measure the performance of the estimator  $\widehat{\beta}_\nu$  defined in (2.16) by the  $\mathcal{W}_\nu$ -risk, that is  $\mathbb{E}\|\widehat{\beta}_\nu - \beta\|_\nu^2$ , provided that  $\beta \in \mathcal{W}_p$  for some  $p \geq \nu \geq 0$ . For an integer  $\nu$  the Sobolev norm  $\|g\|_\nu$  is equivalent to  $\|g\| + \|g^{(\nu)}\|$ , where  $g^{(\nu)}$  denotes the  $\nu$ -th derivative of  $g$  in a weak sense. Consequently, the  $\mathcal{W}_\nu$ -risk reflects the performance of  $\widehat{\beta}_\nu$  and  $\widehat{\beta}_\nu^{(\nu)}$  as estimator of  $\beta$  and  $\beta^{(\nu)}$  respectively.

The  $\mathcal{W}_\nu$ -risk is essentially determined by the deviation of the estimators of  $g_k$  and  $\lambda_k$ ,  $k \in \mathbb{Z}$ , and by the regularization error due to the spectral cut-off. In fact, if

$$\widetilde{\beta}_\nu^\alpha := \sum_{k \in \mathbb{Z}} \beta_k \cdot \mathbb{1}\{\widehat{\lambda}_k / \gamma_k^\nu \geq \alpha\} \cdot \phi_k \quad \text{with} \quad \beta_k := \langle \beta, \phi_k \rangle, \quad k \in \mathbb{Z}, \quad (3.1)$$

then by assuming  $\beta \in \mathcal{W}_p$  for some  $p \geq \nu \geq 0$  we bound the  $\mathcal{W}_\nu$ -risk of  $\widehat{\beta}_\nu$  by

$$\mathbb{E}\|\widehat{\beta}_\nu - \beta\|_\nu^2 \leq 2\{\mathbb{E}\|\widehat{\beta}_\nu - \widetilde{\beta}_\nu^\alpha\|_\nu^2 + \mathbb{E}\|\widetilde{\beta}_\nu^\alpha - \beta\|_\nu^2\}. \quad (3.2)$$

Under the moment condition  $(X, W) \in \mathcal{F}_{\eta, \tau}^8$  defined in (2.8) and  $U \in \mathcal{E}_\eta^4$  we show in the proof of the next proposition that  $\mathbb{E}\|\widehat{\beta}_\nu - \widetilde{\beta}_\nu^\alpha\|_\nu^2$  is bounded up to a universal constant by  $(\alpha^2 \cdot n)^{-1} \cdot \{\sigma^2 + \|\beta\|^2 \cdot \mathbb{E}\|X\|^2\} \cdot \mathbb{E}\|W\|^2 \cdot \eta$  and that the regularization error satisfies  $\mathbb{E}\|\widetilde{\beta}_\nu^\alpha - \beta\|_\nu^2 = o(1)$  provided  $\alpha = o(1)$  and  $(\alpha \cdot n)^{-1} = o(1)$  as  $n \rightarrow \infty$ . The next assertion summarizes the minimal conditions to ensure consistency of the proposed estimator.

**PROPOSITION 3.1 (Consistency).** *Let  $\beta \in \mathcal{W}_p$ ,  $p \geq 0$ . Consider for  $0 \leq \nu \leq p$  the estimator  $\widehat{\beta}_\nu$  given in (2.16) with threshold satisfying  $\alpha = o(1)$  and  $(\alpha^2 \cdot n)^{-1} = o(1)$  as  $n \rightarrow \infty$ . If in addition  $(X, W) \in \mathcal{F}_\eta^8$  and  $U \in \mathcal{E}_\eta^4$ , then we have  $\mathbb{E}\|\widehat{\beta}_\nu - \beta\|_\nu^2 = o(1)$  as  $n \rightarrow \infty$ .*

**REMARK 3.1.** The last result covers the case  $0 = \nu = p$ , i.e., the estimator of  $\beta$  is consistent without an additional smoothness assumption on  $\beta$ . However,  $\widehat{\beta}'_1$  is a consistent estimator of  $\beta'$ , only if  $\beta$  is differentiable, i.e.,  $\beta \in \mathcal{W}_p$ ,  $p \geq 1$ .  $\square$

**Link condition.** In order to obtain a rate of convergence of the regularization error and hence the  $\mathcal{W}_\nu$ -risk we link the smoothness condition on  $\beta$ , i.e., the Sobolev space  $\mathcal{W}_p$ , and the values  $\{\lambda_k\}$  associated to the cross-covariance function of  $(X, W)$ . In fact, the obtainable rate of convergence is essentially determined by the decay of  $(\lambda_k)_{k \in \mathbb{Z}}$  as  $|k| \rightarrow \infty$ , which we first allow to have a general form. Notice that due to the finite second moment of  $X$  the sequence  $(\lambda_k)_{k \in \mathbb{Z}}$  belongs to the set  $\ell_1^+$  of nonnegative summable sequences, i.e.,  $\sum_{k \in \mathbb{Z}} \lambda_k < \infty$ . Thereby,  $\lambda_+ := 1 \vee \max_{k \in \mathbb{Z}} \lambda_k$  is finite and the rescaled sequence  $(\lambda_k/\lambda_+)_{k \in \mathbb{Z}}$  is taking only values in  $(0, 1]$ . It is convenient to choose an index function  $\kappa : (0, 1] \rightarrow \mathbb{R}^+$  (c.f. Nair et al. [2005]), which we always assume here to be a continuous and strictly increasing function with  $\kappa(0+) = 0$ . Then, we require that the sequence  $(\lambda_k/\lambda_+)_{k \in \mathbb{Z}}$  is an element of the subset  $\mathcal{S}_{\kappa, d}$  of  $\ell_1^+$  defined for  $d \geq 1$  by

$$\mathcal{S}_{\kappa, d} := \left\{ (\lambda_k) \in \ell_1^+ : \kappa\left(\frac{\lambda_k}{d \gamma_k^\nu \lambda_+}\right) \leq \gamma_k^{\nu-p} \leq \kappa\left(\frac{d \lambda_k}{\gamma_k^\nu \lambda_+}\right), k \in \mathbb{Z} \right\}, \quad (3.3)$$

where the weights  $\{\gamma_k\}$  are given in (2.11). First we consider this general class of values  $\{\lambda_k\}$ . However, we illustrate condition (3.3) in Section 4 by assuming a ‘‘regular decay’’.

The lower bound as well as the upper bound of the  $\mathcal{W}_\nu$ -risk derived below involve additional conditions on the moments of the random function  $(X, W)$ , which are formalized by using the set  $\mathcal{F}_{\eta, \tau}^m$  defined in (2.8). We suppose in what follows that for some index function  $\kappa(\cdot)$  the random function  $(X, W)$  belongs to the subset  $\mathcal{F}_\kappa^m$  of  $\mathcal{F}_{\eta, \tau}^m$  given by

$$\mathcal{F}_\kappa^m := \left\{ (X, W) \in \mathcal{F}_{\eta, \tau}^m \text{ with associated values } (\lambda_k)_{k \in \mathbb{Z}} \in \mathcal{S}_{\kappa, d} \right. \\ \left. \text{and such that } \mathbb{E}\|X\|^2 \leq \Lambda, \mathbb{E}\|W\|^2 \leq \Lambda \right\} \quad (3.4)$$

for some constants  $d, \eta, \tau, \Lambda \geq 1$  and  $m \in \mathbb{N}$ .

**The lower bound.** It is well-known that in general the hardest one-dimensional subproblem does not capture the full difficulty in estimating the solution of an inverse problem even in case of a known operator (for details see e.g. the proof in Mair and Ruymgaart [1996]). In other words, there does not exist two sequences of slope functions  $\beta_{1, n}, \beta_{2, n} \in \mathcal{W}_p^0$ , which are statistically not consistently distinguishable and which satisfy  $\|\beta_{1, n} - \beta_{2, n}\|_\nu^2 \geq C\psi_n$ , where  $\psi_n$  is the optimal rate of convergence. Therefore we need to consider subsets of  $\mathcal{W}_p^0$  with growing number of elements in order to get the optimal lower bound. More specific, we obtain the following lower bound by applying Assouad’s cube technique (see e.g. Korostolev and Tsybakov [1993] or Chen and Reiß [2008]).

**THEOREM 3.2.** *Assume an  $n$ -sample of  $(Y, X, W)$  satisfying (2.1a) and (2.1b) with  $\sigma > 0$ . Consider  $\mathcal{W}_p^\rho$ ,  $p, \rho > 0$ , as set of slope functions,  $U \in \mathcal{E}_\eta^l$ ,  $l \in \mathbb{N}$ , as set of error terms and  $\mathcal{F}_\kappa^m$ ,  $m \in \mathbb{N}$ , as class of regressors defined in (3.4) for an arbitrary index function  $\kappa$ , constants  $d, \eta, \tau, \Lambda \geq 1$  and  $0 \leq \nu < p$ . Denote by  $\varphi$  the inverse function of  $\kappa$ . Let  $k^* := k^*(n) \in \mathbb{N}$  and  $\delta^* := \delta^*(n) \in (0, 1]$  for some  $\Delta \geq 1$  be chosen such that*

$$\Delta^{-1} \leq \sum_{|k| \leq k^*} \frac{\gamma_{k^*}^{p-\nu}}{n \cdot \varphi(\gamma_k^{\nu-p})} \leq \Delta \quad \text{and} \quad \delta^* := \varphi(\gamma_{k^*}^{\nu-p}). \quad (3.5)$$

*If we assume in addition that  $\eta$  is sufficiently large, then*

$$\inf_{\tilde{\beta}} \sup_{\beta \in \mathcal{W}_p^\rho, (X, W) \in \mathcal{F}_\kappa^m, U \in \mathcal{E}_\eta^l} \left\{ \mathbb{E} \|\tilde{\beta} - \beta\|_\nu^2 \right\} \geq \frac{1}{4} \cdot \min \left\{ \frac{\sigma^2}{d \Delta}, \frac{\rho}{2} \right\} \cdot \frac{\kappa(\delta^*)}{\Lambda}.$$

**REMARK 3.2.** The lower bound in the last result is obtained under the assumption that the class of regressors  $\mathcal{F}_\kappa^m$  and the class of error terms  $\mathcal{E}_\eta^l$  provide a certain complexity, i.e., the uniform bound  $\eta$  allows the moments of  $(X, W)$  to be sufficiently large. In fact, we ensure that for certain slope functions  $\beta \in \mathcal{W}_p^\rho$ , the conditional distribution of the linear prediction error  $Y - \langle \beta, \tilde{W} \rangle$  given the optimal linear instrument  $\tilde{W}$  can be chosen to be Gaussian. This assumption is only needed to simplify the calculation of the distance between distributions corresponding to different slope functions.  $\square$

**The upper bound.** In the following theorem we provide an upper bound for the estimator  $\hat{\beta}_\nu$  defined in (2.16) assuming an index function  $\kappa$  with the additional property that

$$\text{for all } c \geq 1 : \quad \frac{\kappa(c \cdot t)}{\kappa(t)} = O(1) \quad \text{and} \quad \frac{\kappa(t)}{\kappa(t/c)} = O(1) \quad \text{as } t \rightarrow 0. \quad (3.6)$$

The next theorem states that the rate  $\kappa(\delta^*)$  of the lower bound given in Theorem 3.2 provides also an upper bound of the proposed estimator  $\hat{\beta}_\nu$ . We have thus proved that the rate  $\kappa(\delta^*)$  is optimal and hence the estimator  $\hat{\beta}_\nu$  is minimax optimal.

**THEOREM 3.3.** *Assume an  $n$ -sample of  $(Y, X, W)$  satisfying (2.1a) and (2.1b) with  $\sigma > 0$ . Consider  $\mathcal{W}_p^\rho$ ,  $p, \rho > 0$ , as set of slope functions,  $U \in \mathcal{E}_\eta^l$ ,  $l \geq 16$ , as set of error terms and  $\mathcal{F}_\kappa^m$ ,  $m \geq 32$ , as class of regressors defined in (3.4) for an index function  $\kappa$  satisfying (3.6), some constants  $d, \eta, \tau, \Lambda \geq 1$  and  $0 \leq \nu < p$ . Let  $\hat{\beta}_\nu$  be the estimator defined in (2.16). If in addition the threshold  $\alpha := \alpha(n)$  satisfies  $\alpha = 8 d \Lambda \delta^*$ , where  $\delta^* \in (0, 1]$  is given in (3.5) for some  $\Delta \geq 1$ , then we have*

$$\sup_{\beta \in \mathcal{W}_p^\rho, (X, W) \in \mathcal{F}_\kappa^m, U \in \mathcal{E}_\eta^l} \mathbb{E} \|\hat{\beta}_\nu - \beta\|_\nu^2 \leq C \eta d \Delta \cdot [\sigma^2 + \rho \Lambda] \cdot [\Delta \Lambda \kappa(\delta^*) + 1]^4 \cdot \kappa(\delta^*),$$

*where the constant  $C > 0$  does only depend on  $\kappa$ .*

**REMARK 3.3.** We would like to stress, that for integer  $\nu < p$  the Theorem 3.2 and 3.3 show together that  $\kappa(\delta^*)$  is the optimal rate of convergence for the estimation of the  $\nu$ -th derivative  $\beta^{(\nu)}$  of  $\beta$ . Moreover the  $\nu$ -th derivative  $\hat{\beta}_\nu^{(\nu)}$  of the in (2.16) proposed estimator  $\hat{\beta}_\nu$  attains this optimal rate, i.e, is minimax.  $\square$

## 4 Optimality in case of a “regular decay”

In this section we consider two special cases describing a “regular decay” of the values  $\{\lambda_k\}$  associated to the cross-covariance operator  $T_{WX}$  of the random function  $(X, W)$ . In the first example we suppose the values  $\{\lambda_k\}$  descend polynomial, which in case of a linear functional model with exogenous regressor is considered e.g. in Cardot et al. [2003] or Hall and Horowitz [2007]. The second example concerns values  $\{\lambda_k\}$  with exponential decay.

**The finitely smoothing case.** Assume now the values  $\{\lambda_k\}$  associated to the random function  $(X, W)$  have a polynomial decay, that is<sup>1</sup>

$$\lambda_k \asymp |k|^{-2a} \quad \text{for some } a > 0. \quad (4.1)$$

Then straightforward calculus shows the identity  $\mathcal{R}(T_{\widetilde{W}}) = \mathcal{W}_{2a}$ , where  $T_{\widetilde{W}}$  denotes the covariance operator associated to the optimal linear instrument  $\widetilde{W}$  (see the identity (2.9) and its discussion in Section 3). In other words, the operator  $T_{\widetilde{W}}$  acts like integrating  $(2a)$ -times and, hence it is called finitely smoothing. Furthermore, it is easily seen that

$$\forall 0 \leq \nu < p: \quad (\lambda_k)_{k \in \mathbb{Z}} \in S_{\kappa, d} \quad \text{with } \kappa(t) := t^{(p-\nu)/(a+\nu)} \text{ and some } d \geq 1. \quad (4.2)$$

In the proof of the next proposition we shown that the condition (3.5) implies  $\delta^* \asymp n^{-2(a+\nu)/[2(p+a)+1]}$ . Thereby, we have  $\kappa(\delta^*) \asymp n^{-2(p-\nu)/[2(p+a)+1]}$  and hence the lower bound in the next assertion follows from Theorem 3.2.

**PROPOSITION 4.1.** *Let the assumptions of Theorem 3.2 be satisfied with  $\kappa(t) = t^{(p-\nu)/(a+\nu)}$ . Then  $\inf_{\widetilde{\beta}} \sup_{\beta \in \mathcal{W}_p^p, (X, W) \in \mathcal{F}_\kappa^m, U \in \mathcal{E}_\eta^l} \{\mathbb{E} \|\widetilde{\beta} - \beta\|_\nu^2\} \geq C \cdot n^{-2(p-\nu)/[2(p+a)+1]}$  for some  $C > 0$ .*

On the other hand, if the threshold  $\alpha$  in the definition of the estimator  $\widehat{\beta}_\nu$  given in (2.16) is chosen such that  $\alpha \asymp n^{-2(a+\nu)/[2(p+a)+1]}$ . Then by applying Theorem 3.3 the rate  $n^{-2(p-\nu)/[2(p+a)+1]}$  provides up to a constant also the upper bound of the  $\mathcal{W}_\nu$ -risk of the estimator  $\widehat{\beta}_\nu$ , which is summarized in the next proposition. We have thus proved that the rate  $n^{-2(p-\nu)/[2(p+a)+1]}$  is optimal and the proposed estimator  $\widehat{\beta}_\nu$  is minimax optimal. Note that the index function  $\kappa(t) = t^{(p-\nu)/(a+\nu)}$  satisfies the additional condition (3.6).

**PROPOSITION 4.2.** *Let the assumptions of Theorem 3.3 be satisfied with  $\kappa(t) = t^{(p-\nu)/(a+\nu)}$ . Consider the estimator  $\widehat{\beta}_\nu$  defined in (2.16) with threshold  $\alpha = c \cdot n^{-2(a+\nu)/[2(p+a)+1]}$ ,  $c > 0$ . Then we have  $\sup_{\beta \in \mathcal{W}_p^p, (X, W) \in \mathcal{F}_\kappa^m, U \in \mathcal{E}_\eta^l} \{\mathbb{E} \|\widehat{\beta}_\nu - \beta\|_\nu^2\} = O(n^{-2(p-\nu)/[2(p+a)+1]})$ .*

**REMARK 4.1.** We shall emphasize the interesting influence of the parameters  $p$  and  $a$  characterizing the smoothness of  $\beta$  and the smoothing property of  $T_{\widetilde{W}}$  respectively. As we see from Proposition 4.1 and 4.2, if the value of  $a$  increases the obtainable optimal rate of convergence decreases. Therefore, the parameter  $a$  is often called *degree of ill-posedness* (c.f. Natterer [1984]). On the other hand, an increasing of the value  $p$  leads to a faster optimal rate. In other words, as we expect, a smoother slope function can be faster estimated. Finally, the estimation of higher derivatives of the slope function, i.e., increasing of the value of  $\nu$ , is as usual only with a slower optimal rate possible.  $\square$

<sup>1</sup>We write  $a_k \asymp b_k$  if there exists a finite positive constant  $c$  such that  $c^{-1}a_k \leq b_k \leq ca_k$  for all  $k \in \mathbb{Z}$ .

**REMARK 4.2.** There is an interesting issue hidden in the parametrization we have chosen. Consider classical indirect regression with known operator given by the covariance operator  $T_{\widetilde{W}}$  associated to the optimal instrument  $\widetilde{W}$ , i.e.,  $Y = [T_{\widetilde{W}}\beta](Z) + \varepsilon$  where  $Z$  has a uniform distribution on  $[0, 1]$  and  $\varepsilon$  is white noise (for details see e.g. Mair and Ruymgaart [1996]). Then given a  $n$ -sample of  $Y$  the optimal rate of convergence of the  $\mathcal{W}_\nu$ -risk of any estimator of  $\beta$  is of order  $n^{-2(p-\nu)/[2(p+2a)+1]}$ , since  $\mathcal{R}(T_{\widetilde{W}}) = \mathcal{W}_{2a}$  (c.f. Mair and Ruymgaart [1996] or Chen and Reiß [2008]). However, we have shown in Proposition 4.1 and 4.2 that in FLIR the rate  $n^{-2(p-\nu)/[2(p+a)+1]}$  is optimal. Thus comparing both rates we see that in FLIR the covariance operator  $T_{\widetilde{W}}$  has the *degree of ill-posedness*  $a$  while the same operator has in indirect regression a *degree of ill-posedness*  $(2a)$ . In other words in FLIR we do not face the complexity of an inversion of  $T_{\widetilde{W}}$  but only of its square root  $T_{\widetilde{W}}^{1/2}$ . This, roughly speaking, may be seen as a multiplication of the stochastic equation  $Y\widetilde{W} = \langle \beta, \widetilde{W} \rangle \widetilde{W} + \varepsilon\widetilde{W}$  by the inverse of  $T_{\widetilde{W}}^{1/2}$ . Notice that  $T_{\widetilde{W}}$  is also the covariance operator associated to the error term  $\varepsilon\widetilde{W}$ . Thus multiplying to the stochastic equation the inverse of  $T_{\widetilde{W}}^{1/2}$  leads, roughly speaking, to an additive white noise and hence it is then comparable with an indirect regression model with operator given by  $T_{\widetilde{W}}^{1/2}$ . However, the operator  $T_{\widetilde{W}}^{1/2}$  is unknown and thus it has to be estimated from the data.  $\square$

**The infinitely smoothing case.** Suppose now the values  $\{\lambda_k\}$  associated to the regressors  $X$  and  $W$  have an exponential decay, that is

$$\lambda_k \asymp \exp(-|k|^{2a}) \quad \text{for some } a > 0. \quad (4.3)$$

Then it is easy to check that  $\mathcal{R}(T_{\widetilde{W}}) \subset \mathcal{W}_\nu$  for all  $\nu > 0$ , therefore  $T_{\widetilde{W}}$  is called infinitely smoothing. In fact, the transformed values  $\{|\log \lambda_k|^{-1}\}$  satisfy the polynomial condition (4.1). Consequently, by applying the functional calculus we have  $\mathcal{R}(|\log(T_{\widetilde{W}})|^{-1}) = \mathcal{W}_{2a}$ . In other words,  $|\log(T_{\widetilde{W}})|^{-1}$  acts like integrating  $(2a)$ -times. Moreover, it follows that

$$\forall 0 \leq \nu < p: \quad (\lambda_k)_{k \in \mathbb{Z}} \in S_{\kappa, d} \quad \text{with } \kappa(t) := |\log t|^{-(p-\nu)/a} \text{ and some } d \geq 1. \quad (4.4)$$

Let  $\omega$  be the inverse function of  $\omega^{-1}(t) := t \cdot \varphi(t)$ , where  $\varphi$  denotes the inverse function of  $\kappa$ . We show in the proof of the next proposition that in an infinitely smoothing case the condition (3.5) implies  $1/n \asymp \delta^* \kappa(\delta^*)$ . Then it is straightforward to see that  $\delta^* \asymp 1/(n\omega(1/n))$  and  $\kappa(\delta^*) \asymp \omega(1/n)$ . Furthermore, it is shown in Mair [1994] that  $\omega(t) = |\log t|^{-(p-\nu)/a}(1 + o(1))$  as  $t \rightarrow 0$ . Consequently, the lower bound in the next assertion follows again from Theorem 3.2.

**PROPOSITION 4.3.** *Let the assumptions of Theorem 3.2 be satisfied with  $\kappa(t) = |\log t|^{-(p-\nu)/a}$ . Then we have  $\inf_{\widetilde{\beta}} \sup_{\beta \in \mathcal{W}_p^p, (X, W) \in \mathcal{F}_\kappa^m, U \in \mathcal{E}_\eta^l} \{\mathbb{E} \|\widetilde{\beta} - \beta\|_\nu^2\} \geq C \cdot (\log n)^{-(p-\nu)/a}$  for some  $C > 0$ .*

The next proposition states that the rate  $(\log n)^{-(p-\nu)/a}$  of the lower bound in Proposition 4.3 provides up to a constant also the upper bound of the  $W_\nu$ -risk of the estimator  $\widehat{\beta}_\nu$ . We have thus proved that the rate  $(\log n)^{-(p-\nu)/a}$  is optimal and  $\widehat{\beta}_\nu$  is minimax-optimal.

**PROPOSITION 4.4.** *Let the assumptions of Theorem 3.3 be satisfied with  $\kappa(t) = |\log t|^{-(p-\nu)/a}$ . Consider the estimator  $\widehat{\beta}_\nu$  defined in (2.16) with threshold  $\alpha = c \cdot n^{-1/4}$ ,  $c > 0$ . Then we have  $\sup_{\beta \in \mathcal{W}_p^p, (X, W) \in \mathcal{F}_\kappa^m, U \in \mathcal{E}_\eta^l} \{\mathbb{E} \|\widehat{\beta}_\nu - \beta\|_\nu^2\} = O((\log n)^{-(p-\nu)/a})$ .*

**REMARK 4.3.** It seems rather surprising that in opposite to Proposition 4.2 in the last assertion the threshold  $\alpha$  does not depend on the values of  $p$ ,  $\nu$  or  $a$ . This, however, is due to the fact that for  $\alpha = cn^{-1/4}$ ,  $c > 0$ , the  $W_\nu$ -risk of  $\hat{\beta}_\nu$  is of order  $O(n^{-1/2} + |\log n^{-1/4}|^{-(p-\nu)/a}) = O((\log n)^{-(p-\nu)/a})$ . Note, that the parameter  $a$  specifying in condition (4.3) the decay of the values  $\{\lambda_k\}$  describes also in this situation the *degree of ill-posedness*. Finally, a comparison with an indirect regression model as in Remark 4.2 leads to the same findings.  $\square$

## 5 Conclusion and perspectives

Assuming joint second order stationarity of the regressor  $X$  and the instrument  $W$  we derive in this paper the minimax optimal rate of convergence of an estimator of the slope function  $\beta$  and its derivatives provided the covariance operator associated to the optimal linear instrument  $\widetilde{W}$  is in some general form smoothing. This results in its generality cover in particular the case of finitely or infinitely smoothing covariance operators. It is worth pointing out that for establishing the lower bound it is not necessary to assume that the regressor and the instrument are jointly second order stationary. Moreover, the lower bound is derived by assuming a certain complexity of the class of distributions of  $X$  and  $W$ , in particular, it contains a Gaussian model. Therefore, we claim that replacing the optimal linear by the optimal instrument, i.e., the conditional expectation of  $X$  given  $W$ , will not improve the optimal rate of convergence. Indeed, in a Gaussian model both instruments, if they exist, coincide.

Many ideas in this paper can be adapted to the general case without the assumption of joint second order stationarity of the regressor  $X$  and the instrument  $W$ . However, the estimation procedure itself may be different, since a projection onto the in general unknown eigenfunctions of the covariance operator of the optimal linear instrument is not possible. This is subject of ongoing research.

Once this will be established, the open problem of how to choose the threshold  $\alpha$  adaptively from the data will remain in case not knowing the true smoothness of the slope function or not knowing the true link between the covariance operator of the optimal linear instrument and the Sobolev spaces.

## A Appendix: Proofs

### A.1 Proofs of Section 3

We begin by defining and recalling notations to be used in the proofs:

$$X_{ik} := \langle X_i, \phi_k \rangle, \quad W_{ik} := \langle W_i, \phi_k \rangle, \quad T_{n,k} := \frac{1}{n} \sum_{i=1}^n (Y_i W_{ik} - \frac{\hat{c}_k}{\hat{w}_k} \beta_k |W_{ik}|^2),$$

$$c_k = \mathbb{E}[\overline{X}_{ik} W_{ik}], \quad w_k = \text{Var}(W_{ik}), \quad x_k = \text{Var}(X_{ik}), \quad \text{and} \quad \lambda_k = c_k^2 / w_k. \quad (\text{A.1})$$

We shall prove in the end of this section four technical Lemma (A.1 - A.4) which are used in the following proofs.

**Proof of the consistency.**

**PROOF OF PROPOSITION 3.1.** The proof is based on the decomposition (3.2). We show below for some universal constant  $C > 0$  the following bound

$$\mathbb{E}\|\widehat{\beta}_\nu - \widetilde{\beta}_\nu^\alpha\|_\nu^2 \leq \frac{C\eta}{\alpha^2 n} \cdot \mathbb{E}\|W\|^2 \cdot \{\sigma^2 + \|\beta\|^2 \mathbb{E}\|X\|^2\}, \quad (\text{A.2})$$

while in case of  $\|\beta\|_\nu < \infty$  we conclude from Lebesgue's dominated convergence theorem

$$\mathbb{E}\|\widetilde{\beta}_\nu^\alpha - \beta\|_\nu^2 = o(1) \text{ provided } \alpha = o(1) \text{ and } (\alpha n)^{-1} = o(1) \text{ as } n \rightarrow \infty. \quad (\text{A.3})$$

Consequently, the conditions on  $\alpha$  ensure the convergence to zero of the two terms on the right hand side in (3.2) as  $n$  tends to  $\infty$ , which gives the result.

Proof of (A.2). By making use of the notations given in (A.1) it follows that

$$\mathbb{E}\|\widehat{\beta}_\nu - \widetilde{\beta}_\nu^\alpha\|_\nu^2 \leq \frac{1}{\alpha} \sum_{k \in \mathbb{Z}} \mathbb{E} \frac{|\widehat{g}_k - \beta_k \widehat{\lambda}_k|^2}{\widehat{\lambda}_k} \mathbb{1}\{\widehat{\lambda}_k \geq \alpha \gamma_k^\nu\} \leq \frac{1}{\alpha^2} \sum_{k \in \mathbb{Z}} \mathbb{E}|T_{n,k}|^2$$

and hence by using (A.8) in Lemma A.1 we obtain (A.2).

Proof of (A.3). If  $\beta \in W_p$ ,  $p \geq \nu \geq 0$ , then by making use of the relation

$$\mathbb{E}\|\widetilde{\beta}_\nu^\alpha - \beta\|_\nu^2 = \sum_{k \in \mathbb{Z}} \beta_k^2 \cdot \gamma_k^\nu \cdot \mathbb{E}\mathbb{1}\{\widehat{\lambda}_k / \gamma_k^\nu < \alpha\} \leq \sum_{k \in \mathbb{Z}} \beta_k^2 \cdot \gamma_k^\nu = \|\beta\|_\nu^2 \leq \|\beta\|_p^2 < \infty$$

the result follows from Lebesgue's dominated convergence theorem since for each  $k \in \mathbb{Z}$  we claim  $\mathbb{E}\mathbb{1}\{\widehat{\lambda}_k / \gamma_k^\nu < \alpha\} = o(1)$  provided  $\alpha = o(1)$  and  $(\alpha n)^{-1} = o(1)$  as  $n \rightarrow \infty$ . Indeed, there exists  $\alpha_k > 0$  such that for all  $\alpha \leq \alpha_k$  it holds  $\lambda_k \geq 4\tau\alpha\gamma_k^\nu$  and hence by using (A.14) in Lemma A.2 we bound  $\mathbb{E}\mathbb{1}\{\widehat{\lambda}_k / \gamma_k^\nu < \alpha\}$  up to a constant by  $(\alpha n)^{-1} \mathbb{E}\|X\|^2 \{1 + (\alpha n)^{-1} \mathbb{E}\|W\|^2\}$ . Thereby, the conditions on  $\alpha$  imply (A.3) which completes the proof.  $\square$

**Proof of the lower bound.**

**PROOF OF THEOREM 3.2.** Assuming  $\eta$  to be sufficiently large we can pick an i.i.d. sample  $(X_i, W_i) \in \mathcal{F}_{\eta, \tau}^m$ ,  $i = 1, \dots, n$ , of Gaussian random functions such that the associated sequence of values  $(\lambda_k)_{k \in \mathbb{Z}}$  is an element of  $\mathcal{S}_{\kappa, d}$ , i.e.,  $\{(X_i, W_i)\} \subset \mathcal{F}_\kappa^m$ . Consider independent error terms  $\varepsilon_i \sim \mathcal{N}(0, 1)$ ,  $i = 1, \dots, n$ , which are independent of the random functions  $\{(X_i, W_i)\}$ . For  $i = 1, \dots, n$  let  $\widetilde{W}_i$  be the optimal instrument given in (2.7), and denote  $\widetilde{W}_{ik} := \langle \widetilde{W}_i, \phi_k \rangle$ ,  $k \in \mathbb{Z}$ . Note, that  $\widetilde{W}_{ik}$  is a centered random variable with variance  $\lambda_k$ . Let  $\theta = (\theta_k) \in \{-1, 1\}^{2k^*+1}$ , where  $k^* := k^*(n) \in \mathbb{N}$  satisfies (3.5) for some  $\Delta \geq 1$ . Define a  $(2k^* + 1)$ -vector of coefficients  $(b_j)$  such that  $(b_j^2)$  satisfies (A.24) in Lemma A.4. For each  $\theta$  we define a function  $\beta_\theta$  which by (A.26) in Lemma A.4 yields:

$$\beta_\theta := \sum_{|k| \leq k^*} \theta_k b_k \phi_k \in \mathcal{W}_p^p.$$

Define for each  $\theta$  an error term  $U_{\theta i} = \varepsilon_i / 2 + \tau_\theta \langle \beta_\theta, X_i - \widetilde{W}_i \rangle$ . Then  $\{U_{\theta i}\}$  are independent centered Gaussian random variables with variance one for an appropriate chosen  $\tau_\theta$ , and hence  $\{U_{\theta i}\} \subset \mathcal{E}_\eta^m$ . Moreover, we have  $\mathbb{E}[U_{\theta i} W_i(t)] = 0$ , for all  $t \in [0, 1]$  and  $i = 1, \dots, n$ .

Consequently, for each  $\theta$  the random variables  $(Y_i, X_i, W_i)$  with  $Y_i := \langle \beta_\theta, X_i \rangle + \sigma U_{\theta i}$ ,  $i = 1, \dots, n$ , form a sample of the model (2.1a)-(2.1b) and we denote its joint distribution by  $P_\theta$ . Furthermore, for  $|k| \leq k^*$  and each  $\theta$  we introduce  $\theta^{(k)} = (\theta_j^{(k)}) \in \{-1, 1\}^{2k^*+1}$  by  $\theta_j^{(k)} = \theta_j$  for  $k \neq j$  and  $\theta_k^{(k)} = -\theta_k$ . As in case of  $P_\theta$  the conditional distribution of  $Y_i$  given  $\widetilde{W}_i$  is Gaussian with mean  $\langle \beta_\theta, \widetilde{W}_i \rangle = \sum_{|k| \leq k^*} \theta_k b_k \widetilde{W}_{ik}$  and variance  $\sigma_\theta^2 \geq \sigma^2/4$  with  $\sigma_\theta^2 = \sigma_{\theta^{(k)}}^2$  it is easily seen that the log-likelihood of  $P_{\theta^{(k)}}$  w.r.t.  $P_\theta$  is given by

$$\log\left(\frac{dP_{\theta^{(k)}}}{dP_\theta}\right) = -\frac{1}{\sigma_\theta^2} \theta_k \bar{b}_k \sum_{i=1}^n U_{\theta i} \widetilde{W}_{ik} - \frac{1}{\sigma_\theta^2} \theta_k b_k \sum_{i=1}^n \bar{U}_{\theta i} \widetilde{W}_{ik} + \frac{2}{\sigma_\theta^2} b_k^2 \sum_{i=1}^n \widetilde{W}_{ik}^2.$$

Its expectation satisfies  $\mathbb{E}_{P_\theta}[\log(dP_{\theta^{(k)}}/dP_\theta)] \geq -n \cdot b_k^2 \cdot \lambda_k / (2\sigma^2)$  since  $\lambda_k = \text{Var} \widetilde{W}_{ik}$  and  $\sigma_\theta^2 \geq \sigma^2/4$ . In terms of Kullback-Leibler divergence this means  $KL(P_{\theta^{(k)}}, P_\theta) \leq n \cdot b_k^2 \cdot \lambda_k / (2\sigma^2)$ . Since the Hellinger distance  $H(P_{\theta^{(k)}}, P_\theta)$  satisfies  $H^2(P_{\theta^{(k)}}, P_\theta) \leq KL(P_{\theta^{(k)}}, P_\theta)$  it follows from (A.25) in Lemma A.4 that

$$H^2(P_{\theta^{(k)}}, P_\theta) \leq \frac{n}{2\sigma^2} \cdot b_k^2 \cdot \lambda_k \leq 1, \quad |k| \leq k^*. \quad (\text{A.4})$$

Consider the Hellinger affinity  $\rho(P_{\theta^{(k)}}, P_\theta) = \int \sqrt{dP_{\theta^{(k)}} dP_\theta}$ , then we obtain for any estimator  $\widetilde{\beta}$  that

$$\begin{aligned} \rho(P_{\theta^{(k)}}, P_\theta) &\leq \int \frac{|\langle \widetilde{\beta} - \beta_{\theta^{(k)}}, \phi_k \rangle|}{|\langle \beta_\theta - \beta_{\theta^{(k)}}, \phi_k \rangle|} \sqrt{dP_{\theta^{(k)}} dP_\theta} + \int \frac{|\langle \widetilde{\beta} - \beta_\theta, \phi_k \rangle|}{|\langle \beta_\theta - \beta_{\theta^{(k)}}, \phi_k \rangle|} \sqrt{dP_{\theta^{(k)}} dP_\theta} \\ &\leq \left( \int \frac{|\langle \widetilde{\beta} - \beta_{\theta^{(k)}}, \phi_k \rangle|^2}{|\langle \beta_\theta - \beta_{\theta^{(k)}}, \phi_k \rangle|^2} dP_{\theta^{(k)}} \right)^{1/2} + \left( \int \frac{|\langle \widetilde{\beta} - \beta_\theta, \phi_k \rangle|^2}{|\langle \beta_\theta - \beta_{\theta^{(k)}}, \phi_k \rangle|^2} dP_\theta \right)^{1/2}. \end{aligned} \quad (\text{A.5})$$

Due to the identity  $\rho^2(P_{\theta^{(k)}}, P_\theta) = 1 - \frac{1}{2} H^2(P_{\theta^{(k)}}, P_\theta)$  combining (A.4) with (A.5) yields

$$\left\{ \mathbb{E}_{\theta^{(k)}} |\langle \widetilde{\beta} - \beta_{\theta^{(k)}}, \phi_j \rangle|^2 + \mathbb{E}_\theta |\langle \widetilde{\beta} - \beta_\theta, \phi_k \rangle|^2 \right\} \geq \frac{1}{4} b_k^2, \quad |k| \leq k^*.$$

From this we conclude for each estimator  $\widetilde{\beta}$  that

$$\begin{aligned} \sup_{\substack{\beta \in \mathcal{W}_\rho^\beta, U \in \mathcal{E}_\eta^l, \\ (X, W) \in \mathcal{F}_k^n}} \mathbb{E} \|\widetilde{\beta} - \beta\|_\nu^2 &\geq \sup_{\theta \in \{-1, 1\}^{2k^*+1}} \mathbb{E}_\theta \|\widetilde{\beta} - \beta_\theta\|_\nu^2 \\ &\geq \frac{1}{2^{2k^*+1}} \sum_{\theta \in \{-1, 1\}^{2k^*+1}} \sum_{|k| \leq k^*} \gamma_k^\nu \cdot \mathbb{E}_\theta |\langle \widetilde{\beta} - \beta_\theta, \phi_k \rangle|^2 \\ &= \frac{1}{2^{2k^*+1}} \sum_{\theta \in \{-1, 1\}^{2k^*+1}} \sum_{|k| \leq k^*} \frac{\gamma_k^\nu}{2} \cdot \left\{ \mathbb{E}_\theta |\langle \widetilde{\beta} - \beta_\theta, \phi_k \rangle|^2 + \mathbb{E}_{\theta^{(k)}} |\langle \widetilde{\beta} - \beta_{\theta^{(k)}}, \phi_k \rangle|^2 \right\} \\ &\geq \frac{1}{8} \sum_{|k| \leq k^*} \gamma_k^\nu b_k^2 \geq \frac{1}{4} \cdot \min\left\{ \frac{\sigma^2}{d\Delta}, \frac{\rho}{2} \right\} \cdot \frac{\kappa(\delta^*)}{\Lambda}, \end{aligned}$$

where the last inequality follows from (A.27) in Lemma A.4 together with  $\lambda_+ \leq \Lambda$  which completes the proof.  $\square$



**Proof of the upper bound.**

**PROOF OF THEOREM 3.3.** The proof is based on the decomposition (3.2), where we show below under the condition  $(X, W) \in \mathcal{F}_{\eta, \tau}^{32}$ ,  $U \in \mathcal{E}_{\eta}^{16}$  and  $(\lambda_k)_{k \in \mathbb{Z}} \in \mathcal{S}_{\kappa, d}$  for some universal constant  $C > 0$  and  $I := \{k \in \mathbb{Z} : 8\lambda_k > \alpha\gamma_k^\nu\}$  the following two bounds

$$\begin{aligned} \mathbb{E}\|\widehat{\beta}_\nu - \widetilde{\beta}_\nu^\alpha\|_\nu^2 &\leq C \cdot \eta \cdot \left\{ \frac{1}{\alpha n} + d \cdot \sum_{k \in I} \frac{1}{n \cdot \varphi(\gamma_k^{\nu-p})} \right\} \\ &\quad \cdot \{\|\beta\|^2 \cdot \mathbb{E}\|X\|^2 + \sigma^2\} \cdot \left\{ \frac{\mathbb{E}\|X\|^2}{\alpha n} + 1 \right\} \cdot \left\{ \frac{\mathbb{E}\|W\|^2}{\alpha n} + 1 \right\}^3, \end{aligned} \quad (\text{A.6})$$

$$\mathbb{E}\|\widetilde{\beta}_\nu^\alpha - \beta\|_\nu^2 \leq \kappa(d4\tau\alpha) \cdot \|\beta\|_p^2 + \frac{C\eta}{\alpha n} \cdot \mathbb{E}\|X\|^2 \cdot \|\beta\|_p^2 \cdot \left(1 + \frac{\mathbb{E}\|W\|^2}{\alpha n}\right). \quad (\text{A.7})$$

Consequently, for all  $\beta \in \mathcal{W}_p^\rho$  and  $(X, W) \in \mathcal{F}_{\kappa}^{32}$ , i.e.,  $\mathbb{E}\|X\|^2 \leq \Lambda$ ,  $\mathbb{E}\|W\|^2 \leq \Lambda$ , and hence  $\lambda_+ \leq \Lambda$ , follows

$$\mathbb{E}\|\widehat{\beta}_\nu - \beta\|_\nu^2 \leq C \cdot \eta \cdot \left\{ \frac{1}{\alpha n} + d \cdot \sum_{k \in I} \frac{1}{n \cdot \varphi(\gamma_k^{\nu-p})} + \kappa(d4\tau\alpha) \right\} \cdot [\sigma^2 + \rho \cdot \Lambda] \cdot \left[ \frac{\Lambda}{\alpha n} + 1 \right]^4.$$

Let  $k^* := k^*(n) \in \mathbb{N}$  and  $\delta^* := \delta^*(n) \in (0, 1]$  be given by (3.5) for some  $\Delta \geq 1$  then the condition on  $\alpha$ , that is  $\alpha = 8d\Lambda\delta^*$ , implies  $I \subset \{k \in \mathbb{Z} : |k| \leq k^*\}$ . We conclude from (3.5) that  $1/[\alpha \cdot n] \leq \Delta \cdot \kappa(\delta^*)$  and  $\sum_{k \in I} 1/[n \cdot \varphi(\gamma_k^{\nu-p})] \leq \Delta \cdot \kappa(\delta^*)$ , hence that

$$\mathbb{E}\|\widehat{\beta}_\nu - \beta\|_\nu^2 \leq C\eta d\Delta \cdot \frac{\kappa(d4\tau\alpha)}{\kappa(\alpha/(8d\Lambda))} \cdot [\sigma^2 + \rho\Lambda] \cdot \left[ \Delta\Lambda\kappa(\delta^*) + 1 \right]^4 \cdot \kappa(\delta^*).$$

Thereby, the condition (3.6), that is  $\kappa(d4\tau\alpha)/\kappa(\alpha) = O(1)$  and  $\kappa(\alpha)/\kappa(\alpha/(8d\Lambda)) = O(1)$ , as  $\alpha$  tends to zero, implies the result.

Proof of (A.6). By using  $T_{n,k}$  introduced in (A.1) we obtain the identity

$$\mathbb{E}\|\widehat{\beta}_\nu - \widetilde{\beta}_\nu^\alpha\|_\nu^2 = \sum_{k \in \mathbb{Z}} \gamma_k^\nu \cdot \mathbb{E} \left[ \frac{|T_{n,k}|^2 |\widehat{c}_k / \widehat{w}_k|^2 \mathbb{1}\{\widehat{w}_k \geq \alpha\}}{\widehat{\lambda}_k^2} \mathbb{1}\{\widehat{\lambda}_k \geq \alpha\gamma_k^\nu\} \right].$$

We partition the sum into two parts which we estimate separately using the bounds in Lemma A.3. First by using  $\widehat{\lambda}_k \geq \alpha\gamma_k^\nu$  together with (A.22) in Lemma A.3 we bound the sum over  $I^c := \{k \in \mathbb{Z} : 8\lambda_k \leq \alpha\gamma_k^\nu\}$  by

$$\sum_{k \in I^c} \mathbb{E} \left[ \frac{|T_{n,k}|^2 |\widehat{c}_k|^2}{\alpha^2 \widehat{w}_k^2} \mathbb{1}\{\widehat{w}_k \geq \alpha\} \mathbb{1}\{\widehat{\lambda}_k \geq \alpha\gamma_k^\nu\} \right] \leq \frac{C\eta \mathbb{E}\|X\|^2}{n^2 \alpha^2} \cdot \{\|\beta\|^2 \cdot \mathbb{E}\|X\|^2 + \sigma^2\} \cdot \left\{ \frac{\mathbb{E}\|W\|^2}{\alpha n} + 1 \right\}^3.$$

While due to the identity  $\widehat{\lambda}_k = |\widehat{c}_k|^2 / \widehat{w}_k \mathbb{1}\{\widehat{w}_k \geq \alpha\}$  together with (A.21) in Lemma A.3 the sum over  $I := \{k \in \mathbb{Z} : 8\lambda_k > \alpha\gamma_k^\nu\}$  is bounded by

$$\begin{aligned} &\sum_{k \in I} \gamma_k^\nu \cdot \mathbb{E} \left[ \frac{|T_{n,m}|^2}{\widehat{w}_k \cdot \widehat{\lambda}_k} \mathbb{1}\{\widehat{w}_k \geq \alpha\} \mathbb{1}\{\widehat{\lambda}_k \geq \alpha\gamma_k^\nu\} \right] \\ &\leq \sum_{k \in I} \frac{C\eta\gamma_k^\nu}{n\lambda_k} \cdot \{\|\beta\|^2 \cdot \mathbb{E}\|X\|^2 + \sigma^2\} \cdot \left\{ \frac{\mathbb{E}\|X\|^2}{\alpha n} + 1 \right\} \cdot \left\{ \frac{\mathbb{E}\|W\|^2}{\alpha n} + 1 \right\}. \end{aligned}$$

From  $\lambda_+ \geq 1$  it follows that by combining the two parts of the sum we have

$$\begin{aligned} \mathbb{E}\|\widehat{\beta}_\nu - \widetilde{\beta}_\nu^\alpha\|_\nu^2 &\leq C \cdot \eta \cdot \left\{ \frac{1}{\alpha n} + \sum_{k \in I^c} \frac{\gamma_k^\nu \lambda_+}{\lambda_k} \right\} \\ &\quad \cdot \{\|\beta\|^2 \cdot \mathbb{E}\|X\|^2 + \sigma^2\} \cdot \left\{ \frac{\mathbb{E}\|X\|^2}{\alpha n} + 1 \right\} \cdot \left\{ \frac{\mathbb{E}\|W\|^2}{\alpha n} + 1 \right\}^3. \end{aligned}$$

Now the link condition  $(\lambda_k)_{k \in \mathbb{Z}} \in \mathcal{S}_{\kappa, d}$  implies (A.6).

The proof of (A.7) is based on the identity  $\mathbb{E}\|\widetilde{\beta}_\nu^\alpha - \beta\|_\nu^2 = \sum_{k \in \mathbb{Z}} \gamma_k^\nu \beta_k^2 P(\widehat{\lambda}_k < \alpha \gamma_k^\nu)$ , where we partition the sum again into two parts which we estimate separately. First we sum over  $I := \{k \in \mathbb{Z} : \lambda_k < 4\tau \alpha \gamma_k\}$ . Since  $\lambda_+ \geq 1$ , the link condition  $(\lambda_k)_k \in \mathcal{S}_{\kappa, d}$  together with the monotonicity of  $\kappa$  shows that

$$\sum_{k \in I} \gamma_k^\nu \beta_k^2 P(\widehat{\lambda}_k < \alpha \gamma_k^\nu) \leq \sum_{k \in I} \gamma_k^p \beta_k^2 \kappa(d \frac{\lambda_k}{\gamma_k^\nu \lambda_+}) \leq \kappa(d 4\tau \alpha) \cdot \sum_{k \in I} \gamma_k^p \beta_k^2.$$

The sum over  $I^c := \{k \in \mathbb{Z} : \lambda_k \geq 4\tau \alpha \gamma_k\}$  we bound using (A.14) in Lemma A.2, that is

$$\sum_{k \in I^c} \gamma_k^\nu \beta_k^2 P(\widehat{\lambda}_k < \alpha \gamma_k^\nu) \leq \frac{C \eta}{\alpha n} \cdot \mathbb{E}\|X\|^2 \cdot \left\{ 1 + \frac{\mathbb{E}\|W\|^2}{\alpha n} \right\} \sum_{k \in I^c} \gamma_k^\nu \beta_k^2.$$

Combining the two parts of the sum we obtain (A.7), which completes the proof.  $\square$

### Technical assertions.

The following four lemma gather technical results used in the proof of Proposition 3.1, Theorem 3.2 and Theorem 3.3.

**LEMMA A.1.** *Suppose  $(X, W) \in \mathcal{F}_{\eta, \tau}^{4m}$  and  $U \in \mathcal{E}_\eta^{4m}$ ,  $m \in \mathbb{N}$ . Then for some constant  $C > 0$  only depending on  $m$  we have*

$$\sup_{k \in \mathbb{Z}} \left\{ \frac{1}{w_k^m} \cdot \mathbb{E}|T_{n,k}|^{2m} \right\} \leq C \cdot \frac{1}{n^m} \cdot \{\|\beta\|^{2m} \cdot (\mathbb{E}\|X\|^2)^m + \sigma^{2m}\} \cdot \eta, \quad (\text{A.8})$$

$$\sup_{k \in \mathbb{Z}} \mathbb{E} \left| \frac{\widehat{w}_k - w_k}{w_k} \right|^{2m} \leq C \cdot \frac{1}{n^m} \cdot \eta, \quad (\text{A.9})$$

$$\sup_{k \in \mathbb{Z}} \left\{ \frac{\lambda_k^m}{x_k^m} \cdot \mathbb{E} \left| \frac{\widehat{c}_k - c_k}{c_k} \right|^{2m} \right\} \leq C \cdot \frac{1}{n^m} \cdot \eta. \quad (\text{A.10})$$

**PROOF.** Let  $\eta_{ik} := \sum_{l \neq k} \beta_l \overline{X}_{il}$ ,  $\zeta_{ik} := \beta_k \{\overline{X}_{ik} - \overline{W}_{ik} c_k / w_k\}$ ,  $\tau_{ik} := \beta_k \overline{W}_{ik} \{c_k / w_k - \widehat{c}_k / \widehat{w}_k\}$ ,  $i = 1, \dots, n$  and  $k \in \mathbb{Z}$ . Then we have

$$T_{n,k} = \frac{1}{n} \sum_{i=1}^n \{\eta_{ik} + \zeta_{ik} + \tau_{ik} + \sigma U_i\} W_{ik} =: T_1 + T_2 + T_3 + T_4,$$

where we bound below each summand separately, that is

$$\mathbb{E}|T_j|^{2m} \leq C \cdot \frac{w_k^m}{n^m} \cdot \|\beta\|^{2m} \cdot (\mathbb{E}\|X\|^2)^m \cdot \eta, \quad j = 1, 2, 3, \quad (\text{A.11})$$

$$\mathbb{E}|T_4|^{2m} \leq C \cdot \frac{w_k^m}{n^m} \cdot \sigma^{2m} \cdot \eta \quad (\text{A.12})$$

for some  $C > 0$  only depending on  $m$ . Consequently, the inequality (A.8) follows from (A.11) and (A.12). Consider  $T_1$ . For each  $k \in \mathbb{Z}$  the random variables  $(\eta_{ik} \cdot W_{ik})$ ,  $i = 1, \dots, n$ , are independent and identically distributed with mean zero. From Theorem 2.10 in Petrov [1995] we conclude  $\mathbb{E}|T_1|^{2m} \leq Cn^{-m}\mathbb{E}|\eta_{1k}W_{1k}|^{2m}$  for some constant  $C > 0$  only depending on  $m$ . Then we claim that (A.11) follows in case of  $T_1$  from the Cauchy-Schwarz inequality together with  $(X, W) \in \mathcal{F}_{\eta, \tau}^{4m}$ , i.e.,  $\sup_{i,k} \mathbb{E}|X_{ik}/\sqrt{x_k}|^{4m} \leq \eta$  and  $\sup_{i,k} \mathbb{E}|W_{ik}/\sqrt{w_k}|^{4m} \leq \eta$ . Indeed, we have

$$\mathbb{E}|\eta_{1k}W_{1k}|^{2m} \leq \left(\sum_{j \neq k} \beta_j^2\right)^m \sum_{j_1 \neq k} \cdots \sum_{j_m \neq k} \mathbb{E}|W_{ik}|^{2m} \prod_{l=1}^m |X_{1j_l}|^2 \leq \|\beta\|^{2m} \cdot w_k^m \cdot \left(\sum_{j \neq k} x_j\right)^m \cdot \eta.$$

Consider  $T_2$ . (A.11) follows in analogy to the case of  $T_1$  since  $\{\zeta_{ik} \cdot W_{ik}\}$  are independent and identically distributed with mean zero respectively, and  $\mathbb{E}|\zeta_{1k} \cdot W_{1k}|^{2m} \leq C \cdot \beta_k^{2m} \{x_k^m \cdot w_k^m \cdot \eta + |c_k|^{2m} \eta\} \leq C \cdot \|\beta\|^{2m} \cdot (\mathbb{E}\|X\|^2)^m \cdot w_k^m \cdot \eta$ , where  $C > 0$  does only depend on  $m$ . Consider  $T_3$ . We have  $\mathbb{E}|T_3|^{2m} \leq C\beta_k^{2m} \{|c_k|^{2m} \mathbb{E}|\hat{w}_k/w_k - 1|^{2m} + \mathbb{E}|\hat{c}_k - c_k|^{2m}\}$  for some  $C > 0$  only depending on  $m$ , by the identity  $T_3 = \beta_k \{\hat{w}_k c_k/w_k - \hat{c}_k\}$ . Therefore (A.11) in case of  $T_3$  follows from (A.9) and (A.10). Consider  $T_4$ . (A.12) follows in analogy to the case of  $T_1$ , because  $\{\sigma U_i \cdot W_{ik}\}$  are independent and identically distributed with mean zero respectively, and  $\mathbb{E}|\sigma \cdot U_1 \cdot W_{1k}|^{2m} \leq C \cdot \sigma^{2m} \cdot w_k^m \cdot \eta$ , where  $C > 0$  does only depend on  $m$ .

Proof of (A.9) and (A.10). Since  $\{(|W_{ik}|^2/w_k - 1)\}$  and  $\{(\bar{X}_{ik}W_{ik} - c_k)\}$  are independent and identically distributed with mean zero respectively, where  $\mathbb{E}\|W_{1k}|^2/w_k\|^{2m} \leq \eta$  and  $\mathbb{E}|\bar{X}_{1k}W_{1k} - c_k|^{2m} \leq C \cdot x_k^m \cdot w_k^m \cdot \eta$ , for some  $C > 0$  only depending on  $m$ , the result follows by applying Theorem 2.10 in Petrov [1995], which proves the lemma.  $\square$

**LEMMA A.2.** *Let  $(X, W) \in \mathcal{F}_{\eta, \tau}^{4m}$ ,  $m \in \mathbb{N}$ , then for all  $0 < d < 1$  and some constant  $C > 0$  only depending on  $m$  we have*

$$\sup_{k \in \mathbb{Z}} P(\hat{w}_k/w_k < d) \leq C \cdot \frac{1}{(1-d)^{2m}} \cdot \frac{1}{n^m} \cdot \eta. \quad (\text{A.13})$$

Suppose  $(X, W) \in \mathcal{F}_{\eta, \tau}^8$  and let  $I_1 := \{k \in \mathbb{Z} : \lambda_k \geq 4\tau \alpha \gamma_k^\nu\}$ . Then for some universal constant  $C > 0$  we have

$$\sup_{k \in I_1} P(\hat{\lambda}_k < \alpha \gamma_k^\nu) \leq \frac{C \eta}{\alpha n} \cdot \mathbb{E}\|X\|^2 \cdot \left\{1 + \frac{\mathbb{E}\|W\|^2}{\alpha n}\right\}. \quad (\text{A.14})$$

While if  $(X, W) \in \mathcal{F}_{\eta, \tau}^{16}$  and  $I_2 := \{k \in \mathbb{Z} : 8\lambda_k \leq \alpha \gamma_k^\nu\}$ . Then

$$\sup_{k \in I_2} \left\{\frac{\lambda_k^2}{x_k^2} \cdot P(\hat{\lambda}_k \geq \alpha \gamma_k^\nu)\right\} \leq \frac{C \eta}{n^2} \cdot \left\{1 + \frac{\mathbb{E}\|W\|^2}{\alpha n}\right\}^2. \quad (\text{A.15})$$

**PROOF.** Since  $P(\hat{w}_k/w_k < d) \leq P(|\hat{w}_k/w_k - 1| \geq 1-d)$  by applying Markov's inequality the estimate (A.13) follows from (A.9) in Lemma A.1.

The proof of (A.14) is based on the elementary inequality

$$P(\hat{\lambda}_k < \alpha \gamma_k^\nu) \leq P(\hat{w}_k < \alpha) + P(\hat{\lambda}_k < \alpha \gamma_k^\nu \wedge \hat{w}_k \geq \alpha),$$

where we show below for some universal constant  $C > 0$  the following two bounds

$$\sup_{k \in I_1} P(\widehat{w}_k < \alpha) \leq C \cdot \frac{1}{n} \cdot \eta, \quad (\text{A.16})$$

$$\sup_{k \in I_1} P(\widehat{\lambda}_k < \alpha \gamma_k^\nu \wedge \widehat{w}_k \geq \alpha) \leq \frac{C\eta}{\alpha n} \cdot \mathbb{E}\|X\|^2 \cdot \left\{ 1 + \frac{\mathbb{E}\|W\|^2}{\alpha n} \right\} \quad (\text{A.17})$$

which imply together (A.14). Since  $\alpha/w_k \leq \lambda_k/w_k \cdot \alpha \gamma_k^\nu/\lambda_k \leq \tau \alpha \gamma_k^\nu/\lambda_k \leq 1/4$  holds true for all  $k \in I_1$ , the estimate (A.16) follows from (A.13). The proof of (A.17) is based on

$$1 - 2 \frac{\widehat{\lambda}_k}{\lambda_k} \leq \left\{ 4|\widehat{c}_k/c_k - 1|^2 + 1 \right\} \cdot \left\{ \frac{|\widehat{w}_k/w_k - 1|^2}{\widehat{w}_k} + |\widehat{w}_k/w_k - 1| \right\} + 4|\widehat{c}_k/c_k - 1|^2 \quad (\text{A.18})$$

which implies for all  $k \in I_1$  that

$$\begin{aligned} P(\widehat{\lambda}_k < \alpha \gamma_k^\nu \wedge \widehat{w}_k \geq \alpha) &\leq P\left(1/4 \leq 4|\widehat{c}_k/c_k - 1|^2\right) \\ &\quad + P\left(1/4 \leq \left\{ 4|\widehat{c}_k/c_k - 1|^2 + 1 \right\} \cdot \left\{ \frac{|\widehat{w}_k/w_k - 1|^2}{\alpha} + |\widehat{w}_k/w_k - 1| \right\}\right). \end{aligned}$$

Therefore, by applying Markov's inequality together with (A.9) and (A.10) in Lemma A.1 we obtain the estimate (A.17).

The proof of (A.15) is based on the decomposition

$$P(\widehat{\lambda}_k \geq \alpha \gamma_k^\nu) \leq P\left(|\widehat{c}_k/c_k - 1|^2 \frac{\mathbb{1}\{\widehat{w}_k \geq \alpha\}}{\widehat{w}_k/w_k} \geq \frac{\alpha \gamma_k^\nu}{4\lambda_k}\right) + P\left(\frac{\mathbb{1}\{\widehat{w}_k \geq \alpha\}}{\widehat{w}_k/w_k} \geq \frac{\alpha \gamma_k^\nu}{4\lambda_k}\right)$$

which implies for all  $k \in I_2$  together with Markov's inequality that

$$P(\widehat{\lambda}_k \geq \alpha \gamma_k^\nu) \leq \frac{1}{4} \mathbb{E}\left[|\widehat{c}_k/c_k - 1|^4 \frac{\mathbb{1}\{\widehat{w}_k \geq \alpha\}}{|\widehat{w}_k/w_k|^2}\right] + P(\widehat{w}_k/w_k \leq 1/2).$$

Therefore, by using  $1 \leq 2^3\{|\widehat{w}_k/w_k - 1|^4 + |\widehat{w}_k/w_k|^2|\widehat{w}_k/w_k - 1|^2 + |\widehat{w}_k/w_k|^2\}$  we obtain

$$P(\widehat{\lambda}_k \geq \alpha \gamma_k^\nu) \leq 4\mathbb{E}\left[|\widehat{c}_k/c_k - 1|^4 \left\{ \frac{|\widehat{w}_k/w_k - 1|^4}{\alpha^2/w_k^2} + |\widehat{w}_k/w_k - 1|^2 + 1 \right\}\right] + P(\widehat{w}_k/w_k \leq 1/2).$$

Now (A.9) and (A.10) in Lemma A.1 and (A.13) gives (A.15), which completes the proof.  $\square$

**LEMMA A.3.** *Suppose  $(X, W) \in \mathcal{F}_{\eta, \tau}^{32}$  and  $U \in \mathcal{E}_\eta^{16}$ . Let  $I := \{k \in \mathbb{Z} : 8\lambda_k \leq \alpha \gamma^\nu\}$ , then for some universal constant  $C > 0$  we have*

$$\begin{aligned} \sup_{k \in \mathbb{Z}} \left\{ \lambda_k \cdot \mathbb{E}\left[ \frac{|T_{n,k}|^2 |\widehat{c}_k/c_k - 1|^2 \mathbb{1}\{\widehat{w}_k \geq \alpha\}}{\widehat{w}_k} \right] \right\} \\ \leq \frac{C\eta}{n} \cdot \{\|\beta\|^2 \cdot \mathbb{E}\|X\|^2 + \sigma^2\} \cdot \frac{\mathbb{E}\|X\|^2}{n} \cdot \left\{ \frac{\mathbb{E}\|W\|^2}{\alpha n} + 1 \right\}, \quad (\text{A.19}) \end{aligned}$$

$$\begin{aligned} \sup_{k \in \mathbb{Z}} \left\{ w_k^2 \cdot \mathbb{E}\left[ \frac{|T_{n,k}|^4 \mathbb{1}\{\widehat{w}_k \geq \alpha\}}{\widehat{w}_k^4} \right] \right\} \\ \leq \frac{C\eta}{n^2} \cdot \{\|\beta\|^2 \cdot \mathbb{E}\|X\|^2 + \sigma^2\}^2 \cdot \left\{ \frac{\mathbb{E}\|W\|^2}{\alpha n} + 1 \right\}^4, \quad (\text{A.20}) \end{aligned}$$

$$\begin{aligned} & \sup_{k \in \mathbb{Z}} \left\{ \lambda_k \cdot \mathbb{E} \left[ \frac{|T_{n,k}|^2}{\widehat{w}_k \cdot \widehat{\lambda}_k} \mathbb{1}\{\widehat{w}_k \geq \alpha\} \mathbb{1}\{\widehat{\lambda}_k \geq \alpha \gamma_k^\nu\} \right] \right\} \\ & \leq \frac{C \eta}{n} \cdot \{\|\beta\|^2 \cdot \mathbb{E}\|X\|^2 + \sigma^2\} \cdot \left\{ \frac{\mathbb{E}\|X\|^2}{\alpha n} + 1 \right\} \cdot \left\{ \frac{\mathbb{E}\|W\|^2}{\alpha n} + 1 \right\}, \quad (\text{A.21}) \end{aligned}$$

$$\begin{aligned} & \sup_{k \in I} \left\{ \frac{1}{x_k} \cdot \mathbb{E} \left[ \frac{|T_{n,k}|^2 |\widehat{c}_k|^2}{\widehat{w}_k^2} \mathbb{1}\{\widehat{w}_k \geq \alpha\} \mathbb{1}\{\widehat{\lambda}_k \geq \alpha \gamma_k^\nu\} \right] \right\} \\ & \leq \frac{C \eta}{n^2} \cdot \{\|\beta\|^2 \cdot \mathbb{E}\|X\|^2 + \sigma^2\} \cdot \left\{ \frac{\mathbb{E}\|W\|^2}{\alpha n} + 1 \right\}^3. \quad (\text{A.22}) \end{aligned}$$

**PROOF.** Consider the elementary inequality

$$1 \leq 2 \left\{ |\widehat{w}_k/w_k - 1|^2 + |\widehat{w}_k/w_k| |\widehat{w}_k/w_k - 1| + |\widehat{w}_k/w_k| \right\}. \quad (\text{A.23})$$

By applying the Cauchy-Schwarz inequality it follows that

$$\begin{aligned} & \mathbb{E} \left[ \frac{|T_{n,k}|^2 |\widehat{c}_k/c_k - 1|^2 \mathbb{1}\{\widehat{w}_k \geq \alpha\}}{\widehat{w}_k} \right] \\ & \leq 2 \left( \mathbb{E}|T_{n,k}|^4 \right)^{1/2} \left( \mathbb{E}|\widehat{c}_k/c_k - 1|^8 \right)^{1/4} \left\{ \frac{(\mathbb{E}|\widehat{w}_k/w_k - 1|^8)^{1/2}}{\alpha^2} + \frac{(\mathbb{E}|\widehat{w}_k/w_k - 1|^4)^{1/2}}{w_k^2} + \frac{1}{w_k^2} \right\}^{1/2} \end{aligned}$$

and, hence (A.8), (A.9) and (A.10) in Lemma A.1 imply (A.19).

The proof of (A.20) is similar to the proof of (A.19), but uses  $1 \leq 2^7 \{|\widehat{w}_k/w_k - 1|^8 + |\widehat{w}_k/w_k|^4 |\widehat{w}_k/w_k - 1|^4 + |\widehat{w}_k/w_k|^4\}$  rather than (A.23) and we omit the details.

Proof of (A.21). Due to the elementary inequality  $1 \leq 2|\widehat{c}_k/c_k - 1|^2 + 2|\widehat{c}_k/c_k|^2$  we have

$$\mathbb{E} \left[ \frac{|T_{n,k}|^2}{\widehat{w}_k \cdot \widehat{\lambda}_k} \mathbb{1}\{\widehat{w}_k \geq \alpha\} \mathbb{1}\{\widehat{\lambda}_k \geq \alpha \gamma_k^\nu\} \right] \leq 2 \mathbb{E} \left[ \frac{|T_{n,k}|^2 |\widehat{c}_k/c_k - 1|^2 \mathbb{1}\{\widehat{w}_k \geq \alpha\}}{\widehat{w}_k \alpha \gamma_k^\nu} \right] + 2 \frac{\mathbb{E}|T_{n,k}|^2}{c_k^2}.$$

Consequently, (A.19) and (A.8) in Lemma A.1 lead to (A.21).

Proof of (A.22). By using the Cauchy-Schwarz inequality we obtain the decomposition

$$\begin{aligned} & \mathbb{E} \left[ \left| \frac{T_{n,k} \widehat{c}_k}{\widehat{w}_k} \right|^2 \mathbb{1}\{\widehat{w}_k \geq \alpha\} \mathbb{1}\{\widehat{\lambda}_k \geq \alpha \gamma_k^\nu\} \right] \\ & \leq 2 \lambda_k \left( \mathbb{E} \frac{|T_{n,k}|^4 w_k^2 \mathbb{1}\{\widehat{w}_k \geq \alpha\}}{\widehat{w}_k^4} \right)^{1/2} \cdot \left\{ (\mathbb{E}|\widehat{c}_k/c_k - 1|^4)^{1/2} + |P(\widehat{\lambda}_k \geq \alpha \gamma_k^\nu)|^{1/2} \right\}. \end{aligned}$$

Now (A.10) in Lemma A.1, (A.15) in Lemma A.2 and (A.20) imply (A.22), which completes the proof.  $\square$

**LEMMA A.4.** Let  $(\lambda_k)_{k \in \mathbb{Z}}$  be an element of  $\mathcal{S}_{\kappa,d}$  defined in (3.3) with  $\lambda_+ := 1 \vee \max_{k \in \mathbb{Z}} \lambda_k$ . Consider  $k^* \in \mathbb{N}$  and  $\delta^* \in (0, 1]$  given in (3.5) for some  $\Delta \geq 1$ . If we define

$$b_k^2 := \frac{\zeta}{n \cdot \lambda_k}, \quad k \in \mathbb{Z}, \quad \text{with} \quad \zeta := \min \{2\sigma^2, \rho/(d \Delta)\}, \quad (\text{A.24})$$

then we have

$$\frac{n}{2\sigma^2} b_k^2 \lambda_k \leq 1, \quad k \in \mathbb{Z}, \quad (\text{A.25})$$

$$\sum_{|k| \leq k^*} b_k^2 \gamma_k^p \leq \rho, \quad (\text{A.26})$$

$$\sum_{|k| \leq k^*} b_k^2 \gamma_k^\nu \geq \min \left\{ \frac{2\sigma^2}{d\Delta}, \frac{\rho}{(d\Delta)^2} \right\} \cdot \frac{\kappa(\delta^*)}{\lambda_+}. \quad (\text{A.27})$$

**PROOF.** The inequality (A.25) follows trivially by using the definition of  $\zeta$ .

Proof of (A.26). If  $\varphi$  denotes the inverse function of  $\kappa$ , then the link condition  $(\lambda_k) \in \mathcal{S}_{\kappa, d}$ , can be rewritten as

$$d^{-1} \leq |\varphi(\gamma_k^{\nu-p})|^{-1} \frac{\lambda_k}{\gamma_k^\nu \lambda_+} \leq d. \quad (\text{A.28})$$

Thereby, the monotonicity of  $(\gamma_k^\nu)$  together with  $\lambda_+ \geq 1$  implies

$$\sum_{|k| \leq k^*} b_k^2 \gamma_k^p \leq \frac{\zeta}{n} \cdot \sum_{|k| \leq k^*} \frac{\gamma_k^\nu \lambda_+}{\lambda_k} \gamma_k^{p-\nu} \leq \zeta d \cdot \sum_{|k| \leq k^*} \frac{\gamma_{k^*}^{p-\nu}}{n\varphi(\gamma_{k^*}^{\nu-p})}.$$

Thus (A.26) follows from the definition of  $k^*$  given in (3.5), i.e.,  $\sum_{|k| \leq k^*} b_k^2 \gamma_k^p \leq \zeta d \Delta \leq \rho$ .

Proof of (A.27). By using the condition (A.28) together with the definition of  $\delta^*$  we have

$$\sum_{|k| \leq k^*} b_k^2 \gamma_k^\nu \geq \frac{\zeta}{d\lambda_+} \cdot \kappa(\delta^*) \cdot \sum_{|k| \leq k^*} \frac{\gamma_{k^*}^{p-\nu}}{n\varphi(\gamma_{k^*}^{\nu-p})}.$$

Consequently, the definition of  $k^*$  given in (3.5) implies (A.27), which proves the lemma.  $\square$

## A.2 Proofs of Section 4

### The finitely smoothing case.

**PROOF OF PROPOSITION 4.1.** Consider the inverse function  $\varphi(t) = t^{(a+\nu)/(p-\nu)}$  of  $\kappa$ . Then the well-known approximation  $\sum_{k=1}^m k^r \asymp m^{r+1}$  for  $r > 0$  together with the definition of  $\gamma_k$  given in (2.11) implies  $\sum_{|k| \leq k^*} 1/\varphi(\gamma_k^{\nu-p}) \asymp \gamma_{k^*}^{(a+\nu)+1/2}$ . It follows that the condition on  $k^*$  given in (3.5) of Theorem 3.2 can be rewritten as

$$1/n \asymp \gamma_{k^*}^{\nu-p} \left| \sum_{|k| \leq k^*} 1/\varphi(\gamma_k^{\nu-p}) \right|^{-1} \asymp \gamma_{k^*}^{p+a+1/2} = |\varphi(\gamma_{k^*}^{\nu-p})|^{[2(p+a)+1]/[2(a+\nu)]}.$$

From this  $\delta^* := \varphi(\gamma_{k^*}^{\nu-p})$  implies  $\delta^* \asymp n^{-2(a+\nu)/[2(p+a)+1]}$  and  $\kappa(\delta^*) \asymp n^{-2(p-\nu)/[2(p+a)+1]}$ . Consequently, the lower bound in Proposition 4.1 follows by applying Theorem 3.2.  $\square$

**PROOF OF PROPOSITION 4.2.** Since the condition on  $\alpha$  ensures  $\alpha \asymp n^{-2(a+\nu)/[2(p+a)+1]} \asymp \delta^*$  (see the proof of Proposition 4.1) the result follows from Theorem 3.3.  $\square$

**The infinitely smoothing case.**

**PROOF OF PROPOSITION 4.3.** Consider the inverse function  $\varphi(t) = \exp\{-t^{a/(\nu-p)}\}$  of  $\kappa$ . By applying Laplace's Method (c.f. chapter 3.7 in Olver [1974]) the definition of  $\gamma_k$  given in (2.11) implies  $\sum_{|k| \leq k^*} 1/\varphi(\gamma_k^{\nu-p}) \asymp 1/\varphi(\gamma_{k^*}^{\nu-p})$ . It follows that by using  $\delta^* := \varphi(\gamma_{k^*}^{\nu-p})$  the condition on  $k^*$  given in (3.5) of Theorem 3.2 can be rewritten as

$$1/n \asymp \gamma_{k^*}^{\nu-p} \left| \sum_{|k| \leq k^*} 1/\varphi(\gamma_k^{\nu-p}) \right|^{-1} \asymp \gamma_{k^*}^{\nu-p} \varphi(\gamma_{k^*}^{\nu-p}) = \delta^* \kappa(\delta^*),$$

which implies  $\kappa(\delta^*) \asymp \omega(1/n)$ , where  $\omega$  denotes the inverse function of  $\omega^{-1}(t) = t \cdot \varphi(t)$ . Therefore, the lower bound given in Proposition 4.3 follows from Theorem 3.2 together with  $\omega(t) = |\log t|^{-(p-\nu)/a}(1 + o(1))$  as  $t \rightarrow 0$  (c.f. Mair [1994]), which proves the result  $\square$

**PROOF OF PROPOSITION 4.4.** The proof is based on the decomposition (3.2), where we bound the two right hand side terms by (A.2) derived in the proof of Proposition 3.1 and (A.7) shown in the proof of Theorem 3.3 respectively. It follows that,

$$\mathbb{E} \|\widehat{\beta}_\nu - \beta\|_\nu^2 \leq \frac{C\eta}{\alpha^2 n} \cdot \Lambda \cdot \{\sigma^2 + \rho\Lambda\} + \frac{C\eta}{\alpha n} \cdot \Lambda \cdot \rho \cdot \left(1 + \frac{\Lambda}{\alpha n}\right) + C \cdot \kappa(d4\tau\alpha) \cdot \rho$$

for some positive constant  $C$ . Consequently, the condition  $\alpha = c \cdot n^{-1/4}$ ,  $c > 0$  implies  $\mathbb{E} \|\widehat{\beta}_\nu - \beta\|_\nu^2 = O(n^{-1/2}) + O(n^{-3/4}) + O(|\log n^{-1/4}|^{-(p-\nu)/a}) = O((\log n)^{-(p-\nu)/a})$ , which completes the proof.  $\square$

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